SUMS OF SQUARES OF PURE QUATERNIONS

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Abstract

We show that for each $k \geq 1$ there is a division quaternion algebra D of level $s(D) = 2^k$ such that -1 is not the sum of squares of 2^k pure quaternions in D. This answers a question asked by D. W. Lewis (*Rocky Mountain Journal of Mathematics* **19** (1989), 787–92).

The problem of determining the level of quaternion algebras was discussed by D. W. Lewis in [3] and D. B. Leep in [2]. The approach used by Lewis associates with a quaternion algebra $D = \left(\frac{a,b}{F}\right)$ the quadratic form $T_P = \langle a, b, -ab\rangle$ over the field F. Lewis has shown that, for any positive integer $n \in \mathbb{N}$, if $\langle 1\rangle \perp nT_P$ is isotropic over F, then -1 is a sum of n squares of quaternions in D (see [3, lemma 4]). He commented on the converse of this implication and stated that it is true for $n = 2^k - 1,\ k \geq 2$, but for other values of n we do not know.

In this note we show that, in general, this converse statement is not true. We construct an explicit example of a quaternion algebra D over a formally real field K with the property that for $n = 2^k$, the element $-1 \in D$ is a sum of n squares in D, but the quadratic form $\langle 1\rangle \perp nT_P$ is anisotropic over K. We also show that -1 cannot be expressed as a sum of $n-1$ squares in D.

We begin with a refinement of [3, lemma 4].

Lemma 1. *Let n be any positive integer, $D = \left(\frac{a,b}{F}\right)$ and $T_P = \langle a, b, -ab\rangle$. Then the quadratic form $\langle 1\rangle \perp nT_P$ is isotropic over F if and only if -1 can be expressed as a sum of n squares of pure quaternions in D.*

Proof. This is implicit in [3, lemma 4] and [2, theorem 2.2]. But for the sake of completeness we sketch a proof.

The isotropicity of $\langle 1\rangle \perp nT_P$ over F is equivalent to nT_P representing -1 over F, that is, to the existence of $q_1, r_1, s_1, \ldots, q_n, r_n, s_n \in F$ such that

$$-1 = T_P(q_1, r_1, s_1) + \cdots + T_P(q_n, r_n, s_n).$$

Since for a pure quaternion $c = qi + rj + sk$, we have $c^2 = T_P(q, r, s)$, such a representation of -1 exists if and only if there are pure quaternions $a_m = q_m i + r_m j + s_m k$, $1 \leq m \leq n$, satisfying

$$-1 = a_1^2 + \cdots + a_n^2,$$

as desired. ■

Lemma 1 shows that the problem stated by Lewis is equivalent to the following question:

> Let -1 be a sum of n squares of quaternions in D. Is it then true that -1 is a sum of n squares of pure quaternions in D?

This seems to be of particular interest in the case when n is the level of D, so that -1 is expressible as a sum of n but not less squares of quaternions in D. Notice that in the simplest cases, such as Hamilton quaternions or quaternion algebras over the rational function field $\mathbb{R}(X)$, the answer is affirmative.

However, it is not so in the general case. Here is our main result.

Theorem 2. *For any positive integer k there exists a quaternion division algebra D such that -1 is a sum of 2^k squares of quaternions in D but -1 is not a sum of 2^k squares of pure quaternions. Moreover, $s(D) = 2^k$.*

Proof. We will use the ideas of [3, proposition 3]. Let k be a fixed positive integer and write $n := 2^k$. By a theorem of Prestel (see [4, theorem 2.1]), there exists a formally real field $F \subset \mathbb{R}$ such that $P(F) = n + 1$, where $P(F)$ denotes the Pythagoras number of the field F (i.e. the smallest positive integer m such that any sum of squares in F is a sum of m squares). We choose and fix an element $c_0 \in F$ of length $n + 1$ (i.e. c_0 is a sum of $n + 1$ squares but is not a sum of $\leq n$ squares in F) and we consider the quaternion algebra

$$D := \left(\frac{-c_0, X}{F(X)}\right).$$

Observe that the quadratic form $(n + 1)\langle 1\rangle \perp (n - 1)T_P$ is isotropic over $F(X)$. For $T_P = \langle -c_0, X, c_0X\rangle$ and if $c_0 = a_1^2 + \cdots + a_{n+1}^2$, where $a_m \in F \setminus \{0\}$, $m = 1, \ldots, n$, then

$$\begin{aligned} 0 &= a_1^2 + \cdots + a_{n+1}^2 - c_0 \\ &= \Big((n + 1)\langle 1\rangle \perp (n - 1)T_P\Big)(a_1, \ldots, a_{n+1}, 1, 0, \ldots, 0). \end{aligned}$$

By [2, theorem 2.2], it now follows that one can express -1 as a sum of n squares in D.

Now we check that $\langle 1\rangle \perp nT_P$ is anisotropic over $F(X)$. Suppose there are non-zero $x_0, x_m, y_m, z_m \in F(X)$, $m = 1, \ldots, n$ such that

$$\begin{aligned} 0 &= (\langle 1\rangle \perp nT_P)(x_0, x_1, y_1, z_1, \ldots, x_n, y_n, z_n) \\ &= x_0^2 - c_0 \sum_{m=1}^{n} x_m^2 + X \sum_{m=1}^{n} y_m^2 + c_0X \sum_{m=1}^{n} z_m^2. \end{aligned}$$

Hence,

$$X \left(\sum_{m=1}^{n} y_m^2 + c_0 \sum_{m=1}^{n} z_m^2\right) = c_0 \left(\sum_{m=1}^{n} x_m^2\right) - x_0^2, \tag{1}$$

and after clearing denominators we can assume that $x_0, x_m, y_m, z_m \in F[X]$ and not all are zero polynomials. Dividing by an appropriate power X^{2l} of X we may assume that at least one of them does not vanish at zero. Now, the left hand side vanishes at zero, so we get $c_0(\sum_{m=1}^{n} x_m(0)^2) - x_0(0)^2 = 0$. If not all of $x_m(0) = 0$ $(m = 1, \ldots, n)$, then we have

$$\begin{aligned} c_0 &= \frac{x_0(0)^2}{x_1(0)^2 + \cdots + x_n(0)^2} \\ &= \left(\frac{x_0(0)}{x_1(0)^2 + \cdots + x_n(0)^2} \right)^2 \cdot \left(x_1(0)^2 + \cdots + x_n(0)^2 \right). \end{aligned}$$

Thus, c_0 is a sum of n squares in F contrary to the choice of c_0. Therefore, all $x_m(0) = 0$. Hence, $x_0(0) = 0$. Write $x_0 = X\tilde{x}_0, \ldots, x_n = X\tilde{x}_n$ for some $\tilde{x}_m \in F[X]$. Now (1) becomes

$$\sum_{m=1}^{n} y_m^2 + c_0 \sum_{m=1}^{n} z_m^2 = X \left(c_0 \left(\sum_{m=1}^{n} \tilde{x}_m^2 \right) - \tilde{x}_0^2 \right).$$

Setting $X = 0$ we get $\sum_{m=1}^{n} y_m(0)^2 + c_0 \sum_{m=1}^{n} z_m(0)^2 = 0$. Since F is formally real, it follows that all $y_m(0) = z_m(0) = 0$, a contradiction. From Lemma 1 we conclude that -1 is not the sum of n squares of pure quaternions in D.

Now we check that D is a division algebra. If it is not, then the form $\varphi := \langle 1, c_0, -X, -c_0X \rangle$ is isotropic (see, e.g. [1, theorem 2.7, p. 58]). But φ is a Pfister form, so it is isotropic if and only if its pure subform $-T_P$ is isotropic. Thus, T_P is isotropic and it follows that the form $\langle 1 \rangle \perp nT_P$ is isotropic, a contradiction.

The last step in the proof is to show that $s(D) = 2^k$. We have already shown that $s(D) \leq n = 2^k$, so suppose $s(D) < n$. Then there are $x_m, y_m, z_m, t_m \in F[X]$, $m = 1, \ldots, n-1$, such that

$$\sum_{m=1}^{n-1} x_m^2 - c_0 \sum_{m=1}^{n-1} y_m^2 + X \sum_{m=1}^{n-1} z_m^2 + c_0 X \sum_{m=1}^{n-1} x_m^2 = -1$$

and $\sum_{m=1}^{n-1} x_m y_m = \sum_{m=1}^{n-1} x_m z_m = \sum_{m=1}^{n-1} x_m t_m = 0$. Now the argument in [3, proposition 3] applies; multiplying by $\sum_{m=1}^{n-1} y_m^2$ and putting $X = 0$ we get

$$\left(1 + \sum_{m=1}^{n-1} x_m^2 \right) \left(0 + \sum_{m=1}^{n-1} y_m^2 \right) = c_0 \left(\sum_{m=1}^{n-1} y_m^2 \right)^2.$$

Thus, by a theorem of Pfister (see, e.g. [1, chapter 10, proposition 1.7]), c_0 is a sum of n squares in F, a contradiction. ■

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