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On the Total Positivity of Delannoy-Like Triangles

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Abstract

Define an infinite lower triangular matrix $D(e,h) = [d_{n,k}]_{n,k\geq 0}$ by the recurrence

$$d_{0,0} = d_{1,0} = d_{1,1} = 1,\ d_{n,k} = d_{n-1,k-1} + ed_{n-1,k} + hd_{n-2,k-1},$$

where e, h are both nonnegative and $d_{n,k} = 0$ unless $n \geq k \geq 0$. We call $D(e,h)$ the *Delannoy-like triangle*. The entries $d_{n,k}$ count lattice paths from $(0,0)$ to $(n-k,k)$ using the steps $(0,1)$, $(1,0)$ and $(1,1)$ with assigned weights 1, e, and h. Some well-known combinatorial triangles are such matrices, including the Pascal triangle $D(1,0)$, the Fibonacci triangle $D(0,1)$, and the Delannoy triangle $D(1,1)$. Futhermore, the Schröder triangle and Catalan triangle also arise as inverses of Delannoy-like triangles. Here we investigate the total positivity of Delannoy-like triangles. In addition, we show that each row and diagonal of Delannoy-like triangles are all PF sequences.

1 Introduction

The Delannoy number $d(n,k)$ is defined as the number of lattice paths from $(0,0)$ to (n,k) with steps $(0,1)$, $(1,0)$ and $(1,1)$. Banderier and Schwer [1] gave historical background on Delannoy numbers. There is a link between Legendre polynomials and Delannoy numbers [8, 11]. The Delannoy triangle $D = [t_{n,k}]_{n,k\geq 0}$ is an infinite lower triangular matrix defined by $t_{n,k} = d(n-k,k)$. Its entries satisfy the recurrence

$$t_{0,0} = 1,\ t_{n,k} = t_{n-1,k-1} + t_{n-1,k} + t_{n-2,k-1},$$

where $t_{n,k} = 0$ unless $n \geq k \geq 0$. The first few entries of D are as follows:

$$D = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 3 & 1 & & & & \\ 1 & 5 & 5 & 1 & & & \\ 1 & 7 & 13 & 7 & 1 & & \\ 1 & 9 & 25 & 25 & 9 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

An immediate calculation shows that the row sums of Delannoy triangle form the Pell sequence [3]. The unsigned inverse of D is the Schröder triangle (A132372). The central coefficients $t_{2n,n}$ are called the central Delannoy numbers (A001850) and have appeared in several problems, such as the alignments between DNA sequences [17].

In this paper, we study the infinite lower triangular matrix $D(e,h) = [d_{n,k}]_{n,k\geq 0}$ defined by the recurrence

$$d_{0,0} = d_{1,0} = d_{1,1} = 1,\ d_{n,k} = d_{n-1,k-1} + ed_{n-1,k} + hd_{n-2,k-1}, \tag{1}$$

where e, h are both nonnegative and $d_{n,k} = 0$ unless $n \geq k \geq 0$. We call $D(e,h)$ the *Delannoy-like triangle*. The first few rows of a Delannoy-like triangle are as follows:

$$D(e,h) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ e & 1+e+h & 1 & & \\ e^2 & e^2+2e+eh+h & 1+2e+2h & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}.$$

The special cases are Delannoy triangle $D(1,1)$, the Pascal triangle (A007318) $D(1,0)$, and the Fibonacci triangle (A026729) $D(0,1)$.

On the other hand, we let $S(e,h)$ denote the unsigned inverse of $D(e,h)$, i.e., $S(e,h) = MD(e,h)^{-1}M$, where M is the diagonal matrix with diagonal entries alternately 1 and -1.

We call $S(e,h)$ the *generalized Schröder matrix* [18].

$$S(e,h)=\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1+h & 1+e+h & 1 & & \\ 1+2h+eh+2h^2 & e^2+e+3eh+2h^2+2h+1 & 1+2e+2h & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}.$$

Such matrices include the Catalan triangle (A033184) with $e=0$, $h=1$, and Schröder triangle (A132372) with $e=1$, $h=1$.

$$S(1,1)=\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 6 & 10 & 5 & 1 & & \\ 22 & 38 & 22 & 7 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The row sums of $S(1,1)$ are the large Schröder numbers [7].

An infinite matrix is called *totally positive* (*TP*, for short) if its minors of all orders are nonnegative. The theory of totally positive matrices is rich with deep and highly non-trivial results, and has been studied extensively [6, 13]. Brenti [5] showed the total positivity of the Delannoy square (A008288) $[d(n,k)]_{n,k\geq 0}$ by giving a combinatorial interpretation of its minors in terms of nonintersecting paths in a digraph. It is natural to conjecture that Delannoy-like triangles are totally positive. The primary purpose of this paper is to show this is true.

Theorem 1. *The Delannoy-like triangles defined by* (1) *are totally positive matrices.*

We give the proof of this, our main theorem, by showing the total positivity of the generalized Schröder matrix $S(e,h)$ in Section 2. It turns out that the Pascal triangle, the Fibonacci triangle, the Delannoy triangle, the Catalan triangle, and the Schröder triangle are all TP matrices.

In Section 3, we briefly consider the PF sequence (see Section 3 for definitions) in Delannoy-like triangles. With a simple proof, we conclude that each row and diagonal of Delannoy-like triangles all form PF sequences.

2 Proof of Theorem 1

To prove the total positivity of Delannoy-like triangles $D(e,h)$, it suffices to show that the generalized Schröder matrix $S(e,h)$ is TP, because the unsigned inverse of a TP matrix is still TP [13, Prop. 1.6], and $D(e,h)$ is also the unsigned inverse of $S(e,h)$. Actually, since the $D(e,h)$ are Riordan arrays (we will prove it first), so are $S(e,h)$. This implies that $S(e,h)$ possess A-sequences and Z-sequences, since any Riordan array can be characterized by the A- and Z-sequences uniquely [9]. Therefore, our idea is to construct a class of Riordan arrays with A-sequences and Z-sequences having a certain form, and make sure $S(e,h)$ are such matrices. Then $S(e,h)$ is TP if we show this class of Riordan arrays is TP (Claim 2).

We let $(g(x), f(x))$ denote a Riordan array $R = [r_{n,k}]_{n,k\geq 0}$ [14], and $(g(x), f(x))$ is an infinite lower triangular matrix whose generating function of the kth column is $g(x)f^k(x)$ for $k = 0, 1, 2, \ldots$, where $g(x)$ and $f(x)$ are formal power series with $g(0) = 1$ and $f(0) = 0$, $f'(0) \neq 0$. Suppose we multiply the array $(g(x), f(x))$ by a column vector $(b_0, b_1, b_2, \ldots)^T$ with generating function $b(x)$. Then we get a column vector whose generating function is given by $g(x)b(f(x))$. If we identify a sequence with its generating function, the composition rule can be rewritten as

$$(g(x), f(x))b(x) = g(x)b(f(x)). \tag{2}$$

This is called the fundamental theorem for Riordan arrays and this lead to the multiplication rule for the Riordan arrays (see Shapiro et al. [14]):

$$(g(x), f(x))\,(d(x), h(x)) = (g(x)d(f(x)), h(f(x)))\,. \tag{3}$$

The inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = (1/g(\bar{f}(x)), \bar{f}(x)), \tag{4}$$

where $\bar{f}(x)$ is the compositional inverse of $f(x)$, such that $f(\bar{f}(x)) = \bar{f}(f(x)) = x$. The bivariate generating function $R(x,y)$ of the Riordan array R is given by

$$R(x,y) = (g(x), f(x))\frac{1}{1-yx} = \frac{g(x)}{1-yf(x)},$$

following (2).

A Riordan array $R = [r_{n,k}]_{n,k\geq 0}$ can also be characterized by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ such that

$$r_{0,0} = 1,\ r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j},\ r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n,k+j}$$

for $n, k \geq 0$ (see [16] for instance). Call $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ the A- and Z-sequences of R, respectively. Following [6], call

$$J(R) = \begin{bmatrix} z_0 & a_0 & & & & \\ z_1 & a_1 & a_0 & & & \\ z_2 & a_2 & a_1 & a_0 & & \\ z_3 & a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & \cdots & & \ddots \end{bmatrix}$$

the *coefficient matrix* of the Riordan array R. Let $A(x)$ and $Z(x)$ be the generating functions of A-sequence and Z-sequence respectively. Then

$$A(x) = \frac{x}{\bar{f}(x)}; \qquad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right). \tag{5}$$

Now we start by proving that Delannoy-like triangles are Riordan arrays, and deduce the generating functions of A-sequence and Z-sequence of $S(e,h)$. Let $B(x,y)$ denote the bivariate generating function of $D(e,h)$. By (1), we have

$$\begin{aligned}
B(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d_{n,k}x^n y^k \\
&= 1 + x + xy + \sum_{n=2}^{\infty}\sum_{k=0}^{\infty} d_{n,k}x^n y^k \\
&= 1 + x + xy + \sum_{n=2}^{\infty}\sum_{k=0}^{\infty}(d_{n-1,k-1} + ed_{n-1,k} + hd_{n-2,k-1})x^n y^k \\
&= 1 + x + xy + xy(B(x,y) - 1) + ex(B(x,y) - 1) + hx^2 yB(x,y).
\end{aligned}$$

It follows that

$$B(x,y) = \frac{1 + x - ex}{1 - ex - xy - hx^2 y} = \frac{1 + x - ex}{1 - ex}\frac{1}{1 - y\frac{x + hx^2}{1 - ex}}.$$

Thus,

$$D(e,h) = (d(x), h(x)) = \left(\frac{1 + x - ex}{1 - ex}, \frac{x + hx^2}{1 - ex}\right).$$

Then

$$(S(e,h))^{-1} = (1,-x)D(e,h)(1,-x) = (d(-x), -h(-x)) = \left(\frac{1 - x + ex}{1 + ex}, \frac{x - hx^2}{1 + ex}\right),$$

since $S(e,h) = (1,-x)D^{-1}(e,h)(1,-x)$ and (3). From (4) and (5) , we have

$$A^*(x) = \frac{x}{-h(-x)} = \frac{1 + ex}{1 - hx}; \quad Z^*(x) = \frac{1}{-h(-x)}(1 - d(-x)) = \frac{1}{1 - hx},$$

where $A^*(x)$ and $Z^*(x)$ are the generating functions of A-sequence and Z-sequence of $S(e,h)$, respectively.

Next we construct a class of Riordan arrays by expanding the A-sequence of $S(e,h)$ and have the following result.

Lemma 2. *Let $A(x)$ and $Z(x)$ be the generating functions of A-sequence and Z-sequence of Riordan array R. Suppose that $Z(x) = \frac{1+ux}{1-hx}$ and $A(x) = Z(x)(1 + ex)$, where $e, h, u \geq 0$. Then R is TP.*

Proof. The first few rows of the coefficient matrix of R are as follows:

$$J(R) = \begin{bmatrix} 1 & 1 & & & \\ u+h & e+h+u & 1 & & \\ (u+h)h & ue+eh+uh+h^2 & e+h+u & 1 & \\ (u+h)h^2 & (ue+eh+uh+h^2)h & ue+eh+uh+h^2 & e+h+u & \\ \vdots & \vdots & & \cdots & \ddots \end{bmatrix}.$$

A Riordan array is totally positive if its coefficient matrix is TP [6]. That means our next task is to show the total positivity of $J(R)$. Let

$$H = \begin{bmatrix} 1 & & & & \\ h & 1 & & & \\ h^2 & h & 1 & & \\ h^3 & h^2 & h & 1 & \\ \vdots & & \cdots & & \ddots \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & & & & \\ u & e+u & 1 & & & \\ & eu & e+u & 1 & & \\ & & eu & e+u & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}.$$

Then $J(R)$ is obtained from Q by adding h times each row to the succeeding row. It follows that $J(R) = HQ$. Note that the product of two totally positive matrices is TP. Hence, it suffices to show that H and Q are TP. The coefficient matrix of H is

$$J(H) = \begin{bmatrix} h & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & & \ddots \end{bmatrix}.$$

Thus H is TP since $J(H)$ is TP. Note that

$$Q = \begin{bmatrix} 1 & & & & & \\ u & 1 & & & & \\ & u & 1 & & & \\ & & u & 1 & & \\ & & & \ddots & \ddots & \end{bmatrix} \begin{bmatrix} 1 & 1 & & & & \\ & e & 1 & & & \\ & & e & 1 & & \\ & & & e & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}.$$

Clearly, Q is TP if $e, u \geq 0$. $\square$

Therefore, $S(e,h)$ is TP by the case of $u = 0$ in Lemma 2, and so $D(e,h)$ is, too.

3 Remarks

There are various total positivity properties of Riordan arrays. For example, the Pascal triangle P is a totally positive matrix and each row of P is a Pólya frequency sequence. We now consider similar properties of Delannoy-like triangles.

An infinite sequence $(a_n)_{n\geq 0}$ is called a *Pólya frequency sequence* (or shortly, a *PF* sequence) if its Toeplitz matrix $[a_{i-j}]_{i,j\geq 0}$ is TP. Pólya frequency sequences often arise in combinatorics [4]. A fundamental representation theorem of Schoenberg and Edrei states that a sequence $(a_n)_{n\geq 0}$ of real numbers is PF if and only if its generating function has the form

$$\sum_{n\geq 0} a_n z^n = \frac{\prod_{j\geq 1}(1+\alpha_j z)}{\prod_{j\geq 1}(1-\beta_j z)} e^{\gamma z}$$

in some open disk center at the origin, where $\alpha_j, \beta_j, \gamma \geq 0$ and $\sum_{j\geq 1}(\alpha_j + \beta_j) < +\infty$ (see Karlin [10, p. 412], for instance). Aissen, Schoenberg and Whitney showed that a finite sequence of nonnegative numbers is PF if and only if its generating function has only real zeros [10, p. 399].

We let $r_n(x)$ denote the nth row generating function $\sum_{k\geq 0} d_{n,k} x^k$ of a Delannoy-like triangle $D(e,h)$. Then $r_n(x)$ satisfies the recurrence

$$r_0(x) = 1,\ r_1(x) = e + x,\ r_n(x) = (e+x) r_{n-1} + hx r_{n-2}(x).$$

It is easy to check that $r_n(x)$ has only real zeros [12, Theorem 2.1]. Hence each row of a Delannoy-like triangle is a PF sequence.

Moreover, let $m_n(x)$ denote the nth diagonal generating function $\sum_{k\geq 0} d_{n+k,k} x^k$ of $D(e,h)$. Then $m_n(x)$ satisfies the recurrence

$$m_n(x) = \frac{e+hx}{1-x} m_{n-1}(x),$$

where $m_0(x) = 1/(1-x)$, and we have

$$m_n(x) = \frac{(e+hx)^{n-1}}{(1-x)^n}.$$

This means that each diagonal of Delannoy-like triangles is also a PF sequence.

Thus we have the following result:

Theorem 3. *Each row and diagonal of Delannoy-like triangles form PF sequences.*

In addition, the generating function of the nth column of $D(e,h)$ is

$$g(x) f^n(x) = \frac{(1+x-ex)(x+hx^2)^n}{(1-ex)^{n+1}}.$$

Hence each column of a Delannoy-like triangle is a PF sequence if $e = 0, 1$. It follows that each column of the Delannoy triangle, the Pascal triangle and the Fibonacci triangle is a PF sequence. However, not all lines of every Delannoy-like triangle are PF sequences. For example, the central Delannoy numbers, the central coefficient of the Delannoy triangle, is a log-convex sequence [2].

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