

# The Akiyama-Tanigawa algorithm for Bernoulli numbers

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## Abstract

*A direct proof is given for Akiyama and Tanigawa's algorithm for computing Bernoulli numbers. The proof uses a closed formula for Bernoulli numbers expressed in terms of Stirling numbers. The outcome of the same algorithm with different initial values is also briefly discussed.*

## 1 The Algorithm

In their study of values at non-positive integer arguments of multiple zeta functions, S. Akiyama and Y. Tanigawa [1] found as a special case an amusing algorithm for computing Bernoulli numbers in a manner similar to “Pascal’s triangle” for binomial coefficients.

Their algorithm reads as follows: Start with the 0-th row  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$  and define the first row by  $1 \cdot (1 - \frac{1}{2}), 2 \cdot (\frac{1}{2} - \frac{1}{3}), 3 \cdot (\frac{1}{3} - \frac{1}{4}), \dots$  which produces the sequence  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ . Then define the next row by  $1 \cdot (\frac{1}{2} - \frac{1}{3}), 2 \cdot (\frac{1}{3} - \frac{1}{4}), 3 \cdot (\frac{1}{4} - \frac{1}{5}), \dots$ , thus giving  $\frac{1}{6}, \frac{1}{6}, \frac{3}{20}, \dots$  as the second row. In general, denoting the  $m$ -th ( $m = 0, 1, 2, \dots$ ) number in the  $n$ -th ( $n = 0, 1, 2, \dots$ ) row by  $a_{n,m}$ , the  $m$ -th number in the  $(n + 1)$ -st row  $a_{n+1,m}$  is determined recursively by

$$a_{n+1,m} = (m + 1) \cdot (a_{n,m} - a_{n,m+1}).$$

Then the claim is that the 0-th component  $a_{n,0}$  of each row (the “leading diagonal”) is just the  $n$ -th Bernoulli numbers  $B_n$ , where

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{xe^x}{e^x - 1} \left( = \frac{x}{e^x - 1} + x \right).$$

Note that we are using the definition of the Bernoulli numbers in which  $B_1 = \frac{1}{2}$ . This is the definition used by Bernoulli (and independently Seki, published one year prior to Bernoulli). Incidentally, this is more appropriate for the Euler formula  $\zeta(1 - k) = -B_k/k$  ( $k = 1, 2, 3, \dots$ ) for the values of the Riemann zeta function.

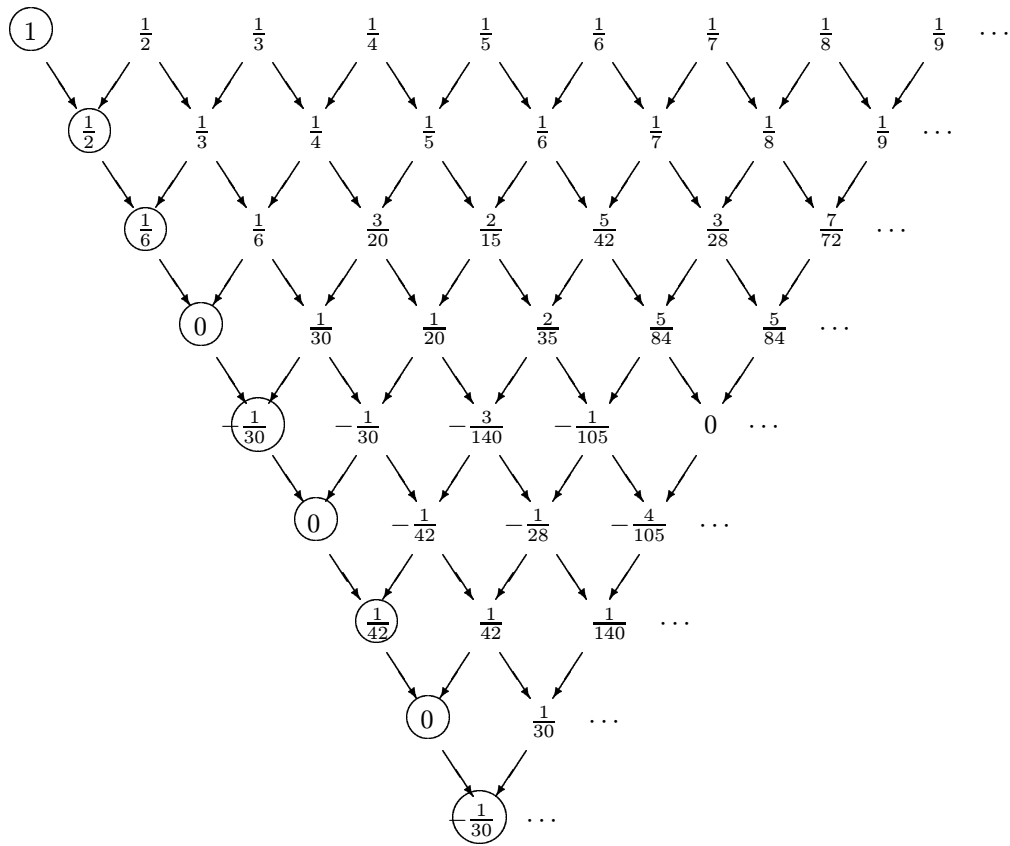


Figure 1: Akiyama-Tanigawa triangle

## 2 Proof

The proof is based on the following identity for Bernoulli numbers, a variant of which goes as far back as Kronecker (see [4]). Here we denote by  $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$  the Stirling number of the second kind:

$$x^n = \sum_{m=0}^n \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} x^{\underline{m}},$$

where  $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$  for  $m \geq 1$  and  $x^{\underline{0}} = 1$ . (We use Knuth's notation [7]. For the Stirling number identities that we shall need, the reader is referred for example to [5].)

**Theorem 1**

$$B_n = \sum_{m=0}^n \frac{(-1)^m m! \left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\}}{m+1}, \quad \forall n \geq 0.$$

We shall give later a proof of this identity for the sake of completeness. Once we have this, the next proposition ensures the validity of our algorithm.

**Proposition 2** *Given an initial sequence  $a_{0,m}$  ( $m = 0, 1, 2, \dots$ ), define the sequences  $a_{n,m}$  ( $n \geq 1$ ) recursively by*

$$a_{n,m} = (m+1) \cdot (a_{n-1,m} - a_{n-1,m+1}) \quad (n \geq 1, m \geq 0). \quad (1)$$

Then

$$a_{n,0} = \sum_{m=0}^n (-1)^m m! \left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} a_{0,m}. \quad (2)$$

*Proof.* Put

$$g_n(t) = \sum_{m=0}^{\infty} a_{n,m} t^m.$$

By the recursion (1) we have for  $n \geq 1$

$$\begin{aligned} g_n(t) &= \sum_{m=0}^{\infty} (m+1)(a_{n-1,m} - a_{n-1,m+1})t^m \\ &= \frac{d}{dt} \left( \sum_{m=0}^{\infty} a_{n-1,m} t^{m+1} \right) - \frac{d}{dt} \left( \sum_{m=0}^{\infty} a_{n-1,m+1} t^{m+1} \right) \\ &= \frac{d}{dt} (t g_{n-1}(t)) - \frac{d}{dt} (g_{n-1}(t) - a_{n-1,0}) \\ &= g_{n-1}(t) + (t-1) \frac{d}{dt} (g_{n-1}(t)) \\ &= \frac{d}{dt} ((t-1) g_{n-1}(t)). \end{aligned}$$

Hence, by putting  $(t-1)g_n(t) = h_n(t)$  we obtain

$$h_n(t) = (t-1) \frac{d}{dt} (h_{n-1}(t)) \quad (n \geq 1),$$

and thus

$$h_n(t) = \left( (t-1) \frac{d}{dt} \right)^n (h_0(t)).$$

Applying the formula (cf. [5, Ch. 6.7 Exer. 13])

$$\left( x \frac{d}{dx} \right)^n = \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} x^m \left( \frac{d}{dx} \right)^m$$

for  $x = t-1$ , we have

$$h_n(t) = \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (t-1)^m \left( \frac{d}{dt} \right)^m h_0(t).$$

Putting  $t = 0$  we obtain

$$\begin{aligned} -a_{n,0} &= \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (-1)^m m! (a_{0,m-1} - a_{0,m}) \\ &= \sum_{m=0}^{n-1} \left\{ \begin{matrix} n \\ m+1 \end{matrix} \right\} (-1)^{m+1} (m+1)! a_{0,m} - \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} (-1)^m m! a_{0,m} \\ &= - \sum_{m=0}^n (-1)^m m! a_{0,m} \left( (m+1) \left\{ \begin{matrix} n \\ m+1 \end{matrix} \right\} + \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \right) \\ &= - \sum_{m=0}^n (-1)^m m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} a_{0,m}. \end{aligned}$$

(We have used the recursion  $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = (m+1) \left\{ \begin{matrix} n \\ m+1 \end{matrix} \right\} + \left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ .) This proves the proposition.

*Proof of Theorem 1.* We show the generating series of the right hand side coincide with that of  $B_n$ . To do this, we use the identity

$$\frac{e^x (e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \frac{x^n}{n!} \quad (3)$$

which results from the well-known generating series for the Stirling numbers (cf. [5, (7.49)])

$$\frac{(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{x^n}{n!}$$

by replacing  $m$  with  $m + 1$  and differentiating with respect to  $x$ . With this, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \frac{(-1)^m m! \{m+1\}^{n+1}}{m+1} \right) \frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m m!}{m+1} \sum_{n=m}^{\infty} \{m+1\} \frac{x^n}{n!} = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{m+1} \frac{e^x (e^x - 1)^m}{m!} \\
&= e^x \sum_{m=0}^{\infty} \frac{(1 - e^x)^m}{m+1} = \frac{e^x}{1 - e^x} \sum_{m=1}^{\infty} \frac{(1 - e^x)^m}{m} \\
&= \frac{e^x}{1 - e^x} (-\log(1 - (1 - e^x))) = \frac{x e^x}{e^x - 1}.
\end{aligned}$$

This proves Theorem 1.

**Remark.** A referee suggested the following interpretation of the algorithm using generating function:

Suppose the first row is  $a_0, a_1, a_2, \dots$ , with ordinary generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Let the leading diagonal be  $b_0 = a_0, b_1, b_2, \dots$ , with exponential generating function

$$\mathbb{B}(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}.$$

Then we have

$$\mathbb{B}(x) = e^x A(1 - e^x).$$

This follows from (2) and (3), the calculation being parallel to that of the proof of Theorem 1. To get the Bernoulli numbers we take  $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, \dots$  with  $A(x) = -\log(1 - x)/x$ , and find  $\mathbb{B}(x) = x e^x / (e^x - 1)$ .

### 3 Poly-Bernoulli numbers

If we replace the initial sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$  by  $1, \frac{1}{2^k}, \frac{1}{3^k}, \frac{1}{4^k}, \dots$  and apply the same algorithm, the resulting sequence is  $(-1)^n D_n^{(k)}$  ( $n = 0, 1, 2, \dots$ ), where  $D_n^{(k)}$  is a variant of ‘‘poly-Bernoulli numbers’’ ([6], [2], [3]): For any integer  $k$ , we define a sequence of numbers  $D_n^{(k)}$  by

$$\frac{Li_k(1 - e^{-x})}{e^x - 1} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{x^n}{n!},$$

where  $Li_k(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^k}$  ( $k$ -th polylogarithm when  $k \geq 1$ ). The above assertion is then a consequence of the following (or, is just a special case of the preceding remark)

**Proposition 3** For any  $k \in \mathbf{Z}$  and  $n \geq 0$ , we have

$$D_n^{(k)} = (-1)^n \sum_{m=0}^n \frac{(-1)^m m! \left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\}}{(m+1)^k}.$$

*Proof.* The proof can be given completely in the same way as the proof of Theorem 1 using generating series, and hence will be omitted.

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(The Bernoulli numbers are A027641/A027642. The table in Figure 1 yields sequences A051714/A051715. Other sequences which mention this paper are A000367, A002445, A026741, A045896, A051712, A051713, A051716, A051717, A051718, A051719, A051720, A051721, A051722, A051723.)

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