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Isomorphic groupoid C^*-algebras associated with different Haar systems

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Abstract. We shall consider a locally compact groupoid endowed with a Haar system ν and having proper orbit space. We shall associate to each appropriate cross section $\sigma : G^{(0)} \to G^F$ for $d_F : G^F \to G^{(0)}$ (where F is a Borel subset of $G^{(0)}$ meeting each orbit exactly once) a C^*-algebra $M^*_\sigma(G, \nu)$. We shall prove that the C^*-algebras associated with different Haar systems are $*$-isomorphic.

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1. Introduction

The reader is referred to Section 2 for the basic defintions and notations we shall use here.

The C^*-algebra of a locally compact groupoid was introduced by J. Renault in [9]. The construction extends the case of a group: the space of continuous functions with compact support on the groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. In order to define the convolution on the groupoid one needs to assume the existence of a Haar system which is an analogue of Haar measure on a group. Unlike the case for groups, Haar systems need not be unique. A result of Paul Muhly, Jean Renault and Dana Williams establishes that the C^*-algebras of G associated with two Haar systems are strongly Morita equivalent [4, Theorem 2.8, p. 10]. If the groupoid G is transitive

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they have proved that the C^*-algebra of G is isomorphic to $C^*(H) \otimes \mathcal{K}(L^2(\mu))$, where H is the isotropy group G_u^u at any unit $u \in G^{(0)}$, μ is an essentially unique measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}(L^2(\mu))$ denotes the compact operators on $L^2(\mu)$ [4, Theorem 3.1, p. 16]. Therefore the C^*-algebras of a *transitive* groupoid G associated with two Haar systems are $*$-isomorphic.

In [8] Arlan Ramsay and Martin E. Walter have associated to a locally compact groupoid G a C^*-algebra denoted $M^*(G,\nu)$. They have considered the universal representation ω of $C^*(G,\nu)$ — the usual C^*-algebra associated to a Haar system $\nu = \{\nu^u, u \in G^{(0)}\}$ (constructed as in [9]). Since every cyclic representation of $C^*(G,\nu)$ is the integrated form of a representation of G, it follows that ω can be also regarded as a representation of $\mathcal{B}_c(G)$, the space of compactly supported bounded Borel functions on G. Arlan Ramsay and Martin E. Walter have used the notation $M^*(G,\nu)$ for the operator norm closure of $\omega(\mathcal{B}_c(G))$. Since ω is an $*$-isomorphism on $C^*(G,\nu)$, we can regard $C^*(G,\nu)$ as a subalgebra of $M^*(G,\nu)$.

Definition 1. A locally compact groupoid G is proper if the map

$$(r,d) : G \to G^{(0)} \times G^{(0)}$$

is proper (i.e., the inverse image of each compact subset of $G^{(0)} \times G^{(0)}$ is compact) [1, Definition 2.1.9].

Throughout this paper we shall assume that G is a second countable locally compact groupoid for which the orbit space is Hausdorff and the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$. Therefore R will be a locally compact groupoid. The fact that R is a closed subset of $G^{(0)} \times G^{(0)}$ and that it is endowed with the product topology is equivalent to the fact R is a proper groupoid.

Throughout this paper by a groupoid with proper orbit space we shall mean a groupoid G for which the orbit space is Hausdorff and the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$.

Let us give an example of a groupoid with proper orbit space that is not a proper groupoid. First let us make some remarks. Any locally compact principal groupoid can be viewed as an equivalence relation on a locally compact space X having its graph $\mathcal{E} \subset X \times X$ endowed with a locally compact topology compatible with the groupoid structure. This topology can be finer than the product topology induced from $X \times X$. $\mathcal{E}$ is proper if and only if $\mathcal{E}$ is endowed with the product topology and $\mathcal{E}$ is closed in $X \times X$. Let $\mathcal{E} \subset X \times X$ be a proper principal groupoid and let Γ be a locally compact group. Then $\mathcal{E} \times \Gamma$ is a groupoid under the following operations:

$$(u,v,x)^{-1} = (v,u,x^{-1})$$
$$(u,v,x)(v,w,y) = (u,w,xy).$$

It is easy to see that $\mathcal{E} \times \Gamma$ is a groupoid with proper orbit space. If Γ is not a compact group, then $\mathcal{E} \times \Gamma$ is not a proper groupoid.

We shall assume that the orbit space of the groupoid G is proper and we shall choose a Borel subset F of $G^{(0)}$ meeting each orbit exactly once and such that $F \cap [K]$

has a compact closure for each compact subset K of $G^{(0)}$. For each appropriate cross section $\sigma : G^{(0)} \to G^F$ for $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, we shall construct a C^*-algebra $M^*_\sigma(G, \nu)$ which can be viewed as a subalgebra of $M^*(G, \nu)$. Given two Haar system $\nu_1 = \{\nu_1^u, u \in G^{(0)}\}$ and $\nu_2 = \{\nu_2^u, u \in G^{(0)}\}$ on G, we shall prove that the C^*-algebras $M^*_\sigma(G, \nu_1)$ and $M^*_\sigma(G, \nu_2)$ are $*$-isomorphic.

For a transitive (or more generally, a locally transitive) groupoid G we shall prove that the C^*-algebras $C^*(G, \nu)$, $M^*(G, \nu)$ and $M^*_\sigma(G, \nu)$ coincide.

If G is a locally transitive groupoid endowed with a Haar system $\{\nu^u, u \in G^{(0)}\}$, then it is the topological disjoint union of its transitivity components $G|_{[u]}$, and $C^*(G, \nu)$ is the direct sum of the $C^*(G|_{[u]}, \nu_{[u]})$, where $\nu_{[u]} = \{\nu^s, s \in [u]\}$. This is a consequence of [2, Theorem 1, p. 10].

For a principal proper groupoid G, we shall prove that

$$C^*(G, \nu) \subset M^*_\sigma(G, \nu) \subset M^*(G, \nu).$$

Let $\pi : G^{(0)} \to G^{(0)}/G$ be the quotient map and let

$$\nu_i = \left\{ \varepsilon_u \times \mu_i^{\pi(u)}, u \in G^{(0)} \right\}, \quad i = 1, 2$$

be two Haar systems on the principal proper groupoid G. We shall also prove that if the Hilbert bundles determined by the systems of measures $\{\mu_i^{\dot{u}}\}_{\dot{u}}$ have continuous bases in the sense of Definition 24, then $*$-isomorphism between $M^*_\sigma(G, \nu_1)$ and $M^*_\sigma(G, \nu_2)$ can be restricted to a $*$-isomorphism between $C^*(G, \nu_1)$ and $C^*(G, \nu_2)$.

2. Basic definitions and notations

For establishing notation, we include some definitions that can be found in several places (e.g., [9], [5]). A groupoid is a set G endowed with a product map

$$(x, y) \to xy \quad [: G^{(2)} \to G]$$

where $G^{(2)}$ is a subset of $G \times G$ called the set of composable pairs, and an inverse map

$$x \to x^{-1} \quad [: G \to G]$$

such that the following conditions hold:

(1) If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$.
(2) $(x^{-1})^{-1} = x$ for all $x \in G$.
(3) For all $x \in G$, $(x, x^{-1}) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$, then $(zx)x^{-1} = z$.
(4) For all $x \in G$, $(x^{-1}, x) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd(x) = r(x)x = x$. Units will usually be denoted by letters as u, v, w while arbitrary elements will be denoted by x, y, z. It is useful to note that a pair (x, y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g., $xy = xz$ iff $y = z$). The fibres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets $A, B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G^A_B = r^{-1}(A) \cap d^{-1}(B)$. The reduction of G to $A \subset G^{(0)}$ is $G|A = G^A_A$. The relation

$u \sim v$ iff $G_v^u \neq \phi$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits and the orbit of a unit u is denoted $[u]$. A groupoid is called transitive iff it has a single orbit. The quotient space for this equivalence relation is called the orbit space of G and denoted $G^{(0)}/G$. We denote by $\pi : G^{(0)} \to G^{(0)}/G, \pi(u) = \dot{u}$ the quotient map. A subset of $G^{(0)}$ is said saturated if it contains the orbits of its elements. For any subset A of $G^{(0)}$, we denote by $[A]$ the union of the orbits $[u]$ for all $u \in A$.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

(1) $x \to x^{-1} \quad [: G \to G]$ is continuous.
(2) $(x, y) \to xy \quad [: G^{(2)} \to G]$ is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

We are exclusively concerned with topological groupoids which are second countable, locally compact Hausdorff. It was shown in [7] that measured groupoids may be assume to have locally compact topologies, with no loss in generality.

If X is a locally compact space, $C_c(X)$ denotes the space of complex-valuated continuous functions with compact support. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets. The space of compactly supported bounded Borel functions on X is denoted by $\mathcal{B}_c(X)$.

For a locally compact groupoid G, we denote by

$$G' = \{x \in G : r(x) = d(x)\}$$

the isotropy group bundle of G. It is closed in G.

Let G be a locally compact second countable groupoid equipped with a Haar system, i.e., a family of positive Radon measures on G, $\{\nu^u,\, u \in G^{(0)}\}$, such that:

(1) For all $u \in G^{(0)}$, $\mathrm{supp}(\nu^u) = G^u$.
(2) For all $f \in C_c(G)$,

$$u \to \int f(x) d\nu^u(x) \;\; [: G^{(0)} \to \mathbf{C}]$$

is continuous.

(3) For all $f \in C_c(G)$ and all $x \in G$,

$$\int f(y) d\nu^{r(x)}(y) \;=\; \int f(xy) d\nu^{d(x)}(y).$$

As a consequence of the existence of continuous Haar systems, $r, d : G \to G^{(0)}$ are open maps ([11]). Therefore, in this paper we shall always assume that $r : G \to G^{(0)}$ is an open map

If μ is a measure on $G^{(0)}$, then the measure $\nu = \int \nu^u d\mu(u)$, defined by

$$\int f(y) d\nu(y) = \int \left(\int f(y) d\nu^u(y) \right) d\mu(u), \quad f \geq 0 \text{ Borel}$$

is called the measure on G induced by μ. The image of ν by the inverse map $x \to x^{-1}$ is denoted ν^{-1}. μ is said to be quasi-invariant if its induced measure ν is equivalent to its inverse, ν^{-1}. A measure belongings to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is invariant.

If μ is a quasi-invariant measure on $G^{(0)}$ and ν is the measure induced on G, then the Radon–Nikodym derivative $\Delta = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of μ.

In order to define the C^*-algebra of a groupoid G, the space $C_c(G)$ of continuous functions with compact support on G, endowed with the inductive limit topology, is made into a topological $*$-algebra and is given the smallest C^*-norm making its representations continuous. In somewhat more detail, for f, $g \in C_c(G)$ the convolution is defined by:

$$f * g(x) = \int f(xy)g(y^{-1})d\nu^{d(x)}(y)$$

and the involution by

$$f^*(x) = \overline{f(x^{-1})}.$$

Under these operations, $C_c(G)$ becomes a topological $*$-algebra.

A representation of $C_c(G)$ is a $*$-homomorphism from $C_c(G)$ into $\mathcal{B}(H)$, for some Hilbert space H, that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $\mathcal{B}(H)$. The full C^*-algebra $C^*(G)$ is defined as the completion of the involutive algebra $C_c(G)$ with respect to the full C^*-norm

$$\|f\| = \sup \|L(f)\|$$

where L runs over all nondegenerate representations of $C_c(G)$ which are continuous for the inductive limit topology.

Every representation $(\mu, G^{(0)} * \mathcal{H}, L)$ [5, Definition 3.20/p. 68] of G can be integrated into a representation, still denoted by L, of $C_c(G)$. The relation between the two representation is:

$$\langle L(f)\xi_1, \xi_2\rangle = \int f(x)\langle L(x)\xi_1(d(x)), \xi_2(r(x))\rangle \Delta^{-\frac{1}{2}}(x)d\nu^u(x)d\mu(u)$$

where $f \in C_c(G)$, $\xi_1, \xi_2 \in \int^{\oplus}_{G^{(0)}} \mathcal{H}(u)d\mu(u)$.

Conversely, every nondegenerate $*$-representation of $C_c(G)$ is obtained in this fashion (see [9] or [5]).

3. The decomposition of a Haar system over the principal groupoid

First we present some results on the structure of the Haar systems, as developed by J. Renault in Section 1 of [10] and also by A. Ramsay and M.E. Walter in Section 2 of [8].

In Section 1 of [10] Jean Renault constructs a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally compact support which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure β_u^u on G_u^u so the integral of F_0 with respect to β_u^u is 1.

Renault defines $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$ (where $x\beta_v^v(f) = \int f(xy)d\beta_v^v(y)$ as usual). If z is another element in G_v^u, then $x^{-1}z \in G_v^v$, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. If K is a compact subset of G, then $\sup\limits_{u,v} \beta_v^u(K) < \infty$. Renault also defines a 1-cocycle δ on G such that for every $u \in G^{(0)}$, $\delta|_{G_u^u}$ is the modular function for β_u^u. δ and $\delta^{-1} = 1/\delta$ are bounded on compact sets in G.

Let

$$R = (r, d)(G) = \{(r(x), d(x)),\ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. This R is the image of G under the homomorphism (r, d), so it is a σ-compact groupoid. With this apparatus in place, Renault describes a decomposition of the Haar system $\{\nu^u, u \in G^{(0)}\}$ for G over the equivalence relation R (the principal groupoid associated to G). He proves that there is a unique Borel Haar system α for R with the property that

$$\nu^u = \int \beta_t^s d\alpha^u(s, t) \text{ for all } u \in G^{(0)}.$$

In Section 2 of [8] A. Ramsay and M.E. Walter prove that

$$\sup_u \alpha^u((r, d)(K)) < \infty, \text{ for all compact } K \subset G$$

For each $u \in G^{(0)}$ the measure α^u is concentrated on $\{u\} \times [u]$. Therefore there is a measure μ^u concentrated on $[u]$ such that $\alpha^u = \varepsilon_u \times \mu^u$, where ε_u is the unit point mass at u. Since $\{\alpha^u, u \in G^{(0)}\}$ is a Haar system, we have $\mu^u = \mu^v$ for all $(u, v) \in R$, and the function

$$u \to \int f(s)\mu^u(s)$$

is Borel for all $f \geq 0$ Borel on $G^{(0)}$. For each u the measure μ^u is quasi-invariant (see Section 2 of [8]). Therefore μ^u is equivalent to $d_*(v^u)$ [6, Lemma 4.5/p. 277].

If η is a quasi-invariant measure for $\{\nu^u, u \in G^{(0)}\}$, then η is a quasi-invariant measure for $\{\alpha^u, u \in G^{(0)}\}$. Also if Δ_R is the modular function associated to $\{\alpha^u, u \in G^{(0)}\}$ and η, then $\Delta = \delta\Delta_R \circ (r, d)$ can serve as the modular function associated to $\{\nu^u, u \in G^{(0)}\}$ and η.

Since $\mu^u = \mu^v$ for all $(u, v) \in R$, the system of measures $\{\mu^u\}_u$ may be indexed by the elements of the orbit space $G^{(0)}/G$.

Definition 2. We shall call the pair of systems of measures

$$(\{\beta_v^u\}_{(u,v)\in R}, \{\mu^{\dot{u}}\}_{\dot{u}\in G^{(0)}/G})$$

(described above) the decomposition of the Haar system $\{\nu^u, u \in G^{(0)}\}$ over the principal groupoid associated to G. Also we shall call δ the 1-cocycle associated to the decomposition.

Remark 3. Let us note that up to trivial changes in normalization, the system of measures $\{\beta_v^u\}$ and the 1-cocycle in the preceding definition are unique. They do not depend on the Haar system, but only on the continuous function F_0.

Lemma 4. *Let G be a locally compact second countable groupoid such that the bundle map $r|_{G'}$ of G' is open. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on G and let $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ be its decomposition over the principal groupoid associated to G. Then for each $f \in C_c(G)$ the function*

$$x \to \int f(y)d\beta_{d(x)}^{r(x)}(y)$$

is continuous on G.

Proof. By Lemma 1.3/p. 6 of [10], for each $f \in C_c(G)$ the function

$$u \to \int f(y) d\beta_u^u(y)$$

is continuous.

Let $x \in G$ and $(x_i)_i$ be a sequence in G converging to x. Let $f \in C_c(G)$ and let g be a continuous extension on G of $y \to f(xy) \quad [: G^{d(x)} \to \mathbf{C}]$. Let K be the compact set

$$(\{x, x_i, i = 1, 2, \dots\}^{-1}\mathrm{supp}(f) \cup \mathrm{supp}(g)) \cap r^{-1}(\{d(x), d(x_i), i = 1, 2, \dots\}).$$

We have

$$\begin{aligned}
\left|\int f(y) d\beta_{d(x)}^{r(x)}(y) - \int f(y) d\beta_{d(x_i)}^{r(x_i)}(y)\right| &= \left|\int f(xy) d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
&= \left|\int g(y) d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
&\leq \left|\int g(y) d\beta_{d(x)}^{d(x)}(y) - \int g(y) d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
&\quad + \left|\int g(y) d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
&\leq \left|\int g(y) d\beta_{d(x)}^{d(x)}(y) - \int g(y) d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
&\quad + \sup_{y \in G_{d(x_i)}^{d(x_i)}} |g(y) - f(x_i y)| \beta_{d(x_i)}^{d(x_i)}(K).
\end{aligned}$$

A compactness argument shows that $\sup_{y \in G_{d(x_i)}^{d(x_i)}} |g(y) - f(x_i y)|$ converges to 0. Also $\left|\int g(y) d\beta_{d(x)}^{d(x)}(y) - \int g(y) d\beta_{d(x_i)}^{d(x_i)}(y)\right|$ converges to 0 because the function $u \to \int f(y) d\beta_u^u(y)$ is continuous. Hence

$$\left|\int f(y) d\beta_{d(x)}^{r(x)}(y) - \int f(y) d\beta_{d(x_i)}^{r(x_i)}(y)\right|$$

converges to 0. □

Proposition 5. *Let G be a second countable locally compact groupoid with proper orbit space. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on G and let $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ be its decomposition over the principal groupoid associated to G. Then for each $g \in C_c(G^{(0)})$, the map*

$$u \to \int g(v) d\mu^{\pi(u)}(v)$$

is continuous.

Proof. Let $g \in C_c(G^{(0)})$ and $u_0 \in G^{(0)}$. Let K_1 be a compact neighborhood of u_0 and K_2 be the support of g. Since G is locally compact and (r, d) is open from G to $(r, d)(G)$, there is a compact subset K of G such that $(r, d)(K)$ contains $(K_1 \times K_2) \cap (r, d)(G)$. Let $F_1 \in C_c(G)$ be a nonnegative function equal to 1 on a compact neighborhood U of K. Let $F_2 \in C_c(G)$ be a function which extends to G

the function $x \to F_1(x)/\int F_1(y)d\beta_{d(x)}^{r(x)}(y)$, $x \in U$. We have $\int F_2(y)d\beta_v^u(y) = 1$ for all $(u,v) \in (r,d)(K)$. Since for all $u \in K_1$,

$$\int g(v)d\mu^{\pi(u)}(v) = \int g(v)\int F_2(y)d\beta_v^u(y)d\mu^{\pi(u)}(v)$$
$$= \int g(d(y))F_2(y)d\nu^u(y),$$

it follows that $u \to \int g(v)d\mu^{\pi(u)}(v)$ is continuous at u_0. □

Remark 6. Let G be a locally compact second countable groupoid with proper orbit space. Let $\{\nu^u, u \in G^{(0)}\}$be a Haar system on G and $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ be its decomposition over the associated principal groupoid. If μ is a quasi-invariant probability measure for the Haar system, then $\mu_1 = \int \mu^{\pi(u)}d\mu(u)$ is a Radon measure which is equivalent to μ. Indeed, let $f \geq 0$ Borel on $G^{(0)}$ such that $\mu(f) = 0$. Since μ is quasi-invariant, it follows that for μ a.a. u, $\nu^u(f \circ d) = 0$, and since $\mu^{\pi(u)}$ is equivalent to $d_*(v^u)$, it follows that $\mu^{\pi(u)}(f) = 0$ for μ a.a. u. Conversely, if $\mu_1(f) = 0$, then $\mu^{\pi(u)}(f) = 0$ for μ a.a. u, and therefore $\nu^u(f \circ d) = 0$. Thus the quasi-invariance of μ implies $\mu(f) = 0$. Thus each Radon quasi-invariant measure is equivalent to a Radon measure of the form $\int \mu^{\dot{u}}d\widetilde{\mu}(\dot{u})$, where $\widetilde{\mu}$ is a probability measure on the orbit space $G/G^{(0)}$.

4. A C^*-algebra associated to a locally compact groupoid with proper orbit space

Let G be a locally compact second countable groupoid with proper orbit space. Let

$$\pi : G^{(0)} \to G^{(0)}/G$$

be the quotient map. Since the quotient space is proper, $G^{(0)}/G$ is Hausdorff.

As we mentioned at the outset, our standing hypothesis that G has a Haar system guarantees that r is open. Consequently, so is the map π.

Applying Lemma 1.1 of [3] to the locally compact second countable spaces $G^{(0)}$ and $G^{(0)}/G$ and to the continuous open surjection $\pi : G^{(0)} \to G^{(0)}/G$, it follows that there is a Borel set F in $G^{(0)}$ such that:

(1) F contains exactly one element in each orbit $[u] = \pi^{-1}(\pi(u))$.
(2) For each compact subset K of $G^{(0)}$, $F \cap [K] = F \cap \pi^{-1}(\pi(K))$ has a compact closure.

For each unit u let us define $e(u)$ to be the unique element in the orbit of u that is contained in F, i.e., $\{e(u)\} = F \cap [u]$. For each Borel subset B of $G^{(0)}$, π is continuous and one-to-one on $B \cap F$ and hence $\pi(B \cap F)$ is Borel in $G^{(0)}/G$. Therefore the map $e : G^{(0)} \to G^{(0)}$ is Borel (for each Borel subset B of $G^{(0)}$, $e^{-1}(B) = [B \cap F] = \pi^{-1}(\pi(B \cap F))$ is Borel in $G^{(0)}$). Also for each compact subset K of $G^{(0)}$, $e(K)$ has a compact closure because $e(K) \subset F \cap [K]$.

Since the orbit space $G^{(0)}/G$ is proper the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

is open and R is closed in $G^{(0)} \times G^{(0)}$. Applying Lemma 1.1 of [3] to the locally compact second countable spaces G and R and to the continuous open surjection

$(r, d) : G \to R$, it follows that there is a *regular cross section* $\sigma_0 : R \to G$. This means that σ_0 is Borel, $(r, d)(\sigma_0(u, v)) = (u, v)$ for all $(u, v) \in R$, and $\sigma_0(K)$ is relatively compact in G for each compact subset K of R.

Let us define $\sigma : G^{(0)} \to G^F$ by $\sigma(u) = \sigma_0(e(u), u)$ for all u. It is easy to note that σ is a cross section for $d : G^F \to G^{(0)}$ and $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. If F is closed, then σ is regular.

Replacing σ by

$$v \to \sigma(e(v))^{-1}\sigma(v)$$

we may assume that $\sigma(e(v)) = e(v)$ for all v. Let us define $q : G \to G_F^F$ by

$$q(x) = \sigma(r(x))x\sigma(d(x))^{-1},\ x \in G.$$

Let $\nu = \{\nu^u : u \in G^{(0)}\}$ be a Haar system on G and let $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ be its decompositions over the principal groupoid . Let δ be the 1-cocycle associated to the decomposition.

Let us denote by $\mathcal{B}_\sigma(G)$ the linear span of the functions of the form

$$x \to g_1(r(x))g(q(x))g_2(d(x))$$

where g_1, g_2 are compactly supported bounded Borel functions on $G^{(0)}$ and g is a bounded Borel function on G_F^F such that if S is the support of g, then the closure of S is compact in G. $\mathcal{B}_\sigma(G)$ is a subspace of $\mathcal{B}_c(G)$, the space of compactly supported bounded Borel functions on G.

If $f_1,\ f_2 \in \mathcal{B}_\sigma(G)$ are defined by

$$f_1(x) = g_1(r(x))g(q(x))g_2(d(x))$$
$$f_2(x) = h_1(r(x))h(q(x))h_2(d(x))$$

then

$$f_1 * f_2(x) = g * h(q(x))g_1(r(x))h_2(d(x))\langle g_2, \overline{h_1}\rangle_{\pi(r(x))}$$
$$f_1^*(x) = \overline{g_2(r(x))g(q(x)^{-1})g_1(d(x))}.$$

Thus $\mathcal{B}_\sigma(G)$ is closed under convolution and involution.

Let ω be the universal representation of $C^*(G, \nu)$ the usual C^*-algebra associated to a Haar system $\nu = \{\nu^u,\ u \in G^{(0)}\}$ (constructed as in [9]). Since every cyclic representation of $C^*(G, \nu)$ is the integrated form of a representation of G, it follows that ω can be also regarded as a representation of $\mathcal{B}_c(G)$, the space of compactly supported bounded Borel functions on G. Arlan Ramsay and Martin E. Walter have used the notation $M^*(G, \nu)$ for the operator norm closure of $\omega(\mathcal{B}_c(G))$. Since ω is an $*$-isomorphism on $C^*(G, \nu)$, we can regarded $C^*(G, \nu)$ as a subalgebra of $M^*(G, \nu)$.

Definition 7. We denote by $M_\sigma^*(G, \nu)$ the operator norm closure of $\omega(\mathcal{B}_\sigma(G))$.

Lemma 8. *Let* $\{\mu_1^{\dot{u}}\}_{\dot{u}}$ *and* $\{\mu_2^{\dot{u}}\}_{\dot{u}}$ *be two systems of measures on* $G^{(0)}$ *satisfying:*

(1) $\mathrm{supp}(\mu_i^{\dot{u}}) = [u]$ *for all* $\dot{u}$, $i = 1, 2$.

(2) *For all compactly supported bounded Borel functions* f *on* $G^{(0)}$ *the function*

$$u \to \int f(v)\mu_i^{\pi(u)}(v)$$

is bounded and Borel.

Then there is a family $\{U_{\dot{u}}\}_{\dot{u}}$ of unitary operators with the following properties:

(1) $U_{\dot{u}} : L^2(\mu_1^{\dot{u}}) \to L^2(\mu_2^{\dot{u}})$ *is a unitary operator for each* $\dot{u} \in G^{(0)}/G$.

(2) *For all bounded Borel functions* f *on* $G^{(0)}$,

$$u \to U_{\pi(u)}(f)$$

is a bounded Borel function with compact support.

(3) *For all bounded Borel functions* f *on* $G^{(0)}$,

$$U_{\pi(u)}(\overline{f}) = \overline{U_{\pi(u)}(f)}.$$

Proof. Using the same argument as in [7] (p. 323) we can construct a sequence $f_1, f_2, \dots$ of real valued bounded Borel function on $G^{(0)}$ such that $\dim(L^2(\mu_1^{\dot{u}})) = \infty$ if and only if $\|f_n\|_2 = 1$ in $L^2(\mu_1^{\dot{u}})$ for $n = 1, 2, \dots$ and then $\{f_1, f_2, \dots\}$ gives an orthonormal basis of $L^2(\mu_1^{\dot{u}})$, while $\dim(L^2(\mu_1^{\dot{u}})) = k < \infty$ if and only if $\|f_n\|_2 = 1$ for $n \le k$, and $\|f_n\|_2 = 0$ for $n > k$ and then $\{f_1, f_2, \dots, f_k\}$ gives an orthonormal basis of $L^2(\mu_1^{\dot{u}})$. Let $g_1, g_2, \dots$ be a sequence with the same properties as $f_1, f_2, \dots$ corresponding to $\{\mu_2^{\dot{u}}\}_{\dot{u}}$. Let us define $U_{\dot{u}} : L^2(\mu_1^{\dot{u}}) \to L^2(\mu_2^{\dot{u}})$ by

$$U_{\dot{u}}(f_n) = g_n \text{ for all } n$$

Then the family $\{U_{\dot{u}}\}_{\dot{u}}$ has the required properties. □

Theorem 9. *Let G be a locally compact second countable groupoid with proper orbit space. Let $\{\nu_i^u, u \in G^{(0)}\}$, $i = 1, 2$ be two Haar systems on G. Let F be a Borel subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a cross section for $d : G^F \to G^{(0)}$ with $\sigma(e(v)) = e(v)$ for all $v \in G^{(0)}$ and such that $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Then the C^*-algebras $M_\sigma^*(G, \nu_1)$ and $M_\sigma^*(G, \nu_2)$ are $*$-isomorphic.*

Proof. Let $(\{\beta_v^u\}, \{\mu_i^{\dot{u}}\})$ be the decompositions of the Haar systems over the principal groupoid. Let δ be the 1-cocycle associated to the decompositions, $i = 1, 2$.

We shall denote by $\langle \cdot, \cdot \rangle_{i,\dot{u}}$ the inner product of $(L^2(G^{(0)}, \delta(\sigma(\cdot))\mu_i^{\dot{u}}))$, $i = 1, 2$.

Let us define $q : G \to G_F^F$ by

$$q(x) = \sigma(r(x)) x \sigma(d(x))^{-1},\ x \in G.$$

We shall define a $*$-homomorphism Φ from $\mathcal{B}_\sigma(G)$ to $\mathcal{B}_\sigma(G)$. It suffices to define Φ on the set of functions on G of the form

$$x \to g_1(r(x)) g(q(x)) g_2(d(x))$$

Let $\{U_{\dot{u}}\}_{\dot{u}}$ be the family of unitary operators with the properties stated in Lemma 8, associated to the systems of measures $\{\delta(\sigma(\cdot))\mu_i^{\dot{u}}\}_{\dot{u}}$, $i = 1, 2$.

Let us define Φ by

$$\Phi(f) = (x \to U_{\pi(r(x))}(g_1)(r(x)) g(q(x)) U_{\pi(d(x))}(g_2)(d(x)))$$

where f is defined by

$$f(x) = g_1(r(x)) g(q(x)) g_2(d(x)).$$

If f_1 and f_2 are defined by

$$f_1(x) = g_1(r(x)) g(q(x)) g_2(d(x))$$
$$f_2(x) = h_1(r(x)) h(q(x)) h_2(d(x))$$

then

$$f_1 * f_2(x) = g * h(q(x))g_1(r(x))h_2(d(x))\langle g_2, \overline{h_1}\rangle_{1,\pi(r(x))}$$

and consequently

$$\begin{aligned}\Phi(f_1 * f_2) &= g * h(q(x))U_{\pi(r(x))}(g_1)(r(x))U_{\pi(r(x))}(h_2)(d(x))\langle g_2, \overline{h_1}\rangle_{1,\pi(r(x))}\\ &= \Phi(f_1) * \Phi(f_2).\end{aligned}$$

Let $\widetilde{\eta}$ be a probability measure on $G^{(0)}/G$ and $\eta_i = \int \mu_i^{\dot{u}} d\widetilde{\eta}(\dot{u}), i = 1, 2$. Let L_1 be the integrated form of a representation $(L, \mathcal{H} * G^{(0)}, \eta_1)$ and L_2 be the integrated form of $(L, \mathcal{H} * G^{(0)}, \eta_2)$. Let B be the Borel function defined by

$$B(u) = L(\sigma(u))$$

and $W : \int_{G^{(0)}}^{\oplus} \mathcal{H}(u) d\eta_1(u) \to \int_{G^{(0)}}^{\oplus} \mathcal{H}(e(u)) d\eta_1(u)$ be defined by

$$W(\zeta) = (u \to B(u)(\zeta(u))).$$

Since every element of $L^2(G^{(0)}, \delta(\sigma(\cdot))\mu_1^{\dot{w}}, \mathcal{H}(e(w)))$ is a limit of linear combinations of elements $u \to a(u)\xi$ with $a \in L^2(G^{(0)}, \delta(\sigma(\cdot))\mu_1^{\dot{w}})$ and $\xi \in \mathcal{H}(e(w))$, we can define a unitary operator

$$V_{\dot{w}} : L^2(G^{(0)}, \delta(\sigma(\cdot))\mu_1^{\dot{w}}, \mathcal{H}(e(w))) \to L^2(G^{(0)}, \delta(\sigma(\cdot))\mu_2^{\dot{w}}, \mathcal{H}(e(w)))$$

by

$$V_{\dot{w}}(u \to a(u)\xi) = U_{\dot{w}}(a)\xi.$$

Let $V : \int_{G^{(0)}}^{\oplus} \mathcal{H}(e(u)) d\eta_1(u) \to \int_{G^{(0)}}^{\oplus} \mathcal{H}(e(u)) d\eta_2(u)$ be defined by

$$V(\zeta) = (u \to V_{\dot{u}}(\zeta(u))).$$

If $\zeta_1, \zeta_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(e(u)) d\eta_1(u)$ and f is of the form

$$f(x) = g_1(r(x))g(q(x))g_2(d(x)),$$

we have

$$\langle WL_1(f)W^*\zeta_1, \zeta_2\rangle = \int\int g(x)\delta(x)^{\frac{-1}{2}}\langle L(x)A_1(\dot{w}), B_1(\dot{w})\rangle d\beta_{e(w)}^{e(w)}(x) d\widetilde{\eta}(\dot{w})$$

where

$$\begin{aligned}A_1(\dot{w}) &= \int g_2(v)\zeta_1(v)\delta(\sigma(v))^{\frac{1}{2}} d\mu_1^{\dot{w}}(v)\\ B_1(\dot{w}) &= \int g_1(u)\zeta_2(u)\delta(\sigma(u))^{\frac{1}{2}} d\mu_1^{\dot{w}}(u).\end{aligned}$$

Moreover, if f is of the form $f(x) = g_1(r(x))g(q(x)g_2(d(x))$ and $\zeta_1, \zeta_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(e(u)) d\eta_2(u)$, then

$$\begin{aligned}\langle VWL_1(f)W^*V^*\zeta_1, \zeta_2\rangle &= \int\int g(x)\delta(x)^{\frac{-1}{2}}\langle L(x)A_2(\dot{w}), B_2(\dot{w})\rangle d\beta_{e(u)}^{e(u)}(x) d\widetilde{\eta}(\dot{w})\\ &= \langle WL_2(\Phi(f))W^*\zeta_1, \zeta_2\rangle\end{aligned}$$

where

$$A_2(\dot{w}) = \int g_2(v)V^*\zeta_1(v)\delta(\sigma(v))^{\frac{1}{2}}d\mu_1^{\dot{w}}(v)$$
$$= \int U_{\dot{v}}(g_2)(v)\zeta_1(v)\delta(\sigma(v))^{\frac{1}{2}}d\mu_2^{\dot{w}}(v)$$
$$B_2(\dot{w}) = \int g_1(v)V^*\zeta_2(v)\delta(\sigma(v))^{\frac{1}{2}}d\mu_1^{\dot{w}}(v)$$
$$= \int U_{\dot{u}}(g_1)(u)\zeta_2(u)\delta(\sigma(u))^{\frac{1}{2}}d\mu_2^{\dot{w}}(u).$$

Therefore $\|L_1(f)\| = \|L_2(\Phi(f))\|$. Consequently we can extend Φ to a $*$-homomorphism between the $M_\sigma^*(G,\nu_1)$ and $M_\sigma^*(G,\nu_2)$. It is not hard to see that Φ is in fact a $*$-isomorphism:

$$\Phi^{-1}(f) = (x \to U^*_{\pi(r(x))}(g_1)(r(x))g(q(x))U^*_{\pi(d(x))}(g_2)(d(x)))$$

for each f of the form

$$f(x) = g_1(r(x))g(q(x))g_2(d(x)). \qquad \square$$

5. The case of locally transitive groupoids

A locally compact *locally transitive* groupoid G is a groupoid for which all orbits $[u]$ are open in $G^{(0)}$. We shall prove that if G is a locally compact second countable locally transitive groupoid endowed with a Haar system ν, then

$$C^*(G,\nu) = M^*(G,\nu) = M_\sigma^*(G,\nu)$$

for any regular cross section σ.

Notation 10. Let $\{\nu^u,\, u \in G^{(0)}\}$ be a fixed Haar system on G. Let μ be a quasi-invariant measure, Δ its modular function, ν_1 be the measure induced by μ on G and $\nu_0 = \Delta^{-\frac{1}{2}}\nu_1$. Let

$$II_\mu(G) = \{f \in L^1(G,\nu_0): \ \|f\|_{II,\mu} < \infty\},$$

where $\|f\|_{II,\mu}$ is defined by

$$\|f\|_{II,\mu} = \sup\left\{\int |f(x)j(d(x))k(r(x))|d\nu_0(x),\ \int |j|^2 d\mu = \int |k|^2 d\mu = 1\right\}.$$

If μ_1 and μ_2 are two equivalent quasi-invariant measures, then

$$\|f\|_{II,\mu_1} = \|f\|_{II,\mu_2},$$

because $\|f\|_{II,\mu} = \|II_\mu(|f|)\|$ for each quasi-invariant measure μ, where II_μ is the one-dimensional trivial representation on μ.

Define

$$\|f\|_{II} = \sup\left\{\|f\|_{II,\mu}: \mu \text{ quasi-invariant Radon measure on } G^{(0)}\right\}.$$

The supremum can be taken over the classes of quasi-invariant measures.

If $\|\cdot\|$ is the full C^*-norm on $C_c(G)$, then (see [8])

$$\|f\| \le \|f\|_{II} \text{ for all } f.$$

Lemma 11. *Let G be a locally compact second countable groupoid with proper orbit space. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on G, let $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ its decomposition over the principal groupoid associated to G and let δ the associated 1-cocycle. If f is a universally measurable function on G, then*

$$\|f\|_{II} \le \sup_{\dot{w}} \left(\int\int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) \right)^2 d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}}.$$

Proof. Each Radon quasi-invariant measure is equivalent with a Radon measure of the form $\int \mu^{\dot{u}} d\widetilde{\mu}(\dot{u})$, where $\widetilde{\mu}$ is a probability measure on the orbit space $G/G^{(0)}$. Therefore for the computation of $\|.\|_{II}$ it is enough to consider only the quasi-invariant measures of the form $\mu = \int \mu^{\dot{u}} d\widetilde{\mu}(\dot{u})$, where $\widetilde{\mu}$ is a probability measure on $G^{(0)}/G$. It is easy to see that the modular function of $\int \mu^{\dot{u}} d\widetilde{\mu}(\dot{u})$ is $\Delta = \delta$.

Let $j, k \in L^2(G^{(0)}, \mu)$ with $\int |j|^2 d\mu = \int |k|^2 d\mu = 1$. We have

$$\begin{aligned}
&\int\int\int\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) |j(v)||k(u)| d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) d\widetilde{\mu}(\dot{w}) \\
&\quad \le \int \left(\int\int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) \right)^2 d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int\int |j(v)|^2 |k(u)|^2 d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}} d\widetilde{\mu}(\dot{w}) \\
&\quad \le \sup_{\dot{w}} \left(\int\int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) \right)^2 d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}} \\
&\qquad \cdot \int \left(\int |j(v)|^2 d\mu^{\dot{w}}(v) \right)^{\frac{1}{2}} \left(|k(u)|^2 d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}} d\widetilde{\mu}(\dot{w}) \\
&\quad \le \sup_{\dot{w}} \left(\int\int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) \right)^2 d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\|f\|_{II} \le \sup_{\dot{w}} \left(\int\int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) \right)^2 d\mu^{\dot{w}}(v) d\mu^{\dot{w}}(u) \right)^{\frac{1}{2}}. \qquad \square$$

If G is locally transitive, each orbit $[u]$ is open in $G^{(0)}$. Each measure $\mu^{\dot{u}}$ is supported on $[u]$. Since $([u])$ is a partition of $G^{(0)}$ into open sets, it follows that there is a unique Radon measure m on $G^{(0)}$ such that the restriction of m at $C_c([u])$ is $\mu^{\dot{u}}$ for each $[u]$.

Corollary 12. *Let G be a locally compact second countable locally transitive groupoid endowed with a Haar system $\nu = \{\nu^u, u \in G^{(0)}\}$. Let f be a universally measurable function such that $\|f\|_{II} < \infty$.*

(1) *If $(f_n)_n$ is a uniformly bounded sequence of universally measurable functions supported on a compact set, and if $(f_n)_n$ converges pointwise to f, then $(f_n)_n$ converges to f in the norm of $C^*(G, \nu)$.*

(2) *If $(f_n)_n$is an increasing sequence of universally measurable nonnegative functions on G that converges pointwise to f, then $(f_n)_n$ converges to f in the norm of $C^*(G,\nu)$.*

Proof. Let $(\{\beta_v^u\},\{\mu^{\dot{u}}\})$ be the decomposition of the Haar system over the principal groupoid associated to G and δ the associated 1-cocycle. Let m be the unique measure such that restriction of m at $C_c([u])$ is $\mu^{\dot{u}}$ for each $[u]$. Let $(f_n)_n$be a sequence of universally measurable functions supported on a compact set K. Let

$$M=\sup_{u,v}\beta_u^v(K^{-1})$$

and let us assume that $(f_n)_n$ converges pointwise to f. By Lemma 11,

$$\|f-f_n\|_{II}\le\sup_{\dot{w}}\left(\int\int\left(\int|f(x)-f_n(x)|\delta(x)^{-\frac{1}{2}}d\beta_v^u(x)\right)^2d\mu^{\dot{w}}(v)d\mu^{\dot{w}}(u)\right)^{\frac{1}{2}},$$

hence

$$\begin{aligned}\|f-f_n\|_{II}&\le\sup_{\dot{w}}M\left(\int\int\left(\int|f(x)-f_n(x)|^2d\beta_v^u(x)\right)d\mu^{\dot{w}}(v)d\mu^{\dot{w}}(u)\right)^{\frac{1}{2}}\\&\le M\left(\int\int\left(\int|f(x)-f_n(x)|^2d\beta_u^v(x)\right)dm(v)dm(u)\right)^{\frac{1}{2}}.\end{aligned}$$

If $\|\cdot\|$ denotes the C^*-norm, then

$$\lim_n\|f-f_n\|\le\lim_n\|f-f_n\|_{II}=0,$$

because

$$\int\int\left(\int|f(x)-f_n(x)|^2d\beta_u^v(x)\right)dm(v)dm(u)$$

converges to zero, by the Dominated Convergence Theorem.

Let $(f_n)_n$be an increasing sequence of universally measurable nonnegative functions that converges pointwise to f. Since

$$\begin{aligned}\|f-f_n\|_{II}&\le\sup_{\dot{w}}\left(\int\int\left(\int|f(x)-f_n(x)|\delta(x)^{-\frac{1}{2}}d\beta_v^u(x)\right)^2d\mu^{\dot{w}}(v)d\mu^{\dot{w}}(u)\right)^{\frac{1}{2}}\\&\le\left(\int\int\left(\int|f(x)-f_n(x)|\delta(x)^{-\frac{1}{2}}d\beta_v^u(x)\right)^2dm(v)dm(u)\right)^{\frac{1}{2}}\end{aligned}$$

it follows that

$$\lim_n\|f-f_n\|_{II}=0.$$

□

Proposition 13. *Let G be a locally compact second countable* locally transitive *groupoid endowed with a Haar system $\nu=\{\nu^u,u\in G^{(0)}\}$. Then any function in $\mathcal{B}_c(G)$, the space of compactly supported bounded Borel functions on G, can be viewed as an element of $C^*(G,\nu)$.*

Proof. Let $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ be the decomposition of the Haar system over the principal groupoid associated to G, and δ the associated 1-cocycle. Let m be the unique measure such that restriction of m at $C_c([u])$ is $\mu^{\dot{u}}$ for each $[u]$. Let m be a dominant for the family $\{\mu^{\dot{u}}\}$. Let ν_1 be the measure on G defined by

$$\int f(x)d\nu_1(x) = \left(\int\int\left(\int f(x)d\beta_u^v(x)\right)dm(v)dm(u)\right)$$

for all Borel nonnegative functions f. If $f \in \mathcal{B}_c(G)$, then f is the limit in $L^2(G,\nu_1)$ of a sequence, $(f_n)_n$, in $C_c(G)$ that is supported on some compact set K supporting f. If we write

$$M = \sup_{u,v} \beta_u^v(K^{-1}),$$

then

$$\begin{aligned}\|f - f_n\|_{II} &\leq \sup_{\dot{w}} M\left(\int\int\left(\int |f(x) - f_n(x)|^2 d\beta_v^u(x)\right)d\mu^{\dot{w}}(v)d\mu^{\dot{w}}(u)\right)^{\frac{1}{2}}\\ &\leq M\left(\int\int\left(\int |f(x) - f_n(x)|^2 d\beta_u^v(x)\right)dm(v)dm(u)\right)^{\frac{1}{2}}.\end{aligned}$$

If $\|\|$ denotes the C^*-norm, then

$$\lim_n \|f - f_n\| \leq \lim_n \|f - f_n\|_{II} = 0.$$

Thus f can be viewed as an element in $C^*(G,\nu)$. □

The following is an immediate consequence of Proposition 13:

Proposition 14. *If G is a locally compact second countable locally transitive groupoid endowed with a Haar system $\{\nu^u,\, u \in G^{(0)}\}$ with bounded decomposition, then*

$$C^*(G,\nu) = M^*(G,\nu).$$

Remark 15. Let G be locally compact locally transitive groupoid. Let F be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. It is easy to see that F is a closed subset of G and that F is a discrete space. Let $\sigma : G^{(0)} \to G^F$ be a *regular cross section* of d_F. Let us endow $\bigcup_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ with the topology induced from $G^{(0)} \times G_F^F \times G^{(0)}$. The topology of $\bigcup_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ is locally compact because $\bigcup_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ is a closed subset of the locally compact space $G^{(0)} \times G_F^F \times G^{(0)}$. With the operations

$$\begin{aligned}(u,x,v)(v,y,w) &= (u,xy,w)\\ (u,x,v)^{-1} &= (v,x^{-1},u),\end{aligned}$$

$\bigcup_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ becomes a groupoid. Define $\phi : G \to \bigcup_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ by

$$\phi(x) = (r(x), \sigma(r(x))x\sigma(d(x))^{-1}, d(x))$$

and note that ϕ is a Borel isomorphism which carries compact sets to relatively compact sets.

Lemma 16. *Let G be locally compact second countable locally transitive groupoid. Let F be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a regular cross section of d_F. Then any compactly supported bounded Borel function on G is the pointwise limit of a uniformly bounded sequence $(f_n)_n$ of Borel functions supported on a compact set supporting f, having the property that each f_n is a linear combination of functions of the form*

$$x \to g_1(r(x))g(\sigma(r(x))x\sigma(d(x))^{-1})g_2(d(x))$$

where g_1, g_2 are compactly supported bounded Borel functions on $G^{(0)}$ and g is a compactly supported bounded Borel function on G_F^F.

Proof. Endow $\bigcup\limits_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ with the topology induced from $G^{(0)} \times G_F^F \times G^{(0)}$ as in Remark.15. The topology of $\bigcup\limits_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ is locally compact. Any compactly supported Borel bounded function on $G^{(0)} \times G_F^F \times G^{(0)}$ is pointwise limit of uniformly bounded sequences $(f_n)_n$ of Borel functions supported on a compact set, such that each function f_n is a linear combination of functions of the form

$$(u, x, v) \to g_1(u)g(x)g_2(v)$$

where g_1, g_2 are compactly supported bounded Borel functions on $G^{(0)}$ and g is a compactly supported bounded Borel function on G_F^F. Consequently, any compactly supported bounded Borel function on $\bigcup\limits_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ has the same property. Since $\phi : G \to \bigcup\limits_{[u]}[u] \times G_{e(u)}^{e(u)} \times [u]$ defined by

$$\phi(x) = (r(x), \sigma(r(x))x\sigma(d(x))^{-1}, d(x))$$

is a Borel isomorphism which carries compact sets to relatively compact sets, it follows that any compactly supported bounded Borel function on G can be represented as a pointwise limit of a uniformly bounded sequence $(f_n)_n$ of Borel functions supported on a compact set supporting f, having the property that each f_n is a linear combination of functions of the form

$$x \to g_1(r(x))g(\sigma(r(x))x\sigma(d(x))^{-1})g_2(d(x)). \qquad \square$$

Corollary 17. *Let G be locally compact second countable locally transitive groupoid. Let F be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a regular cross section of d_F. Then the linear span of the functions of the form*

$$x \to g_1(r(x))g(\sigma(r(x))x\sigma(d(x))^{-1})g_2(d(x))$$

where $g_1, g_2 \in \mathcal{B}_c(G^{(0)})$ and $g \in \mathcal{B}_c(G_F^F)$, is dense in the full C^*-algebra of G.

Proof. Let f be a function on G, defined by

$$f(x) = g_1(r(x))g(\sigma(r(x))x\sigma(d(x))^{-1})g_2(d(x))$$

where $g_1, g_2 \in \mathcal{B}_c(G^{(0)})$ and $g \in \mathcal{B}_c(G_F^F)$. Then f lies in $\mathcal{B}_c(G)$, and so may be viewed as an element of the $C^*(G, \nu)$, as we note in Proposition 13. Each $f \in \mathcal{B}_c(G)$ (in particular in $C_c(G)$) is the limit (pointwise and consequently in the C^*-norm according to Corollary 12) of a uniformly bounded sequence $(f_n)_n$ of Borel functions supported on a compact set supporting f, having the property that each f_n is a linear combination of functions of the required form. □

Proposition 18. *Let G be a locally compact second countable locally transitive groupoid endowed with a Haar system $\{\nu^u,\, u \in G^{(0)}\}$. Let F be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a regular cross section of d_F. Then*

$$C^*(G,\nu) = M^*(G,\nu) = M^*_\sigma(G,\nu).$$

Proof. We have proved that $C^*(G,\nu) = M^*(G,\nu)$. From the preceding corollary, it follows that the linear span of the functions of the form

$$x \to g_1(r(x))g(\sigma(r(x))x\sigma(d(x))^{-1})g_2(d(x))$$

where $g_1, g_2 \in \mathcal{B}_c(G^{(0)})$ and $g \in \mathcal{B}_c(G^F_F)$ is dense in $C^*(G,\nu)$. But this space is contained in $\mathcal{B}_\sigma(G)$. Therefore $C^*(G,\nu) = M^*(G,\nu) = M^*_\sigma(G,\nu)$. □

6. The case of principal proper groupoids

Notation 19. Let G be a locally compact second countable groupoid with proper orbit space. Let F be a Borel subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a cross section for $d_F : G^F \to G^{(0)}, d_F(x) = d(x)$ with $\sigma(e(v)) = e(v)$ for all $v \in G^{(0)}$ and such that $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Let $q : G \to G^F_F$ be defined by

$$q(x) = \sigma(r(x))x\sigma(d(x))^{-1}$$

We shall endow G^F_F with the quotient topology induced by q. We shall denote by $\mathbf{C}_\sigma(G)$ the linear span of the functions of the form

$$x \to g_1(r(x))g(\sigma(r(x))x\sigma(d(x))^{-1})g_2(d(x))$$

where $g_1, g_2 \in C_c(G^{(0)})$ and g continuous on G^F_F such that its support is relatively compact in G.

Proposition 20. *Using Notation* 19*, if the space of continuous functions* (*with the respect to the quotient topology induced by q*) *with relatively compact support on G^F_F separates the points of G^F_F, then $\mathbf{C}_\sigma(G)$ is dense in $C_c(G)$* (*for the inductive limit topology*)*. In particular, if the quotient topology induced by q on G^F_F is a locally compact* (*Hausdorff*) *topology, then $\mathbf{C}_\sigma(G)$ is dense in $C_c(G)$.*

Proof. If the space of continuous functions on G^F_F (with the respect to the quotient topology induced by q) having relatively compact support separates the points of G^F_F, then $\mathbf{C}_\sigma(G)$ separates the points of G. By Stone–Weierstrass Theorem, it follows that $\mathbf{C}_\sigma(G)$ is dense in $C_c(G)$ (for the inductive limit topology). □

Proposition 21. *Let G be a locally compact principal groupoid. If G is proper, then the quotient topology induced by q on G^F_F is a locally compact* (*Hausdorff*) *topology. Consequently, $\mathbf{C}_\sigma(G)$ is dense in $C_c(G)$ for the inductive limit topology* (*we use Notation* 19)*.*

Proof. Let $\pi : G \to G^{(0)}/G$ be the canonical projection. Let us note that for a principal groupoid the condition

$$q(x) = q(y)$$

is equivalent to

$$\pi(r(x)) = \pi(r(y)).$$

First we shall prove that the topology on G_F^F is Hausdorff. Let $(x_i)_i$ and $(y_i)_i$ be two nets with $q(x_i) = q(y_i)$ for every i. Let us suppose that $(x_i)_i$ converges to x and $(y_i)_i$ converges to y. Then

$$\lim \pi(r(x_i)) = \lim \pi(r(y_i)) = \pi(r(x)) = \pi(r(y)).$$

Hence $q(x) = q(y)$, and therefore the topology on G_F^F is Hausdorff. We shall prove that q is open. If $(z_i)_i$ is a net converging to $q(x)$ in G_F^F, then $\pi \circ r(z_i)$ converges to $\pi \circ r(x)$. Since

$$\pi \circ r : G \to G^{(0)}/G$$

is an open map, there is a net $(x_i)_i$ converging to x, such that $\pi \circ r(x_i) = \pi \circ r(z_i)$, and consequently $q(x_i) = q(z_i) = z_i$. Hence q is an open map and the quotient topology induced by q on G_F^F is locally compact. □

Theorem 22. *Let G be a locally compact second countable groupoid with proper orbit space. Let F be a Borel subset of $G^{(0)}$ meeting each orbit exactly once. Let $\sigma : G^{(0)} \to G^F$ be a cross section for $d : G^F \to G$ such that $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Let us assume that the quotient topology induced by q on G_F^F is a locally compact (Hausdorff) topology. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on G. Then*

$$C^*(G, \nu) \subset M_\sigma^*(G, \nu) \subset M^*(G, \nu).$$

Proof. From Proposition 20, $\mathbf{C}_\sigma(G)$ is dense in $C_c(G)$ for the inductive limit topology and hence is dense in $C^*(G, \nu)$. Since $\mathbf{C}_\sigma(G) \subset \mathcal{B}_\sigma(G)$, it follows that $C^*(G, \nu) \subset M_\sigma^*(G, \nu)$. □

Corollary 23. *Let G be a locally compact second countable principal proper groupoid. Let F be a Borel subset of $G^{(0)}$ meeting each orbit exactly once. Let $\sigma : G^{(0)} \to G^F$ be a cross section for $d : G^F \to G$ such that $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on G. Then*

$$C^*(G, \nu) \subset M_\sigma^*(G, \nu) \subset M^*(G, \nu).$$

Proof. Applying Proposition 21, we obtain that the quotient topology induced by q on G_F^F is a locally compact (Hausdorff) topology. Therefore G satisfies the hypothesis of Theorem 22. □

Definition 24. Let $\{\mu^{\dot{u}}\}_{\dot{u}}$ be a system of measures on $G^{(0)}$ satisfying:

(1) $\mathrm{supp}(\mu^{\dot{u}}) = [u]$ for all $\dot{u}$.
(2) For all compactly supported continuous functions f on $G^{(0)}$ the function

$$u \to \int f(v)\mu^{\pi(u)}(v)$$

is continuous.

We shall say that the Hilbert bundle determined by the system of measures $\{\mu^{\dot{u}}\}_{\dot{u}}$ has a continuous basis if there is sequence $f_1, f_2, \dots$ of real valued *continuous* functions on $G^{(0)}$ such that $\dim(L^2(\mu^{\dot{u}})) = \infty$ if and only if $\|f_n\|_2 = 1$ in $L^2(\mu^{\dot{u}})$ for $n = 1, 2, \dots$ and then $\{f_1, f_2, \dots\}$ gives an orthonormal basis of $L^2(\mu^{\dot{u}})$, while

$\dim(L^2(\mu^{\dot{u}})) = k < \infty$ if and only if $\|f_n\|_2 = 1$ for $n \le k$, and $\|f_n\|_2 = 0$ for $n > k$ and then $\{f_1, f_2, \dots, f_k\}$ gives an orthonormal basis of $L^2(\mu^{\dot{u}})$.

Remark 25. Let $\{\mu_1^{\dot{u}}\}_{\dot{u}}$ and $\{\mu_2^{\dot{u}}\}_{\dot{u}}$ be two systems of measures on $G^{(0)}$ satisfying:

(1) $\mathrm{supp}(\mu_i^{\dot{u}}) = [u]$ for all $\dot{u}$, $i = 1, 2$.

(2) For all compactly supported continuous functions f on $G^{(0)}$ the function

$$u \to \int f(v)\mu_i^{\pi(u)}(v)$$

is continuous.

Let us assume that the Hilbert bundles determined by the systems of measures $\{\mu_i^{\dot{u}}\}_{\dot{u}}$ have continuous bases. Let $f_1.f_2, \dots$ be a continuous basis for Hilbert bundle determined by $\{\mu_1^{\dot{u}}\}_{\dot{u}}$ and let $g_1, g_2, \dots$ be a continuous basis for Hilbert bundle determined by $\{\mu_2^{\dot{u}}\}_{\dot{u}}$. Let us define a unitary operator $U_{\dot{u}} : L^2(\mu_1^{\dot{u}}) \to L^2(\mu_2^{\dot{u}})$ by

$$U_{\dot{u}}(f_n) = g_n \text{ for all } n.$$

Then the family $\{U_{\dot{u}}\}_{\dot{u}}$ has the following properties:

(1) For all bounded Borel functions f on $G^{(0)}$,

$$u \to U_{\pi(u)}(f)$$

is a bounded Borel function with compact support.

(2) For all bounded Borel functions f on $G^{(0)}$,

$$U_{\pi(u)}(\bar{f}) = \overline{U_{\pi(u)}(f)}.$$

(3) For all compactly supported continuous functions f on $G^{(0)}$ there is a sequence $(h_n)_n$ of compactly supported continuous functions on $G^{(0)}$ such that

$$\sup_{\dot{u}} \int |U_{\dot{u}}(f) - h_n|^2 d\mu_2^{\dot{u}} \to 0\,(n \to \infty).$$

Indeed, we can define

$$h_n(v) = \sum_{k=1}^{n} g_k(v) \int f(u) f_k(u) d\mu_1^{\pi(v)}(u).$$

Remark 26. Let G be a locally compact second countable groupoid with proper orbit space. Let F be a Borel subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let us assume that $F \cap [K]$ has a compact closure for each compact subset K of $G^{(0)}$, and let $\sigma : G^{(0)} \to G^F$ be a cross section for $d_F : G^F \to G^{(0)}$ such that $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Let us endow G_F^F with the quotient topology induced by $q : G \to G_F^F$

$$q(x) = \sigma(r(x)) x \sigma(d(x))^{-1},\ x \in G.$$

If g is continuous on G_F^F and has relatively compact support in G, and if g_1, g_2 are two functions on $G^{(0)}$ with the property that there is two sequences $(h_n^1)_n$ and $(h_n^2)_n$ of compactly supported continuous functions on $G^{(0)}$ such that

$$\sup_{\dot{u}} \int |g_i - h_n^i|^2 d\mu_2^{\dot{u}} \to 0\,(n \to \infty)$$

for $i = 1, 2$, then

$$x \xrightarrow{f} g_1(r(x)) g(\sigma(r(x)) x \sigma(d(x))^{-1}) g_2(d(x))$$

can be viewed as an element of $C^*(G,\nu)$. Indeed, it is easy to see that

$$\|f-(h_n^1\circ r)(g\circ q)(h_n^2\circ d)\|_{II}\to 0\ (n\to\infty).$$

Proposition 27. *Let G be a locally compact second countable principal proper groupoid. Let $\nu_i=\{\nu_i^u, u\in G^{(0)}\}$, $i=1,2$, be two Haar systems on G and let $(\{\beta_v^u\},\{\mu_i^{\dot u}\})$ be the corresponding decompositions over the principal groupoid. If the Hilbert bundles determined by the systems of measures $\{\mu_i^{\dot u}\}_{\dot u}$ have continuous bases, then the C^*-algebras $C^*(G,\nu_1)$ and $C^*(G,\nu_2)$ are $*$-isomorphic.*

Proof. We use Notation 19. From Proposition 20, $\mathbf{C}_\sigma(G)$ is dense in $C_c(G)$ for the inductive limit topology and hence is dense in $C^*(G,\nu_1)$. We shall define a $*$-homomorphism Φ from $\mathbf{C}_\sigma(G)$ to $C^*(G,\nu_2)$. It suffices to define Φ on the set of functions on G of the form

$$x\to g_1(r(x))g(q(x))g_2(d(x))$$

where $g_1,g_2\in C_c(G^{(0)})$ and g continuous on G_F^F having relatively compact support in G. Let $\{U_{\dot u}\}_{\dot u}$ be the family of unitary operators with the properties stated in Remark 25 associated to the systems of measures $\{\mu_i^{\dot u}\}_{\dot u}$, $i=1,2$.

Let us define Φ by

$$\Phi(f)=(x\to U_{\pi(r(x))}(g_1)(r(x))g(q(x))U_{\pi(d(x))}(g_2)(d(x)))$$

where f is defined by

$$f(x)=g_1(r(x))g(q(x))g_2(d(x))$$

with $g_1,g_2\in C_c(G^{(0)})$ and g continuous on G_F^F having relatively compact support in G.

As noted in Remark 26, the functions of the form $\Phi(f)$ can be viewed as elements of $C^*(G,\nu_2)$. With the same argument as in the proof of Theorem 9, it follows that Φ can be extended to $*$-isomorphism between $C^*(G,\nu_1)$ and $C^*(G,\nu_2)$. □

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