Journal of Lie Theory
Volume **14** (2004) 215–226

Asymptotic Products and Enlargibility of Banach-Lie Algebras

Daniel Beltiţă

Communicated by K.-H. Neeb

Abstract. The paper provides a "standard" proof of a local theorem on enlargibility of Banach-Lie algebras. A particularly important special case of that theorem is that a Banach-Lie algebra is enlargible provided it has a dense locally finite subalgebra. The theorem is due to V. Pestov, who proved it by techniques of nonstandard analysis. The present proof uses a theorem concerning enlargibility of asymptotic products of contractive Banach-Lie algebras.
2000 MSC: Primary 22E65; Secondary 17B65, 46B08
Keywords: asymptotic product; enlargible Banach-Lie algebra

1. Introduction

The present paper provides a "standard" proof of an enlargibility criterion (see Corollary 4.4 below) discovered by V. Pestov in [Pe88] and [Pe92], where it is proved by techniques of nonstandard analysis. A particularly important special case of this criterion is the fact that a real Banach-Lie algebra is enlargible (i.e., it is the Lie algebra of some Banach-Lie group) whenever it has a dense locally finite subalgebra; see Corollary 1 in [Pe88] and Remark 4.5 below. The latter fact turns out to play an essential role in connection with the very existence of groups corresponding to pseudo-restricted Lie algebras, eventually leading to a class of Banach-Lie groups which possess quite natural complex homogeneous spaces (see [Be02]). Other interesting applications of the enlargibility criterion of [Pe88] and [Pe92] can be found in [Pe93].

The main ingredients in the present proof of the aforementioned criterion are a slight sharpening of a result in [GN01] concerning quotient Banach-Lie groups, the notion of enlargibility radius (see Definition 2.1 below), and the asymptotic products. Thus, these three ingredients allow us to give a new example of the general principle that the use of nonstandard analysis is equivalent to the ultraproduct approach (see e.g., page 27 in [HM83]). We note that the idea of ultraproduct had been previously exploited in connection with the Lie algebras. See e.g., [Fr82] (or §20 in [BS01]), as well as [CGM].

ISSN 0949–5932 / $2.50

The structure of the paper is as follows. In §2 we introduce the notion of enlargibility radius of a Banach-Lie algebra with respect to a closed subalgebra (Definition 2.1), and prove an important property in the case when the subalgebra is actually an ideal (Theorem 2.6).

In §3 we are concerned with lower estimates for the enlargibility radius of an asymptotic product of Banach-Lie algebras (Theorem 3.1). As a consequence, we get an enlargibility criterion for asymptotic products (Corollary 3.9).

In §4, we use Theorems 2.6 and 3.1 to prove lower estimates for the enlargibility radius of a Banach-Lie algebra $\mathfrak{g}$ in terms of a local system of closed subalgebras whose union is dense in $\mathfrak{g}$ (Theorem 4.3). As a special case (see Corollary 4.4), we then recover the Local Theorem on Enlargibility of [Pe88] and [Pe92].

We now introduce some notation and terminology. For any Banach-Lie group G, we denote its Lie algebra by $\mathbf{L}(G)$. If $\varphi\colon G \to H$ is a homomorphism of Banach-Lie groups, then $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$ denotes the corresponding homomorphism of Banach-Lie algebras. By *contractive* Banach-Lie algebra we mean a real Banach-Lie algebra $\mathfrak{g}$ equipped with a fixed norm $\|\cdot\|$ which defines the topology of $\mathfrak{g}$ and has the property that $\|[x,y]\| \le \|x\|\cdot\|y\|$ for all $x,y\in\mathfrak{g}$. In this case, for every $R>0$ we denote $B_{\mathfrak{g}}(0,R) = \{x\in\mathfrak{g} \mid \|x\| < R\}$.

Now let J be a directed set and $\{\mathfrak{g}_j\}_{j\in J}$ a family of Banach spaces over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Then

$$\ell^\infty\big(\{\mathfrak{g}_j\}_{j\in J}\big) := \Big\{x = (x_j)_{j\in J} \in \prod_{j\in J}\mathfrak{g}_j \mid \|x\| := \sup_{j\in J}\|x_j\| < \infty\Big\}$$

(with componentwise defined addition and scalar multiplication) is in turn a Banach space over $\mathbb{K}$, and

$$c_0\big(\{\mathfrak{g}_j\}_{j\in J}\big) := \Big\{(x_j)_{j\in J} \in \ell^\infty\big(\{\mathfrak{g}_j\}_{j\in J}\big) \mid \lim_{j\in J}\|x_j\| = 0\Big\}$$

is a closed subspace of $\ell^\infty\big(\{\mathfrak{g}_j\}_{j\in J}\big)$. Thus the quotient

$$\mathfrak{g} := \ell^\infty\big(\{\mathfrak{g}_j\}_{j\in J}\big)/c_0\big(\{\mathfrak{g}_j\}_{j\in J}\big)$$

is a Banach space over $\mathbb{K}$ (with the quotient norm), and we call it the *asymptotic product* of the family $\{\mathfrak{g}_j\}_{j\in J}$. If moreover $\mathfrak{g}_j$ is a contractive Banach-Lie algebra for each $j\in J$, then it is clear that $\ell^\infty\big(\{\mathfrak{g}_j\}_{j\in J}\big)$ is a contractive Banach-Lie algebra (with componentwise defined bracket) and $c_0\big(\{\mathfrak{g}_j\}_{j\in J}\big)$ is a closed ideal of $\ell^\infty\big(\{\mathfrak{g}_j\}_{j\in J}\big)$, so that the asymptotic product $\mathfrak{g}$ has a natural structure of contractive Banach-Lie algebra.

For later reference, we now recall a simple result which allows us to compute the norms of elements in asymptotic products.

Lemma 1.1. *Let J be a directed set and $\{\mathfrak{Y}\}_{j\in J}$ a family of Banach spaces over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Denote $\widetilde{\mathfrak{Y}} = \ell^\infty\big(\{\mathfrak{Y}_j\}_{j\in J}\big)$, $\widetilde{\mathfrak{Y}}_0 = c_0\big(\{\mathfrak{Y}_j\}_{j\in J}\big)$ and $\mathfrak{Y} = \widetilde{\mathfrak{Y}}/\widetilde{\mathfrak{Y}}_0$. Then for all $\tilde a = (a_j)_{j\in J}\in\widetilde{\mathfrak{Y}}$ the norm of $\tilde a + \widetilde{\mathfrak{Y}}_0 \in \mathfrak{Y}$ can be computed by*

$$\|\tilde a + \widetilde{\mathfrak{Y}}_0\| = \limsup_{j\in J}\|a_j\| = \inf_{i\in J}\sup_{i\le j\in J}\|a_j\|.$$

Proof. The proof of Proposition A.6.1 (at page 343) in [ER00] extends word by word. ∎

Also for later reference, we state a well-known property of the *Baker-Campbell-Hausdorff series* $\mathcal{H}(\cdot,\cdot)$.

Lemma 1.2. *For every* $\varepsilon > 0$ *there exists* $\delta \in (0,\varepsilon)$ *such that for every contractive Banach-Lie algebra* $\mathfrak{g}$ *and every* $x, y \in B_{\mathfrak{g}}(0,\delta)$ *we have*

$$\|\mathcal{H}(x,y)\| < \varepsilon.$$

Proof. Use e.g., Lemma 1 and formula (13) in no. 2 in §7 in Chapter II in [Bo72]. ∎

2. Enlargibility radii

Definition 2.1. Let $\mathfrak{g}$ be a contractive Banach-Lie algebra. If $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}$, we define the *enlargibility radius of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$ as the supremum $r_{\mathfrak{h}}(\mathfrak{g})$ of the set of all real numbers $R > 0$ such that there exist a real Banach-Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ and a subgroup H of G such that

$$\exp_G|_{B_{\mathfrak{g}}(0,R)} \text{ is injective and } \exp_G\bigl(\mathfrak{h} \cap B_{\mathfrak{g}}(0,R)\bigr) = H \cap \exp_G\bigl(B_{\mathfrak{g}}(0,R)\bigr),$$

where the supremum of an empty set is defined to be 0.

If $\mathfrak{h} = \{0\}$, we denote simply $r_{\mathfrak{h}}(\mathfrak{g}) = r(\mathfrak{g})$ and call it the *enlargibility radius* of $\mathfrak{g}$. ∎

Definition 2.2. Let G be a connected real Banach-Lie group such that $\mathbf{L}(G) = \mathfrak{g}$ is equipped with a norm making it into a contractive Banach-Lie algebra. For every $R > 0$ such that $\exp_G|_{B_{\mathfrak{g}}(0,R)}$ is injective (that is, $0 < R \le r(\mathfrak{g})$ in the terminology of Definition 2.1), we denote

$$V_{G,R} = \exp_G\bigl(B_{\mathfrak{g}}(0,R)\bigr)$$

and define

$$\log_{G,R}\colon G \to \mathfrak{g} \cup \{\infty\}$$

by

$$\log_{G,R} a = \begin{cases} \bigl(\exp_G|_{B_{\mathfrak{g}}(0,R)}\bigr)^{-1}(a) & \text{if } a \in V_{G,R}, \\ \infty & \text{if } a \in G \setminus V_{G,R}. \end{cases}$$

Furthermore, we extend the norm of $\mathfrak{g}$ to a function $\|\cdot\|\colon \mathfrak{g} \cup \{\infty\} \to [0,\infty]$ with $\|\infty\| = \infty$.

It is clear that, if $0 < R_1 \le R < r(\mathfrak{g})$, then $\log_{G,R_1} = \log_{G,R}$ on V_{G,R_1}. ∎

Remark 2.3. (a) A contractive Banach-Lie algebra $\mathfrak{g}$ is enlargible if and only if $r(\mathfrak{g}) > 0$. Moreover, let us denote by $\Pi(\mathfrak{g})$ the period group of $\mathfrak{g}$ and

$$\delta := \inf\{\|\gamma\| \mid 0 \neq \gamma \in \Pi(\mathfrak{g})\}.$$

We recall that $\Pi(\mathfrak{g})$ is an additive subgroup of the center of $\mathfrak{g}$ and, if $\widetilde{G}$ is a simply connected Banach-Lie group whose Lie algebra is $\mathfrak{g}$, then $\Pi(\mathfrak{g})$ equals the set of all elements x in the center of $\mathfrak{g}$ with $\exp_{\widetilde{G}} x = \mathbf{1}$. Now, Lemma III.11 and the remark following it in [GN01] show that

$$\min\{\pi, r(\mathfrak{g})\} = \min\{\pi, \delta/2\}.$$

In fact, that lemma implies that $\min(\pi, \delta/2) \leq r(\mathfrak{g})$. In particular, if $\delta/2 \geq \pi$, then $r(\mathfrak{g}) \geq \pi$. Actually, the aforementioned facts from [GN01] show that, if $\delta/2 \leq \pi$, then $\delta/2$ is the supremum of the set of all real numbers $R > 0$ such that $\exp_{\widetilde{G}} |_{B_{\mathfrak{g}}(0,R)}$ is injective. Thus, if $\delta/2 \leq \pi$, then Remark 2.4 bellow shows that $r(\mathfrak{g}) = \delta/2$. Consequently, $\min\{\pi, r(\mathfrak{g})\} = \min\{\pi, \delta/2\}$ as claimed.

(b) In connection with Definition 2.1, we note that, if G is a Banach-Lie group with $\mathbf{L}(G) = \mathfrak{g}$ (a contractive Banach-Lie algebra), $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}$ and H is a subgroup of G such that for some real number $R > 0$ we have

$$\exp_G |_{B_{\mathfrak{g}}(0,R)} \text{ is injective and } \exp_G\big(\mathfrak{h} \cap B_{\mathfrak{g}}(0,R)\big) = H \cap \exp_G\big(B_{\mathfrak{g}}(0,R)\big),$$

then H is actually a *Lie subgroup* of G in the sense of Definition I.4 (b) in [GN01]. In fact, H is locally closed (since $\mathfrak{h} \cap B_{\mathfrak{g}}(0,R)$ is closed in $B_{\mathfrak{g}}(0,R)$ and H is a subgroup), hence it is closed by Proposition 2.1 in Chapter I in [Ho65]. Now the assumption $\exp_G\big(\mathfrak{h} \cap B_{\mathfrak{g}}(0,R)\big) = H \cap \exp_G\big(B_{\mathfrak{g}}(0,R)\big)$ easily implies that

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid \exp_G(\mathbb{R}x) \subseteq H\},$$

and then H is a Lie subgroup of G (see Remark I.5 in [GN01]).

(c) It follows by Proposition 2.5 in [LT66] that for every finite-dimensional contractive Banach-Lie algebra $\mathfrak{g}$ we have $r(\mathfrak{g}) \geq 2\pi$. ■

Remark 2.4. If $\mathfrak{g}$ is a contractive Banach-Lie algebra, $0 < R < r(\mathfrak{g})$ and $\widetilde{G}$ is a connected *simply connected* Banach-Lie group with $\mathbf{L}(\widetilde{G}) = \mathfrak{g}$, then $\exp_{\widetilde{G}} |_{B_{\mathfrak{g}}(0,R)}$ is injective. In fact, by the very definition of $r(\mathfrak{g})$ (see Definition 2.1), there exists a Banach-Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ and $\exp_G |_{B_{\mathfrak{g}}(0,R)}$ injective. Now let G_0 be the connected $\mathbf{1}$-component of G. Since $\mathbf{L}(G_0) = \mathfrak{g}$, we have a covering map $p\colon \widetilde{G} \to G_0$ such that $p \circ \exp_{\widetilde{G}} = \exp_G$. Since $\exp_G |_{B_{\mathfrak{g}}(0,R)}$ is injective, it then follows that $\exp_{\widetilde{G}} |_{B_{\mathfrak{g}}(0,R)}$ is in turn injective. ■

Remark 2.5. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are contractive Banach-Lie algebras and there exists an injective homomorphism of Banach-Lie algebras $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ with $\|\varphi\| \leq 1$, then we have $r(\mathfrak{g}_2) \leq r(\mathfrak{g}_1)$. (This is a slight improvement of assertion $(***)$ at page 22 in [EK64]). Using Lemma II.1 in [GN01] and the preceding Remark 2.3 (b), one can actually prove that $r_{\mathfrak{h}}(\mathfrak{g}_2) \leq r_{\varphi^{-1}(\mathfrak{h})}(\mathfrak{g}_1)$ provided $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}_2$. ■

The next statement is, in some respects, a slight sharpening of Theorem II.2 in [GN01], expressed in terms of enlargibility radii.

Theorem 2.6. *For every real number $\varepsilon > 0$ there exists another real number $\eta > 0$ such that the following assertion holds: If $\mathfrak{g}$ is a contractive Banach-Lie algebra and $\mathfrak{h}$ is a closed ideal of $\mathfrak{g}$ such that $r_{\mathfrak{h}}(\mathfrak{g}) \geq \varepsilon$, then $r(\mathfrak{g}/\mathfrak{h}) \geq \eta$.*

Proof. Let R_0 be an arbitrary real number with

$$0 < R_0 < \min\{\varepsilon, (1/2)\log 2\},$$

so that the Baker-Campbell-Hausdorff series $\mathcal{H}(x,y)$ is convergent whenever $x, y \in B_{\mathfrak{k}}(0, R_0)$ and $\mathfrak{k}$ is a contractive Banach-Lie algebra (see Proposition 1 in no. 2 in §7 in [Bo72]).

Then for every contractive Banach-Lie algebra $\mathfrak{g}$ and every closed ideal $\mathfrak{h}$ of $\mathfrak{g}$ with $r_{\mathfrak{h}}(\mathfrak{g}) \geq \varepsilon$ we have $r_{\mathfrak{h}}(\mathfrak{g}) > R_0$. It then follows by Remark 2.3 (b) that there exists a Banach-Lie group G with a Lie subgroup H such that $\mathbf{L}(G) = \mathfrak{g}$, $\mathbf{L}(H) = \mathfrak{h}$, $\exp_G|_{B_{\mathfrak{g}}(0,R_0)}$ is injective and $\exp_G\big(\mathfrak{h} \cap B_{\mathfrak{g}}(0,R_0)\big) = H \cap \exp_G\big(B_{\mathfrak{g}}(0,R_0)\big)$. Clearly we may assume that H is connected. Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, it then follows that H is a normal subgroup of G. On the other hand, since $\mathfrak{g}/\mathfrak{h}$ is equipped with the quotient norm, it follows that for all $R > 0$ we have $Q\big(B_{\mathfrak{g}}(0,R)\big) = B_{\mathfrak{g}/\mathfrak{h}}(0,R)$, where $Q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the quotient map.

Now denote $V = B_{\mathfrak{g}/\mathfrak{h}}(0,R_0)$ and $W = B_{\mathfrak{g}/\mathfrak{h}}(0,\eta)$, where η stands for the value of δ given by Lemma 1.2 for $\varepsilon = R_0$. Also denote $U = B_{\mathfrak{g}}(0,R_0)$ and $A = B_{\mathfrak{g}}(0,\eta)$, so that $\mathcal{H}(A \times A) \subseteq U$ according to the choice of η. Also, $Q(A) = W$. It then follows as in the proof of Theorem II.2 in [GN01] that the mapping

$$E\colon B_{\mathfrak{g}/\mathfrak{h}}(0,\eta) \to G/H, \quad E\big(Q(X)\big) := q(\exp_G X),$$

is correctly defined and an analytic homomorphism of local analytic groups. Here $q\colon G \to G/H$ stands for the quotient map, and we think of $B_{\mathfrak{g}/\mathfrak{h}}(0,\eta)$ as a local analytic group with the Baker-Campbell-Hausdorff multiplication. Thus, in view of the theorem on extension of analytic structure (see [Sw65], page 213), it follows that there exists a Banach-Lie group K with $\mathbf{L}(K) = \mathfrak{g}/\mathfrak{h}$ *and* $\exp_K|_{B_{\mathfrak{g}/\mathfrak{h}}(0,\eta)}$ *injective*, hence $\eta \leq r(\mathfrak{g}/\mathfrak{h})$ according to Definition 2.1. ■

3. Enlargibility of asymptotic products

Here is the main result of the present section.

Theorem 3.1. *For every real number $\varepsilon > 0$ there exists another real number $\eta > 0$ such that the following assertion holds: If J is a directed set and $\{\mathfrak{g}_j\}_{j\in J}$ is a family of contractive Banach-Lie algebras with the asymptotic product $\mathfrak{g}$ and such that $\inf\limits_{j\in J} r(\mathfrak{g}_j) \geq \varepsilon$, then $r(\mathfrak{g}) \geq \eta$.* ■

We now establish some notation needed in the proof of Theorem 3.1.

Notation 3.2. In the present section, until the proof of Theorem 3.1, we keep the notation in its statement, assuming that $\inf\limits_{j\in J} r(\mathfrak{g}_j) \geq \varepsilon$. We fix an arbitrary real number R_0 with $0 < R_0 < \varepsilon$.

For each $j \in J$, it then follows by Remark 2.3 (a) that the Banach-Lie algebra $\mathfrak{g}_j$ is enlargible, and we denote by G_j a connected simply connected Banach-Lie group with $\mathbf{L}(G_j) = \mathfrak{g}_j$. ■

Remark 3.3. It follows by Proposition III.12 (ii)–(iii) in [GN01] and the preceding Remark 2.3 (a) that there exist a connected Banach-Lie group G and a group homomorphism $\psi: G \to \prod_{j \in J} G_j$ such that, if

$$\pi_k: \ell^\infty(\{\mathfrak{g}_j\}_{j \in J}) \to \mathfrak{g}_k \text{ and } p_k: \prod_{j \in J} G_j \to G_k$$

are the canonical projection maps whenever $k \in J$, then the following assertions hold.

(i) We have $\mathbf{L}(G) = \ell^\infty(\{\mathfrak{g}_j\}_{j \in J}) := \widetilde{\mathfrak{g}}$.
(ii) The homomorphism ψ is injective and continuous.
(iii) For every $k \in J$ we have $\mathbf{L}(p_k \circ \psi) = \pi_k$.
(iv) For every $x = (x_j)_{j \in J} \in \widetilde{\mathfrak{g}}$ we have $\psi(\exp_G x) = (\exp_{G_j} x_j)_{j \in J}$. ■

Notation 3.4. Until the proof of Theorem 3.1 we shall keep the notation of Remark 3.3. Using Definition 2.2, we introduce the following subset of G:

$$H := \{h \in G \mid \lim_{j \in J} \|(\log_{G_j, R} \circ p_j \circ \psi)(h)\| = 0\},$$

where $0 < R \leq R_0$. Note that the definition of H does not depend on the choice of R (see the remark concluding Definition 2.2).

We also denote

$$\mathfrak{h} := c_0(\{\mathfrak{g}_j\}_{j \in J}). \quad ■$$

Remark 3.5. We recall from the Introduction that $\mathfrak{h} = c_0(\{\mathfrak{g}_j\}_{j \in J})$ is a closed ideal of the contractive Banach-Lie algebra $\widetilde{\mathfrak{g}} = \ell^\infty(\{\mathfrak{g}_j\}_{j \in J}$. In this connection, we shall eventually see from Lemmas 3.6–8 that H is a normal Lie subgroup of G corresponding to the ideal $\mathfrak{h}$ of $\widetilde{\mathfrak{g}} = \mathbf{L}(G)$. ■

Lemma 3.6. *The set H is a subgroup of G.*

Proof. Let $h_1, h_2 \in H$ and fix $\delta > 0$ given by Lemma 1.2 for $\varepsilon = R_0$. We may suppose that $0 < \delta < R_0$. It then follows by Notation 3.4 that, for $i \in \{1, 2\}$, we have $\lim_{j \in J} \|(\log_{G_j, \delta} \circ p_j \circ \psi)(h_i)\| = 0$, hence there exists $j_0 \in J$ such that

$$(\forall j \in J,\, j \geq j_0) \qquad \|(\log_{G_j, \delta} \circ p_j \circ \psi)(h_i)\| < \delta \text{ for } i \in \{1, 2\}.$$

Then for all $j \in J$ with $j \geq j_0$ we have

$$\begin{aligned}(\log_{G_j, R_0} \circ p_j \circ \psi)(h_1 h_2) &= \log_{G_j, R_0}\big((p_j \circ \psi)(h_1) \cdot (p_j \circ \psi)(h_2)\big) \\ &= \mathcal{H}\big((\log_{G_j, \delta} \circ p_j \circ \psi)(h_1), (\log_{G_j, \delta} \circ p_j \circ \psi)(h_2)\big),\end{aligned}$$

where the latter equality follows in view of the choice of δ. Now, since

$$\lim_{j\in J} \|(\log_{G_j,\delta} \circ p_j \circ \psi)(h_i)\| = 0$$

for $i \in \{1,2\}$, it easily follows by Lemma 1.2 that $\lim_{j\in J} \|(\log_{G_j,R_0} \circ p_j \circ \psi)(h_1 h_2)\| = 0$. Thus $h_1 h_2 \in H$.

Finally, note that for every $h \in G$ and $j \in J$ we have

$$(\log_{G_j,R_0} \circ p_j \circ \psi)(h^{-1}) = \begin{cases} -(\log_{G_j,R_0} \circ p_j \circ \psi)(h) & \text{if } h \in V_{G,R_0}, \\ \infty & \text{if } h \in G \setminus V_{G,R_0}, \end{cases}$$

which implies at once that $h \in H$ is equivalent to $h^{-1} \in H$, and the proof ends.■

Lemma 3.7. *The subgroup H of G is normal.*

Proof. Since the group G is connected, it clearly suffices to show that $gHg^{-1} \subseteq H$ whenever $g \in V_{G,R_0}$. Let us fix such an element g and denote $x := \log_{G,R_0} g$, so that $x \in B_{\widetilde{\mathfrak{g}}}(0,R_0)$ and $\exp_G x = g$. Also denote $g_j := (p_j \circ \psi)(g) \in G_j$ whenever $j \in J$. By Remark 3.3 (iii)–(iv) we then have $g_j \in V_{G_j,R_0}$, hence denoting $x_j := \log_{G_j,R_0} g_j$ we get $x_j \in B_{\mathfrak{g}_j}(0,R_0)$ and $\exp_{G_j} x_j = g_j$.

Now let $h \in H$ arbitrary, so that $\lim_{j\in J} \|(\log_{G_j,R_0} \circ p_j \circ \psi)(h)\| = 0$. For each $j \in J$ we have

$$\begin{aligned}
(\log_{G_j,R_0/\mathrm{e}^{R_0}} &\circ p_j \circ \psi)(ghg^{-1}) \\
&= \log_{G_j,R_0/\mathrm{e}^{R_0}}\big((p_j \circ \psi)(g) \cdot (p_j \circ \psi)(h) \cdot (p_j \circ \psi)(g)^{-1}\big) \\
&= \log_{G_j,R_0/\mathrm{e}^{R_0}}\big(g_j \cdot (p_j \circ \psi)(h) \cdot g_j^{-1}\big) \\
&= \log_{G_j,R_0/\mathrm{e}^{R_0}}\big((\exp_{G_j} x_j) \cdot (p_j \circ \psi)(h) \cdot (\exp_{G_j} x_j)^{-1}\big) \\
&= \mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_j} x_j} \log_{G_j,R_0}\big((p_j \circ \psi)(h)\big).
\end{aligned}$$

For the latter equality, see Notation 3.4 and note that, since the Banach-Lie algebra $\mathfrak{g}_j$ is contractive, we have $\|\mathrm{ad}_{\mathfrak{g}_j} x_j\| \le \|x_j\| < R_0$, hence $\|\mathrm{e}^{-\mathrm{ad}_{\mathfrak{g}_j} x_j}\| \le \mathrm{e}^{\|\mathrm{ad}_{\mathfrak{g}_j} x_j\|} \le \mathrm{e}^{R_0}$, which in turn implies $\mathrm{e}^{-\mathrm{ad}_{\mathfrak{g}_j} x_j}\big(B_{\mathfrak{g}_j}(0,R_0/\mathrm{e}^{R_0})\big) \subseteq B_{\mathfrak{g}_j}(0,R_0)$.

Using the inequality $\|\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_j} x_j}\| \le \mathrm{e}^{R_0}$, it follows from the above equalities that

$$(\forall j \in J) \qquad \|(\log_{G_j,R_0/\mathrm{e}^{R_0}} \circ p_j \circ \psi)(ghg^{-1})\| \le \mathrm{e}^{R_0} \cdot \|(\log_{G_j,R_0} \circ p_j \circ \psi)(h)\|.$$

Now, since $h \in H$, we get $ghg^{-1} \in H$ (see Notation 3.4) as desired. ■

Lemma 3.8. *For every $R \in (0,R_0]$ we have*

$$\exp_G|_{B_{\widetilde{\mathfrak{g}}}(0,R)} \textit{ injective and } \exp_G\big(\mathfrak{h} \cap B_{\widetilde{\mathfrak{g}}}(0,R)\big) = H \cap V_{G,R}.$$

Proof. The fact that $\exp_G|_{B_{\widetilde{\mathfrak{g}}}(0,R)}$ is injective follows by Remark 3.3 (i), (ii), (iv), along with the fact that $\exp_{G_j}|_{B_{\mathfrak{g}_j}(0,R)}$ is injective for all $j \in J$ (according to the choice of R_0).

The proof of the other assertion has two steps.

1° We first prove that $\exp_G(\mathfrak{h}) \subseteq H$. To this end, let $x = (x_j)_{j\in J} \in \mathfrak{h} = c_0(\{\mathfrak{g}_j\}_{j\in J})$. Then

$$\lim_{j\in J} \|x_j\| = 0, \tag{1}$$

hence there exists $j_0 \in J$ such that

$$(\forall j \in J,\ j \geq j_0) \qquad \|x_j\| < R. \tag{2}$$

On the other hand, for every $j \in J$, we have $(p_j \circ \psi)(\exp_G x) = \exp_{G_j} x_j$ by Remark 3.3 (iv). Then (2) implies that

$$(\forall j \in J,\ j \geq j_0) \qquad (\log_{G_j,R} \circ p_j \circ \psi)(\exp_G x) = x_j. \tag{3}$$

It then follows by (1), (3) and Notation 3.4 that $\exp_G x \in H$.

2° It follows by Step 1° that $\exp_G\big(\mathfrak{h} \cap B_{\mathfrak{g}}(0,R)\big) \subseteq H \cap V_{G,R}$.

To prove the converse inclusion, take $h \in H \cap V_{G,R}$ arbitrary. Since $h \in V_{G,R}$, there exists a unique $x = (x_j)_{j\in J} \in B_{\widetilde{\mathfrak{g}}}(0,R)$ such that $h = \exp_G x$ (see the beginning of the present proof, and also Remark 3.3 (i)). The fact that $x \in B_{\widetilde{\mathfrak{g}}}(0,R)$ shows that for every $j \in J$ we have $\|x_j\| < R$.

Now note that, in Step 1°, we actually proved that (2) ⇒ (3), more precisely that $(\log_{G_j,R} \circ p_j \circ \psi)(\exp_G x) = x_j$ provided $\|x_j\| < R$. This fact shows that, in the present situation, we have

$$(\forall j \in J) \qquad (\log_{G_j,R} \circ p_j \circ \psi)(h) = x_j.$$

But $h \in H$, so by Notation 3.4 we have $\lim\limits_{j\in J} \|x_j\| = 0$, that is, $x \in \mathfrak{h}$. Thus $h = \exp_G x$ with $x \in \mathfrak{h} \cap B_{\widetilde{\mathfrak{g}}}(0,R)$, which concludes the proof of the desired equality. ∎

Proof of Theorem 3.1. For R_0 as in in Notation 3.2, we get by Remark 3.3, Notation 3.4, and Lemmas 3.6–8 that there exist a Banach-Lie group G and a subgroup H of G such that $\mathbf{L}(G) = \widetilde{\mathfrak{g}} := \ell^\infty(\{\mathfrak{g}_j\}_{j\in J})$, the function $\exp_G|_{B_{\widetilde{\mathfrak{g}}}(0,R_0)}$ is injective and $\exp_G\big(\mathfrak{h} \cap B_{\widetilde{\mathfrak{g}}}(0,R_0)\big) = H \cap \exp_G\big(B_{\widetilde{\mathfrak{g}}}(0,R_0)\big)$, where $\mathfrak{h} = c_0(\{\mathfrak{g}_j\}_{j\in J})$ as in Notation 3.4.

Thus $R_0 \leq r_{\mathfrak{h}}(\widetilde{\mathfrak{g}})$ by Definition 2.1. It then follows by Theorem 2.6 that there exists $\eta > 0$ depending only on R_0 such that $\eta \leq r(\widetilde{\mathfrak{g}}/\mathfrak{h})$. But $\widetilde{\mathfrak{g}}/\mathfrak{h} = \mathfrak{g}$ (see the definition of asymptotic products in the Introduction), hence $\eta \leq r(\mathfrak{g})$, as desired. ∎

Corollary 3.9. *If J is a directed set and $\{\mathfrak{g}_j\}_{j\in J}$ is a family of contractive Banach-Lie algebras such that $\inf\limits_{j\in J} r(\mathfrak{g}_j) > 0$, then the asymptotic product of the family $\{\mathfrak{g}_j\}_{j\in J}$ is an enlargible Banach-Lie algebra.*

Proof. Use Theorem 3.1 and Remark 2.3 (a). ∎

4. The local theorem on enlargibility

In the present section, we make use of the previous results on asymptotic products to prove the Local Theorem on Enlargibility of V. Pestov (see [Pe88] and [Pe92]). The next two lemmas claim their origins from a typical ultraproduct method of proof. (For this method, see the comments following the proof of Proposition 6.2 in [He80].)

Lemma 4.1. *Let $\mathfrak{g}$ be a contractive Banach-Lie algebra, J a directed set and $\{\mathfrak{g}_j\}_{j\in J}$ a family of closed subalgebras of $\mathfrak{g}$ such that the following conditions hold.*

(j) *If $j_1, j_2 \in J$ and $j_1 \le j_2$, then $\mathfrak{g}_{j_1} \subseteq \mathfrak{g}_{j_2}$.*

(jj) *We have $\mathfrak{g} = \bigcup\limits_{j\in J} \mathfrak{g}_j$.*

If $\widehat{\mathfrak{g}}$ stands for the asymptotic product of the family $\{\mathfrak{g}_j\}_{j\in J}$, then there exists an isometric homomorphism of Banach-Lie algebras $\psi\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$.

Proof. Let $a \in \mathfrak{g}$. According to the hypothesis (jj), there exists $j_0 \in J$ with $a \in \mathfrak{g}_{j_0}$. Define

$$a_j = \begin{cases} a & \text{if } j \ge j_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(a_j)_{j\in J} \in \ell^\infty(\{\mathfrak{g}_j\}_{j\in J})$, and it is easy to see that

$$\psi(a) := (a_j)_{j\in J} + c_0(\{\mathfrak{g}_j\}_{j\in J}) \in \widehat{\mathfrak{g}}$$

does not depend on the choice of j_0 and moreover $\|\psi(a)\| = \|a\|$ (see Lemma 1.1). Furthermore, it is clear that $\psi\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ is a Lie algebra homomorphism. ∎

Lemma 4.2. *Let $\mathfrak{g}$ be a contractive Banach-Lie algebra, I a directed set and $\{\mathfrak{h}_i\}_{i\in I}$ a family of closed subalgebras of $\mathfrak{g}$ such that the following hypotheses hold.*

(i) *If $i_1, i_2 \in I$ and $i_1 \le i_2$ then $\mathfrak{h}_{i_1} \subseteq \mathfrak{h}_{i_2}$.*

(ii) *We have $\mathfrak{g} = \overline{\bigcup\limits_{i\in I} \mathfrak{h}_i}$.*

Then there exist a directed set J and a family $\{\mathfrak{g}_j\}_{j\in J}$ of closed subalgebras of $\mathfrak{g}$ such that the following conditions hold.

(j) *If $j_1, j_2 \in J$ and $j_1 \le j_2$, then $\mathfrak{g}_{j_1} \subseteq \mathfrak{g}_{j_2}$.*

(jj) *We have $\mathfrak{g} = \bigcup\limits_{j\in J} \mathfrak{g}_j$.*

(jjj) *For every $j \in J$ there exists a sequence $i_0(j) \le i_1(j) \le \cdots$ in I such that, if $\widehat{\mathfrak{h}}_j$ stands for the asymptotic product of the family $\{\mathfrak{h}_{i_n(j)}\}_{n\in\mathbb{N}}$, then there exists an isometric homomorphism of Banach-Lie algebras $\varphi_j\colon \mathfrak{g}_j \to \widehat{\mathfrak{h}}_j$.*

Proof. The proof has several stages.

1° We define $J := \{j:\mathbb{N} \to I \mid j(0) \le j(1) \le \cdots\}$ endowed with the partial ordering

$$j_1 \le j_2 \iff (\forall n \in \mathbb{N}) \qquad j_1(n) \le j_2(n).$$

It is easy to check that J is a directed set, since I is directed.

For each $j \in J$ we define

$$\mathfrak{g}_j := \{a \in \mathfrak{g} \mid \big(\exists\, (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{h}_{j(n)}\big) \quad \lim_{n\to\infty} \|a_n - a\| = 0\}.$$

Since $\mathfrak{h}_{j(0)} \subseteq \mathfrak{h}_{j(1)} \subseteq \cdots$, it is easy to check that

$$\mathfrak{g}_j = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{h}_{j(n)}}$$

which implies at once that $\mathfrak{g}_j$ is a closed subalgebra of $\mathfrak{g}$.

2° We now show that the family $\{\mathfrak{g}_j\}_{j\in J}$ satisfies conditions (j) and (jj). Condition (j) is straightforward. To check condition (jj), let $a \in \mathfrak{g}$ arbitrary.

By hypothesis (ii), there exist $i_0, i_1, i_2, \ldots \in I$ and $a_0 \in \mathfrak{h}_{i_0}, a_1 \in \mathfrak{h}_{i_1}, \ldots$ such that $\lim_{n\to\infty} \|a_n - a\| = 0$. To construct $j \in J$ with $a \in \mathfrak{g}_j$, first define $j(0) = i_0$. Then pick $j(1) \in I$ such that $j(1) \ge j(0)$ and $j(1) \ge i_1$, so that $a_1 \in \mathfrak{h}_{i_1} \subseteq \mathfrak{h}_{j(1)}$. If $j(0) \le \cdots \le j(k)$ have been constructed, let $j(k+1) \in I$ with $j(k+1) \ge j(k)$ and $j(k+1) \ge i_{k+1}$, so that $a_{k+1} \in \mathfrak{h}_{i_{k+1}} \subseteq \mathfrak{h}_{j(k+1)}$, and so on. Since $\lim_{n\to\infty} \|a_n - a\| = 0$ and $a_n \in \mathfrak{h}_{j(n)}$ for all $n \in \mathbb{N}$, we get $a \in \mathfrak{g}_j$ according to the definition of $\mathfrak{g}_j$ at stage 1°.

Consequently, $\mathfrak{g} = \bigcup_{j\in J} \mathfrak{g}_j$, which is just condition (jj).

3° To check condition (jjj), let $j \in J$ be arbitrary, and define $i_n(j) = j(n)$ for all $n \in \mathbb{N}$. If $\widehat{\mathfrak{h}}_j$ denotes the asymptotic product of the family $\{\mathfrak{h}_{j(n)}\}_{n\in\mathbb{N}}$, we define

$$\varphi: \mathfrak{g}_j \to \widehat{\mathfrak{h}}_j = \ell^\infty(\{\mathfrak{h}_{j(n)}\}_{n\in\mathbb{N}})/c_0(\{\mathfrak{h}_{j(n)}\}_{n\in\mathbb{N}})$$

in the following way: for arbitrary $a \in \mathfrak{g}_j$, there exists $(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{h}_{j(n)}$ with $\lim_{n\to\infty} \|a_n - a\| = 0$. Then $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\{\mathfrak{h}_{j(n)}\}_{n\in\mathbb{N}})$, so that we may define

$$\varphi_j(a) := (a_n)_{n\in\mathbb{N}} + c_0(\{\mathfrak{h}_{j(n)}\}_{n\in\mathbb{N}}) \in \widehat{\mathfrak{h}}_j.$$

Then it is clear that $\varphi_j(a)$ does not depend on the choice of the sequence $(a_n)_{n\in\mathbb{N}}$, and Lemma 1.1 shows that

$$\|\varphi_j(a)\| = \limsup_{n\to\infty} \|a_n\| = \lim_{n\to\infty} \|a_n\| = \|a\|.$$

Also, it is clear that $\varphi_j: \mathfrak{g}_j \to \widehat{\mathfrak{h}}_j$ is a Lie algebra homomorphism. ∎

Concerning the equality $\mathfrak{g}_j = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{h}_{j(n)}}$ in stage 1° of the proof of Lemma 4.2, it is noteworthy that a related fact is noted in Remark 1.2 in [Me71].

We now come to the main result of the present paper.

Theorem 4.3. *For every real number $\varepsilon > 0$ there exists another real number $\eta > 0$ such that $r(\mathfrak{g}) \geq \eta$ provided $\mathfrak{g}$ is a contractive Banach-Lie algebra, I is a directed set and $\{\mathfrak{h}_i\}_{i\in I}$ is a family of closed subalgebras of $\mathfrak{g}$ such that $\inf_{i\in I} r(\mathfrak{h}_i) \geq \varepsilon$ and the following conditions hold:*

(i) *If $i_1, i_2 \in I$ and $i_1 \leq i_2$ then $\mathfrak{h}_{i_1} \subseteq \mathfrak{h}_{i_2}$.*

(ii) *We have $\mathfrak{g} = \overline{\bigcup_{i\in I} \mathfrak{h}_i}$.*

Proof. Let J and $\{\mathfrak{g}_j\}_{j\in J}$ given by Lemma 4.2. By Theorem 3.1, condition (jjj) in Lemma 4.2 and Remark 2.5 we get $\varepsilon_1 \leq \inf_{j\in J} r(\mathfrak{g}_j)$, where the real number $\varepsilon_1 > 0$ depends only on ε.

On the other hand, by Theorem 3.1, Lemma 4.1 and Remark 2.5 again, we have $\eta \leq r(\mathfrak{g})$, for some real number $\eta > 0$ depending only on ε_1, that is, on ε. ■

The following result is just the Local Theorem on Enlargibility proved in [Pe92].

Corollary 4.4. *A contractive Banach-Lie algebra $\mathfrak{g}$ is enlargible if and only if there exist a directed set I and a family $\{\mathfrak{h}_i\}_{i\in I}$ of closed subalgebras of $\mathfrak{g}$ satisfying the following conditions:*

(i) *If $i_1, i_2 \in I$ and $i_1 \leq i_2$ then $\mathfrak{h}_{i_1} \subseteq \mathfrak{h}_{i_2}$.*

(ii) *We have $\mathfrak{g} = \overline{\bigcup_{i\in I} \mathfrak{h}_i}$.*

(iii) *We have $\inf_{i\in I} r(\mathfrak{h}_i) > 0$.*

Proof. If $\mathfrak{g}$ is enlargible, we may let the family $\{\mathfrak{h}_i\}_{i\in I}$ consist in $\mathfrak{g}$ alone. For the converse assertion, use Theorem 4.3 and Remark 2.3 (a). ■

For the sake of completeness, we conclude by mentioning the result contained in Corollary 3.5 in [Pe92] (or Corrolaire 1 in [Pe88]).

Remark 4.5. It follows by Remark 2.3 (c) that condition (iii) in Corollary 4.4 is satisfied if each $\mathfrak{h}_i$ is finite dimensional. Consequently a Banach-Lie algebra is enlargible provided it has a dense locally finite subalgebra. ■

Acknowledgment: I thank Professor K.-H. Neeb for kindly drawing my attention to the paper [GN01], and to Professor V.G. Pestov for sending me, upon my request, a lot of offprints of his papers on enlargibility. I also thank the referee for a number of useful suggestions and interesting remarks.

References

[Be02] Beltiţă, D., *Complex homogeneous spaces of pseudo-restricted groups*, Math. Res. Lett., to appear.

[Bs01] Beltiţă, D., and M. Şabac, "Lie algebras of bounded operators," Operator Theory: Advances and Applications, vol. **120**, Birkhäuser Verlag, Basel, 2001.

[Bo72] Bourbaki, N., "Groupes et algébres de Lie," Hermann, Paris, 1972.

[CGM] Cuenca Mira, J. A., A. García Martín, and C. Martínez Gonzalez, *Structure theory for L^*-algebras*, Math. Proc. Camb. Phil. Soc. **107** (1990), 361–365.

[ER00] Effros, E. G., and Z.-J. Ruan, "Operator spaces," London Math. Soc. Monographs, vol. **23**, Clarendon Press, Oxford, 2000.

[EK64] van Est, W. T., and Th. J. Korthagen, *Non-enlargible Lie algebras*, Indag. Math. **26**= Proc. Kon. Ned. Akad. v. Wet. **A67** (1964), 15–31.

[Fr82] Frunză, Şt., *Generalized weights for operator Lie algebras*, in: "Spectral theory," Banach Center Publications, vol. **8**, Warszawa, 1982, pp. 281–287.

[GN01] Glöckner, H., and K.-H. Neeb, *Banach-Lie quotients, enlargibility, and universal complexifications*, J. reine angew. Math., to appear.

[He80] Heinrich, S., *Ultraproducts in Banach space theory*, J. reine angew. Math. **313** (1980), 72–104.

[HM83] Henson, C. W., and L. C. Moore, Jr., *Nonstandard analysis and the theory of Banach spaces*, in: A. E. Hurd (ed.), "Nonstandard analysis—recent developments," Lecture Notes in Mathematics, vol. **983**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983, pp. 27–112.

[Ho65] Hochschild, G., "The structure of Lie groups," Holden-Day, San Francisco, 1965.

[LT66] Lazard, M., and J. Tits *Domaines d'injectivité de l'application exponentielle*, Topology **4** (1965/1966), 315–322.

[Me71] Meyer-Nieberg,P.,*QuasitriangulierbareOperatoren und invarianteUntervektorräume stetiger linearer Operatoren*, Arch. Math. (Basel) **22** (1971), 186–199.

[Pe88] Pestov, V. G., *Fermeture nonstandard des algèbres et groupes de Lie banachiques*, C. R. Acad. Sci. Paris Sèr. I Math. **306** (1988), 643–645.

[Pe92] —, *Nonstandard hulls of Banach-Lie groups and algebras*, Nova J. Algebra Geom. **1** (1992), 371–381.

[Pe93] —, *Enlargeable Banach-Lie algebras and free topological groups*, Bull. Austral. Math. Soc. **48** (1993), 13–22.

[Sw65] Świerczkowski, S., *Embedding theorems for local analytic groups*, Acta Math. **114** (1965), 207–235.

Daniel Beltiţă
Institute of Mathematics "Simion Stoilow"
of the Romanian Academy
P.O. Box 1-764
RO-70700 Bucharest
Romania
dbeltita@imar.ro

Received December 02, 2002
and in final form April 7, 2003