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On the Distribution of Perfect Totients

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Abstract

In this paper, we study the sum of iterates of the Euler function.

1 Introduction

For every positive integer n we put $\phi(n)$ for the Euler function of n. If $k \ge 1$ we put $\phi^{(k)}(n)$ for the kth iterate of the Euler function evaluated in n and $\kappa(n)$ for the smallest positive integer n such that $\phi^{(\kappa(n))}(n) = 1$. Let

$$F(n) = \sum_{k=1}^{\kappa(n)} \phi^{(k)}(n).$$

Positive integers n such that $F(n) = n$ are called *perfect totients* and were introduced in [10] and studied also in [6] and [12]. Let $\mathcal{M}$ be the set of all perfect totients. This set contains all powers of 3 so it is certainly infinite. In the recent paper [12] it was shown that $\mathcal{M}$ is of asymptotic density zero. More precisely, if we write $\mathcal{M}(x) = \mathcal{M} \cap [1, x]$, then it was shown in Theorem 2.2 in [12] – a little bit more than – that the estimate

$$\#\mathcal{M}(x) \le \frac{x}{(\log x)^{1+o(1)}} \tag{1}$$

holds as $x \to \infty$. The above estimate (1) is too weak to allow one to decide whether the sum

$$\sum_{m \in \mathcal{M}} \frac{1}{m} \tag{2}$$

is finite. Here, we prove a stronger upper bound on $\#\mathcal{M}(x)$ than (1) which in particular implies that the sum of the series (2) is convergent.

Theorem 1. *The estimate*

$$\#\mathcal{M}(x) \le \frac{x}{(\log x)^{2+o(1)}} \tag{3}$$

holds as $x \to \infty$.

It was also shown in [12] that the inequality

$$|F(n) - n| > (\log n)^{\ln 2 + o(1)} \tag{4}$$

holds on a set of positive integers n of asymptotic density 1. Here, we improve this to:

Theorem 2. *Let $\varepsilon(x)$ be any function defined on the positive real numbers x with values in the positive real numbers which is decreasing for large x and $\lim_{x\to\infty} \varepsilon(x) = 0$. Then the inequality*

$$n - F(n) > \varepsilon(n)n \tag{5}$$

holds on a set of positive integers n of asymptotic density 1.

Let $\mathcal{U}(x) = \{F(n) \le x\}$. In [12], it was shown that $\#\mathcal{U}(x) \gg (\log x)^2$ and it was asked to show that $\log(\#\mathcal{U}(x))/\log\log x$ tends to infinity with x. We prove:

Theorem 3. *The estimate*

$$\log(\#\mathcal{U}(x)) \gg \frac{(\log\log x)^2}{\log\log\log\log x} \tag{6}$$

holds for all positive integers x.

Throughout this paper, we use $\log x = \max\{\ln x, 1\}$, where ln is the natural logarithm. Further, if $k \ge 1$, we write $\log_k x$ for the kth fold iterate of the function log. We omit the subscript when $k = 1$. We use the Vinogradov symbols $\ll$, $\gg$ and $\asymp$, as well as the Landau symbols O and o with their regular meanings. The constants of convergence implied by them may depend on some fixed parameters such as K (see Section 2.1). For a positive integer n, we use $P(n)$ for its largest prime factor with the convention that $P(1) = 1$, $\omega(n)$ for the number of distinct prime factors of n and $\nu_2(n)$ for the 2-adic order of n; i.e., the largest non-negative integer k such that $2^k \mid n$. We use p, q and r to denote prime numbers and $c_0, c_1, \ldots$ to denote positive constants which are absolute.

2 Proof of Theorem 1

The proof of inequality (1) in [12] is based on the fact that if we put $\mathcal{V}(x) = \{\phi(n) \le x\}$, then $\#\mathcal{V}(x) \le x/(\log x)^{1+o(1)}$ (see [4] and [8]) together with the observation that if $n \in \mathcal{M}(x)$, then $n = v + F(v)$ for some $v \in \mathcal{V}(x)$ (namely, $v = \phi(n)$).

For our proof of Theorem 1, we take a different approach and we exploit the numbers $\nu_2(\phi^{(2)}(n))$ and $\nu_2(\phi^{(3)}(n))$ for $n \in \mathcal{M}(x)$. Before we start, we record a result which might be of independent interest.

2.1 An auxiliary result

Let $K > 0$ be any fixed constant. Put

$$\mathcal{N}(K,x) = \{n \le x \ : \ \nu_2(\phi^{(2)}(n)) \le K \log_2 x\}.$$

Lemma 4. *(i) The estimate*

$$\sum_{n\in\mathcal{N}(K,x)} \frac{1}{n} = (\log x)^{o(1)}$$

holds as $x \to \infty$.

(ii) Let $\pi(K,x) = \#\{p \le x \ : \ p-1 \in \mathcal{N}(K,x)\}$. Then

$$\pi(K,x) \le \frac{x}{(\log x)^{2+o(1)}}$$

holds as $x \to \infty$.

Proof. (i) Put $L = 10\log_2 x$. Let $\mathcal{N}_1$ be the set of $n \in \mathcal{N}(K,x)$ with $\omega(n) \ge L$ and note that

$$\begin{aligned} S_1 &= \sum_{n\in\mathcal{N}_1} \frac{1}{n} \le \sum_{\substack{n\le x\\ \omega(n)\ge L}} \frac{1}{n} \le \sum_{k\ge L} \sum_{\substack{n\le x\\ \omega(n)=k}} \frac{1}{n} \le \sum_{k\ge L} \frac{1}{k!}\left(\sum_{p^\alpha\le x} \frac{1}{p^\alpha}\right)^k \\ &\le \sum_{k\ge L} \frac{1}{k!}(\log_2 x + c_0)^k \ll \sum_{k\ge L}\left(\frac{e\log_2 x + ec_0}{k}\right)^k \\ &\le \sum_{k\ge L}\left(\frac{e\log_2 x + ec_0}{L}\right)^k \ll \left(\frac{e\log_2 x + ec_0}{L}\right)^L \ll 1. \end{aligned} \tag{7}$$

In the above inequalities, we used the multinomial formula, the unique factorization, the estimate $k! \gg (k/e)^k$ which follows from Stirling's formula, as well as the known fact that the estimate

$$\sum_{p\le x}\sum_{\alpha\ge 1}\frac{1}{p^\alpha} \le \log_2 x + c_0$$

holds for all x with some absolute constant c_0.

Put $y = (\log x)^2$ and let $\mathcal{N}_2$ be the subset of $n \in \mathcal{N}(K,x)$ having two distinct prime factors p and q such that $\gcd(p-1, q-1)$ is a multiple of some prime $r > y$. Writing $n = pqm$ for some positive integer m, we see that the sum S_2 of the reciprocals of $n \in \mathcal{N}_2$ satisfies

$$\begin{aligned} S_2 &= \sum_{n\in\mathcal{N}_2}\frac{1}{n} \le \sum_{y<r\le x}\ \sum_{\substack{p\le x,\ q\le x\\ r\,|\,\gcd(p-1,q-1)}}\ \sum_{m\le x/pq}\frac{1}{pqm} \\ &\le \sum_{y<r\le x}\frac{1}{2}\left(\sum_{\substack{p\le x\\ p\equiv 1 \pmod r}}\frac{1}{p}\right)^2\left(\sum_{m\le x}\frac{1}{m}\right) \ll \log x(\log_2 x)^2\sum_{y<r}\frac{1}{r^2} \\ &\ll \frac{\log x(\log_2 x)^2}{y\log y} = \frac{\log_2 x}{\log x} \ll 1, \end{aligned} \tag{8}$$

where we used the known estimate

$$\sum_{\substack{p\le t\\ p\equiv 1 \pmod b}} \frac{1}{p} \ll \frac{\log_2 t}{\phi(b)}$$

which holds uniformly for $1 \le b \le t$ (see Lemma 1 of [1] or the inequality (3.1) in [3]), as well as the fact that

$$\sum_{t\le p} \frac{1}{p^2} \ll \frac{1}{t\log t}$$

which follows by partial summation from the Prime Number Theorem.

Now we deal with the numbers $n \in \mathcal{N}_3 = \mathcal{N}(K,x)\backslash(\mathcal{N}_1 \cup \mathcal{N}_2)$. Let z be such that $\log_2 z = \log_2 x/\log_3 x$. For a positive integer n and a real number t write $\omega_{>t}(n)$ for the number of distinct prime factors $p > t$ of n. Let $n \in \mathcal{N}_3$ and write it as $n = abc$, where

(i) all prime factors of a are $\le z$;

(ii) all prime factors p of b are $> z$ but $\omega_{>y}(p-1) < \log_2 p/(\log_3 p)^2$;

(iii) all prime factors p of c are $> z$ and $\omega_{>y}(p-1) \ge \log_2 p/(\log_3 p)^2$.

Note that if $p \mid c$, then

$$\omega_{>y}(p-1) \ge \frac{\log_2 p}{(\log_3 p)^2} \ge \frac{\log_2 z}{(\log_3 z)^2} > \frac{\log_2 x}{(\log_3 x)^3}$$

for large x. Further, for $t > 2$ we have that $\omega_{>t}(n) \le \nu_2(\phi(n))$. Furthermore, since $n \notin \mathcal{N}_2$, we have that

$$\begin{aligned}\sum_{p\mid c}\omega_{>y}(p-1) &\le \sum_{p\mid n}\omega_y(p-1) = \omega_{>y}\left(\prod_{p\mid n}(p-1)\right)\\ &\le \nu_2\left(\phi\left(\prod_{p\mid n}(p-1)\right)\right) \le \nu_2(\phi^{(2)}(n)) \le K\log_2 x.\end{aligned}$$

We thus get that

$$\omega(c)\frac{\log_2 x}{(\log_3 x)^3} < \sum_{p\mid c}\omega_{>y}(p-1) \le K\log_2 x,$$

therefore

$$\omega(c) < K(\log_3 x)^3.$$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be the subsets of all possible values of a, b and c, respectively. Thus,

$$S_3 = \sum_{n\in\mathcal{N}_3}\frac{1}{n} \le ABC, \tag{9}$$

where

$$A = \sum_{a\in\mathcal{A}} \frac{1}{a}, \qquad B = \sum_{b\in\mathcal{B}} \frac{1}{b} \qquad \text{and} \qquad C = \sum_{c\in\mathcal{C}} \frac{1}{c}. \tag{10}$$

Since the union of $\mathcal{N}_i$ for $i = 1,\ 2$ and 3 covers $\mathcal{N}(K, x)$ it follows, by estimates (7), (8), (9) and (10), that in order to establish (i) it suffices to show that

$$\max\{A, B, C\} \le (\log x)^{o(1)} \qquad \text{as } x \to \infty. \tag{11}$$

Clearly $\mathcal{A} \subset \{a \le x \ :\ P(a) \le z\}$ and $\mathcal{C} \subset \{c \le x \ :\ \omega(c) \le K(\log_3 x)^3\}$. Hence,

$$\begin{aligned} A &\le \prod_{p\le z}\left(\sum_{\alpha\ge 0}\frac{1}{p^\alpha}\right) \le \exp\left(\sum_{p\le z}\sum_{\alpha\ge 1}\frac{1}{p^\alpha}\right) \le \exp(\log_2 z + c_0) \\ &\ll \log z = (\log x)^{1/\log_3 x} = (\log x)^{o(1)} \qquad \text{as } x \to \infty, \end{aligned} \tag{12}$$

while by an argument similar to the one used to bound S_1 (see estimate (7)), we get

$$\begin{aligned} C &\le \sum_{k\le K(\log_3 x)^3}\ \sum_{\substack{c\le x\\ \omega(c)=k}} \frac{1}{c} \le \sum_{k\le K(\log_3 x)^3}\left(\frac{e\log_2 x + ec_0}{k}\right)^k \\ &\ll (\log_3 x)^3(e\log_2 x + ec_0)^{K(\log_3 x)^3} \\ &= (\log x)^{O((\log_3 x)^4/\log_2 x)} = (\log x)^{o(1)} \qquad \text{as } x \to \infty. \end{aligned} \tag{13}$$

For $\mathcal{B}$, let $f(t) = (\log t)^{3\log_3 t}$ and note that

$$\begin{aligned} f(z) = \exp(3\log_2 z \log_3 z) &> \exp(2\log_2 z\log_3 x) = \exp(2\log_2 x) = (\log x)^2 \\ &= y \qquad \text{for large } x, \end{aligned}$$

therefore all the primes p dividing $b \in \mathcal{B}$ belong to the set

$$\mathcal{P} = \{p \ :\ \omega_{>f(p)} < \log_2 p/(\log_3 p)^2\}$$

for large values of x. We show that

$$\sum_{p\in\mathcal{P}} \frac{1}{p} = O(1). \tag{14}$$

Note that once the above estimate (14) is proved, then

$$B = \sum_{b\in\mathcal{B}}\frac{1}{b} \le \prod_{p\in\mathcal{P}}\left(\sum_{\alpha\ge 0}\frac{1}{p^\alpha}\right) \le \exp\left(\sum_{p\in\mathcal{P}}\sum_{\alpha\ge 1}\frac{1}{p^\alpha}\right) = \exp(O(1)) = O(1),$$

which together with estimates (12) and (13) implies estimate (11) and completes the proof of (i).

Thus, it remains to prove estimate (14). For this, let $t > 0$ and put $\mathcal{P}(t) = \mathcal{P} \cap [1, t]$. We estimate the counting function $\#\mathcal{P}(t)$ of $\mathcal{P}$. Let $p \in \mathcal{P}(t)$. Let $\mathcal{P}_1 = \{p \le t \ : \ P(p-1) \le t^{1/\log_2 t}\}$. By results from [2] (see also Chapter III.5 of [13]), it follows that

$$\#\mathcal{P}_1(t) \le t \exp(-(1+o(1)) \log_2 t \log_3 t) = o\left(\frac{t}{(\log t)^2}\right) \qquad \text{as } t \to \infty. \tag{15}$$

For $p \in \mathcal{P}_2 = \mathcal{P}(t) \backslash \mathcal{P}_1$, we write $p - 1 = q\ell$, where $q = P(p-1)$ and ℓ is some positive integer. Fix ℓ. Then $q \le t/\ell$ is such that both linear forms q and $q\ell + 1$ are primes. By Brun's sieve (see, for example, Theorem 2.3 in [5]), the number of such primes q is

$$\ll \frac{t}{\ell(\log(t/\ell))^2} \left(\frac{\ell}{\phi(\ell)}\right)^2 \le \frac{t(\log_2 t)^4}{\ell(\log t)^2},$$

where we used the minimal order $\phi(\ell)/\ell \gg 1/\log_2 t$ of the Euler function in the interval $[1, t]$ as well as the fact that

$$\log\left(\frac{t}{\ell}\right) \ge \log q \ge \log(t^{1/\log_2 t}) = \frac{\log t}{\log_2 t}.$$

Since ℓ is a divisor of $p - 1$, we get that if we write $\ell = \ell_1 \ell_2$, where $P(\ell_1) \le f(t)$ and every prime factor of ℓ_2 is $> f(t)$, then

$$\omega(\ell_2) = \omega_{>f(t)}(p-1) \le \omega_{<f(p)}(p-1) \le \frac{\log_2 p}{(\log_3 p)^2} \le \frac{\log_2 t}{(\log_3 t)^2}$$

for large t. Hence, summing up over all possible ℓ's we get

$$\#\mathcal{P}_2(t) \le \frac{t(\log_2 t)^4}{(\log t)^2} L_1 L_2, \tag{16}$$

where

$$L_1 = \sum_{P(\ell_1) \le f(t)} \frac{1}{\ell_1} \qquad \text{and} \qquad L_2 = \sum_{\substack{\ell_2 \le t \\ \omega(\ell_2) \le \log_2 t/(\log_3 t)^2}} \frac{1}{\ell_2}. \tag{17}$$

Clearly, by the argument used to bound A (see (12)), we have

$$\begin{aligned} L_1 &= \sum_{P(\ell_1) \le f(t)} \frac{1}{\ell_1} \le \exp\left(\sum_{p \le f(t)} \sum_{\alpha \ge 1} \frac{1}{p^\alpha}\right) \le \exp\left(\log_2 f(t) + c_0\right) \\ &= \exp(\log_3 t + \log_4 t + c_0 + \log 3) \ll \log_2 t \log_3 t, \end{aligned} \tag{18}$$

while by the argument used to bound S_1 (see (7)) or C (see (13)) we have

$$\begin{aligned} L_2 &\le \sum_{\omega(\ell_2) \le \log_2 t/(\log_3 t)^2} \frac{1}{\ell_2} \le \sum_{k \le \log_2 t/(\log_3 t)^2} \left(\frac{e \log_2 t + ec_0}{k}\right)^k \\ &\ll \log_2 t (e \log_2 t + ec_0)^{\log_2 t/(\log_3 t)^2} = \exp(O(\log_2 t/\log_3 t)) \\ &= (\log t)^{o(1)} \qquad \text{as } t \to \infty. \end{aligned} \tag{19}$$

Now estimates (18), (19) and (16) show that

$$\#\mathcal{P}_2 \le \frac{t}{(\log t)^{2+o(1)}} \qquad \text{as } t \to \infty, \tag{20}$$

and since $\mathcal{P}_i$ with $i = 1$ and 2 cover $\mathcal{P}(t)$, we get, by estimates (15) and (20), that

$$\#\mathcal{P}(t) \le \#\mathcal{P}_1 + \#\mathcal{P}_2 \le \frac{t}{(\log t)^{2+o(1)}} \qquad \text{as } t \to \infty. \tag{21}$$

Estimate (14) follows now from the above estimate (21) by partial summation, which completes the proof of (i).

(ii) Let $\mathcal{R}(K, x) = \{p \le x \ : \ p - 1 \in \mathcal{N}(K, x)\}$. Let $y = x^{1/\log_2 x}$ and $\mathcal{R}_1 = \{p \le x \ : \ P(p-1) \le y\}$. Estimate (15) shows that

$$\#\mathcal{R}_1 \le x \exp(-(1 + o(1)) \log_2 x \log_3 x) = o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x \to \infty. \tag{22}$$

For $p \in \mathcal{R}_2 = \mathcal{R}(K, x)\backslash\mathcal{R}_1$, we write $p - 1 = q\ell$, where $q = P(p - 1)$. The argument used in the proof of (i) to bound $\mathcal{P}_2$ shows that for a fixed ℓ the number of choices for $q \le x/\ell$ is

$$\ll \frac{x(\log_2 x)^4}{\ell(\log x)^2}.$$

Upon noticing that $\ell \mid p - 1$ implies that $\ell \in \mathcal{N}(K, x)$, we get, by using (i), that

$$\#\mathcal{R}_2 \le \frac{x(\log_2 x)^4}{(\log x)^2} \sum_{\ell \in \mathcal{N}(K,x)} \frac{1}{\ell} = \frac{x}{(\log x)^{2+o(1)}} \qquad \text{as } x \to \infty. \tag{23}$$

Since

$$\pi(K, x) \le \#\mathcal{R}_1 + \#\mathcal{R}_2,$$

(ii) follows from estimates (22) and (23). □

2.2 The Proof of Theorem 1

We start by sieving off a few sets of positive integers $n \le x$ of cardinalities $O(x/(\log x)^2)$. We ignore the following positive integers $n \le x$:

(i) positive integers $n \le x$ with $P(n) \le y = x^{1/\log_2 x}$. By the results from [2] or Theorem XX in [13] we get, as in the estimates of $\#\mathcal{P}_1$ or $\#\mathcal{R}_1$ in Lemma 4, that the number of such n does not exceed

$$x \exp(-(1 + o(1)) \log_2 x \log_3 x) = o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x \to \infty.$$

(ii) positive integers $n \le x$ for which there exists a prime $q > (\log x)^2$ such that $q^2 \mid n$. It is clear that the number of such positive integers does not exceed

$$\sum_{(\log x)^2 < q \le x^{1/2}} \frac{x}{q^2} \ll \frac{x}{(\log x)^2 \log_3 x} = o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x \to \infty.$$

(iii) positive integers $n \le x$ not satisfying (i) and (ii) above such that if we put $z = y^{1/\log_2 x}$, then $n = Pm$, where $P = P(n)$, and $P(p-1) < z$. Fix the number m. Since $\log P/\log z \ge \log P/\log z \ge \log_2 x$, it follows again by the results from [2] or Theorem XX in [13], that the number of possible values for P is

$$\le \frac{x}{m} \exp(-(1+o(1)) \log_2 \log_3 x) = o\left(\frac{x}{m(\log x)^3}\right) \qquad \text{as } x \to \infty$$

and uniformly in $m \le x/P \le x/y$. Thus, summing up over all the possible values of $m \le x$, we get that the total number of such integers n does not exceed

$$\frac{x}{(\log x)^3} \sum_{m \le x} \frac{1}{m} \ll \frac{x}{(\log x)^2}$$

if x is sufficiently large.

(iv) positive integers $n \le x$ not satisfying (i)–(iii) such that $q^2 \mid P-1$ for some $q \ge (\log x)^3$, where $P = P(n)$. Write again $n = Pm$. For fixed values of m and q, the number of such choices for $P \le x/m$, even neglecting the fact that it is prime, is at most x/mq^2. This shows that the totality of such integers n does not exceed

$$\sum_{m \le x/y} \sum_{(\log x)^3 \le q \le (x/m)^{1/2}} \frac{x}{mq^2} \le x \left(\sum_{m \le x} \frac{1}{m}\right) \left(\sum_{(\log x)^3 \le q} \frac{1}{q^2}\right)$$
$$\ll x \log x \int_{(\log x)^3}^{\infty} \frac{dt}{t^2} = O\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x \to \infty.$$

(v) positive integers $n \le x$ not of the form (i)–(iv) such that if we write again $n = Pm$, where $P = P(n)$, then there exists a prime $q > (\log x)^2$ with the property that $q \mid \gcd(P-1, \phi(m))$. Since n is not like in (ii), it follows that q^2 does not divide n. Thus, there must exist a prime factor r of m such that $q \mid r-1$. Fixing q, r and P, we get that the number of $n \le x$ which are multiples of Pr does not exceed x/Pr. Hence, the totality of such integers n does not exceed

$$\sum_{(\log x)^2 \le q \le x^{1/2}} \sum_{\substack{q \mid \gcd(P-1, r-1) \\ Pr \le x}} \frac{x}{Pr} \le x \sum_{(\log x)^2 \le q \le x^{1/2}} \frac{1}{2} \left(\sum_{\substack{p \equiv 1 \pmod q \\ p \le x}} \frac{1}{p} \right)^2$$
$$\ll x(\log_2 x)^2 \sum_{(\log x)^2 \le q \le x^{1/2}} \frac{1}{q^2} \ll \frac{x \log_2 x}{(\log x)^2}.$$

(vi) positive integers n not of the form (i)–(v) such that if we write $n = Pm$ with $P = P(n) > P(m)$, then $m \equiv -1 \pmod{2^M}$, where $M = \lfloor 5 \log_2 x \rfloor$. For each such fixed m, the number of possible choices for $P \le x/m$ is

$$\pi\left(\frac{x}{m}\right) \le \frac{x}{m\log(x/m)} \le \frac{x}{m\log y} = \frac{x\log_2 x}{m\log x}.$$

Summing up over all the $m \le x$ of the form $m = 2^M\lambda - 1$ for some $\lambda \ge 1$, we get that the totality of such n does not exceed

$$\begin{aligned} \frac{x\log_2 x}{\log x} \sum_{1\le\lambda\le x/2^M} \frac{1}{2^M\lambda - 1} &\ll \frac{x\log_2 x}{2^M\log x} \sum_{1\le\lambda\le x} \frac{1}{\lambda} \\ \ll \frac{x\log_2 x}{2^M} \ll \frac{x\log_2 x}{(\log x)^{5\log 2}} &= o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x \to \infty. \end{aligned}$$

From now on, we work with the set $\mathcal{M}'$ of the perfect totients $n \le x$ not satisfying any of the conditions (i)–(vi) above. Note that if $n > 2$, then $F(n)$ is always odd. Hence, n is odd.

Let $\mathcal{M}_1$ be the subset of $n \in \mathcal{M}'$ such that $\nu_2(\phi^{(2)}(n)) \ge 2M$. We now make the observation that if m is an even number, then $\nu_2(\phi(m)) \ge \nu_2(m)$ *except* when m is a power of 2. Further, note that if $\phi^{(\ell)}(m)$ is a power of 2, then $\phi^{(\ell+i)}(m)$ is also a power of 2 for all $i \ge 0$. Thus, letting $\kappa_1(n)$ be the largest positive integer $\ell \ge 2$ such that $\phi^{(\ell)}(n)$ is not a power of 2, from the above remark we get that for $n \in \mathcal{M}_1$ we have

$$2M \le \nu_2(\phi^{(2)}(n)) \le \nu_2(\phi^{(3)}(n)) \le \ldots \le \nu_2(\phi^{(\kappa_1(n)+1)}(n)).$$

Writing $\phi^{(\kappa_1(n)+1)}(n) = 2^\beta$ with some $\beta \ge 2M$, we get

$$\sum_{k=\kappa_1(n)+1}^{\kappa(n)} \phi^{(k)}(n) = \sum_{i=0}^{\beta} 2^i = 2^{\beta+1} - 1.$$

Since n is a perfect totient, we get the following congruence

$$\begin{aligned} n - \phi(n) + 1 &= 1 + \sum_{k=2}^{\kappa(n)} \phi^{(k)}(n) = 1 + \sum_{k=2}^{\kappa_1(n)} \phi^{\ell}(n) + \sum_{k=\kappa_1(n)+1}^{\kappa(n)} \phi^{(k)}(n) \\ &= \sum_{\ell=2}^{\kappa_1(n)} \phi^{(k)}(n) + 2^{\beta+1} \equiv 0 \pmod{2^{2M}}. \end{aligned} \tag{24}$$

Write $n = Pm$, where $P > \max\{P(m), y\}$ for large x because n is neither as in (i) nor as in (ii). So

$$n - \phi(n) + 1 = Pm - (P-1)\phi(m) + 1 = P(m - \phi(m)) + (\phi(m) + 1).$$

Note that $m > 2$ if x is large enough. Indeed, since n is odd we get that if $m \le 2$, then $m = 1$ and $n = P$, therefore

$$P \ge \phi(P) + \phi(\phi(P)) = P - 1 + \phi(P - 1),$$

leading to $1 \ge \phi(P - 1)$; thus, $P \le 3$, which is impossible for large x since $P > y$. Since $m > 2$, we get that $m - \phi(m) > 0$ and $\phi(m) + 1$ is odd. Thus, for fixed odd $m > 1$ the congruence

$$P(m - \phi(m)) + \phi(m) + 1 \equiv 0 \pmod{2^{2M}}$$

puts P into a certain residue class a_m modulo 2^{2M}. Since $P \le x/m$, it follows that the number of possibilities for P is $\pi(x/m; 2^{2M}, a_m)$. By a result of Montgomery and Vaughan [11], we know that

$$\pi(x/m; 2^{2M}, a_m) \le \frac{2x}{m\phi(2^{2M}) \log(x/m2^{2M})}. \tag{25}$$

Note that

$$\frac{x}{m2^{2M}} \ge \frac{y}{(\log x)^{10 \ln 2}} > y^{1/2} \tag{26}$$

if x is sufficiently large. Hence, inequalities (25) and (26) lead to

$$\pi(x/m; 2^{2M}, a_m) \ll \frac{x}{m2^{2M} \log y} \ll \frac{x \log_2 x}{m(\log x)^{1+10 \ln 2}}. \tag{27}$$

Summing up over all the possible choices for m we get that

$$\begin{aligned} \#\mathcal{M}_1 &\le \sum_{m \le x/y} \pi(x/m; 2^{2M}, a_m) \ll \frac{x \log_2 x}{(\log x)^{1+10 \ln 2}} \sum_{m \le x/y} \frac{1}{m} \\ &\ll \frac{x \log_2 x}{(\log x)^{10 \ln 2}} = o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x \to \infty. \end{aligned} \tag{28}$$

Now let $\mathcal{M}_2$ be the subset of $n \in \mathcal{M}' \backslash \mathcal{M}_1$ such that $\nu_2(\phi^{(3)}(n)) < 4M$. Let $n = mP$. Since $n \notin \mathcal{M}_1$, it follows that $\nu_2(\phi^{(2)}(m)) \le \nu_2(\phi^{(2)}(n)) < 2M$. In particular, $m \in \mathcal{N}(10, x)$. Further, since

$$\log_2(x/m) \ge \log_2 y \ge (\log_2 x)/2$$

holds for large x and all $m < x/y$, we get

$$\nu_2(\phi^{(2)}(P - 1)) \le \nu_2(\phi^{(3)}(n)) < 4M \le 20 \log_2 x \le 40 \log_2(x/m).$$

Thus, $P \in \mathcal{N}(40, x/m)$. Fixing m, it follows by Lemma 4 (ii) that the number of possibilities for P is

$$\le \frac{x}{m(\log(x/m))^{2+o(1)}} \le \frac{x}{m(\log y)^{2+o(1)}} \le \frac{x}{m(\log x)^{2+o(1)}} \qquad \text{as } x \to \infty$$

uniformly in $m \le x/y$. Summing up over all $m \in \mathcal{N}(10, x)$ and using Lemma 4 (i), we get that

$$\#\mathcal{M}_2 \le \frac{x}{(\log x)^{2+o(1)}} \sum_{m \in \mathcal{N}(10,x)} \frac{1}{m} \le \frac{x}{(\log x)^{2+o(1)}} \qquad \text{as } x \to \infty. \tag{29}$$

From now on, we look at positive integers in $\mathcal{M}_3 = \mathcal{M}' \backslash (\mathcal{M}_1 \cup \mathcal{M}_2)$. Note that if $n \in \mathcal{M}_3$ then $4M \le \nu_2(\phi^{(3)}(n))$.

For $n \in \mathcal{M}_3$, we write $n = Pm$, where $P = P(n) > P(m)$, and $P - 1 = Q\ell$, where $Q = P(P-1)$. Note that we again have $m \in \mathcal{N}(10, x)$. Observe also that $Q > z$ because n is not like in (iii). Since $z > (\log x)^3 > (\log x)^2$ for large x and n is not like in (iv) or (v), we get that Q does not divide $\phi(m)\ell$. Let d be the largest divisor of $P - 1$ which is divisible only by primes dividing $\phi(m)$. Thus, $P(d) < (\log x)^2$ and d is a divisor of ℓ. Write $\ell = ds$. From now on we fix m, the number d which consists only of prime factors of $\phi(m)$ smaller than $(\log x)^2$, and the number s which is coprime to $d\phi(m)$. Notice that $\phi(n) = (P-1)\phi(m) = Qsd\phi(m)$, every prime factor of d divides $\phi(m)$, and Qs is coprime to $\phi(m)$. Thus, $\phi(\phi(n)) = (Q-1)\phi(s)d\phi(\phi(m))$.

An argument identical to the one used to derive congruence (24) gives

$$n - \phi(n) - \phi(\phi(n)) + 1 \equiv 0 \pmod{2^{4M}}$$

for $n \in \mathcal{M}_3$. Note that

$$\begin{aligned}
&n - \phi(n) - \phi(\phi(n)) + 1 \\
&= P(m - \phi(m)) + (\phi(m) + 1) - (Q-1)\phi(s)d\phi(\phi(m)) \\
&= (Qsd + 1)(m - \phi(m)) + (\phi(m) + 1) - (Q-1)\phi(s)d\phi(\phi(m)) \\
&= Q(sd(m - \phi(m)) - \phi(s)d\phi(\phi(m))) + (m - \phi(m)) + (\phi(m) + 1) \\
&+\phi(s)d\phi(\phi(m)) \\
&= Q(sd(m - \phi(m)) - \phi(s)d\phi(\phi(m))) + (m + 1 + \phi(s)d\phi(\phi(m))).
\end{aligned}$$

Put

$$C_{m,d,s} = sd(m - \phi(m)) - \phi(s)d\phi(\phi(m)) \quad \text{and} \quad D_{m,d,s} = m + 1 + \phi(s)d\phi(\phi(m)).$$

Observe that $C_{m,d,s} \neq 0$ for large x. Indeed, if $C_{m,d,s} = 0$, we then get

$$\begin{aligned}
n - \phi(n) - \phi(\phi(n)) + 1 &= m + 1 + \phi(s)d\phi(\phi(m)) \\
&= m + 1 + sd(m - \phi(m)) \\
&\le m + (P-1)\frac{m}{Q} \le \frac{mP}{y} + \frac{mP}{z} \\
&\le \frac{2mP}{z}.
\end{aligned} \tag{30}$$

However, since $n \in \mathcal{M}(x)$, we get that

$$n - \phi(n) - \phi(\phi(n)) + 1 \ge \phi(\phi(\phi(n))) \gg \frac{n}{(\log_2 x)^3} = \frac{mP}{(\log_2 x)^3}. \tag{31}$$

The last inequality above follows from applying the minimal order of the Euler function in the interval $[1, x]$ three times as

$$\frac{\phi^{(3)}(n)}{n} \gg \frac{\phi^{(2)}(n)}{n \log_2 x} \gg \frac{\phi(n)}{n(\log_2 x)^2} \gg \frac{1}{(\log_2 x)^3}$$

for $n \leq x$. Comparing estimates (30) and (31), we get that

$$\frac{2mP}{z} \gg \frac{mP}{(\log_2 x)^3},$$

leading to $z \ll (\log_2 x)^3$, which is impossible for large x.

Hence, $C_{m,d,s} \neq 0$ and

$$C_{m,d,s}Q + D_{m,d,s} \equiv 0 \pmod{2^{4M}}. \tag{32}$$

Let $\mathcal{M}_4$ be the subset of $\mathcal{M}_3$ such that 2^{2M} divides both $C_{m,d,s}$ and $D_{m,d,s}$. Then 2^{2M} divides also $C_{m,d,s} + D_{m,d,s} = sd(m - \phi(m)) + m + 1$. Note that $m - \phi(m)$ is odd because $m > 1$ is odd. Further, since n is not like in (vi), it follows that if we write $m+1 = 2^{\alpha}m_1$ where m_1 is odd, then $\alpha \leq M$. Since $sd(m-\phi(m))+m+1$ is a multiple of 2^{2M}, we get that $sd = 2^{\alpha}s_1d_1$, where s_1 and d_1 are both odd. Further note that if $m > 3$, then $\phi(\phi(m))$ is even, therefore $d = 2^{\alpha}d_1$ and $s = s_1$. If $m = 3$, then $\phi(\phi(m)) = 1$, therefore $d = 1$ and $s = 2^{\alpha}s_1$. Fixing m and d (hence, also α and d_1) the congruence $sd(m - \phi(m)) + m + 1 \equiv 0 \pmod{2^{2M}}$ leads to $s_1d_1(m - \phi(m)) + m_1 \equiv 0 \pmod{2^{2M-\alpha}}$. Hence, s_1 belongs to a certain odd residue class $b_{m,d}$ modulo 2^M. We assume that $b_{m,d}$ is the smallest positive integer in this class. Since $P - 1 = 2^{\alpha}d_1s_1Q$, $P \leq x/m$ and both P and Q are primes, it follows, by Brun's method (see again Theorem 2.3 in [5]), that the number of possibilities for $Q \leq x/(m2^{\alpha}s_1d_1)$ when m, d_1 and s_1 are fixed is

$$\ll \frac{x}{m\phi(2^{\alpha}s_1d_1)(\log(x/msd))^2}$$

Since $x/msd \geq Q \geq z$, we get, by using again the minimal order of the Euler function in the interval $[1, x]$, that the above number is bounded from above by

$$\ll \frac{x\log_2 x}{msd(\log z)^2} \leq \frac{x(\log_2 x)^5}{m2^{\alpha}s_1d_1(\log x)^2} \ll \frac{x(\log_2 x)^5}{ms_1d_1(\log x)^2}.$$

Note that α is uniquely determined by m alone. Summing up first over all $m \leq x/y$ and in $\mathcal{N}(10, x)$, then over all odd $d_1 \mid \phi(m)$ such that $P(d_1) \leq (\log x)^2$, and finally over those $s_1 \leq x/(2^{\alpha}zmd_1)$ with $s_1 \equiv b_{m,d} \pmod{2^M}$, we get

$$\begin{aligned}
\#\mathcal{M}_4 &\ll \frac{x(\log_2 x)^5}{(\log x)^2} \sum_{\substack{m \leq x/y \\ m \in \mathcal{N}(5,x)}} \frac{1}{m} \sum_{\substack{d_1 \mid \phi(m) \\ P(d_1) \leq (\log x)^2}} \frac{1}{d_1} \sum_{0 \leq \lambda \leq x/(2^{M+\alpha}ms_1d_1)} \frac{1}{b_{m,d} + \lambda 2^M} \\
&\leq \frac{x(\log_2 x)^5}{(\log x)^2} \left(\sum_{m \in \mathcal{N}(10,x)} \frac{1}{m} \sum_{\substack{d_1 \equiv 1 \pmod 2 \\ p \mid d_1 => p \mid \phi(m)}} \frac{1}{d_1} \right) \left(1 + \frac{1}{2^M} \sum_{1 \leq \lambda \leq x} \frac{1}{\lambda} \right) \\
&\leq \frac{x(\log_2 x)^5}{(\log x)^2} \left(\sum_{m \in \mathcal{N}(10,x)} \frac{1}{m} \frac{\phi(m)}{\phi(\phi(m))} \right) \left(1 + O\left(\frac{\log x}{2^M} \right) \right) \\
&\ll \frac{x(\log_2 x)^6}{(\log x)^2} \sum_{m \in \mathcal{N}(10,x)} \frac{1}{m} \leq \frac{x}{(\log x)^{2+o(1)}} \qquad \text{as } x \to \infty,
\end{aligned} \tag{33}$$

where in the above inequalities we used the obvious fact that if m is a positive integer then

$$\sum_{\substack{d\ge 1\\ p\,|\,d => p\,|\,m}} \frac{1}{d} = \prod_{p\,|\,m}\left(\sum_{\alpha\ge 0}\frac{1}{p^{\alpha}}\right) = \prod_{p\,|\,m}\left(1-\frac{1}{p}\right)^{-1} = \frac{m}{\phi(m)}, \tag{34}$$

the inequality $\phi(m)/\phi(\phi(m)) \ll \log_2 x$ which follows from the minimal order of the Euler function in the interval $[1,x]$, as well as Lemma 4 (i).

Finally, let $\mathcal{M}_5 = \mathcal{M}_3\backslash\mathcal{M}_4$. If $n\in\mathcal{M}_5$, then $n = Pm,\ P = P(n) > P(m),\ P-1 = Qsd$, and $\gamma = \nu_2(\gcd(C_{m,d,s}, D_{m,d,s})) < M$. Let $C' = C_{m,d,s}/2^{\gamma}$ and $D' = D_{m,d,s}/2^{\gamma}$. Congruence (32) shows that $C'Q + D' \equiv 0 \pmod{2^M}$, therefore Q is in a certain residue class $e_{m,d,s}$ modulo 2^M. Assume that $e_{m,d,s}$ is the smallest positive integer in this congruence class. Then $Q = 2^M\lambda + e_{m,d,s} \le x/(mds)$ is a prime such that $P = sdQ+1 = 2^M sd\lambda + (sde_{m,d,s}+1)$ is also a prime. By Brun's method again, the number of such possibilities for fixed m, d and s is

$$\ll \frac{x}{mds2^M(\log(x/(mds2^M)))^2}\left(\frac{\phi(sde_{m,d,s}(sde_{m,d,s}+1))}{sde_{m,d,s}(sde_{m,d,s}+1)}\right)^2.$$

Using again the minimal order of the Euler function in the interval $[1,x]$ as well as the fact that

$$\frac{x}{mds2^M} \ge \frac{Q}{2^M} \ge \frac{z}{2^M} \ge z^{1/2}$$

for large x, we get that the above number is at most

$$\ll \frac{x(\log_2 x)^2}{mds2^M(\log z)^2} \ll \frac{x(\log_2 x)^6}{mds2^M(\log x)^2}.$$

Summing up the above inequality over all the choices of $m \le x/y$ in $\mathcal{N}(10,x)$, d a divisor of $\phi(m)$ with $P(d)\le(\log x)^2$, and $s \le x/(zmd)$ coprime to $\phi(m)$, we get that

$$\begin{aligned}\#\mathcal{M}_5 &\ll \frac{x(\log_2 x)^6}{2^M(\log x)^2}\sum_{\substack{m\le x/y\\ m\in\mathcal{N}(10,x)}}\frac{1}{m}\sum_{\substack{d\ge 1\\ p\,|\,d=>p\,|\,\phi(m)}}\frac{1}{d}\sum_{s\le x/(zmd)}\frac{1}{s}\\ &\ll \frac{x(\log_2 x)^6}{2^M\log x}\sum_{m\in\mathcal{N}(10,x)}\frac{1}{m}\frac{\phi(m)}{\phi(\phi(m))} \ll \frac{x(\log_2 x)^7}{2^M\log x}\sum_{m\in\mathcal{N}(10,x)}\frac{1}{m}\\ &\ll \frac{x}{(\log x)^{1+10\ln 2+o(1)}} = o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x\to\infty. \end{aligned} \tag{35}$$

In the above estimates, we used again identity (34), the minimal order of the Euler function in the interval $[1,x]$ as well as Lemma 4 (i). Since $\mathcal{M}_4$ and $\mathcal{M}_5$ cover $\mathcal{M}_3$, we get from estimates (33) and (35) that

$$\#\mathcal{M}_3 = o\left(\frac{x}{(\log x)^2}\right) \qquad \text{as } x\to\infty. \tag{36}$$

which together with the estimates (28), (29) and (i)–(vi) completes the proof of Theorem 1. □

3 Proof of Theorem 2

Let x be a large positive real number. Since $\varepsilon(x)$ decreases and tends to infinity arbitrarily slowly, we may assume that $\varepsilon(x) \geq 2/\log_3 x$ for if not we may replace $\varepsilon(x)$ by $\max\{\varepsilon(x), 2/\log_3 x\}$. It was shown in [7] that there exists a positive constant c_1 such that for all $n \leq x$ except $o(x)$ of them $\phi(n)$ is a multiple of all the primes $p \leq c_1 \log_2 x/\log_3 x$. Thus,

$$\phi^{(2)}(n) \leq e^{-\gamma}(1+o(1))\frac{\phi(n)}{\log_3 x} \qquad \text{as } x \to \infty$$

with $o(x)$ exceptions n. Here, γ is the Euler constant. Since $\phi(m) \leq m/2$ whenever m is even, we get that

$$\sum_{k=2}^{\kappa(n)} \phi^{(k)}(n) \leq \phi^{(2)}(n) \sum_{k=2}^{\kappa(n)} \frac{1}{2^{k-2}} < 2\phi^{(2)}(n) < \frac{2n}{\log_3 x}$$

holds for all $n \leq x$ with $o(x)$ exceptions as $x \to \infty$. We thus get that

$$n - F(n) \geq (n - \phi(n)) - \frac{2n}{\log_3 x}.$$

Since $n - \phi(n)$ counts the number of positive integers $k \leq n$ which are not coprime to n, it follows that $n - \phi(n) \geq n/p(n)$ where $p(n)$ is the smallest prime factor of n. Since the number of $n \leq x$ for which $p(n) > (2\varepsilon(x))^{-1}$ is $O(x/\log((2\varepsilon(x))^{-1})) = o(x)$ as $x \to \infty$, we get that $p(n) \leq (2\varepsilon(x))^{-1}$ with $o(x)$ exceptions as $x \to \infty$. Thus, except for $o(x)$ such $n \leq x$, we have

$$n - F(n) \geq \frac{n}{p(n)} - \frac{2n}{\log_3 x} \geq 2n\left(\varepsilon(x) - \frac{1}{\log_3 x}\right) \geq \varepsilon(x)n.$$

Since $n \log n \geq x$ holds for all $n \leq x$ with $O(x/\log x) = o(x)$ exceptions and $\varepsilon(x)$ is decreasing, we get that the inequality

$$n - F(n) \geq \varepsilon(n \log n)n$$

holds on a set of n of asymptotic density 1, which implies the desired conclusion since the function $\varepsilon(x)$ is arbitrary, subject to the conditions that it is decreasing for large x and tends to zero when x tends to infinity. □

4 Proof of Theorem 3

We let x be large and put $s = c_2 \log_2 x \log_3 x/\log_4 x$ where $c_2 > 0$ is some absolute constant to be chosen later. Let $L = \lfloor \sqrt{\log x} \rfloor$ and consider the set of integers

$$\mathcal{W} = \{\prod_{p \leq s} p^{\alpha_p} \ : \ \alpha_p \in [L,\ 2L] \text{ for all } p \leq s\}.$$

Let $M = \prod_{p \leq s} p$. If $n \in \mathcal{W}$ then

$$n \leq M^{2L} \leq \exp((2+o(1))sL) = \exp(O((\log x)^{1/2} \log_2 x \log_3 x)) = x^{o(1)}$$

as $x \to \infty$, therefore

$$F(n) = \sum_{k\geq 1} \phi^{(k)}(n) \leq \phi(n) \sum_{k\geq 1} \frac{1}{2^{k-1}} \leq 2n = x^{o(1)} \qquad \text{as } x \to \infty.$$

In particular $F(\mathcal{W}) \subset \mathcal{U}(x)$ holds for large x. Note also that

$$\#\mathcal{W} \geq (L+1)^{\pi(s)} = \exp((1+o(1))s \log L/\log s),$$

therefore

$$\log(\#\mathcal{W}) \gg \frac{s \log L}{\log s} \gg \frac{(\log_2 x)^2}{\log_4 x}.$$

From the above considerations, it follows that Theorem 3 will follow provided that we can show that if $c_2 > 0$ is suitably chosen and x is large, then F restricted to $\mathcal{W}$ is one-to-one.

We take a closer look at $F(n)$ for $n \in \mathcal{W}$. Note that as long as $M \mid \phi^{(k)}(n)$, we have that $\phi^{(k+1)}(n) = (\phi(M)/M)\phi^{(k)}(n)$. Since clearly $M \mid \phi^{(k)}(n)$ for all $k \leq L-1$, it follows that if we put $\delta = M/\phi(M)$, then

$$\phi^{(k)}(n) = \frac{\phi(n)}{\delta^{k-1}} = \frac{n}{\delta^k}$$

holds for all $k = 1,\ 2, \ldots,\ L$. Thus, for $n \in \mathcal{W}$ we have

$$\begin{aligned} F(n) &= \sum_{k\leq L} \phi^{(k)}(n) + \sum_{k>L} \phi^{(k)}(n) = n \sum_{k=1}^{L} \delta^{-k} + O\left(\phi^{(L+1)}(n) \sum_{j\geq 0} \frac{1}{2^j}\right) \\ &= n\left(\frac{1-\delta^{-(L+1)}}{1-\delta^{-1}}\right) + O\left(\frac{n}{\delta^L}\right) = n\left(\frac{1}{1-\delta^{-1}} + O\left(\frac{1}{\delta^L}\right)\right). \end{aligned} \tag{37}$$

Note that $\delta = \prod_{p\leq s}(1-1/p)^{-1} \asymp \log s$. Suppose now that $F(n_1) = F(n_2)$ for two distinct integers $n_1,\ n_2$ in $\mathcal{W}$. From the above relation (37) we get that

$$\frac{1}{1-\delta^{-1}}(n_1 - n_2) = O\left(\frac{n_1+n_2}{\delta^L}\right)$$

and since $\delta \to \infty$ when $x \to \infty$, we get that $1/2 < n_1/n_2 < 2$ holds when x is sufficiently large. We now get that

$$|n_1 n_2^{-1} - 1| \ll \delta^{-L}. \tag{38}$$

Note that $n_1 n_2^{-1} \neq 1$ is a rational number of the form $\prod_{p\leq s} p^{\delta_p}$ with some integer exponents $\delta_p \in [-L, L]$. Using a linear form in logarithms due to Matveev (see Corollary 2.3 of [9]), we get that

$$\log |n_1 n_2^{-1} - 1| \geq -c_3^{\pi(s)} \Omega \log L, \tag{39}$$

where

$$\Omega = \prod_{p\leq s} \log p \leq (\log s)^{\pi(s)}.$$

Taking logarithms in estimate (38) and using (39), we get

$$L \log \delta \leq (c_3 \log s)^{\pi(s)} \log L,$$

therefore

$$\sqrt{\log x} \le (c_3 \log s)^{\pi(s)} (\log_2 x)^2.$$

Taking logarithms again we get

$$(\log_2 x)/2 - \log_3 x \le \pi(s) \log_2 s + O(1) \le (1 + o(1)) \frac{s \log_2 s}{\log s} + O(1).$$

Recalling the definition of s, we see that if we choose $c_2 = 1/3$, then the above inequality is impossible for large enough values of x. This completes the proof of Theorem 3. □

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References

[1] N. L. Bassily, I. Kátai and M. Wijsmuller, On the prime power divisors of the iterates of the Euler function, *Publ. Math. (Debrecen)* **55** (1999), 17–32.

[2] E. R. Canfield, P. Erdős and C. Pomerance, On a problem of Oppenheim concerning "Factorisatio Numerorum", *J. Number Theory*, **17** (1983), 1–28.

[3] P. Erdős, A. Granville, C. Pomerance and S. Spiro, On the normal behavior of the iterates of some arithmetic function, in *Analytic Number Theory* Birkhäuser, Boston, 1990, pp. 165–204.

[4] K. Ford, On the distribution of totients, *Ramanujan J.* **2** (1998), 67–151.

[5] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[6] D. E. Iannucci, D. Moujie and G. L. Cohen, On perfect totients numbers, *J. Integer Sequences* **6** (2003), Article 03.4.5.

[7] F. Luca and C. Pomerance, On some problems of Mąkowski-Schinzel and Erdős concerning the arithmetical functions ϕ and σ, *Colloq. Math.* **92** (2002), 111–130.

[8] H. Maier and C. Pomerance, On the number of distinct values of the Euler function, *Acta Arith.* **49** (1988), 263–275.

[9] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk. Ser. Math.* **64** (2000), 125–180; English translation in *Izv. Math.* **64** (2000), 1217–1269.

[10] A. L. Mohan and D. Suryanarayana, Perfect totient numbers, in *Number Theory (Proc. 3rd Matscience Conf. Misore 1981)*, Lect. Notes in Math. **938**, Springer-Verlag, New York, 1982, 101–105.

[11] H. L. Montgomery and R. C. Vaughan, The large sieve, *Mathematika* **20** (1973), 119–134.

[12] I. E. Shparlinski, On the sum of iterations of the Euler function, *J. Integer Sequences* **9** (2006), Article 06.1.6.

[13] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, Cambridge, 1995.

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