Journal of Integer Sequences, Vol. 9 (2006),
Article 06.4.8

Infinite Sets of Integers Whose Distinct Elements Do Not Sum to a Power

Artūras Dubickas and Paulius Šarka
Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
Vilnius LT-03225
Lithuania
arturas.dubickas@maf.vu.lt
paulius.sarka@mif.vu.lt

Abstract

We first prove two results which both imply that for any sequence B of asymptotic density zero there exists an infinite sequence A such that the sum of any number of distinct elements of A does not belong to B. Then, for any $\varepsilon > 0$, we construct an infinite sequence of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ satisfying $a_n < K(\varepsilon)(1+\varepsilon)^n$ for each $n \in \mathbb{N}$ such that no sum of some distinct elements of A is a perfect square. Finally, given any finite set $U \subset \mathbb{N}$, we construct a sequence A of the same growth, namely, $a_n < K(\varepsilon, U)(1+\varepsilon)^n$ for every $n \in \mathbb{N}$ such that no sum of its distinct elements is equal to uv^s with $u \in U$, $v \in \mathbb{N}$ and $s \geqslant 2$.

1 Introduction

Let $B = \{b_1 < b_2 < b_3 < \dots\}$ be an infinite sequence of positive integers. In this note we are interested in the following two questions.

- For which B there exists an infinite sequence of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ such that $a_{i_1} + \dots + a_{i_m} \notin B$ for every $m \in \mathbb{N}$ and any distinct elements $a_{i_1}, \dots, a_{i_m} \in A$?
- In the case when the answer is 'yes', how dense the sequence A can be?

In his paper [2], F. Luca considered the case when B is the set of all perfect squares $\{1, 4, 9, 16, 25, 36, \dots\}$ and of all perfect powers $\{1, 4, 8, 9, 16, 25, 27, 32, 36, \dots\}$. He showed that in both cases the answer to the first question is 'yes'. In particular, it was observed in [2] that the sum of any distinct Fermat numbers $2^{2^n} + 1$, $n = 1, 2, \dots$, is not a perfect square. Moreover, it was proved that the sum of any distinct numbers of the form $a^{p_1 p_2 \cdots p_n} + 1$, $n = n_0, n_0 + 1, \dots$, where $a \geqslant 2$ is an integer, p_k is the kth prime number and $n_0 = n_0(a)$ is an effectively computable constant, cannot be a perfect power.

2 Sets with asymptotic density zero

We begin with the following observation (see also [1]) which settles the first of the two problems stated above for every set B satisfying $\limsup_{n\to\infty}(b_{n+1} - b_n) = \infty$.

Theorem 2.1. *Let $m \in \mathbb{N}$ and let $B = \{b_1 < b_2 < b_3 < \dots\}$ be an infinite sequence of positive integers satisfying $\limsup_{n\to\infty}(b_{n+1} - mb_n) = \infty$. Then there exists an infinite sequence of positive integers A such that every sum over some elements of A, at most m of which are equal, is not in B.*

Proof. Take the smallest positive integer ℓ such that $b_{\ell+1} - b_\ell \geqslant 2$, and set $a_1 := b_\ell + 1$. Then $a_1 \notin B$. Suppose we already have a finite set $\{a_1 < a_2 < \dots < a_k\}$ such that all possible $(m+1)^k - 1$ nonzero sums $\delta_1 a_1 + \dots + \delta_k a_k$, where $\delta_1, \dots, \delta_k \in \{0, 1, \dots, m\}$, do not belong to B. Put $a_{k+1} := b_l + 1$, where l is the smallest positive integer for which $b_{l+1} - mb_l \geqslant 1 + m + m(a_1 + \dots + a_k)$ and $b_l \geqslant a_k$. Such an l exists, because $\limsup_{n\to\infty}(b_{n+1} - mb_n) = \infty$.

Clearly, $b_l \geqslant a_k$ implies that $a_{k+1} > a_k$. In order to complete the proof of the theorem (by induction) it suffices to show that no sum of the form $\delta_1 a_1 + \dots + \delta_k a_k + \delta_{k+1} a_{k+1}$, where $\delta_1, \dots, \delta_{k+1} \in \{0, 1, \dots, m\}$, lies in B. If $\delta_{k+1} = 0$, this follows by our assumption, so suppose that $\delta_{k+1} \geqslant 1$. Then $\delta_1 a_1 + \dots + \delta_k a_k + \delta_{k+1} a_{k+1}$ is greater than $a_{k+1} - 1 = b_l$ and smaller than

$$1 + m(a_1 + \dots + a_k + a_{k+1}) \leqslant b_{l+1} - mb_l - m + ma_{k+1} = b_{l+1} - mb_l - m + m(b_l + 1) = b_{l+1},$$

so it is not in B, as claimed. □ □

Recall that the *upper asymptotic density* $\overline{d}(B)$ of the sequence B is defined as

$$\overline{d}(B) = \limsup_{N\to\infty} \frac{\#\{n \in \mathbb{N} : b_n \leqslant N\}}{N}$$

(see, e.g., 1.2 in [4]). Similarly, the *lower asymptotic density* $\underline{d}(B)$ is defined as $\underline{d}(B) = \liminf_{N\to\infty} N^{-1}\#\{n \in \mathbb{N} : b_n \leqslant N\}$. If $\overline{d}(B) = \underline{d}(B)$, then the common value $d(B) = \overline{d}(B) = \underline{d}(B)$ is said to be the *asymptotic density* of B.

Evidently, if B has asymptotic density zero then, for any positive integer k, there are infinitely many positive integers N such that the numbers $N + 1, N + 2, \dots, N + k$ do not lie in B. This implies that the condition $\limsup_{n\to\infty}(b_{n+1} - b_n) = \infty$ holds. Hence, by Theorem 2.1 with $m = 1$, for any sequence B of asymptotic density zero there exists an infinite sequence A such that the sum of any number of distinct elements of A is not in B. It is well-known that the sequence of perfect powers has asymptotic density zero, so such an A as claimed exists for $B = \{1, 4, 8, 9, 16, 25, 27, 32, 36, \dots\}$.

For $m \geqslant 2$, it can very often happen that $b_{n+1} < mb_n$ for every $n \in \mathbb{N}$. For such a set B Theorem 2.1 is not applicable. However, its conclusion is true for any set B of asymptotic density zero.

Theorem 2.2. *Let $m \in \mathbb{N}$ and let B be an infinite sequence of positive integers of asymptotic density zero. Then there exists an infinite sequence of positive integers A such that every sum over some elements of A, at most m of which are equal, is not in B.*

Proof. Once again, take the smallest positive integer ℓ such that $b_{\ell+1} - b_\ell \geqslant 2$, and put $a_1 := b_\ell + 1$. Then $a_1 \notin B$. Suppose we already have a finite set $\{a_1 < a_2 < \dots < a_k\}$ such that all possible $(m+1)^k - 1$ nonzero sums $\delta_1 a_1 + \dots + \delta_k a_k$, where $\delta_1, \dots, \delta_k \in \{0, 1, \dots, m\}$, do not belong to B. It suffices to prove that there exists an integer a_{k+1} greater than a_k such that, for every $i \in \{1, \dots, m\}$, the sum $ia_{k+1} + \delta_k a_k + \dots + \delta_1 a_1$, where $\delta_1, \dots, \delta_k \in \{0, 1, \dots, m\}$, is not in B.

Suppose that $B = \{b_1 < b_2 < b_3 < \dots\}$. For any $h \in \mathbb{N}$, the set $\{hb_1 < hb_2 < hb_3 < \dots\}$ will be denoted by hB. Put $B_i := \frac{m!}{i} B$ for $i = 1, 2, \dots, m$. Since $d(B_i) = 0$ for each $i = 1, \dots, m$, we have $d(B_1 \cup \dots \cup B_m) = 0$. Thus, for any $v > m!(mS+1)$, where $S := a_1 + \dots + a_k$, there is an integer $u > m!a_k$ such that the interval $[u, u+v]$ is free of the elements of the set $B_1 \cup \dots \cup B_m$.

Put $a_{k+1} := \lfloor u/m! \rfloor + 1$. Clearly, $a_{k+1} > a_k$. Furthermore, for any $i \in \{1, \dots, m\}$, no element of B_i lies in $[u, u+v]$. Thus there is a nonnegative integer $j = j(i)$ such that $m!b_j/i < u$ and $m!b_{j+1}/i > u+v$. (Here, for convenience of notation, we assume that $b_0 = 0$.) Hence $ia_{k+1} > iu/m! > b_j$ and

$$ia_{k+1} + mS < ia_{k+1} + imS \leqslant i(u/m! + 1 + mS) < i(u+v)/m! < b_{j+1}.$$

In particular, these inequalities imply that, for each $i \in \{1, \dots, m\}$, the sum $ia_{k+1} + \delta_k a_k + \dots + \delta_1 a_1$, where $\delta_1, \dots, \delta_k \in \{0, 1, \dots, m\}$, is between $b_{j(i)} + 1$ and $b_{j(i)+1} - 1$, hence it is not in B. This completes the proof of the theorem. □ □

Several examples illustrating Theorem 2 will be given in Section 5. In particular, for any $\varepsilon > 0$, there is a set $B \subset \mathbb{N}$ with asymptotic density $d(B) < \varepsilon$ such that for any infinite set $A \subseteq \mathbb{N}$ some of its distinct elements sum to an element lying in B. On the other hand, there are sets $B \subseteq \mathbb{N}$ with asymptotic density 1 for which there exists an infinite set A whose distinct elements do not sum to an element lying in B.

3 Infinite sets whose elements do not sum to a square

The second question concerning the 'densiest' sequence A for a fixed B seems to be much more subtle. It seems likely that this question is very difficult already for the above mentioned sequence of perfect squares $\{1, 4, 9, 16, 25, 36, \dots\}$. The example of Fermat numbers $2^{2^n} + 1$, $n = 1, 2, \dots$, given above is clearly not satisfactory, because this sequence grows very rapidly.

In this sense, much better is the sequence 2^{2n-1}, $n = 1, 2, \dots$. The sum of its distinct elements

$$2^{2n_1-1} + \dots + 2^{2n_l-1} = 2^{2n_1-1}(1 + 4^{n_2-n_1} + \dots + 4^{n_l-n_1}),$$

where $1 \leqslant n_1 < \dots < n_l$, is not a perfect square, because it is divisible by 2^{2n_1-1}, but not divisible by 2^{2n_1}.

Smaller, but still of exponential growth, is the sequence $2 \cdot 3^n$, $n = 0, 1, 2, \dots$. No sum of its distinct elements is a perfect square, because

$$2(3^{n_1} + \dots + 3^{n_l}) = 2 \cdot 3^{n_1}(1 + 3^{n_2-n_1} + \dots + 3^{n_l-n_1}) = h^2$$

implies that n_1 is even, so $2(1 + 3^{n_2-n_1} + \dots + 3^{n_l-n_1})$ must be a square too. However, this number is of the form $3k + 2$ with integer k, so it is not a perfect square.

A natural way to generate an infinite sequence whose distinct elements do not sum to square is to start with $c_1 = 2$. Then, for each $n \in \mathbb{N}$, take the smallest positive integer c_{n+1} such that no sum of the form $c_{n+1} + \delta_n c_n + \dots + \delta_1 c_1$, where $\delta_1, \dots, \delta_n \in \{0, 1\}$, is a perfect square. Clearly, $c_2 = 3$, $c_3 = 5$. Then, as $6 + 3 = 3^2$, $7 + 2 = 3^2$, $8 + 5 + 3 = 4^2$, $9 = 3^2$, we obtain that $c_4 = 10$, and so on. In the following table we give the first 18 elements of this sequence:

n	c_n	$\log c_n$	n	c_n	$\log c_n$
1	2	0.6931	10	2030	7.6157
2	3	1.0986	11	3225	8.0786
3	5	1.6094	12	8295	9.0234
4	10	2.3025	13	15850	9.6709
5	27	3.2958	14	80642	11.2977
6	38	3.6375	15	378295	12.8434
7	120	4.7874	16	1049868	13.8641
8	258	5.5529	17	3031570	14.9245
9	907	6.8101	18	12565348	16.3464

Here, the values of $\log c_n$ are truncated at the fourth decimal place. At the first glance, they suggest that the limit $\liminf_{n\to\infty} n^{-1} \log c_n$ is positive. If so, then the sequence c_n, $n = 1, 2, 3, \dots$, is of exponential growth too. It seems that the sequence c_n, $n = 1, 2, 3, \dots$, i.e.,

$$2, 3, 5, 10, 27, 38, 120, 258, 907, 2030, 3225, 8295, 15850, 80642, 378295, 1049868, \dots$$

was not studied before. At least, it is not given in N.J.A. Sloane's on-line encyclopedia of integer sequences `http://www.research.att.com/~njas/sequences/`. We thus raise the following problem.

- Determine whether $\liminf_{n\to\infty} n^{-1} \log c_n$ is zero or a positive number.

In the opposite direction, one can easily show that $c_n < 4^n$ for each $n \geqslant 1$. Here is the proof of this inequality by induction (due to a referee). Suppose that $c_n < 4^n$. If $c_{n+1} \leqslant c_n + 4^n$, then $c_{n+1} < 4^n + 4^n < 4^{n+1}$. Otherwise, for each $j = 1, 2, \dots, 4^n$, there exists a set $I = I_j \subseteq \{1, 2, \dots, n\}$ such that $c_n + j + S(I) = s_j^2$, where $S(I) := \sum_{i \in I} c_i$ and $s_j \in \mathbb{N}$. There are 2^n different subsets I of $\{1, 2, \dots, n\}$, so the set $\{4^n - 2^n, \dots, 4^n - 1, 4^n\}$ with $2^n + 1$ elements contains some two indices $j < j'$ for which the corresponding subsets I (and so the values for $S(I)$) are equal. Subtracting $c_n + j + S(I) = s_j^2$ from $c_n + j' + S(I) = s_{j'}^2$, we deduce that $j' - j = (s_{j'} - s_j)(s_{j'} + s_j)$. Since $j' - j \leqslant 2^n$, we have $s_{j'} + s_j \leqslant 2^n$, i.e., $s_{j'} \leqslant 2^n - 1$. Hence

$$4^n - 2^n < j' < c_n + j' + S(I) = s_{j'}^2 \leqslant (2^n - 1)^2 = 4^n - 2^{n+1} + 1,$$

a contradiction.

Of course, $c_n < 4^n$ implies that $\limsup_{n \to \infty} n^{-1} \log c_n < \log 4$. Our next theorem shows that, for any fixed positive ε, there is a sequence $A = \{a_1 < a_2 < a_3 < \dots\}$ whose distinct elements do not sum to a square and whose growth is small in the sense that $\limsup_{n \to \infty} n^{-1} \log a_n < \varepsilon$.

Theorem 3.1. *For any $\varepsilon > 0$ there is a positive constant $K = K(\varepsilon)$ and an infinite sequence $A = \{a_1 < a_2 < a_3 < \dots\} \subset \mathbb{N}$ satisfying $a_n < K(1 + \varepsilon)^n$ for each $n \in \mathbb{N}$ such that the sum of any number of distinct elements of A is not a perfect square.*

Proof. Fix a prime number p to be chosen later and consider the following infinite set

$$A := \{gp^{2m} + p^{2m-1} : g \in \{0, 1, \dots, p-2\},\ m \in \mathbb{N}\}.$$

Each element of A in base p can be written as $\overline{g100\dots0}$ with $2m - 1$ zeros, where the 'digit' g is allowed to be zero. So all the elements of A are distinct.

First, we will show that the sum of any distinct elements of A is not a perfect square. Assume that there exists a sum S which is a perfect square. Suppose that for every $t = 1, 2, \dots, l$ the sum S contains $s_t > 0$ elements of the form $gp^{2m_t} + p^{2m_t - 1}$, where $g \in \{0, 1, \dots, p-2\}$ and $1 \leqslant m_1 < m_2 < \dots < m_l$. Clearly, $s_t \leqslant p - 1$. Let us write S in the form

$$S = s_1 p^{2m_1 - 1} + h_1 p^{2m_1} + s_2 p^{2m_2 - 1} + h_2 p^{2m_2} + \dots + s_l p^{2m_l - 1} + h_l p^{2m_l}$$
$$= p^{2m_1 - 1}(s_1 + h_1 p + \dots + s_l p^{2m_l - 2m_1} + h_l p^{2m_l - 2m_1 + 1}) = p^{2m_1 - 1}(s_1 + pH).$$

Now, since $s_1 \in \{1, \dots, p-1\}$ and since H is an integer, we see that S is divisible by $p^{2m_1 - 1}$, but not by p^{2m_1}, so it is not a perfect square.

It remains to estimate the size of the nth element a_n of A. Write n in the form $n = (p-1)(m-1) + r$, where $r \in \{1, \dots, p-2, p-1\}$ and $m \geqslant 1$ is an integer. Suppose that the elements of A are divided into consecutive equal blocks with $p - 1$ elements in each block. Then all the elements of the mth block are of the form $\overline{g100\dots0}$ (with $2m - 1$ zeros), where $g = 0, 1, \dots, p-2$. Hence the nth element of A, where $n = (p-1)(m-1) + r$, is precisely the rth element of the mth block, i.e., $a_n = a_{(p-1)(m-1)+r} = (r-1)p^{2m} + p^{2m-1}$. It follows that

$$a_n \leqslant (p-2)p^{2m} + p^{2m-1} < p^{2m+1} = p^{2(n-r)/(p-1)+3} < p^{2n/(p-1)+3} = p^3 e^{(2n \log p)/(p-1)}.$$

Clearly, $(2\log p)/(p-1) \to 0$ as $p \to \infty$. Thus, for any $\varepsilon > 0$, there exists a prime number p such that $e^{(2\log p)/(p-1)} < 1+\varepsilon$. Take the smallest such a prime $p = p(\varepsilon)$. Setting $K(\varepsilon) := p(\varepsilon)^3$, we obtain that $a_n < K(\varepsilon)(1+\varepsilon)^n$ for each $n \in \mathbb{N}$. □ □

4 Infinite sets whose elements do not sum to a power

Observe that distinct elements of the sequence $2 \cdot 6^n$, $n = 0, 1, 2, \dots$, cannot sum to a perfect power. Indeed,

$$S = 2(6^{n_1} + \dots + 6^{n_l}) = 2^{n_1+1}3^{n_1}(1 + 6^{n_2-n_1} + \dots + 6^{n_l-n_1}),$$

where $0 \leqslant n_1 < \dots < n_l$, is not a perfect power, because $n_1 + 1$ and n_1 are exact powers of 2 and 3 in the prime decomposition of S. So if $S > 1$ were a kth power, where k is a prime number (which can be assumed without loss of generality), then both $n_1 + 1$ and n_1 must be divisible by k, a contradiction.

This example is already 'better' than the example $a^{p_1 p_2 \cdots p_n} + 1$, $n = n_0, n_0 + 1, \dots$, given in [2] not only because it is completely explicit, but also because the sequence $2 \cdot 6^n$, $n = 0, 1, 2, \dots$, grows slower.

As above, we can also consider the sequence $2, 3, 10, 18, \dots$, starting with $e_1 = 2$, whose each 'next' element $e_{n+1} > e_n$, where $n \geqslant 1$, is the smallest positive integer preserving the property that no sum of the form $\delta_1 e_1 + \dots + \delta_n e_n + e_{n+1}$, where $\delta_1, \dots, \delta_n \in \{0, 1\}$, is a perfect power. By an argument which is slightly more complicated than the one given for c_n, one can prove again that $e_n < 4^n$ for n large enough.

However, our aim is to prove the existence of the sequence whose nth element is bounded from above by $K(\varepsilon)(1+\varepsilon)^n$ for $n \in \mathbb{N}$. For this, we shall generalize Theorem 2 as follows:

Theorem 4.1. *Let U be the set of positive integers of the form $q_1^{\alpha_1} \dots q_k^{\alpha_k}$, where $q_1, \dots, q_k$ are some fixed prime numbers and $\alpha_1, \dots, \alpha_k$ run through all nonnegative integers. Then, for any $\varepsilon > 0$, there is a positive constant $K = K(\varepsilon, U)$ and an infinite sequence $A = \{a_1 < a_2 < a_3 < \dots\} \subset \mathbb{N}$ satisfying $a_n < K(1+\varepsilon)^n$ for $n \in \mathbb{N}$ such that the sum of any number of distinct elements of A is not equal to uv^s with positive integers u, v, s such that $u \in U$ and $s \geqslant 2$.*

In particular, Theorem 3 with $U = \{1\}$ implies a more general version of Theorem 2 with 'perfect square' replaced by 'perfect power'.

Proof. Fix two prime numbers p and q satisfying $p < q < 2p$. Here, the prime number p will be chosen later, whereas, by Bertrand's postulate, the interval $(p, 2p)$ always contains at least one prime number, so we can take q to be any of those primes. Consider the following infinite set

$$A := \{gp^{m+1}q^m + p^m q^{m-1} : g \in \{1, \dots, p-1\},\ m \in \mathbb{N}\}.$$

The inequality $p^{m+2}q^{m+1} + p^{m+1}q^m > (p-1)p^{m+1}q^m + p^m q^{m-1}$ implies that all the elements of A are distinct. Also, as above, by dividing the sequence A into consecutive equal blocks with $p-1$ elements each, we find that

$$a_n = rp^{m+1}q^m + p^m q^{m-1}$$

for $n = (p-1)(m-1) + r$, where $m \in \mathbb{N}$ and $r \in \{1, \dots, p-2, p-1\}$.

Assume that there exists a sum S of some distinct a_n which is of the form uv^s. Without loss of generality we may assume that $s \geqslant 2$ is a prime number. Suppose that for every $t = 1, 2, \dots, l$ the sum S contains $s_t > 0$ elements of the form $gp^{m_t+1}q^{m_t} + p^{m_t}q^{m_t-1}$, where $g \in \{1, \dots, p-1\}$ and $1 \leqslant m_1 < m_2 < \dots < m_l$. Clearly, $s_t \leqslant p-1$, so, in particular, $1 \leqslant s_1 \leqslant p-1$. Then, as above, $S = p^{m_1}q^{m_1-1}(s_1 + pqH)$ with an integer H. If $q > p > q_k$, then $p, q \notin U$, so the equality $uv^s = p^{m_1}q^{m_1-1}(s_1 + pqH)$ implies that $s|m_1$ and $s|(m_1 - 1)$, a contradiction.

Using $a_n = rp^{m+1}q^m + p^m q^{m-1}$, where $n = (p-1)(m-1) + r$ and $p < q < 2p$, we find that

$$a_n < (p-1)q^{2m+1} + q^{2m-1} < q^{2m+2} < (2p)^{2(n-r)/(p-1)+4} < (2p)^4 e^{(2n\log(2p))/(p-1)}.$$

For any $\varepsilon > 0$, there exists a positive number p_ε such that $e^{(2\log(2p))/(p-1)} < 1 + \varepsilon$ for each $p > p_\varepsilon$. Take the smallest prime number $p = p(\varepsilon)$ greater than $\max\{p_\varepsilon, q_k\}$, and put $K(\varepsilon, q_k) = K(\varepsilon, U) := 2p(\varepsilon)^4$. Then $a_n < K(\varepsilon, U)(1+\varepsilon)^n$ for each $n \in \mathbb{N}$, as claimed. □ □

5 Concluding remarks

We do not give any lower bounds for the nth element a_n of the 'densiest' sequence $A = \{a_1 < a_2 < \dots\}$ whose distinct elements do not sum to a square or, more generally, to a power. As a first step towards solution of this problem, it would be of interest to find out whether every infinite sequence of positive integers A which has a positive asymptotic density (i.e., $d(A) > 0$) contains some elements that sum to a square. It is essential that we can only sum distinct elements of A, because, for any nonempty set $A \subset \mathbb{N}$, there is a sumset $A + A + \dots + A$ which contains a square. In this direction, we can mention the following result of T. Schoen [3]: if A is a set of positive integers with asymptotic density $d(A) > 2/5$ then the sumset $A + A$ contains a perfect square. For more references on sumset related results see the recent book [5] of T. Tao and V. H. Vu.

A 'finite version' of the problem on the 'densiest' set whose elements do not sum to a square was recently considered by J. Cilleruelo [1]. He showed that there is an absolute positive constant c such that, for any positive integer $N \geqslant 2$, there exists a subset A of $\{1, 2, \dots, N\}$ with $\geqslant cN^{1/3}$ elements whose distinct elements do not sum to a perfect square. In fact, by taking the largest prime number $p \leqslant N^{1/3}$, we see that the set $A := \{p, p^2 + p, 2p^2 + p, \dots, (p-2)p^2 + p\}$ with $p-1$ element is a subset of $\{1, 2, \dots, N\}$. Since any sum of distinct elements of A is divisible by p, but not by p^2, we conclude that no sum of distinct elements of the set A of cardinality $p - 1 \geqslant \frac{1}{2}N^{1/3}$ is a perfect power.

Notice that in this type of questions not everything is determined by the density of B. In fact, there are some 'large' sets B for which there is a 'large' set A whose elements do not sum to an integer lying in B. For example, for the set of all odd positive integers $B = \{1, 3, 5, 7, \dots\}$ whose density $d(B)$ is $1/2$, the 'densiest' set A whose elements do not sum to an odd number is the set of all even positive integers $\{2, 4, 6, 8, \dots\}$ with density $d(A) = 1/2$. On the other hand, taking $B = \{2, 4, 6, 8, \dots\}$, we see that no infinite sequence A as required exists. Moreover, if B is the set of all positive integers divisible by m, where $m \in \mathbb{N}$ is large, then the density $d(B) = 1/m$ is small. However, by a simple argument modulo m, it is easy to see that there is no infinite set $A \subset \mathbb{N}$ (and even no set A with $\geqslant m$ distinct positive integers) with the property that its distinct elements always sum to a number lying outside B. Indeed, if $a_1, \dots, a_m \in \mathbb{N}$ then either at least two of the following m numbers $S_j := \sum_{i=1}^{j} a_i$, where $j = 1, \dots, m$, say, S_u and S_v ($u < v$, $u, v \in \{1, \dots, m\}$) are equal modulo m or $m | S_t$, where $t \in \{1, \dots, m\}$. Therefore, either their difference $S_v - S_u = a_{u+1} + a_{u+2} + \dots + a_v$ or $S_t = a_1 + \dots + a_t$ is divisible by m. In both cases, there is a sum of distinct elements of $\{a_1, a_2, \dots, a_m\}$ that lies in B.

It follows that if, for an infinite set $B \subset \mathbb{N}$, there exists an infinite sequence of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ for which $a_{i_1} + \dots + a_{i_m} \notin B$ for every $m \in \mathbb{N}$ and any distinct elements $a_{i_1}, \dots, a_{i_m} \in A$, then B must have the following property. For each $m \in \mathbb{N}$ there are infinitely many $k \in \mathbb{N}$ such that $km \notin B$.

This necessary condition is not sufficient. Take, for instance, $B := \mathbb{N} \setminus \{j^2 : j \in \mathbb{N}\}$. Then, for each $m \in \mathbb{N}$, there are infinitely many positive integers k, say, $k = \ell^2 m$, where $\ell = 1, 2, \dots,$ such that $km = (\ell m)^2 \notin B$. However, there does not exist an infinite set of positive integers $A = \{a_1 < a_2 < a_3 < \dots\}$ such that for any $n \in \mathbb{N}$ and any distinct $a_{i_1}, \dots, a_{i_n} \in A$ the sum $a_{i_1} + \dots + a_{i_n}$ is a perfect square. See, e.g., the proposition in the same paper [2], where this was proved in a more general form with 'perfect square' replaced by 'perfect power'.

Given any infinite set $B \subset \mathbb{N}$, put $K := \mathbb{N} \setminus B$. Our first question stated in the introduction can be also formulated in the following equivalent form.

- For which $K = \{k_1 < k_2 < k_3 < \dots\} \subset \mathbb{N}$ there exists an infinite subsequence of $\{k_{i_1} < k_{i_2} < k_{i_3} < \dots\}$ of K such that all possible sums over its distinct elements lie in K?

Theorem 2.1 implies that if $d(K) = 1$ then such a subsequence exists. On the other hand, take the sequence K of positive integers that are not divisible by m with asymptotic density $d(K) = 1 - 1/m$ (which is 'close' to 1 if m is 'large'). Then such a subsequence does not exist despite of $d(K)$ being large. Finally, set $D := \{2^{2^j} : j \in \mathbb{N}\}$ and suppose that K is the set of all possible finite sums over distinct elements of D. Then $d(K)$ is easily seen to be 0, but for K such a subsequence exists, e.g., D.

6 Acknowledgments

We are most grateful to the referee of this paper, who not only carefully read the paper, but also made several useful suggestions and also supplied us with a few recent references. We thank A. Stankevičius, who computed the first 18 elements of the sequence c_n, $n = 1, 2, \dots$. This research was supported in part by the Lithuanian State Studies and Science Foundation.

References

[1] J. Cilleruelo, Solution of the Problem 38, *Gaceta de la Real Sociedad Matematica Española* **9** (2006), 455–460.

[2] F. Luca, Infinite sets of positive integers whose sums are free of powers, *Rev. Colomb. Matem.* **36** (2002), 67–70.

[3] T. Schoen, On sets of natural numbers whose sumset is free of squares, *J. Comb. Theory, Ser. A* **88** (1999), 385–388.

[4] O. Strauch and Š. Porubský, *Distribution of Sequences: A Sampler*, Schriftenreihe der Slowakischen Akademie der Wissenschaften, Peter Lang, 2005.

[5] T. Tao and V. H. Vu, *Additive Combinatorics,* CUP, 2006.

2000 *Mathematics Subject Classification*: Primary 11A99; Secondary 11B05, 11B99.
Keywords: infinite sequence, perfect square, power, asymptotic density, sumset.

Received November 13 2006; revised version received December 4 2006. Published in *Journal of Integer Sequences*, December 4 2006.

Return to Journal of Integer Sequences home page.