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A Note on a Theorem of Schumacher

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Abstract

In this paper, we give a short new proof of a recent result due to Schumacher concerning an extension of Faulhaber's identity for the Bernoulli numbers. Our approach follows from basic manipulations involving the ordinary generating function for the Bernoulli polynomials in the context of the Zeon algebra.

1 Introduction

Zeon algebra [20, Chap. 5], [27, Chap. 2] and Grassmann algebra [5, Chap. 1], [10, Chap. 1], [26, Chap. 3], [27, Chap. 2] are efficient tools towards proving combinatorial identities. In the context of the Zeon algebra, examples include a criterion for ergodicity of Markov chains [13], alternative proofs of Spivey's identity for Bell numbers [22], the one-variable Faà di Bruno formula [22], identities involving Stirling numbers of the second kind, Bernoulli numbers, and Bernoulli polynomials [23, 24, 25]. Building on ideas from Grassmann algebra we mention, e.g., proofs of theorems of the matrix-tree type [1], representation of the generating function for hyperforests in hypergraphs [4], Cayley-type identities [6], Lindström-Gessel-Viennot lemma, and Schur functions [7].

In this paper, we will give yet another example of the utility of the Zeon algebra by giving a new, simple, and short proof of an extension of Faulhaber's identity for the Bernoulli numbers [14, Chap. 6] obtained recently by Schumacher [28]. Another compelling feature of

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our proof is that it does not assume the usual Faulhaber formula a priori, as in the proof given by Schumacher. More precisely, we will show that

$$\sum_{i=0}^{\lfloor x\rfloor} i^n = \frac{x^{n+1}}{n+1} + (-1)^n \frac{B_{n+1}}{n+1} + \frac{1}{n+1}\sum_{j=1}^{n+1}(-1)^j \binom{n+1}{j} B_j(\{x\})x^{n-j+1}, \tag{1}$$

using the Zeon algebra [20, 22]. Throughout this work, we let $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the natural, rational, and real numbers, respectively. We define $\mathbb{N}_0 := \{0\}\cup\mathbb{N}$ and $\mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$. For $x \in \mathbb{R}_0^+$, we write $\lfloor x\rfloor$ for the floor of x and $\{x\}$ for the fractional part of x. In Eq. (1) we take $x \in \mathbb{R}_0^+$ and $n \in \mathbb{N}_0$.

We remark that there are other examples concerning extensions of Faulhaber's identity in different contexts [2, 3, 8, 9, 11, 12, 15, 16, 17, 18, 19, 21].

Before we continue, we establish the basic underlying algebraic setup needed to give the proof of Eq. (1).

2 Basic definitions: Zeon algebra and the Grassmann-Berezin integral

Definition 1. The *Zeon algebra* $\mathcal{Z}_n \supset \mathbb{R}$ is defined as the associative algebra generated by the collection $\{\varepsilon_i\}_{i=1}^n$ $(n < \infty)$ and the scalar $1 \in \mathbb{R}$, such that $1\varepsilon_i = \varepsilon_i = \varepsilon_i 1$, $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$ $\forall$ i, j and $\varepsilon_i^2 = 0$ $\forall$ i.

For $\{i, j, \ldots, k\} \subset \{1, 2, \ldots, n\}$ and $\varepsilon_{ij\cdots k} \equiv \varepsilon_i\varepsilon_j \cdots \varepsilon_k$ the most general element with n generators ε_i can be written as (with the convention of sum over repeated indices implicit)

$$\phi_n = a + a_i\varepsilon_i + a_{ij}\varepsilon_{ij} + \cdots + a_{12\cdots n}\varepsilon_{12\cdots n} = \sum_{\mathbf{i}\in 2^{[n]}} a_{\mathbf{i}}\varepsilon_{\mathbf{i}},$$

with a, a_i, a_{ij}, $\ldots$, $a_{12\cdots n} \in \mathbb{R}$, $2^{[n]}$ being the power set of $[n] := \{1, 2, \ldots, n\}$, and $1 \leq i < j < \cdots \leq n$. We refer to a as the body of ϕ_n and write $b(\phi_n) = a$ and to $\phi_n - a$ as the soul such that $s(\phi_n) = \phi_n - a$. This terminology is borrowed from the literature on superanalysis [10, Chap. 1].

Definition 2. The *Grassmann-Berezin integral* on $\mathcal{Z}_n$, denoted by $\int$, is the linear functional $\int : \mathcal{Z}_n \to \mathbb{R}$ such that (we use throughout this work the compact notation $d\nu_n := d\varepsilon_n \cdots d\varepsilon_1$)

$$d\varepsilon_i d\varepsilon_j = d\varepsilon_j d\varepsilon_i, \ \int \phi_n(\hat{\varepsilon}_i)d\varepsilon_i = 0, \text{ and} \int \phi_n(\hat{\varepsilon}_i)\varepsilon_i d\varepsilon_i = \phi_n(\hat{\varepsilon}_i),$$

where $\phi_n(\hat{\varepsilon}_i)$ means any element of $\mathcal{Z}_n$ with no dependence on ε_i. Multiple integrals are iterated integrals, i.e.,

$$\int f(\phi_n)d\nu_n = \int \cdots \left(\int\left(\int f(\phi_n)d\varepsilon_n\right)d\varepsilon_{n-1}\right)\cdots d\varepsilon_1.$$

We now extend some of the constructions of previous work on Zeons and Bernoulli numbers [24] to the context of Bernoulli polynomials [14, Chap. 7], [29, Chap. 4]. Let us write $\mathbb{Q}[[x,z]]$ for the ring of formal power series in the variables x and z over $\mathbb{Q}$. We recall the generating function for the Bernoulli polynomials $B_j(x)$ in $\mathbb{Q}[[x,z]]$, i.e.,

$$\frac{e^{xz}}{\sum_{i=0}^{\infty}\frac{z^i}{(i+1)!}} = \frac{ze^{xz}}{e^z-1} = \sum_{j=0}^{\infty} B_j(x)\frac{z^j}{j!} \tag{2}$$

and, making the change $z \mapsto -z$ in Eq. (2), we get

$$\frac{e^{(1-x)z}}{\sum_{i=0}^{\infty}\frac{z^i}{(i+1)!}} = \frac{ze^{(1-x)z}}{e^z-1} = \sum_{j=0}^{\infty} B_j(x)\frac{(-z)^j}{j!}. \tag{3}$$

Note that $B_j(0) \equiv B_j$ are the Bernoulli numbers.

Following the strategy of our previous work [22, 24], we consider Eqs. (2) and (3) in the context of the Zeon algebra with the replacement $z \to \phi_k \equiv \varphi_k := \varepsilon_1 + \cdots + \varepsilon_k$. Therefore, we get

$$\frac{e^{x\varphi_k}}{\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}} = \sum_{j=0}^{k} B_j(x)\frac{\varphi_k^j}{j!} \tag{4}$$

and

$$\frac{e^{(1-x)\varphi_k}}{\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}} = \sum_{j=0}^{k} B_j(x)\frac{(-\varphi_k)^j}{j!}, \tag{5}$$

using that $\varphi_k^{k+1} = 0 \ \forall\ k \geq 1$. We observe that

$$b\left(\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}\right) = 1 \neq 0 \tag{6}$$

and, hence, $\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}$ is invertible in $\mathcal{Z}_k$.

Now, integrating Eq. (4) in the Zeon algebra and using Definition 2, we get the representation of the Bernoulli polynomials

$$\int \frac{e^{x\varphi_k}}{\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}} d\nu_k = \sum_{j=0}^{k}\frac{B_j(x)}{j!}\int \varphi_k^j d\nu_k = B_k(x)$$

$\forall\ k \geq 1$.

Similarly, integrating Eq. (5) in the Zeon algebra, we get

$$\int \frac{e^{(1-x)\varphi_k}}{\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}} d\nu_k = \sum_{j=0}^{k}(-1)^j\frac{B_j(x)}{j!}\int \varphi_k^j d\nu_k = (-1)^k B_k(x) \tag{7}$$

$\forall\ k \geq 1$.

3 Proof of Eq. (1)

We are now ready to prove Eq. (1). We start with the following identity

$$e^{(\lfloor x\rfloor+1)\varphi_{n+1}}-e^{\varphi_{n+1}}=\sum_{i=1}^{\lfloor x\rfloor}e^{i\varphi_{n+1}}\left(e^{\varphi_{n+1}}-1\right)=\varphi_{n+1}\sum_{i=1}^{\lfloor x\rfloor}e^{i\varphi_{n+1}}\left(\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}\right).$$

Note that Eq. (6) can be used and we can write

$$\frac{e^{(\lfloor x\rfloor+1)\varphi_{n+1}}-e^{\varphi_{n+1}}}{\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}}=\varphi_{n+1}\sum_{i=1}^{\lfloor x\rfloor}e^{i\varphi_{n+1}}.$$

Next, using the Grassmann-Berezin integration of Definition 2, we have

$$\int\frac{e^{(\lfloor x\rfloor+1)\varphi_{n+1}}-e^{\varphi_{n+1}}}{\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}}d\nu_{n+1}=\int\varphi_{n+1}\sum_{i=1}^{\lfloor x\rfloor}e^{i\varphi_{n+1}}d\nu_{n+1}. \tag{8}$$

We observe that

$$\int\varphi_{n+1}\sum_{i=1}^{\lfloor x\rfloor}e^{i\varphi_{n+1}}d\nu_{n+1}=(n+1)\sum_{i=1}^{\lfloor x\rfloor}\int e^{i\varphi_{n}}d\nu_{n}=(n+1)\sum_{i=1}^{\lfloor x\rfloor}i^{n}. \tag{9}$$

Using Eq. (7) and $x=\lfloor x\rfloor+\{x\}$, we obtain

$$\begin{aligned}
\int\frac{e^{(\lfloor x\rfloor+1)\varphi_{n+1}}}{\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}}d\nu_{n+1} &= \int\frac{e^{(x-\{x\}+1)\varphi_{n+1}}}{\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}}d\nu_{n+1}\\
&= \sum_{k=0}^{n+1}\frac{x^k}{k!}\int\frac{e^{(1-\{x\})\varphi_{n+1}}}{\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}}\varphi_{n+1}^{k}d\nu_{n+1}\\
&= x^{n+1}+\sum_{k=0}^{n}\binom{n+1}{k}x^k\int\frac{e^{(1-\{x\})\varphi_{n-k+1}}}{\sum_{j=0}^{n-k+1}\frac{\varphi_{n-k+1}^{j}}{(j+1)!}}d\nu_{n-k+1}\\
&= x^{n+1}+\sum_{k=0}^{n}\binom{n+1}{k}(-1)^{n-k+1}B_{n-k+1}(\{x\})x^k\\
&= x^{n+1}+\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k}B_{k}(\{x\})x^{n-k+1} \qquad (10)
\end{aligned}$$

making the change of variables $n-k+1\mapsto k$ to obtain the last equality. Finally, taking $x=0$ in Eq. (7), we get

$$\int\frac{e^{\varphi_{n+1}}}{\sum_{j=0}^{n+1}\frac{\varphi_{n+1}^{j}}{(j+1)!}}d\nu_{n+1}=(-1)^{n+1}B_{n+1}. \tag{11}$$

Collecting the results in Eqs. (9), (10), (11), and going back to Eq. (8), we arrive at the desired result, i.e., Eq. (1).

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