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On Arithmetic Progressions of Integers with a Distinct Sum of Digits

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Abstract

Let $b \geq 2$ be a fixed integer. Let $s_b(n)$ denote the sum of digits of the nonnegative integer n in the base-b representation. Further let q be a positive integer. In this paper we study the length k of arithmetic progressions $n, n+q, \ldots, n+q(k-1)$ such that $s_b(n), s_b(n+q), \ldots, s_b(n+q(k-1))$ are (pairwise) distinct. More specifically, let $L_{b,q}$ denote the supremum of k as n varies in the set of nonnegative integers $\mathbb{N}$. We show that $L_{b,q}$ is bounded from above and hence finite. Then it makes sense to define $\mu_{b,q}$ as the smallest $n \in \mathbb{N}$ such that one can take $k = L_{b,q}$. We provide upper and lower bounds for $\mu_{b,q}$. Furthermore, we derive explicit formulas for $L_{b,1}$ and $\mu_{b,1}$. Lastly, we give a constructive proof that $L_{b,q}$ is unbounded with respect to q.

1 Introduction

Let $b \geq 2$ be a fixed integer and let $s_b(n)$ denote the sum of digits of the nonnegative integer n in the base-b representation. Further let q a positive integer, we are interested in the length k of arithmetic progressions $n, n+q, \ldots, n+q(k-1)$ such that the integers $s_b(n), s_b(n+q), \ldots, s_b(n+q(k-1))$ are (pairwise) distinct.

There are known results on the asymptotic behavior of the sum of digits function [2, 4], and about its distribution along arithmetic progressions [3, 5]. But, to our knowledge, this particular problem has not been studied before.

More specifically, let $L_{b,q}$ denote the supremum of k as n varies in the set of nonnegative integers $\mathbb{N}$. We show that $L_{b,q}$ is bounded from above and hence finite. As a consequence, it makes sense to define $\mu_{b,q}$ as the smallest $n \in \mathbb{N}$ such that one can take $k = L_{b,q}$. Then, we provide upper and lower bounds for $\mu_{b,q}$. Everything allows for an effective computation of $L_{b,q}$ and $\mu_{b,q}$ by checking a finite number of candidates, though this is feasible in a short amount of time only for small values of b and q. Furthermore, we derive explicit formulas for $L_{b,1}$ and $\mu_{b,1}$. Lastly, we give a constructive proof that $L_{b,q}$ is unbounded with respect to q, in the sense that $\sup_{q \in \mathbb{N}^+} L_{b,q} = +\infty$.

2 Bounds for $L_{b,q}$ and $\mu_{b,q}$

Theorem 1. *Let m be the least positive integer such that*

$$m(b-1)+1 \le \left\lfloor \frac{b^m}{q} \right\rfloor. \tag{1}$$

Then $L_{b,q} \le 2\lfloor b^m/q \rfloor$.

Proof. Let $n \in \mathbb{N}$, $k \in \mathbb{N}^+$ and $A := \{n + qi : i = 0, 1, \ldots, k-1\}$ such that $s_b(n)$, $s_b(n+q)$, $\ldots$, $s_b(n + q(k-1))$ are distinct. For any $t \in \mathbb{N}$ we define $A_t := A \cap [tb^m, (t+1)b^m - 1]$. For convenience, take $M := \lfloor b^m/q \rfloor$. Then for all nonnegative integers $t < u$ the following statements are true:

(i). $|A_t| \le M$.

(ii). $A_t, A_u \neq \varnothing \;\Rightarrow\; \forall v \in \mathbb{N}, t < v < u \quad |A_v| = M$.

(iii). $|A_t| = M,\ t \not\equiv -1 \pmod b \;\Rightarrow\; \forall u \in \mathbb{N}, u > t \quad A_u = \varnothing$.

(iv). $|A_t| = M,\ t \not\equiv 0 \pmod b \;\Rightarrow\; \forall u \in \mathbb{N}, u < t \quad A_u = \varnothing$.

(i). For all $a \in A_t$ we have $s_b(a) = s_b(t) + s_b(a \bmod b^m)$ and therefore

$$s_b(A_t) := \{s_b(a) : a \in A_t\} \subseteq \{s_b(t), s_b(t)+1, \ldots, s_b(t) + m(b-1)\}, \tag{2}$$

so by the hypotheses $|A_t| = |s_b(A_t)| \le m(b-1) + 1 \le M$.

(ii). Since A_t, A_u are nonempty we have $n < vb^m$ and $(v+1)b^m - 1 < n + q(k-1)$. Then

$$\begin{aligned} |A_v| &= \big|\{qi : i = 0, \ldots, k-1\} \cap [vb^m - n, (v+1)b^m - n - 1]\,\big| \\ &= \big|q\mathbb{N} \cap [vb^m - n, (v+1)b^m - n - 1[\,\big| \ge \left\lfloor \frac{b^m}{q} \right\rfloor = M, \end{aligned} \tag{3}$$

because $\big|q\mathbb{N} \cap [x, y]\big| \ge \lfloor (y - x + 1)/q \rfloor$ for any integers $y \ge x \ge 0$. From point (i) it follows that $|A_v| = M$.

(iii). We have $s_b(t+1) = s_b(t) + 1$. Suppose by contradiction that A_{t+1} is nonempty, so there exists $a := \min(A_{t+1})$. Now $s_b(a) = s_b(t) + 1 + s_b(a \bmod b^m)$ and, since $|A_t| = M$ implies $s_b(A_t) = \{s_b(t), s_b(t)+1, \ldots, s_b(t) + m(b-1)\}$, then necessarily $s_b(a \bmod b^m) = m(b-1)$, so that $a = (t+2)b^m - 1$. In fact, $s_b(a \bmod b^m) \le m(b-1)$ and if we suppose $s_b(a \bmod b^m) < m(b-1)$ then $s_b(a) \in s_b(A_t)$, in contradiction to our standing hypotheses. But $q \le \frac{1}{M} b^m \le b^{m-1}$, so $a - q \ge (t+1)b^m$ and $a - q \in A_{t+1}$, a contradiction. In conclusion $A_{t+1} = \varnothing$ and, since $q < b^m$, this implies $A_u = \varnothing$.

(iv). Note that $t \ge 1$, we have $s_b(t-1) = s_b(t) - 1$. Suppose that A_{t-1} is nonempty, so there exists $a := \max(A_{t-1})$. Then $s_b(a) = s_b(t) - 1 + s_b(a \bmod b^m)$ and, since $|A_t| = M$ implies $s_b(A_t) = \{s_b(t), s_b(t)+1, \ldots, s_b(t) + m(b-1)\}$, then it must be $s_b(a \bmod b^m) = 0$ that is $a = (t-1)b^m$. But $a + q \le tb^m - 1$ so $a + q \in A_{t-1}$, a contradiction. Thus $A_{t-1} = \varnothing$ and, since $q < b^m$, it follows that $A_u = \varnothing$.

The sets $\{A_t\}_{t=0}^\infty$ form a partition of A, hence $A = \bigcup_{t \in \mathbb{N}} A_t$. On the other hand, for the statements proved, we have that at most two of the sets $\{A_t\}_{t=0}^\infty$ are nonempty and their cardinality is less than or equal to M. In conclusion $k = |A| \le 2M$. □

Corollary 2. $L_{b,1} = 2b$, $\mu_{2,1} = 14$ *and* $\mu_{b,1} = b^3 - b$ *if* $b \geq 3$.

Proof. From Theorem 1 we know that $L_{b,1} \leq 2b$. It is easy to verify that $L_{2,1} = 4$ and $\mu_{2,1} = 14$ (OEIS A000120). If $b \geq 3$ then $L_{b,1} = 2b$ and $\mu_{b,1} \leq b^3 - b$ because

$$s_b(b^3 - b + i) = \begin{cases} 2(b-1) + i, & \text{if} \quad i = 0, 1, \ldots, b-1; \\ i - b + 1, & \text{if} \quad i = b, b+1, \ldots, 2b-1. \end{cases} \tag{4}$$

Now let $n < b^3 - b$ be a nonnegative integer. Write $n = d_2 b^2 + d_1 b + d_0$ with $d_0, d_1, d_2 \in \{0, 1, \ldots, b-1\}$. If $d_1 \neq b-1$ then let m be the least integer greater than or equal to n and not divisible by b. We have $m \leq n+1$ and $s_b(m) = s_b(m+b-1)$. If $d_1 = b-1$ and $d_0 = 0$ then $d_2 \leq b-2$, since $n < b^3 - b$, and so $s_b(n) = s_b(n+2b-2)$. If $d_1 = b-1$ and $d_0 \neq 0$ then let $h := (d_2+1)b^2$. We have $h \leq n+b-1$ and $s_b(h+1) = s_b(h+b)$. In any case we have found two integers u, v such that $n \leq u < v \leq n+2b-1$ and $s_b(u) = s_b(v)$. Therefore $\mu_{b,1} \geq b^3 - b$ and actually $\mu_{b,1} = b^3 - b$. □

Theorem 3. $L_{b,bq} = L_{b,q}$ *and* $\mu_{b,bq} = b\mu_{b,q}$.

Proof. For all $n, i \in \mathbb{N}$ and $d \in \{0, 1, \ldots, b-1\}$ we have $s_b(bn+d+bqi) = s_b(n+qi)+d$. Then for any $k \in \mathbb{N}$ we have that $s_b(bn+d), s_b(bn+d+bq), \ldots, s_b(bn+d+(k-1)bq)$ are distinct if and only if $s_b(bn), s_b(bn+bq), \ldots, s_b(bn+(k-1)bq)$ are distinct, which in turn holds if and only if $s_b(n), s_b(n+q), \ldots, s_b(n+(k-1)q)$ are distinct. In conclusion $L_{b,bq} = L_{b,q}$ and $\mu_{b,bq} = b\mu_{b,q}$. □

Theorem 4. $\mu_{b,q} > b^{\frac{L_{b,q}-b}{b-1}} - q(L_{b,q} - 1)$.

Proof. Let $r := \lfloor \log_b(\mu_{b,q} + q(L_{b,q} - 1)) \rfloor + 1$. Since $s_b(\mu_{b,q}), s_b(\mu_{b,q}+q), \ldots s_b(\mu_{b,q}+q(k-1))$ are distinct and less than or equal to $r(b-1)$, then

$$L_{b,q} \leq r(b-1) + 1 < (b-1)\log_b(\mu_{b,q} + q(L_{b,q} - 1)) + b. \tag{5}$$

Solving (5) for $\mu_{b,q}$ we get $\mu_{b,q} > b^{\frac{L_{b,q}-b}{b-1}} - q(L_{b,q} - 1)$. □

Theorem 5. $\mu_{b,q} \leq b^3[q(L_{b,q} - 1)]^2 - 1$.

Proof. Take $r := \lfloor \log_b(q(L_{b,q} - 1)) \rfloor + 1$, $\mu := (\mu_{b,q} \bmod b^r)$, $m := \lfloor \mu_{b,q}/b^r \rfloor$, so that $\mu_{b,q} = mb^r + \mu$, and let $i \in \{0, 1, \ldots, L_{b,q} - 1\}$. If $\mu + qi < b^r$ then

$$s_b(\mu_{b,q} + qi) = s_b(m) + s_b(\mu + qi), \tag{6}$$

else if $\mu + qi \geq b^r$ then

$$s_b(\mu_{b,q} + qi) = s_b(m+1) + s_b(\mu + qi - b^r), \tag{7}$$

for the fact that $\mu + qi \leq 2b^r - 2$. Define $n := (b^{r+1} - 1)b^r + \mu$, if $\mu + qi < b^r$ then

$$s_b(n+qi) = s_b(b^{r+1} - 1) + s_b(\mu + qi) = (r+1)(b-1) + s_b(\mu + qi) > r(b-1). \tag{8}$$

On the other hand, if $\mu + qi \geq b^r$ then

$$s_b(n+qi) = s_b(b^{r+1}) + s_b(\mu + qi - b^r) = 1 + s_b(\mu + qi - b^r) \leq r(b-1), \tag{9}$$

because $\mu + qi - b^r \leq b^r - 2$ and so $s_b(\mu + qi - b^r) \leq r(b-1) - 1$. In conclusion $s_b(n), s_b(n+q), \ldots, s_b(n + q(L_{b,q} - 1))$ are distinct and

$$\mu_{b,q} \leq n \leq b^{2r+1} - 1 \leq b^3[q(L_{b,q} - 1)]^2 - 1, \tag{10}$$

this completes our proof. □

3 Arbitrarily large values of $L_{b,q}$

For the next theorem we need a lemma about the sum of digits of multiples of $b^r - 1$, $r \in \mathbb{N}^+$. Similar results has been considered by Stolarsky [6, Lemma 2.2] in the case $b = 2$, and by Balog and Dartyge [1, Lemma 1] for a generic base b.

Lemma 6. *If $r \in \mathbb{N}^+$ then $s_b((b^r - 1)i) = r(b-1)$ for all $i = 1, 2, \ldots, b^r - 1$.*

Proof. Let t be the greatest nonnegative integer such that $b^t \mid i$. If $i_0 := b^{-t} i$ then there exist $d_0, d_1, \ldots, d_{r-1} \in \{0, 1, \ldots, b-1\}$ with $d_0 \neq 0$ such that $i_0 = \sum_{j=0}^{r-1} d_j b^j$ is the base-b representation of i_0. We have

$$\begin{aligned}(b^r - 1)i_0 &= \sum_{j=r}^{2r-1} d_{j-r} b^j - \sum_{j=0}^{r-1} d_j b^j \\ &= \sum_{j=r+1}^{2r-1} d_{j-r} b^j + (d_0 - 1)b^r + b^r - \sum_{j=0}^{r-1} d_j b^j \\ &= \sum_{j=r+1}^{2r-1} d_{j-r} b^j + (d_0 - 1)b^r + \sum_{j=1}^{r-1} (b - 1 - d_j) b^j + (b - d_0). \end{aligned} \tag{11}$$

Then it is straightforward that

$$s_b((b^r - 1)i_0) = \sum_{j=r+1}^{2r-1} d_{j-r} + (d_0 - 1) + \sum_{j=1}^{r-1} (b - 1 - d_j) + (b - d_0) = r(b-1). \tag{12}$$

The claim follows from $s_b((b^r - 1)i) = s_b((b^r - 1)i_0)$. □

Theorem 7. $\sup_{q \in \mathbb{N}^+} L_{b,q} = +\infty$.

Proof. Let $n \in \mathbb{N}$ and $q, k \in \mathbb{N}^+$ such that $s_b(n), s_b(n+q), \ldots, s_b(n + q(k-1))$ are distinct. If t, r are the least positive integers such that $n + qk < b^t$ and $(b-1)b^{r-1} \geq k$ then we define

$$\begin{aligned} n' &= \big((b^t - 1)b^{2r} + b^{2r-1} + (b^r - 1)((b-1)b^{r-1} - k + 1)\big)b^t + n \\ q' &= (b^r - 1)b^t + q. \end{aligned} \tag{13}$$

For any $i = 0, 1, \ldots, k$ we have

$$n' + q'i = \big((b^t - 1)b^{2r} + b^{2r-1} + (b^r - 1)((b-1)b^{r-1} + i - k + 1)\big)b^t + n + qi, \tag{14}$$

and then

$$s_b(n' + q'i) = s_b\big((b^t - 1)b^{2r} + b^{2r-1} + (b^r - 1)((b-1)b^{r-1} + i - k + 1)\big) + s_b(n + qi). \tag{15}$$

If $i \leq k - 1$ then $(b^r - 1)((b-1)b^{r-1} + i - k + 1) < (b-1)b^{2r-1}$ and

$$s_b(n' + q'i) = t(b-1) + 1 + s_b\big((b^r - 1)((b-1)b^{r-1} + i - k + 1)\big) + s_b(n + qi), \tag{16}$$

so Lemma 6 implies that

$$s_b(n' + q'i) = (t + r)(b - 1) + 1 + s_b(n + qi) > (t + r)(b - 1). \tag{17}$$

On the other hand, if $i = k$ then

$$\begin{aligned} s_b(n' + q'k) &= s_b\big(b^{2r+t} + b^{r-1} - 1\big) + s_b(n + qk) = 1 + (r-1)(b-1) + s_b(n + qk) \\ &\leq 1 + (t + r - 1)(b - 1) \leq (t + r)(b - 1). \end{aligned} \tag{18}$$

Therefore $s_b(n'), s_b(n' + q), \ldots, s_b(n' + qk)$ are distinct, it follows that $L_{b,q'} > L_{b,q}$ and hence $\sup_{q \in \mathbb{N}^+} L_{b,q} = +\infty$. □

4 Further developments

Thanks to Theorem 1 we know that $L_{b,q}$ is not too large when q is small compared to b, e.g., if $q \leq \frac{1}{2}b$ then $L_{b,q} \leq 2b^2$. Actually, it is likely that for small q there exist explicit formulas for $L_{b,q}$ and $\mu_{b,q}$ analogous to those of Corollary 2. However, when q is much larger than b the question becomes more difficult. Theorem 1 and 5 allowed us, with the aid of a personal computer, to calculate some values of $\mu_{b,q}$ and $L_{b,q}$.

$\mu_{b,q}, L_{b,q}$	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$	$q = 6$	$q = 7$	$q = 8$	$q = 9$
$b = 2$	$14, 4$	$28, 4$	$58, 4$	$56, 4$	$242, 6$	$116, 4$	$109, 5$	$112, 4$	$994, 6$
$b = 3$	$24, 6$	$24, 3$	$72, 6$	$234, 5$	$705, 9$	$72, 3$	$697, 10$	$18, 3$	$216, 6$
$b = 4$	$60, 8$	$56, 8$	$60, 3$	$240, 8$	$1004, 8$	$244, 4$	$977, 13$	$224, 8$	$239, 4$

Table 1: Values of $\mu_{b,q}$ and $L_{b,q}$ for $b = 2, 3, 4$ and $q = 1, 2, \ldots, 9$.

Through these series of numerical experiments we have reason to believe that the upper bound given by Theorem 1 is in some sense "astronomical" and surely can be improved. Similarly, also the upper and lower bounds for $\mu_{b,q}$ can be improved.

Many other questions remain unsolved. For instance, let $\kappa_{b,q}$ be the function sending the nonnegative integer n to the maximum $k \in \mathbb{N}$ such that $s_b(n), s_b(n + q), \ldots, s_b(n + q(k-1))$ are distinct. We proved that the function $\kappa_{b,q}$ is bounded. Is $\kappa_{b,q}$ definitely periodic? Does it present a fractal behavior? What is its Fourier expansion? What is its mean, variance, etc.? On the other side, for which $n \in \mathbb{N}$ is $\kappa_{b,q}(n)$ particularly small? These and other questions will be the subject of future investigations.

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