Journal of Integer Sequences, Vol. 14 (2011),
Article 11.8.7

The p-adic Valuation of the ASM Numbers

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Abstract

A classical formula of Legendre gives the p-adic valuation for factorials as a finite sum of values of the floor function. This expression can be used to produce a formula for the p-adic valuation of n as a finite sum of periodic functions. An analogous result is established for the p-adic valuation of the ASM numbers. This sequence counts the number of alternating sign matrices.

1 Introduction

Let $n \in \mathbb{N}$ and p be a prime. The highest power of p that divides n, called the *p-adic valuation of* n, is denoted by $\nu_p(n)$. The elementary formula

$$\nu_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor \tag{1}$$

is a classical result which appears in most texts in number theory. Observe that, for each fixed value of n, there are only finitely many non-zero terms in (1). An alternative form was given by Legendre [3] in the form

$$\nu_p(n!) = \frac{n - S_p(n)}{p-1}, \tag{2}$$

where $S_p(n)$ denotes the sum of the digits of n in base p.

The formula (1) can be used to express the p-adic valuation of n as

$$\nu_p(n) = \sum_{j=1}^{\infty} \left(\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{n-1}{p^j} \right\rfloor \right). \tag{3}$$

Each summand in (3) is a periodic function of period p^j.

The goal of this paper is to describe the p-adic valuations of a sequence that count a famous class of matrices. The special cases $p = 2$ and 3 have been described in [4]. This paper extends, to arbitrary primes p, the interesting patterns found in [4].

An *alternating sign matrix* is an array of 0, 1 and -1, such that the entries of each row and column add up to 1 and the non-zero entries of a given row/column alternate. After a fascinating sequence of events, D. Zeilberger [5] proved that the numbers of such matrices is given by

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{4}$$

In particular, the numbers A_n are integers: not an obvious fact. They are called the ASM numbers and form sequence A005130 in Sloane's *Encylopedia*.

The story behind this formula and its many combinatorial interpretations are given in D. Bressoud's book [1].

The main result presented here is a formula for the p-adic valuation of A_n similar to (3).

Theorem 1. *Let $n \in \mathbb{N}$ and $p \geq 5$ be a prime. Define*

$$Per_{j,p}(n) = \begin{cases} 0, & \text{if} \quad 0 \leq n \leq \left\lfloor \frac{p^j+1}{3} \right\rfloor; \\ n - \left\lfloor \frac{p^j+1}{3} \right\rfloor, & \text{if} \quad \left\lfloor \frac{p^j+1}{3} \right\rfloor + 1 \leq n \leq \frac{p^j-1}{2}; \\ \left\lfloor \frac{2p^j+1}{3} \right\rfloor - n, & \text{if} \quad \frac{p^j+1}{2} \leq n \leq \left\lfloor \frac{2p^j+1}{3} \right\rfloor; \\ 0, & \text{if} \quad \left\lfloor \frac{2p^j+1}{3} \right\rfloor + 1 \leq n \leq p^j - 1. \end{cases} \tag{5}$$

Then

$$\nu_p(A_n) = \sum_{j=1}^{\infty} Per_{j,p}\left(n \mod p^j\right). \tag{6}$$

Observations.

1) For a fixed prime p, define $r = r(n,p)$ by the inequalities $p^r \leq n < p^{r+1}$. Then $n = \lfloor \log n / \log p \rfloor$. The number n admits a representation $n = up^r + v$, with $1 \leq u \leq p-1$ and $0 \leq v \leq p^r - 1$. The index u is given by $u = \lfloor n/p^r \rfloor$.

2) The j-th term in the series (6) is a periodic function of period p^j.

3) For fixed $n \in \mathbb{N}$, the series (6) reduces to a finite sum. Indeed, with r as above,

$$\left\lfloor \frac{p^{r+2}+1}{3} \right\rfloor \geq \frac{p^{r+2}+1}{3} - 1 \geq p^{r+1} > n \tag{7}$$

so the sum ends after $j = r+1$.

4) The form of the series (6) was found empirically. It would be desirable to develop a method that gives a series of this type for a large class of sequences. The goal is to produce an expansion of the form

$$\nu_p(a_n) = \sum_{j=1}^{\infty} \alpha_{j,n} \phi_{j,p}(n) \tag{8}$$

where $\phi_{j,p}$ is a function of period p^j. A procedure to determine the coefficients $\alpha_{j,n}$ directly from the sequence $\{a_n\}$ and the functions $\{\phi_{j,p}\}$ should be developed. Moreover, it is required that, for fixed $n \in \mathbb{N}$, the series (8) contains only finitely non-vanishing terms.

5) The authors of [2] present an analytic formula for $\nu_p(A_n)$ from which they obtain the asymptotic behavior of this function. A Fourier-series based approach is presented which reveals the dominant terms and also periodic fluctuations. A study of the expressions discussed in this paper and the results in [2] will be reported later.

The proof of the theorem is based on the observation that $\nu_p(A_1) = 0$ and $\mathrm{Per}_{j,p}(1) = 0$, showing that both sides of (6) agree at $n = 1$ coupled with recurrences satisfied by $\nu_p(A_n)$ and $\mathrm{Per}_{j,p}(n)$. These are given by

$$\nu_p(A_n) - \nu_p(A_{n-1}) = \frac{1}{p-1} \left(S_p(2n-2) + S_p(2n-1) - S_p(3n-2) - S_p(n-1) \right) \tag{9}$$

and

$$\mathrm{Per}_{j,p}(n) - \mathrm{Per}_{j,p}(n-1) = \begin{cases} 0, & \text{if} \quad 0 \leq n \leq \left\lfloor \frac{p^j+1}{3} \right\rfloor; \\ 1, & \text{if} \quad \left\lfloor \frac{p^j+1}{3} \right\rfloor + 1 \leq n \leq \frac{p^j-1}{2}; \\ 0, & \text{if} \quad n = \frac{p^j+1}{2}; \\ -1, & \text{if} \quad \frac{p^j+3}{2} \leq n \leq \left\lfloor \frac{2p^j+1}{3} \right\rfloor; \\ 0, & \text{if} \quad \left\lfloor \frac{2p^j+1}{3} \right\rfloor + 1 \leq n \leq p^j - 1. \end{cases} \tag{10}$$

The proof shows that the right-hand side of (9) matches that of (10).

The study of the arithmetic aspects of the sequence A_n was initiated in [4], where the case of $\nu_2(A_n)$ was considered. It is shown that $\nu_2(A_n)$ vanishes precisely when n is a *Jacobstahl number* J_m. These numbers are defined by the recurrence $J_m = J_{m-1} + 2J_{m-2}$ with initial conditions $J_0 = 1$ and $J_1 = 1$. The main result is the existence of a well-defined algorithm to produce the graph of $\nu_2(A_n)$ on the interval $[J_m, J_{m+1}]$ from its value on the two previous intervals $[J_{m-2}, J_{m-1}] \cup [J_{m-1}, J_m]$. In the situation considered here, the partial sums of the series (6) give approximations to $\nu_p(A_n)$.

Section 2 describes data that motivated the main theorem. Section 3 establishes the recurrence for the valuation of A_n and Section 4 the corresponding recurrence for the function $\mathrm{Per}_{j,p}$. Section 5 presents the proof of the main result.

2 An experimental illustration of the main theorem

In this section the procedure employed to find the main result is described in the case of the valuation $\nu_p(A_n)$ for $p = 5$. The first 100 values of $\nu_5(A(n))$ are given below (each row is of length 10)

0	0	0	0	0	0	0	0	1	2
3	4	4	3	2	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	1	2	3	4	4	3	2
1	0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	18	20
22	24	24	22	20	18	16	15	14	13
12	11	10	9	8	7	6	5	4	3
2	1	0	1	2	3	4	4	3	2
1	0	0	0	0	0	0	0	0	0

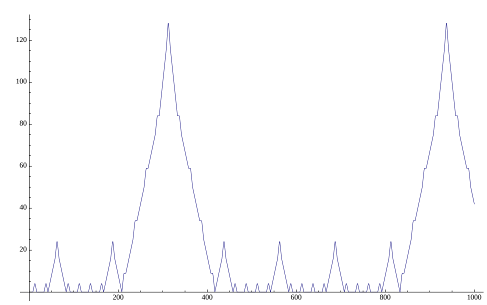

Figure 1: The 5-adic valuation of A_n

The observation employed to find the main theorem is based on the string

$$0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 2\ 3\ 4\ 4\ 3\ 2\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \tag{1}$$

and its central role in the function $\nu_2(n)$. An analytic expression for the string is given by

$$\mathrm{Per}_{2,5}(n) := \begin{cases} 0, & \text{if } 0 \leq n \leq 8; \\ n-8, & \text{if } 9 \leq n \leq 12; \\ 17-n, & \text{if } 13 \leq n \leq 16; \\ 0, & \text{if } 17 \leq n \leq 24; \end{cases} \tag{2}$$

and this is now extended to a periodic function of period 5^2 by $\text{Per}_{2,5}(n \bmod 5^2)$.

The function

$$\nu_5(A_n) - \text{Per}_{2,5}(n \bmod 5^2) \tag{3}$$

has values given by

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17	18
19	20	20	19	18	17	16	15	14	13
12	11	10	9	8	7	6	5	4	3
2	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

This data suggests the function

$$\text{Per}_{3,5}(n) := \begin{cases} 0, & \text{if } 0 \leq n \leq 42; \\ n - 42, & \text{if } 43 \leq n \leq 62; \\ 83 - n, & \text{if } 63 \leq n \leq 82; \\ 0, & \text{if } 83 \leq n \leq 124; \end{cases} \tag{4}$$

and extend $\text{Per}_{3,5}$ to a periodic function of period 5^3. This empirical procedure leads to the functions $\text{Per}_{j,p}(n)$ defined Theorem 1.

3 An analytic formula

For a prime p, introduce the notation

$$f_p(j) := \nu_p(j!). \tag{1}$$

Lemma 2. *Let p be a prime. Then the p-adic valuation of A_n satisfies*

$$\nu_p(A_n) = \nu_p(A_{n-1}) + f_p(3n-2) + f_p(n-1) - f_p(2n-2) - f_p(2n-1). \tag{2}$$

Proof. This follows directly by combining the initial value $A_1 = 1$ with the expression

$$\nu_p(A_n) = \sum_{j=0}^{n-1} f_p(3j+1) - \sum_{j=0}^{n-1} f_p(n+j) \tag{3}$$

and the corresponding one for $\nu_p(A_{n-1})$. $\square$

Legendre's formula (2) gives the result of Theorem 2 in terms of the function S_p.

Corollary 3. *The p-adic valuation of A_n is given by*

$$\nu_p(A_n) = \frac{1}{p-1}\left(\sum_{j=0}^{n-1} S_p(n+j) - \sum_{j=0}^{n-1} S_p(3j+1)\right). \tag{4}$$

Summing the recurrence (2) and using $A_1 = 1$ we obtain an alternative expression for the p-adic valuation of A_n.

Proposition 4. *The p-adic valuation of A_n is given by*

$$\nu_p(A_n) = \frac{1}{p-1}\sum_{j=1}^{n-1}\left(S_p(2j) + S_p(2j+1) - S_p(3j+1) - S_p(j)\right). \tag{5}$$

This gives a recurrence for the p-adic valuation of A_n.

Theorem 5. *The p-adic valuation of A_n satisfies*

$$\nu_p(A_n) - \nu_p(A_{n-1}) = \frac{1}{p-1}\left(S_p(2n-2) + S_p(2n-1) - S_p(3n-2) - S_p(n-1)\right).$$

4 The recurrence for $\mathrm{Per}_{j,p}(n)$

The explicit formulas for $\mathrm{Per}_{j,p}(n)$ can be used to give a proof of (10). The only cases that require special attention are when n and $n-1$ are on different intervals of the definition. For instance, if $n = \left\lfloor \frac{p^j+1}{3} \right\rfloor + 1$, then $\mathrm{Per}_{j,p}(n) = n - \left\lfloor \frac{p^j+1}{3} \right\rfloor = 1$ and $\mathrm{Per}_{j,p}(n-1) = 0$. The verification of all the cases is elementary.

5 The proof of the main theorem

Introduce the notation

$$L_1(n,p) = S_p(2n-2) + S_p(2n-1) - S_p(3n-2) - S_p(n-1), \tag{1}$$

with the convention that $S_p(x) = 0$ if $x < 0$ and

$$L_2(n,p) = \sum_{j=1}^{\infty} g_j(n,p) \tag{2}$$

where

$$g_j(n,p) = \begin{cases} 0, & \text{if} \quad 0 \le n \bmod p^j \le \left\lfloor \frac{p^j+1}{3} \right\rfloor; \\ p-1, & \text{if} \quad \left\lfloor \frac{p^j+1}{3} \right\rfloor + 1 \le n \bmod p^j \le \frac{p^j-1}{2}; \\ 0, & \text{if} \quad n \bmod p^j = \frac{p^j+1}{2}; \\ -(p-1) & \text{if} \quad \frac{p^j+3}{2} \le n \bmod p^j \le \left\lfloor \frac{2p^j+1}{3} \right\rfloor; \\ 0, & \text{if} \quad \left\lfloor \frac{2p^j+1}{3} \right\rfloor + 1 \le n \bmod p^j \le p^j - 1. \end{cases} \tag{3}$$

The statement of the main theorem is the identity

$$L_1(n,p) = L_2(n,p) \text{ for } n \in \mathbb{N}. \tag{4}$$

The proof is achieved by induction on the number of digits in the expansion of n in base p. Write $n = up^r + v$, with $1 \leq u \leq p-1$ and $0 \leq v \leq p^r - 1$. The base case shows that $L_1(n,p) = L_2(n,p)$ for $1 \leq n \leq p-1$ and the inductive step is based on the identities $L_1(n,p) = L_1(v,p) + E(n,p)$ and $L_2(n,p) = L_2(n,p) + E(n,p)$, with the *same* function E in both cases. This completes the proof.

Observe that, for fixed $n \in \mathbb{N}$, the series in (2) is actually a finite sum. Terms with index $j \geq 2 + \lfloor \log n / \log p \rfloor$ vanish. Thus, if $p^r \leq n < p^{r+1}$,

$$L_2(n,p) = \sum_{j=1}^{r+1} g_j(n,p). \tag{5}$$

The proof of (4) is by induction on the number of digits of n in base p. The basic case is considered first: it deals with $n \in \mathbb{N}$ that have a single digit; that is, $1 \leq n \leq p-1$.

- **Base case**. Assume that $1 \leq n \leq p-1$.

The bound $p \geq 5$ implies that for $j \geq 2$

$$1 \leq n \leq p-1 < \frac{p^2-2}{3} \leq \left\lfloor \frac{p^j+1}{3} \right\rfloor. \tag{6}$$

It follows that the sum in (2) contains a single term. It is required to show that

$$L_1(n,p) = \begin{cases} 0, & \text{if} \quad 0 \leq n \leq \left\lfloor \frac{p+1}{3} \right\rfloor; \\ p-1, & \text{if} \quad \left\lfloor \frac{p+1}{3} \right\rfloor + 1 \leq n \leq \frac{p-1}{2}; \\ 0, & \text{if} \quad n = \frac{p+1}{2}; \\ 1-p, & \text{if} \quad \frac{p+3}{2} \leq n \leq \left\lfloor \frac{2p+1}{3} \right\rfloor; \\ 0, & \text{if} \quad \left\lfloor \frac{2p+1}{3} \right\rfloor + 1 \leq n \leq p-1. \end{cases} \tag{7}$$

This identity is verified by considering the position of n in $[0, p-1]$.

Case 1.1. Observe that $3n-2 < p$ is equivalent to $n \leq \left\lfloor \frac{p+1}{3} \right\rfloor$. Under this condition the terms $2n-2$, $2n-1$, $3n-2$ and $n-1$ have a single digit in base p. Therefore

$$L_1(n,p) = (2n-2) + (2n-1) - (3n-2) - (n-1) = 0. \tag{8}$$

Case 1.2. Assume $\left\lfloor \frac{p+1}{3} \right\rfloor < n \leq \frac{p-1}{2}$. Then $2n-2 < 2n-1 \leq p-2$ and $p \leq 3n-2 < 2p$. Therefore the numbers $2n-2$, $2n-1$ and $n-1$ have a single digit in base p. The representation of $3n-2$ is $3n-2 = 1 \cdot p + (3n-2-p)$. It follows that $S_p(2n-2) = 2n-2$, $S_p(2n-1) = 2n-1$, $S_p(n-1) = n-1$, and $S_p(3n-2) = 3n-p-1$. The identity (7) follows from these values.

Case 1.3. If $n = \frac{p+1}{2}$, then the terms involved in (7) are

$$S_p(2n-2) = p-1,\ S_p(2n-1) = 1,\ S_p(3n-2) = 1 + (3n-2-p) \text{ and } S_p(n-1) = n-1.$$

The identity (7) follows from these values.

The remaining cases can be obtained by similar arguments. The proof of (7) is now complete establishing the base case of the main theorem.

• **Inductive step**. For fixed $n \in \mathbb{N}$, recall that $r \in \mathbb{N}$ is defined by the inequalities $p^r \leq n < p^{r+1}$; that is,

$$r = \left\lfloor \frac{\log n}{\log p} \right\rfloor. \tag{9}$$

Write $n = up^r + v$, with $1 \leq u < p$ and $0 \leq v \leq p^r - 1$. The main step of the proof is to produce a reduction formula that relates $L_1(n,p)$ to $L_1(v,p)$.

The next lemma illustrates one case in complete detail. The common assumption is that $n \in \mathbb{N}$ satisfies $p^r \leq n < p^{r+1}$. By convention, $S_p(n) = 0$ if $n < 0$.

Lemma 6. *For $n \in \mathbb{N}$,*

$$S_p(2n-2) = S_p(2v-2) + T_1(n,p) \tag{10}$$

where $T_1(n,p)$ is given in the table below.

v	u	$T_1(n,p)$
$v = 0$	$1 \leq u \leq \frac{p-1}{2}$	$2u - 2 + r(p-1)$
$v = 0$	$\frac{p+1}{2} \leq u \leq p-1$	$2u - p - 1 + r(p-1)$
$1 \leq v \leq \frac{p^r+1}{2}$	$1 \leq u \leq \frac{p-1}{2}$	$2u$
$1 \leq v \leq \frac{p^r+1}{2}$	$\frac{p+1}{2} \leq u \leq p-1$	$2u - p + 1$
$\frac{p^r+3}{2} \leq v \leq p^r - 1$	$1 \leq u \leq \frac{p-3}{2}$	$2u$
$\frac{p^r+3}{2} \leq v \leq p^r - 1$	$\frac{p-1}{2} \leq u \leq p-1$	$2u - p + 1$

The values of $T_1(n,p)$.

Proof. Start with $2n-2 = 2up^r + 2v - 2$. The discussion of $S_p(2n-2)$ is divided into cases according to $2v-2$.

Case 1: $v = 0$. Then $2n-2 = 2up^r - 2 = (2u-1)p^r + (p^r - 2)$. The term $p^r - 2$ does not contribute to the power p^r and the bounds $1 \leq 2u-1 \leq 2p-3$ yield two separate cases:

SubCase 1.1: $1 \leq 2u-1 \leq p-1$. In this case $S_p(2u-1) = 2u-1$ and it follows that

$$S_p(2n-2) = 2u - 1 + S_p(p^r - 2). \tag{11}$$

The identity

$$p^r - 2 = (p-2) + (p-1)p + (p-1)p^2 + \cdots (p-1)p^{r-1} \tag{12}$$

produces $S_p(p^r-2) = r(p-1) - 1$. Therefore

$$S_p(2n-2) = 2u - 2 + r(p-1). \tag{13}$$

SubCase 1.2: $p \leq 2u-1 \leq 2p-3$. The expression

$$\begin{aligned} 2n-2 &= (2u-1)p^r + (p^r-2) \\ &= p^{r+1} + (2u-1-p)p^r + (p^r-2) \end{aligned}$$

gives

$$\begin{aligned} S_p(2n-2) &= 1 + (2u-1-p) + S_p(p^r-2) \\ &= (2u-p-1) + r(p-1). \end{aligned}$$

This completes the case $v=0$.

Case 2. This considers the situation where $0 < 2v-2 < p^r$. Then the representation of $2v-2$ in base p does not produce carries to the position of p^r. The discussion of

$$2n-2 = 2up^r + 2v - 2 \tag{14}$$

is divided, as before, into two subcases according to the value of $2u$.

SubCase 2.1.: $1 \leq 2u \leq p-1$. Then (14) gives

$$S_p(2n-2) = 2u + S_p(2v-2). \tag{15}$$

SubCase 2.2: $p \leq 2u \leq 2p-1$. Equation (14) is now written as

$$2n-2 = p^{r+1} + (2u-p)p^r + (2v-2). \tag{16}$$

It follows from here that $S_p(2n-2) = 1 + (2u-p) + S_p(2v-2)$.

Case 3. The last possibility is $p^r \leq 2v-2 < 2p^r$. The expression

$$2n-2 = (2u+1)p^r + (2v-2-p^r) \tag{17}$$

leads to two subcases:

SubCase 3.1: $2u+1 \leq p-1$. Then $2u+1 < p$ and $S_p(2n-2) = 2u+1+S_p(2v-2-p^r)$. Now, from $2v-2 = p^r + (2v-2-p^r)$, it follows that $S_p(2v-2-p^r) = S_p(2v-2) - 1$. Therefore $S_p(2n-2) = 2u + S_p(2v-2)$.

SubCase 3.2: $p \leq 2u+1 \leq 2p-1$. Proceeding as before gives $S_p(2n-2) = 2u-p+1+S_p(2v-2)$. All the cases have now been considered and the proof is complete. $\square$

The other terms appearing in the expression for L_1 have similar reductions. These are stated next. The proofs are ommitted since they are similar to the one presented above.

Lemma 7. *Let $n \in \mathbb{N}$. Then $S_p(2n-1) = S_p(2v-1) + T_2(n,p)$ where $T_2(n,p)$ is given in the table below.*

v	u	$T_2(n,p)$
$v = 0$	$1 \le u \le \frac{p-1}{2}$	$2u - 1 + r(p-1)$
$v = 0$	$\frac{p+1}{2} \le u \le p-1$	$2u - p + r(p-1)$
$1 \le v \le \frac{p^r-1}{2}$	$1 \le u \le \frac{p-1}{2}$	$2u$
$1 \le v \le \frac{p^r-1}{2}$	$\frac{p+1}{2} \le u \le p-1$	$2u - p + 1$
$\frac{p^r+1}{2} \le v \le p^r - 1$	$1 \le u \le \frac{p-3}{2}$	$2u$
$\frac{p^r+1}{2} \le v \le p^r - 1$	$\frac{p-1}{2} \le u \le p-1$	$2u - p + 1$

The values of $T_2(n,p)$.

Lemma 8. *Let $n \in \mathbb{N}$. Then $S_p(n-1) = S_p(v-1) + T_3(n,p)$ where $T_3(n,p)$ is given below.*

v	$T_3(n,p)$
$v = 0$	$u - 1 + r(p-1)$
$1 \le v \le p^r - 1$	u

The values of $T_3(n,p)$.

Lemma 9. *Let $n \in \mathbb{N}$. Then $S_p(3n-2) = S_p(3v-2) + T_4(n,p)$ where $T_4(n,p)$ is given in the table below.*

v	u	$T_4(n,p)$
$v=0$	$2 \leq 3u-1 \leq p-1$	$3u-2+r(p-1)$
$v=0$	$p \leq 3u-1 \leq 2p-1$	$3u-p-1+r(p-1)$
$v=0$	$2p \leq 3u-1 \leq 3p-4$	$3u-2p+r(p-1)$
$1 \leq 3v-2 \leq p^r-1$	$1 \leq 3u \leq p-1$	$3u$
$1 \leq 3v-2 \leq p^r-1$	$p \leq 3u \leq 2p-1$	$3u-p+1$
$1 \leq 3v-2 \leq p^r-1$	$2p \leq 3u \leq 3p-3$	$3u-2p+2$
$p^r \leq 3v-2 \leq 2p^r-1$	$1 \leq 3u+1 \leq p-1$	$3u$
$p^r \leq 3v-2 \leq 2p^r-1$	$p \leq 3u+1 \leq 2p-1$	$3u-p+1$
$p^r \leq 3v-2 \leq 2p^r-1$	$2p \leq 3u+1 \leq 3p-2$	$3u-2p+2$
$2p^r \leq 3v-2 \leq 3p^r-1$	$1 \leq 3u+2 \leq p-1$	$3u$
$2p^r \leq 3v-2 \leq 3p^r-1$	$p \leq 3u+2 \leq 2p-1$	$3u-p+1$
$2p^r \leq 3v-2 \leq 3p^r-1$	$2p \leq 3u+2 \leq 3p-1$	$3u-2p+1$

The values of $T_4(n,p)$.

Corollary 10. *The information given above, shows that*

$$L_1(n,p) = L_1(v,p) + \left[T_1(n,p) + T_2(n,p) - T_3(n,p) - T_4(n,p)\right]. \tag{18}$$

The next step is to obtain a relation between $L_2(n,p)$ and $L_2(v,p)$.

Recall that r is defined by $p^r \leq n < p^{r+1}$. It follows that the inequality $p^{r+1} < \lfloor (p^j+1)/3 \rfloor$ holds for $j \geq r+2$, yielding

$$n \bmod p^j = n < p^{r+1} < \lfloor (p^j+1)/3 \rfloor. \tag{19}$$

Thus the corresponding term in (2) vanishes. For indices $1 \leq j \leq r$, the relation $n = up^r + v$, gives $n \bmod p^j = v \bmod p^j$. Therefore $L_2(n,p)$ and $L_2(v,p)$ differ only in the last term. Morever, the bound $n < p^{r+1}$ implies that $n \bmod p^{r+1}$ is simply n. These observations are recorded in the next proposition.

Proposition 11. *Let $n \in \mathbb{N}$. Recall the definition $r = r(n,p) := \lfloor \log n / \log p \rfloor$, so that $p^r \leq n < p^{r+1}$. Then, the identity*

$$L_2(n,p) = L_2(v,p) + g_{r+1}(n,p) \tag{20}$$

holds, with

$$g_{r+1}(n,p) = \begin{cases} 0, & \textit{if} \quad 0 \leq n \leq \left\lfloor \frac{p^{r+1}+1}{3} \right\rfloor; \\ p-1, & \textit{if} \quad \left\lfloor \frac{p^{r+1}+1}{3} \right\rfloor + 1 \leq n \leq \frac{p^{r+1}-1}{2}; \\ 0, & \textit{if} \quad n = \frac{p^{r+1}+1}{2}; \\ -(p-1) & \textit{if} \quad \frac{p^{r+1}+3}{2} \leq n \leq \left\lfloor \frac{2p^{r+1}+1}{3} \right\rfloor; \\ 0, & \textit{if} \quad \left\lfloor \frac{2p^{r+1}+1}{3} \right\rfloor + 1 \leq n \leq p^{r+1}-1. \end{cases} \tag{21}$$

The next result completes the proof of the main theorem.

Theorem 12. *With the notations established above*

$$T_1(n,p) + T_2(n,p) - T_3(n,p) - T_4(n,p) = g_{r+1}(n,p). \tag{22}$$

Proof. The proof is presented in detail for the case $\lfloor (p^{r+1}+1)/3 \rfloor + 1 \leq n \leq (p^{r+1}-1)/2$. The other cases are similar.

From the representation $n = up^r + v$ and the bounds considered above, it follows that

$$0 \leq v \leq p^r - 1 \text{ and } p \leq 3u + \text{Spill}(3v-2) \leq \tfrac{3}{2}(p-1). \tag{23}$$

Assume that $v > 0$. The case $v = 0$ can be treated by similar methods. The spill of $3v-2$ is defined as the contribution of $3v-2$ to the power p^r in its expansion on base p. The bounds $-2 \leq 3v - 2 < 3p^r$ shows that the spill is between 0 and 2. The details are given for the case where the spill is 0. The analysis for the case of spill 1 and 2 is analogous.

If the spill is 0, then $0 \leq 3v-2 \leq p^r - 1$ and $p \leq 3u \leq 2p-1$. The table of values for T_4 gives $T_4(n,p) = 3u - p + 1$. The bound $n \leq \frac{1}{2}(p^{r+1}-1) = \frac{p-1}{2}p^r + \frac{1}{2}(p^r-1)$ imply that $u \leq \frac{p-1}{2}$. Since the spill of $3v-2$ is 0 it follows that $3v \leq p^r + 1$. Now, $p \geq 5$, therefore $v \leq \frac{1}{3}(p^r+1) \leq \frac{1}{2}(p^r-1)$. This gives $T_1(n,p) = 2u$, $T_2(n,p) = 2u$ and $T_3(n,p) = u$. The result is

$$T_1(n,p) + T_2(n,p) - T_3(n,p) - T_4(n,p) = 2u + 2u - [u + 3u - p + 1] = p - 1.$$

This is the value of $g_{r+1}(n,p)$. The remaining cases follow the same pattern. □

The main theorem has now been established.

6 Acknowledgements

The second author wishes to thank the partial support of NSF-DMS 0713836. The work of the first author was also funded, as a graduate student, by the same grant.

References

[1] D. Bressoud. *Proofs and Confirmations: the Story of the Alternating Sign Matrix Conjecture*. Cambridge University Press, 1999.

[2] C. Heuberger and H. Prodinger. A precise description of the p-adic valuation of the number of alternating sign matrices. *Int. J. Number Theory*, **7** (2011), 57–69.

[3] A. M. Legendre. *Théorie des Nombres*. Firmin Didot Frères, Paris, 1830.

[4] X. Sun and V. Moll. The p-adic valuation of sequences counting alternating sign matrices. *J. Integer Seq.*, **12** (2009), Article 09.3.8.

[5] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Combin.*, **3** (1996), 1–78.

2010 *Mathematics Subject Classification*: Primary 05A10; Secondary 11B75, 11Y55
Keywords: Alternating sign matrices, valuations, recurrences, digit count.

(Concerned with sequence A005130.)

Received February 3 2011; revised version received July 18 2011; September 8 2011. Published in *Journal of Integer Sequences*, October 16 2011.

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