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Real theta-characteristics on real projective curves

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Abstract. Here we prove the existence of non-locally free real theta-characteristics on any real reduced projective curve.

1. Introduction

Let X be a complex reduced and projective curve and $\mathcal{F}$ a rank one torsion-free sheaf on X; here "rank one" means that for every irreducible component T of X there is a non-empty open subset U of X such that $\mathcal{F}|U \in \mathrm{Pic}(U)$. As in [6], Note 2.15, we will say that $\mathcal{F}$ is a complex theta-characteristic (or just a theta-characteristic) if there is an isomorphism $j : \mathcal{F} \to \mathcal{H}om(\mathcal{F}, \omega_X)$. We do not fix the isomorphism j in this definition because it is uniquely determined up to an invertible element of $H^0(X, \mathcal{O}_X)$. Now assume that X is real, i.e., it is defined over $\mathrm{Spec}(\mathbf{R})$. A real structure on X is uniquely determined by an anti-holomorphic involution $\sigma : X \to X$; alternatively, see σ as the map induced by the action of the generator the Galois group $\mathbf{Z}/2\mathbf{Z}$ of the field extension $\mathbf{C}/\mathbf{R}$. Notice that σ induces a permutation of order at most two of the set $\mathrm{Sing}(X)$ of all singular points of X and of the set of all irreducible components of X. We have $X(\mathbf{R}) = \{P \in X(\mathbf{C}) : \sigma(P) = P\}$. A complex theta-characteristic $\mathcal{F}$ will be called *strongly real* if the sheaf $\mathcal{F}$ is defined over $\mathrm{Spec}(\mathbf{R})$ and *real* if the complex sheaf $\mathcal{F}$ is isomorphic over $\mathrm{Spec}(\mathbf{C})$ to its complex conjugate $\sigma^*(\mathcal{F})$. On many real curves strongly real theta-characteristics do not exist (see Remark 2 for the case of a smooth real curve of genus zero) and hence most of our existence results will only be for real theta-characteristics. For the existence of strongly real theta-characteristics, see Remark 1.

Let $\mathcal{F}$ be a rank one torsion-free sheaf on the reduced and projective curve X. Set $\mathrm{Sing}(\mathcal{F}) := \{P \in X : \mathcal{F}$ is not locally free at $P\}$. Since every torsion-free coherent module on a one-dimensional regular local ring is free, we have $\mathrm{Sing}(\mathcal{F}) \subseteq \mathrm{Sing}(X)$. We will say that $\mathcal{F}$ is *completely singular* if $\mathrm{Sing}(\mathcal{F}) = \mathrm{Sing}(X)$. We will say that $\mathcal{F}$ is *freely full* if there is a reduced projective curve C, $L \in \mathrm{Pic}(C)$ and a proper birational morphism $f : C \to X$ such that $\mathcal{F} \cong f_*(L)$. Every locally free $\mathcal{F}$ is freely full: just take $C = X$ and f the identity. If X has only ordinary nodes or

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ordinary cusps as singularities, then every rank one torsion-free sheaf on X is freely full (Remark 5).

Theorem 1. *Let (X, σ) be a reduced and projective real curve. Then there is a completely singular and freely full real theta-characteristic on (X, σ).*

Then we discuss the existence of even or odd real theta-characteristics (see Theorems 2 and 3).

2. Proof of Theorem 1

Remark 1. Let (X, σ) be a reduced and projective real curve and $L \in \mathrm{Pic}(X)(\mathbf{C})$. L is σ-invariant (i.e., $\sigma^*(L) \cong L$) if and only if $L \in \mathrm{Pic}(X)(\mathbf{R})$. Let D be a Cartier divisor on X supported by smooth points of X. D is σ-invariant (i.e., $\sigma(D) = D$) if and only if it is defined over $\mathrm{Spec}(\mathbf{R})$. Hence the projectivity of X implies that L is defined over $\mathrm{Spec}(\mathbf{R})$ if and only if it is associated to a σ-invariant Cartier divisor supported by X_{reg}. If X is geometrically connected and $X(\mathbf{R}) \neq \emptyset$, then L is defined over $\mathrm{Spec}(\mathbf{R})$ ([7], Ex. 1.17). If X is smooth, then the converse hold ([3], Prop. 3.1). Let $f : C \to X$ be the normalization map. Equip C with the real structure induced by σ. We just saw that if every connected component of $C(\mathbf{C})$ has a real point, then any completely singular real theta-characteristic on X is strongly real (use also Lemma 1).

Remark 2. There are two schemes defined over $\mathrm{Spec}(\mathbf{R})$ and whose extension to $\mathrm{Spec}(\mathbf{C})$ is isomorphic to $\mathbf{P}^1_{\mathbf{C}}$: $\mathbf{P}^1_{\mathbf{R}}$ and the smooth plane conic $\{x^2 + y^2 = -1\}$ ([7], Ex. 1.10). The latter real form of $\mathbf{P}^1_{\mathbf{C}}$ has no real point; following [3], p. 178, we will denote with N or with (N, σ) the form of $\mathbf{P}^1_{\mathbf{C}}$ with $N(\mathbf{R}) = \emptyset$. Since every line bundle on $N(\mathbf{C})$ is uniquely determined by its degree, every line bundle on N is σ-invariant. Since for any algebraic scheme C the sheaf ω_C is defined over the base field of C and $\deg(\omega_N) = -2$, every even degree line bundle on N is defined over $\mathrm{Spec}(\mathbf{R})$. Since $N(\mathbf{R}) = \emptyset$, every divisor of N defined over $\mathrm{Spec}(\mathbf{R})$ has even degree. Hence no odd degree line bundle on (N, σ) is defined over $\mathrm{Spec}(\mathbf{R})$. Thus N has no strongly real theta-characteristic.

Remark 3. Let X be reduced projective curve admitting a locally free theta-characteristic L. In particular X is assumed to be Gorenstein. Then for every irreducible component T of X we have $\deg(\omega_X|T) = 2\deg(L|T)$. Hence $\deg(\omega_X|T)$ is even. If X is reducible, this is a strong restriction on X. For instance, if X has only ordinary nodes as singularities, then T must intersects the other components of X in an even number of points. Thus for any $g \geq 2$ there are stable curves without any locally free complex theta-characteristic.

Remark 4. Let X, C be reduced and projective curves and $f : C \to X$ a birational morphism. By Riemann-Roch for any rank one torsion free sheaf A on C we have $\deg(f_*(A)) = \deg(A) + p_a(X) - p_a(C)$.

Remark 5. Let X be a reduced and projective curve whose only singularities are ordinary nodes and ordinary cusps. Fix any $P \in \mathrm{Sing}(X)$. There is a full classification of all rank one torsion-free sheaves, M, on the completion $\hat{\mathcal{O}}_{X,P}$ of the local ring $\mathcal{O}_{X,P}$: either M is free or it is isomorphic to the maximal ideal of $\hat{\mathcal{O}}_{X,P}$ ([1], p. 24, or [2]). Let $\mathcal{F}$ be a rank one torsion-free sheaf on X and $f : C \to X$

the partial normalization of X in which we normalize exactly the points of $\mathrm{Sing}(\mathcal{F})$. Set $x := \mathrm{card}(\mathrm{Sing}(\mathcal{F}))$. Hence $p_a(C) = p_a(X) - x$. Set $L := f^*(\mathcal{F})/\mathrm{Tors}(f^*(\mathcal{F}))$. Since a torsion-free finitely generated module over a regular local ring is free, L is a line bundle. The canonical map $\mathcal{F} \to f_*f^*(\mathcal{F})$ induces a map $u : \mathcal{F} \to f_*(L)$ which is an isomorphism outside $\mathrm{Sing}(\mathcal{F})$. By the local classification of modules on $\hat{\mathcal{O}}_{X,P}$, we see that u is an isomorphism. By Remark 4 we have $\deg(\mathcal{F}) = \deg(L) + x$. Now assume that X is real (say, with a real structure determined by σ) and that $\mathcal{F}$ is σ-invariant. Thus $\mathrm{Sing}(\mathcal{F})$ is σ-invariant and hence it is defined over $\mathrm{Spec}(\mathbf{R})$. Thus C and f are defined over $\mathrm{Spec}(\mathbf{R})$. Call η the associated real structure on C. Since $f^*(\mathcal{F})$ is η-invariant, L is η-invariant. Now assume that $\mathcal{F}$ is defined over $\mathrm{Spec}(\mathbf{R})$. Then L is defined over $\mathrm{Spec}(\mathbf{R})$.

Remark 6. Let $f : C \to X$ a birational morphism between reduced projective curves and L a rank one torsion-free sheaf on C. The coherent sheaf $f_*(L)$ has rank one and it is torsion-free. By Riemann-Roch we have $\deg(f_*(L)) = \deg(L) + p_a(X) - p_a(C)$.

Lemma 1. *Let $f : C \to X$ be a finite birational map between reduced and projective curves. Fix $L \in \mathrm{Pic}(C)$ and set $\mathcal{F} := f_*(L)$. L is a theta-characteristic on C if and only if $\mathcal{F}$ is a theta-characteristic on X.*

Proof. By Remark 4 we have $\deg(\mathcal{F}) = \deg(L) + p_a(X) - p_a(C)$. By Riemann-Roch we have $\deg(\omega_X) = 2p_a(X) - 2$ even if X is not Gorenstein. Furthermore, by the local duality for locally Cohen- Macaulay schemes the sheaves $\mathcal{F}$ and $Hom(\mathcal{F}, \omega_X)$ have the same degree if and only if $\deg(L) = p_a(C) - 1$ ([1], Prop. 3.1.6, part 2). Hence $\mathcal{F}$ is a theta-characteristic if and only if $\deg(L) = p_a(C) - 1$ and there is a morphism $u : \mathcal{F} \to Hom(\mathcal{F}, \omega_X)$ which is non-zero at the general point of each irreducible component of X. By [5], Ex. III.7.2 (a), we have $f^!(\omega_X) \cong \omega_C$. Assume that L is a theta-characteristic. The isomorphism $L \to Hom_C(L, \omega_C)$ induces a morphism $u : f_*(L) \to f_*(Hom_C(L, \omega_C)) = f_*(Hom_C(L, f^!(\omega_X)))$ which is an isomorphism at the general point of each irreducible component of X. By [5], Ex. III.6.10, we have $f_*(Hom_C(L, f^!(\omega_X))) \cong Hom_Y(f_*(L, \omega_X))$. Thus $f_*(L)$ is a theta-characteristic. The proof of the other implication is similar. □

Remark 7. Let (X, σ) be a reduced and projective real curve and $f : C \to X$ the normalization. Let η be the real structure on C induced by σ. Let T be an irreducible component of X such that $\sigma(T) \neq T$, U the normalization of T and V the normalization of $\sigma(T)$. Thus $\eta(U) = V$ and $\eta(V) = U$. U and V are connected components of C and $Y := U \cup V$ (disjoint union) has a real structure induced by η and called again η. We have $Y(\mathbf{C}) = U(\mathbf{C}) \cup V(\mathbf{C})$ (disjoint union) and $Y(\mathbf{R}) = \emptyset$. Y is irreducible over $\mathrm{Spec}(\mathbf{R})$ but not over $\mathrm{Spec}(\mathbf{C})$. We have $\mathrm{Pic}(Y)(\mathbf{C}) \cong \mathrm{Pic}(U)(\mathbf{C}) \times \mathrm{Pic}(V)(\mathbf{C})$ and $L = (M, R) \in \mathrm{Pic}(U)(\mathbf{C}) \times \mathrm{Pic}(V)(\mathbf{C})$ is η-invariant if and only if $M \cong \eta^*(L)$, i.e., if and only if $L \cong \eta^*(M)$. Thus a line bundle on Y is η-invariant if and only if it is defined over $\mathrm{Spec}(\mathbf{R})$ and every η-invariant line bundle on Y has even dimensional cohomology groups. By [3], Cor. 4.3, U has a complex theta-characteristic. Hence Y has a real theta-characteristic (and even a theta-characteristic defined over $\mathrm{Spec}(\mathbf{R})$) and every real theta-characteristic on Y is even.

Proof of Theorem 1. Let $f : C \to X$ be the normalization map and η the real structure on C induced by σ. Let A be a connected component of C such that

$\eta(A) = A$. Thus (A, η) is a smooth and connected real curve. By [3], Cor. 4.3, (A, η) has an η-invariant theta-characteristic. Let U be a connected component of C such that $\eta(U) \neq U$. By Remark 7 the real curve $U \cup \eta(U)$ has a real theta-characteristic. Thus C has a real theta-characteristic. By Lemma 1 the completely singular full sheaf $f_*(L)$ is a real theta-characteristic. □

A theta-characteristic $\mathcal{F}$ on the reduced and projective curve X is said to be *even* (resp. *odd*) if the integer $h^0(X, \mathcal{F})$ is even (resp. odd). Now we will discuss the notion of even and odd theta-characteristic when the corresponding torsion-free sheaf is not locally free.

Remark 8. Let $\{C_t\}_{t \in T}$ be a flat family of reduced projective curves parametrized by an integral variety T and $\{L_t\}_{t \in T}$ a flat family of locally free theta-characteristics on this family of curves. By [4], Th. 1.10, the congruence class modulo two of the integer $h^0(X, L_t)$ does not depend from t. Let X be a reduced and projective curve and assume the existence of a flat family $f_t : C_t \to X$ of birational morphisms. Since each L_t is locally free, the family $\{f_{t*}(L_t)\}_{t \in T}$ is a flat family of rank one torsion-free sheaves on X. By Lemma 1 each $f_{t*}(L_t)$ is a theta-characterstic. Since $h^0(X, f_{t*}(L_t)) = h^0(C_t, L_t)$, we obtain that the parity of the integer $h^0(X, f_{t*}(L_t))$ does not depend from t. In this sense the parity of freely full theta-characteristics is constant in equidesingularizable families. This is nice in the case in which X has only ordinary nodes or ordinary cusps because in this case by the classification of torsion-free modules over A_i-singularities, $i = 1, 2$, ([1], p. 24, or [2]) not only each theta-characteristic is freely full but "equidisigularizable" means "with the same singular support": just take as C_t the partial normalization of X at the singular points of the theta-characteristics of the family.

Theorem 2. *Let (X, σ) be a reduced and projective real curve. Then there is a completely singular and freely full even real theta-characteristic on (X, σ).*

Proof. Let $f : C \to X$ be the normalization. By Lemma 1 it is sufficient to prove the existence of a theta-characteristic L on C such that $h^0(C, L)$ is even. Let η be the real structure on C induced by σ. Let A be a connected component of C such that $\eta(A) = A$. By [3], Prop. 5.1, A admits an even theta-characteristic for the real structure of A induced by η. Let U be a connected component of C such that $\eta(U) \neq U$. By Remark 7 η induces a real structure for $U \cup \eta(U)$ and $U \cup \eta(U)$ admits an even real theta-characteristic. Since a theta-characteristic on C is just given assigning a theta-characteristic on each connected component of C, we are done. □

Theorem 3. *Let (X, σ) be a reduced and projective real curve. Assume the existence of at least one irreducible component T of X such that $\sigma(T) = T$ and the normalization A of T has $\mathrm{Pic}^0(A)(\mathbf{R})$ not connected. Then there is a completely singular and freely full odd real theta-characteristic on (X, σ).*

Proof. Let $f : C \to X$ be the normalization. As in the proof of Theorem 2 it is sufficient to show the existence of a real theta-characteristic L on C such that $h^0(A, L|A)$ is odd and, if $A \neq C$, $h^0(C \backslash A, L|C \backslash A)$ is even. By Theorem 2 it is sufficient to show the existence of an odd real theta-characteristic on A. This is true by our assumption on A and [3], Prop. 5.1. □

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