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The Atiyah–Patodi–Singer theorem for perturbed Dirac operators on even-dimensional manifolds with bounded geometry

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Abstract. This paper establishes conditions under which one can prove an Atiyah–Patodi–Singer index theorem for perturbed Dirac operators on complete noncompact even-dimensional manifolds with boundary. This index theorem introduces into index theory spectral invariants of self-adjoint perturbed Dirac operators on noncompact manifolds.

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1. Introduction

A perturbed Dirac operator is the sum of a first-order elliptic differential operator of Dirac type and a vector-bundle map (the perturbation). The paper of C. Callias [5] stimulated the publication of many papers on the Fredholm index theory of perturbed Dirac operators. The index theory of these operators has played a role

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in mathematical physics [4, 13] and geometry [10]. References to many of the papers on the subject appear in [8]. A heat equation derivation of an index formula for a class of Fredholm perturbed Dirac operators appears in [9].

The present paper uses this heat equation method to prove an index formula for a class of perturbed Dirac operators satisfying Atiyah–Patodi–Singer boundary conditions on complete noncompact even-dimensional manifolds with boundary. The boundary operators associated with these operators are self-adjoint perturbed Dirac operators on the odd-dimensional complete noncompact boundaries. Their eta invariants appear in the index formula and thus introduce into index theory the spectral invariants of perturbed Dirac operators on noncompact manifolds.

Sections 2 and 3 introduce the manifolds and operators we study. Briefly the manifolds and vector bundles in this paper have bounded geometry, and the Dirac operators' perturbations grow sufficiently rapidly and regularly. Section 4 outlines the tools that can help identify rates of growth for the spectra of self-adjoint perturbed Dirac operators. Section 5 outlines the results of [9] that provide essential estimates related to the domains of powers of self-adjoint perturbed Dirac operators. Throughout the paper we show that natural constructions on manifolds with cylindrical or linearly expanding ends lead to operators that satisfy the assumptions of our theorems. Most of the proofs that the operators so constructed satisfy our hypotheses are straightforward, but the discussion of these issues for the hypotheses leading to the estimates in Section 5 is more involved.

Section 6 discusses the properties of and constructions on normal neighborhoods that are permitted by bounded geometry. Section 7 introduces the Atiyah–Patodi–Singer boundary conditions and proves some properties of vector-bundle sections supported near the boundary of the manifold. Section 8 provides the estimates on heat kernels that show that our perturbed Dirac operators are Fredholm and that are the foundation for the expression of an index formula in terms of an asymptotic expansion involving locally defined quantities. Section 9 summarizes our assumptions and proves our index formula. Section 10 comments on some properties of the eta invariants of self-adjoint perturbed Dirac operators.

2. The manifolds

The operators whose index theory is discussed in this paper live on even-dimensional, not necessarily compact, complete, oriented, Riemannian manifolds with not necessarily compact boundaries. Let M denote the manifold with boundary, and let N denote the boundary. We assume throughout that, in M, N has a collar neighborhood of uniform width $N \times [0, \delta_b]$ on which all structures we use are product structures. (All references to a collar neighborhood include the assumption that all structures are products.) We also assume that all of our manifolds have bounded geometry. We use the discussion of bounded geometry given in [19], where earlier references can be found, and we extend it to manifolds with boundaries having collar neighborhoods by taking it to mean that the manifold's double has bounded geometry. We denote the double of M by M^*.

Definition 2.1. A complete Riemannian manifold is said to have bounded geometry if it has positive injectivity radius and if the curvature tensor and its covariant derivatives are each uniformly bounded.

The following proposition appears in [19].

Proposition 2.2. *A Riemannian manifold M has bounded geometry if and only if: one can choose a fixed ball B in Euclidean space that serves as the domain of a normal coordinate system at every $x \in M$, and the Christoffel symbols of M (viewed as a family parametrized by $x \in M$ and their indices) lie in a bounded subset of the Fréchet space $C^\infty(B)$.*

One section of this paper relies on calculations on N and on the double of M, M^*, that are special to manifolds with cylindrical or linearly expanding ends. Both are examples of manifolds with warped-product ends. A manifold with warped-product end is separated by a hypersurface H into a compact submanifold with boundary H and an end diffeomorphic to $[r_0, \infty) \times H$ for some positive r_0. The metric on the $[r_0, \infty) \times H$ piece is of the form

$$dr \otimes dr + h^2(r) \cdot g_H,$$

where g_H is a Riemannian metric on H and $h(r)$ is a positive function called the warping factor. When h is constant, the end is called cylindrical. When $h(r) = r$, the end is called linearly expanding. Any compact manifold with boundary can be extended to have a cylindrical end by attaching a cylinder to the boundary H. Euclidean space, with H the unit sphere, has a linearly expanding end. Any compact manifold with boundary can be extended to have a linearly expanding end by attaching a cylinder to the boundary and altering the metric on the cylinder.

A Euclidean half-space is an example of a noncompact manifold whose double and whose boundary have linearly expanding ends. Other examples of manifolds whose doubles and whose boundaries have cylindrical or linearly expanding ends can be constructed as follows. Start with a compact oriented manifold with boundary. In this manifold choose a hypersurface H satisfying two conditions: H has nonempty boundary equal to H's intersection with the original manifold's boundary; and H separates the original manifold into two pieces. (Examples of such H include, but are not limited to, the boundary relative to the original manifold, of a small neighborhood of a point on the original manifold's boundary.) Choose a metric in which the boundary of the manifold and the boundary of H have compatible collar neighborhoods and in which H has a neighborhood isometric to the product of H with an interval. Remove one of the two pieces into which H divides the manifold, and replace this piece by $[r_0, \infty) \times H$. This piece can be given a cylindrical metric, or that metric can be altered to provide a linearly expanding end.

3. Operators on manifolds without boundary

In this section we describe perturbed operators of Dirac type (henceforth called perturbed Dirac operators) on complete noncompact even-dimensional manifolds without boundary. We also describe self-adjoint perturbed Dirac operators on complete noncompact odd-dimensional manifolds without boundary. We show that when the even-dimensional manifold is the double of a manifold with boundary (having a collar neighborhood), then the self-adjoint perturbed Dirac operator on the boundary can be viewed as the restriction of the perturbed Dirac operator on the even-dimensional manifold to the boundary directions.

Our goal in this paper is the index theory of certain perturbed Dirac operators on even-dimensional manifolds with boundary. However the definition of such operators depends on boundary conditions whose expression depends on the spectra of perturbed Dirac operators on manifolds without boundary. For this reason we introduce the operators on manifolds without boundary in this section and develop their spectral theory in the following section before introducing the operators on manifolds with boundary.

Let $\mathcal{M}$ be a complete, noncompact, oriented, Riemannian manifold without boundary. In discussing the structure of Dirac operators and perturbed Dirac operators, it is simplest to assume that the operators are acting on their natural domains of smooth vector-bundle sections with compact support. At some point rigorous analysis requires that we pass to the closures of these operators. For each operator that we consider on manifolds without boundary, this closure is the only closed extension whose adjoint's domain contains all smooth compactly supported sections [6].

Using the terminology of [14], let S be a complex Dirac bundle over $\mathcal{M}$, and let $\mathcal{D}$ be the associated Dirac operator. When $\mathcal{M}$ is even-dimensional, we use the grading $S = S^+ \oplus S^-$ determined by the volume form. Then $\mathcal{D}$ decomposes as $\mathcal{D} = \mathcal{D}^\pm$ mapping sections of $S^\pm$ to sections of $S^\mp$.

On $\mathcal{M}$ let E be a Hermitian vector bundle that has a connection compatible with its metric. We assume that E is graded, $E = E_0 \oplus E_1$, and that all of E's structure respects this grading. We let $\mathcal{A}$ denote a smooth vector-bundle map $E_0 \to E_1$ that is invertible off some compact subset of $\mathcal{M}$. Throughout the paper we use the same notation for a vector-bundle map that we use for the associated map on sections.

Using tensor-product connections, we extend $\mathcal{D}$ to sections of $E \otimes S$. We use the notation D_j for the Dirac operator mapping sections of $E_j \otimes S^+$ to sections of $E_j \otimes S^-$ in the even-dimensional case and mapping sections of $E_j \otimes S$ to sections of $E_j \otimes S$ in the odd-dimensional case. In the even-dimensional case we use the notation $A_\pm$ for the vector-bundle map $\mathcal{A} \otimes I \colon E_0 \otimes S^\pm \to E_1 \otimes S^\pm$. In the odd-dimensional case $\widetilde{A}$ denotes the vector-bundle map $\mathcal{A} \otimes I \colon E_0 \otimes S \to E_1 \otimes S$.

Assumption 3.1. We assume throughout the paper that the vector bundle $E \otimes S$ has bounded geometry in the sense (of [19]) that the Christoffel symbols for $E \otimes S$ lie in a bounded subset of the Fréchet space referred to in Proposition 2.2.

Definition 3.2. On our even-dimensional $\mathcal{M}$ a perturbed Dirac operator is an operator of the form

$$\widetilde{D}_A = \begin{pmatrix} A_+ & -D_1^* \\ D_0 & A_-^* \end{pmatrix}$$

from sections of $E_0 \otimes S^+ \oplus E_1 \otimes S^-$ to sections of $E_1 \otimes S^+ \oplus E_0 \otimes S^-$.

Notation 3.3. On even-dimensional $\mathcal{M}$ we also use the notations

$$\widetilde{D} = \begin{pmatrix} 0 & -D_1^* \\ D_0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & \widetilde{D}^* \\ \widetilde{D} & 0 \end{pmatrix}$$

$$\widetilde{A} = \begin{pmatrix} A_+ & 0 \\ 0 & A_-^* \end{pmatrix} \qquad A = \begin{pmatrix} 0 & \widetilde{A}^* \\ \widetilde{A} & 0 \end{pmatrix}$$

as well as

$$D_A = \begin{pmatrix} 0 & (\widetilde{D}_A)^* \\ \widetilde{D}_A & 0 \end{pmatrix} \quad \text{and} \quad \Delta_A = (D_A)^2.$$

Definition 3.4. On an odd-dimensional $\mathcal{M}$ a self-adjoint perturbed Dirac operator is an operator of the form

$$D_A = \begin{pmatrix} D_0 & \widetilde{A}^* \\ \widetilde{A} & -D_1 \end{pmatrix}$$

from sections of $E_0 \otimes S \oplus E_1 \otimes S$ to sections of $E_0 \otimes S \oplus E_1 \otimes S$. The terminology "self-adjoint" is justified by our previous reference to [6]: with domain the smooth compactly supported sections, this operator is esssentially self-adjoint.

Notation 3.5. On odd-dimensional $\mathcal{M}$ we use the notations

$$D = \begin{pmatrix} D_0 & 0 \\ 0 & -D_1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \widetilde{A}^* \\ \widetilde{A} & 0 \end{pmatrix} \quad \text{and} \quad \Delta_A = (D_A)^2.$$

Suppose that our even-dimensional manifold is the double of a complete noncompact manifold M with noncompact boundary N having a collar neighborhood of uniform width. Assume that the vector bundles and vector-bundle maps have product structures in this collar neighborhood. By this we mean that all structures are independent of the variable normal to N in the collar neighborhood. We use the notation ∂_u to denote differentiation ("directional derivative") in the direction normal to N and pointing into M on the collar neighborhood. We use e_0 for the element the corresponds to the inward unit normal in the bundle of Clifford algebras associated with the tangent bundle. We assume that M and N are oriented so that N's orientation followed by the inward unit normal agrees with M's orientation. As noted before we use the conventions of [14], including the volume elements referred to there as the complex volume elements. Under the above conventions, in a collar neighborhood of N, we can identify S^+ and S^- with the pullback (under the projection from the collar to N) of a single complex Dirac bundle S over N. The identification proceeds as follows. Let $\{\partial_1, \dots, \partial_{m-1}\}$ denote an orthonormal basis of the tangent space to N at a point in N, and let $\{e_1, \dots, e_{m-1}\}$ denote the associated Clifford algebra elements. Define the action of e_i on S^+ to be the Clifford action over M of $e_i e_0$, and define the Clifford action of e_i on S^- to be the Clifford action over M of $e_0 e_i$. The Clifford action of e_0 maps S^+ to S^- and intertwines these actions. Its inverse is the Clifford action of $-e_0$.

Changing the order of summands in the range, we can rewrite $\widetilde{D}_A$ on M as

$$\begin{pmatrix} D_0 & A_-^* \\ A_+ & -D_1^* \end{pmatrix}.$$

In coordinates on the collar neighborhood, this operator equals

$$\begin{pmatrix} e_0\partial_u & 0 \\ 0 & -e_0\partial_u \end{pmatrix} + \begin{pmatrix} \sum e_i\partial_i & A_-^* \\ A_+ & -\sum e_i\partial_i \end{pmatrix},$$

which can be rewritten

$$\begin{pmatrix} e_0\partial_u & 0 \\ 0 & -e_0\partial_u \end{pmatrix} + \begin{pmatrix} e_0 \sum e_i e_0 \partial_i & A_-^* \\ A_+ & -e_0 \sum -e_0 e_i \partial_i \end{pmatrix}.$$

Under our conventions for identifying S^+ and S^- with a single Dirac bundle pulled back from N, we can write the above as

$$\begin{pmatrix} \partial_u & 0 \\ 0 & \partial_u \end{pmatrix} + \begin{pmatrix} D_0 & \widetilde{A}^* \\ \widetilde{A} & -D_1 \end{pmatrix}$$

as a map from sections of $E_0 \otimes S \oplus E_1 \otimes S$ to sections of $E_0 \otimes S \oplus E_1 \otimes S$ over the collar neighborhood.

Remark 3.6. Later in this paper we will start with an even-dimensional manifold M with boundary and, on it, a perturbed Dirac operator $\widetilde{D}_A$ taking the form described in Definition 3.2. We will form the double of M, M^*, by using the identity map between the boundaries of two copies of M to attach a copy of M to a second copy of M with the opposite orientation. Both copies of M will carry bundles $E \otimes S$. At the boundaries we will attach E to itself by the identity map, which will intertwine the maps $\mathcal{A}$ from the two copies. Over the second copy of M, we will give S the Dirac-bundle structure determined by defining the action of a tangent vector on the associated fiber of S to be the negative of the corresponding action over the first copy of M. At the boundary we will attach the first copy of S to the second copy of S by the Clifford action of e_0 (as defined over the first copy of M). Due to the choice of orientations, the decomposition $S = S^+ \oplus S^-$ will be well-defined over M^*. Due to the choice of actions, S will have a well-defined Dirac-bundle structure over M^*, and the Dirac operator on M will extend to a Dirac operator on M^*. It will follow that the operator $\widetilde{D}_A$ on M will extend naturally to an operator $\widetilde{D}_A$ on M^*.

4. The spectrum

In this section we identify conditions under which Δ_A has discrete spectrum and conditions under which $\{\lambda \in \sigma(\Delta_A)\colon |\lambda| \leq L\}$ has growth satisfying a polynomial bound as $L \to \infty$. Here Δ_A refers to any of the operators $(D_A)^2$ defined on manifolds without boundary in Section 3. The arguments, given in detail in the even-dimensional case in [9], extend without complication to the odd-dimensional case.

Writing

$$\Delta_A = D^2 + (DA + AD) + A^2,$$

let

$$R(D_A) = (DA + AD) + A^2.$$

Let $\mu(R(D_A))(x)$ denote the infimum of the spectrum of the self-adjoint vector-bundle map $R(D_A)$ at the point x, with $\mu(A^2)(x)$ having an analogous meaning. Throughout this paper we make the following assumption about the operators D_A.

Assumption 4.1. As $x \to \infty$ in $\mathcal{M}$, $\mu(A^2)(x) \to \infty$ and $\mu(R(D_A))(x) \to \infty$.

Lemma 4.2. *If the $R(D_A)$ part of Δ_A satisfies Assumption 4.1, then Δ_A has empty essential spectrum, and thus $(1 + \Delta_A)^{-1}$ is compact.*

Proof. The argument is standard and may be found, for example, in [8]. □

Examples of operators satisfying Assumption 4.1 arise naturally on manifolds with warped-product ends. Detailed calculations appear in [9], but the idea is as follows. Assume that, on the end, E takes the form of the pullback of a bundle from the end's cross-section to the end. Assume that A is the product of a function $j(r)$ (where r is the variable representing the direction orthogonal to the cross-section) and the pullback of an invertible vector-bundle map on the cross-section. Noting that the individual entries in $DA+AD$ are commutators, we see that A^2 is a positive vector-bundle map that grows as $j^2(r)$, while $DA + AD$ has terms (not necessarily positive) with norms growing as $j'(r)$ and terms (not necessarily positive) with norms growing as $j(r)/h(r)$. The former terms arise from commutators with differentiation in the r-direction, while the latter terms arise from commutators with differentiation in the cross-section direction, which introduce the reciprocal of the warping factor $h(r)$. Thus Assumption 4.1 is satisfied when $j^2(r)$ exhibits growth that dominates the growth of $j'(r)$ and of $j(r)/h(r)$. On a cylindrical or linearly expanding end, any positive power of r will have this property.

To study the growth rate of the discrete spectrum, we introduce the following notation.

Notation 4.3. For $L \geq 1$ let $N(\Delta_A; L)$ denote the cardinality of

$$\{\lambda \in \sigma(\Delta_A) \colon |\lambda| \leq L\},$$

the number of eigenvalues, counting multiplicity, of Δ_A that have absolute value no greater than L.

One "expects" $N(\Delta_A; L)$ to grow with L at the same rate as the volume of $\{(x, \xi) \in T^*\mathcal{M} \colon |\xi|^2 + \mu(R(D_A))(x) \leq L\}$ does. Arguments based on Neumann comparison and certain estimates prove a theorem that realizes the expectation. The Neuman comparison argument, given in detail in the even-dimensional case in [9], extends without change to the odd-dimensional case. We outline the argument here. The estimates of [9] on cylindrical and linearly expanding ends also extend without change. We recall them.

Let $\mathcal{M}$ be a complete, noncompact, oriented, Riemannian $\overline{m}$-dimensional manifold without boundary and with bounded geometry. The manifold $\mathcal{M}$ may be either odd-dimensional or even-dimensional. Let the operators D, D_A, and Δ_A be as in Section 3. Let ∇ denote the operator formed from the gradient by two-by-two matrix constructions so that on $\mathcal{M}$ the formula known as the Weitzenböck formula [18] or the general Bochner formula [14] gives

$$D^2 = \nabla^*\nabla + \mathcal{K}(x),$$

where $\mathcal{K}(x)$ is the curvature term. By our assumption of bounded geometry and Assumption 4.1, it is possible to choose a positive constant λ_0 so that for every $x \in \mathcal{M}$, $\mathcal{K}(x) + R(D_A)(x) + \lambda_0 \geq 1$.

Definition 4.4. Choose a constant λ_0 so that for every $x \in \mathcal{M}$

$$\mathcal{K}(x) + R(D_A)(x) + \lambda_0 \geq 1.$$

Let W be an $\overline{m}$-dimensional submanifold of $\mathcal{M}$ having smooth compact boundary. Define the operator $\Delta_A[W] + \lambda_0$ to be the self-adjoint operator associated with the quadratic form

$$q_A(\alpha, \beta) = \langle \nabla\alpha, \nabla\beta \rangle + \langle (\mathcal{K} + R(D_A) + \lambda_0)^{1/2}\alpha, (\mathcal{K} + R(D_A) + \lambda_0)^{1/2}\beta \rangle.$$

Here the form domain is the set of sections α that are L^2 on the interior of W and for which $\nabla\alpha$, taken in the distribution sense on interior(W), and $(\mathcal{K}+R(D_A)+\lambda_0)^{1/2}\alpha$ are L^2 on the interior of W. The operator $\Delta[W]+1$ is the self-adjoint operator associated with the quadratic form

$$q(\alpha,\beta)=\langle\nabla\alpha,\nabla\beta\rangle+\langle\alpha,\beta\rangle,$$

where ∇ is interpreted as above and where the form domain is as above with the condition involving $\mathcal{K}+R(D_A)+\lambda_0$ omitted. Similarly $\Delta[W]$ arises from the quadratic form $\langle\nabla\alpha,\nabla\beta\rangle$, with ∇ as above. In all cases the notation $\langle -,-\rangle$ refers to the L^2 inner product.

Theorem 4.5 ([9]). *Suppose that a smooth compact hypersurface divides $\mathcal{M}$ into $\mathcal{M}_1$ and $\mathcal{M}_2$. Then, in the sense of Section XIII.15 of [15],*

$$(\Delta_A[\mathcal{M}_1]+\lambda_0)\oplus(\Delta_A[\mathcal{M}_2]+\lambda_0)\leq\Delta_A+\lambda_0$$

and thus

$$N(\Delta_A+\lambda_0;L)\leq N(\Delta_A[\mathcal{M}_1]+\lambda_0;L)+N(\Delta_A[\mathcal{M}_2]+\lambda_0;L).$$

Proof. By [6] the operator $\Delta_A+\lambda_0$ defined on smooth compactly supported sections is essentially self-adjoint. Thus its closure, whose spectrum $N(\Delta_A+\lambda_0;L)$ describes, equals its Friedrichs extension. (See, e.g., Section X.3 of [15].) On smooth compactly supported sections the quadratic form defining the Friedrichs extension equals the form q_A of Definition 4.4. The conclusions of this theorem follow from the discussion of Neumann comparison represented by Propositions 3 and 4 in Section XIII.15 of [15] and by the discussion surrounding those propositions. □

Theorem 4.6 ([9]). *Let W and λ_0 be as in Definition 4.4. If W is compact, there is a constant C, depending on W, E, and the Dirac operator, such that*

$$N(\Delta_A[W]+\lambda_0;L)\leq N(\Delta[W]+1;L)\leq C\cdot L^{\overline{m}/2}.$$

Regardless of whether W is compact, if there is $L_1>L$ such that $\forall x\in W$,

$$\mathcal{K}+R(D_A)(x)+\lambda_0\geq L_1,$$

then $N(\Delta_A[W]+\lambda_0;L)=0$.

Proof. The inequality involving $L^{\overline{m}/2}$ is a direct consequence of Weyl's theorem, and the other statements follow directly from the definitions of the operators involved. A detailed proof appears in [9]. □

Natural assumptions on the auxiliary vector bundle and the perturbation permit the application of the preceding theorem to show that the spectrum of Δ_A has polynomial growth on manifolds with cylindrical or linearly expanding ends. Detailed calculations appear in [9]. We summarize the results here. In each case the calculations are based on breaking the manifold into a compact piece and the end, which is itself broken into infinitely many compact pieces. By Assumption 4.1 and the last statement of the preceding theorem, for any given L, only finitely many compact pieces contribute to $N(\Delta_A;L)$. In the cylindrical case the pieces are isometric, and the contribution of each is bounded by a constant multiple of $L^{\overline{m}/2}$. In the linearly expanding case each piece is a scaled version of the preceding piece, with a scale factor of 2. In either case suppose that, on the end, the auxiliary vector bundle is pulled back from a vector bundle on the end's cross-section Y and all of the vector bundle's structure is constant in the r-direction. Suppose that for $(r,y) \in [r_0,\infty)\times Y$, $\mu(R(D_A))(r,y) \geq r$. This assumption holds for sufficiently large r if the perturbation takes the form discussed in the paragraph following Lemma 4.2 and if $j(r) = r^n$ for $n > 1/2$. Then on a manifold with cylindrical end, $N(\Delta_A; L)$ satisfies a growth bound of the form $L^{(\overline{m}/2)+1}$. On a manifold with linearly expanding end, the growth bound can be taken to be of the form $L^{(3\overline{m}+1)/2}$.

5. Domains of powers

Many of the estimates in this paper depend on having the operator Δ_A on the double of the manifold M satisfy the following assumption involving L^2 inner products.

Assumption 5.1. For each positive integer k there is a positive constant C_k such that for every $w \in \text{domain}((\Delta_A)^k)$

$$\langle (\Delta_A)^k w, (\Delta_A)^k w\rangle + \langle w,w\rangle \geq C_k(\langle D^{2k}w, D^{2k}w\rangle + \langle A^{2k}w, A^{2k}w\rangle).$$

Section 6 of [9] establishes one way to prove that such an estimate holds and uses that approach to show that the estimate holds for a natural class of perturbed Dirac operators on manifolds with cylindrical or linearly expanding ends. For completeness we summarize the general arguments here, and we give a discussion of their application on manifolds with cylindrical or linearly expanding ends that is more detailed than the discussion in [9]. Because the discussion applies equally to even-dimensional and odd-dimensional manifolds, we denote the manifold by $\mathcal{M}$.

Notation 5.2. For G a vector-bundle map on a manifold $\mathcal{M}$ and for $x \in \mathcal{M}$, let $s[G](x)$ denote the supremum of the spectrum of $(G^*G)^{1/2}(x)$ and let $i[G](x)$ denote the infimum of the spectrum of $(G^*G)^{1/2}(x)$. If G_1 and G_2 are vector-bundle maps, we write $G_1 = o(G_2)$ if $s[G_1](x)/i[G_2](x) \to 0$ as $x \to \infty$ in $\mathcal{M}$.

The following proposition suggests what terms should be grouped in proving the estimate we desire for $(\Delta_A)^k$. In the proposition the inner products are L^2 inner products on vector-bundle sections over $\mathcal{M}$, and expressions of the form $[T_1,T_2]$ refer to commutators.

Proposition 5.3 ([9]). *Suppose that Δ_A has empty essential spectrum. Assume that $DA+AD$ is $o(A^2)$, with $\mathcal{M}'$ a compact subset of $\mathcal{M}$ off of which*

$$s[DA+AD]/i[A^2] \leq 1/4.$$

Supppose that off some compact subset $\mathcal{M}''$ of $\mathcal{M}$, $\mathcal{A}$ is the product of a scalar-valued function and a unitary-valued vector-bundle map. Finally assume that for each positive integer k there is a compact subset $\mathcal{M}_k$ of $\mathcal{M}$ containing $\mathcal{M}'\bigcup\mathcal{M}''$ and satisfying the following condition: for each smooth section w with compact support in $\mathcal{M}\setminus\mathcal{M}_k$,

$$2\langle D^{2k-1}R(D_A)D^{2k-1}w,w\rangle + \langle R(D_A)D^{4(k-1)}R(D_A)w,w\rangle$$
$$+ (1/3)\langle R(D_A)A^{4(k-1)}R(D_A)w,w\rangle + \langle [D^{2k-1},[D^{2k-1},R(D_A)]]w,w\rangle \geq 0,$$

and

$$2\langle DA^{2(k-1)}R(D_A)A^{2(k-1)}Dw,w\rangle + (1/3)\langle R(D_A)A^{4(k-1)}R(D_A)w,w\rangle$$
$$+ (1/3)\langle D^2A^{4(k-1)}D^2w,w\rangle + \langle [D,[D,A^{2(k-1)}R(D_A)A^{2(k-1)}]]w,w\rangle \geq 0.$$

Then for each positive integer k there is a positive constant C'_k such that for all smooth sections w compactly supported in $\mathcal{M} \setminus \mathcal{M}_k$,

$$\langle (\Delta_A)^k w, (\Delta_A)^k w\rangle + \langle w, w\rangle \geq C'_k(\langle D^{2k}w, D^{2k}w\rangle + \langle A^{2k}w, A^{2k}w\rangle).$$

Proof. The proof, which is given in [9], is largely a straightforward calculation involving induction on k. Among the roles played by the proposition's hypotheses are the following. The hypothesis on the spectrum of Δ_A permits the decomposition (respecting domains of other operators) of the Hilbert space into a subspace where $\|(\Delta_A)^k \Delta_A v\| \geq \|\Delta_A v\|$ and a subspace where $\|v\| \geq \|\Delta_A v\|$. This decomposition permits the passage from an estimate of the form

$$\begin{aligned}\langle (\Delta_A)^k \Delta_A w, (\Delta_A)^k \Delta_A w\rangle + &\langle \Delta_A w, \Delta_A w\rangle \geq \\ &C'_k(\langle D^{2k}\Delta_A w, D^{2k}\Delta_A w\rangle + \langle A^{2k}\Delta_A w, A^{2k}\Delta_A w\rangle)\end{aligned}$$

for w with compact support in the complement of $\mathcal{M}_k$ (the inductive hypothesis applied to the value k and $\Delta_A w$) to an estimate of the form

$$\begin{aligned}\langle (\Delta_A)^k \Delta_A w, (\Delta_A)^k \Delta_A w\rangle + &\langle w, w\rangle \geq \\ &C(\langle D^{2k}\Delta_A w, D^{2k}\Delta_A w\rangle + \langle A^{2k}\Delta_A w, A^{2k}\Delta_A w\rangle).\end{aligned}$$

The hypotheses involving commutators and inequalities permit the transformation of the right side of this inequality into a form more closely related to the proposition's conclusion (for the power $k+1$). The hypothesis on the structure of $\mathcal{A}$ lets us commute scalar-valued even powers of A past the vector-bundle map $R(D_A)$. Because $DA + AD$ is $o(A^2)$, $\langle R(D_A)A^{4k}R(D_A)w, w\rangle$ is greater than or equal to some positive constant multiple of $\|A^{2k+2}w\|^2$. □

The following proposition captures natural assumptions under which Assumption 5.1 holds.

Proposition 5.4 ([9])**.** *Suppose that Δ_A has empty essential spectrum and that $\mathcal{M}$ has a collection of compact subsets $\mathcal{M}_k$ for which the conclusion of Proposition 5.3 holds. Then for each positive integer k there is a positive constant C_k such that for all smooth compactly supported sections w over $\mathcal{M}$,*

$$\langle (\Delta_A)^k w, (\Delta_A)^k w\rangle + \langle w, w\rangle \geq C_k(\langle D^{2k}w, D^{2k}w\rangle + \langle A^{2k}w, A^{2k}w\rangle).$$

Thus for positive integers k every $w \in \mathrm{domain}((\Delta_A)^k)$ (not necessarily smooth and compactly supported) satisfies $D^{2k}w \in L^2$ and $A^{2k}w \in L^2$.

Proof. The proof is given in detail in Section 6 of [9]. The last statement in the proposition is a consequence of the estimate in the proposition and the theorem [6] that smooth compactly supported sections form a core for $(\Delta_A)^k$. The proof of the inequality is again by induction on k. The arguments are more involved than those used in the proof of Proposition 5.3, but in essence they are the following. Form a cover by two open sets, one of which has compact closure and the other of which has closure contained in the complement of $\mathcal{M}_k$. With the use of an appropriate partition of unity subordinate to this cover, we can separate terms to which the estimate of Proposition 5.3 applies from the remaining terms, which are supported on some fixed compact set that depends on only the open cover and partition of unity. Covering this open set by a finite collection of normal neighborhoods and applying a partition of unity subordinate to this cover, we can use estimates related to Gårding's inequality to complete the argument. □

Example 5.5. We end this section with an outline of the arguments showing that the hypotheses of Proposition 5.3 are satisfied for appropriate $\mathcal{A}$ on a manifold $\mathcal{M}$ with cylindrical or linearly expanding end. We focus our comments and calculations on the end. Let r denote the end's "longitudinal variable" associated with the direction orthogonal to the end's compact cross-section. Suppose that $\mathcal{A}$ is the product of r^p, for some $p > 1/2$, with the pullback from the cross-section to the end of a unitary-valued vector-bundle map. Note that this second factor is independent of r and that, for the A associated with this $\mathcal{A}$, A^2 is a scalar-valued function. The vector-bundle map $AD+DA$ is $o(A^2)$ because $\frac{d}{dr}r^p$ is $o(r^{2p})$ and because the factor of A that is independent of r is the pullback from the cross-section to the end, in which the cross-section has constant size or expands as r grows. The reasoning in Section 4 shows that Δ_A has empty essential spectrum.

The other hypotheses of Proposition 5.3 involve commutators of higher powers of terms of Δ_A. We show that these hypotheses are satisfied in our examples by choosing appropriate families of coordinate neighborhoods and using Gårding's inequality. We choose a periodic open cover and periodic subordinate partition of unity for the r-half-line. (By periodic we mean that the open cover and partition of unity arise by choosing a finite open cover of the circle by sets diffeomorphic to intervals, choosing a partition of unity subordinate to this cover, choosing a covering map from the line to the circle, and then using the components of preimages of the circle's open sets to create the open cover of the half-line, with subordinate partition of unity formed analogously using the pullback of functions.) We also choose a finite cover of the cross-section by normal neighborhoods (chosen so that the coordinates and transition maps extend smoothly to the neighborhoods' closures), and we choose a partition of unity subordinate to this cover. The natural diffeomorphism between the end and the product of the half-line with the cross-section determines a cover of the end by open sets diffeomorphic to sets of the form $I \times W$, where I runs through the intervals we are using to cover the half-line and W runs through the open subsets of the cross-section that we are using. Similarly the products of functions from the partitions of unity form a partition of unity subordinate to this cover.

In the case of a cylindrical end, we assign coordinates in each of these open sets $I \times W$ by combining the standard coordinate r of $\mathbb{R}$ with normal coordinates for W. The linearly expanding case requires somewhat more care. For the copy W_r of W in the cross-section at $r \in \mathbb{R}$, there is an open subset $\widetilde{W}_r$ of Euclidean space that provides normal coordinates via the exponential map $\exp_r\colon \widetilde{W}_r \to W_r$. Note that $\{\vec{x}\colon \vec{x} \in \widetilde{W}_r\} = \{\frac{r}{s}\vec{y}\colon \vec{y} \in \widetilde{W}_s\}$. For each pair (I, W) we choose $r_0 \in I$ (for concreteness let r_0 be the midpoint of I) and assign coordinates $r \in I$ and $\vec{x} \in \widetilde{W}_{r_0}$ to the open set $\{(r, v)\colon r \in I \text{ and } v \in W_r\}$ via the map $I \times \widetilde{W}_{r_0} \to \{(r, v)\colon r \in I \text{ and } v \in W_r\}$ defined by $(r, \vec{x}) \mapsto (r, \exp_r(\frac{r}{r_0}\vec{x}))$. On each chart we also need coordinates on our vector bundles. These coordinates arise from choosing an orthonormal basis for the fiber over $(r_0, \vec{0})$, using parallel translation along rays from the origin to provide bases in each fiber over $\{(r_0, \vec{x})\colon \vec{x} \in \widetilde{W}_{r_0}\}$, and pulling these bases back under the projection from $(r, \vec{x})$ to $(r_0, \vec{x})$.

Each $I \times \widetilde{W}_{r_0}$ that arises is a bounded Euclidean domain with the k-extension property (terminology as in [16]). Although there is no upper bound on the diameters of the $I \times \widetilde{W}_{r_0}$ that arise, there is a uniform upper bound on the distortion (terminology as in [19]) associated with our coordinate map. One consequence is that, for each power of of our manifold's Dirac operator, when the power is written in the coordinates described above, there is a constant of uniform ellipticity that is independent of the coordinate neighborhood.

Because our neighborhoods have uniformly bounded length in the r-direction, powers of r satisfy a strong o-property: if $k < \ell$, the supremum of r^k over a given neighborhood divided by the infimum of r^ℓ over the same neighborhood goes to zero as the neighborhood "goes to infinity in our collection of neighborhoods" via r-translation. Powers of A satisfy an analogous property. The same result is true if we replace the single neighborhood by a finite set of neighborhoods and translate that set of neighborhoods to infinity through our collection of neighborhoods.

For Ω a bounded open subset of Euclidean space with the k-extension property let $H_0^k(\Omega)$ denote the closure, in the Fourier-transform-defined order-k Sobolev norm, of the set of smooth vector-bundle sections with compact support in Ω. We denote this norm by $\|w\|_{0,k}$ for $k > 0$ and by $\|w\|$ for $k = 0$. For $0 \le k < \ell$, and for arbitrary $\epsilon > 0$, there exists a C_ϵ such that for all smooth compactly supported w

$$\|w\|_{0,k}^2 \le \epsilon \|w\|_{0,\ell}^2 + C_\epsilon \|w\|^2.$$

The C_ϵ response to ϵ arises from choosing a large enough ball in the space associated with the Fourier transform variable, and thus the C_ϵ response to ϵ can be chosen to be independent of Ω.

For us Ω will be the Euclidean open set identified by the coordinate map with one of our neighborhoods. Let D denote the manifold's Dirac operator, expressed in these coordinates. Gårding's inequality (or the elliptic estimate) states that for each k and each bounded Ω there exist constants c_1 and c_2 such that

$$\|D^k w\|^2 + c_1 \|w\|^2 \ge c_2 \|w\|_{0,k}^2$$

for all $w \in H_0^k(\Omega)$. The independence of Ω discussed previously helps show that these constants can be chosen to be independent of Ω. (See, e.g., Section 8.2.3 of [16] for a statement and proof of Gårding's inequality. In addition to the use of a constant of uniform ellipticity and a C_ϵ response to ϵ that are independent of Ω, we note the following step in making the estimates in the proof independent of Ω. The proof uses approximation of the highest order operators by constant coefficient operators. Done on sets that are "small enough," this approximation is "good enough." As Ω is scaled in the cross-section direction, the estimates remain "good enough" as the "small enough" sets are scaled by the same factor in the cross-section direction.)

To reduce our analysis of the expressions in Proposition 5.3 to analysis on domains Ω, we use our partition of unity $\{\psi_i\}$ in the following way. Let $\rho_i = (\psi_i^2 / \sum_j \psi_j^2)^{1/2}$. Note that $0 \le \rho_i \le 1$, that $\sum_i \rho_i^2 = 1$, and that for each k there is a bound independent of i on the pointwise norms of all partial derivatives of ρ_i having order $\le k$. Here the partial derivatives are calculated in the coordinates of any neighborhood Ω_j having nonempty intersection with the support of ρ_i. The ρ_i provide the following decomposition of the fiberwise inner product on vector-bundle sections over $\mathcal{M}$:

$$\langle \alpha, \beta \rangle_x = \sum_i \rho_i^2(x) \langle \alpha, \beta \rangle_x = \sum_i \langle \rho_i \alpha, \rho_i \beta \rangle_x.$$

Consider the first expression that Proposition 5.3 assumes is nonnegative. Noting that the second term on the left of the inequality is nonnegative, we focus on the remaining terms, which we rewrite

$$2\langle R(D_A)D^{2k-1}w, D^{2k-1}w\rangle + (1/3)\langle A^{2(k-1)}R(D_A)w, A^{2(k-1)}R(D_A)w\rangle$$
$$+ \langle [D^{2k-1}, R(D_A)]w, D^{2k-1}w\rangle + \langle D^{2k-1}w, [D^{2k-1}, R(D_A)]w\rangle.$$

Applying $\sum_i \rho_i^2$, we get

$$\sum_i (2\langle R(D_A)D^{2k-1}\rho_i w, D^{2k-1}\rho_i w\rangle + (1/3)\langle A^{2k-1}R(D_A)\rho_i w, A^{2k-1}R(D_A)\rho_i w\rangle$$
$$+ \langle [D^{2k-1}, R(D_A)]\rho_i w, D^{2k-1}\rho_i w\rangle + \langle D^{2k-1}\rho_i w, [D^{2k-1}, R(D_A)]\rho_i w\rangle)$$

plus

$$\sum_i (2\langle [\rho_i, R(D_A)D^{2k-1}]w, [\rho_i, D^{2k-1}]w\rangle$$
$$+ \langle [\rho_i, [D^{2k-1}, R(D_A)]]w, [\rho_i, D^{2k-1}]w\rangle + \langle [\rho_i, D^{2k-1}]w, [\rho_i, [D^{2k-1}, R(D_A)]]w\rangle)$$

plus

$$\sum_i (2\langle [\rho_i, R(D_A)D^{2k-1}]w, D^{2k-1}\rho_i w\rangle$$
$$+ \langle [\rho_i, [D^{2k-1}, R(D_A)]]w, D^{2k-1}\rho_i w\rangle + \langle [\rho_i, D^{2k-1}]w, [D^{2k-1}, R(D_A)]\rho_i w\rangle)$$

plus

$$\sum_i (2\langle R(D_A)D^{2k-1}\rho_i w, [\rho_i, D^{2k-1}]w\rangle$$
$$+ \langle [D^{2k-1}, R(D_A)]\rho_i w, [\rho_i, D^{2k-1}]w\rangle + \langle D^{2k-1}\rho_i w, [\rho_i, [D^{2k-1}, R(D_A)]]w\rangle).$$

In the last three sums, for each w that is not immediately preceded by a ρ_i, we replace w by $\sum_j \rho_j^2 w$. In every case we get an expression of the form $S\sum_j \rho_j^2 w$, where S is an operator. We replace each of these expressions by $\sum_j(\rho_j S\rho_j w + [S, \rho_j]\rho_j w)$. These steps affect the preceding sum in the following way: the first summation over i does not change; each term in the second summation over i changes to have the form $\sum_{j,\ell}\langle T\rho_j w, Q\rho_\ell w\rangle$; each term in the third summation over i takes the form $\sum_j\langle T\rho_j w, Q\rho_i w\rangle$; and each term in the final summation over i takes the form $\sum_\ell\langle T\rho_i w, Q\rho_\ell w\rangle$. (In these expressions T and Q represent operators.) Note that there is a bound, independent of i, on the number of neighborhoods that intersect the neighborhood associated with ρ_i, and so there is a bound, independent of i, on the number of terms indexed by j and/or ℓ that make a nonzero contribution to a term in which ρ_i appears (possibly within the operator). (Our notation for an individual term $\langle T\rho_j w, Q\rho_\ell w\rangle$ includes the cases where j and/or ℓ equals the value of i that was the original index for the term.)

To each of the inner products $\langle T\rho_j w, Q\rho_\ell w\rangle$ involving operators T and Q originally associated with neighborhood i, we associate a growth factor $a_i(T, Q)$ determined (up to a uniformly bounded factor) by the following conditions. In the neighborhood i that we are working in, the operator T (whether it is a vector-bundle map or a higher-order differential operator) is a constant multiple of an operator T' chosen as follows. Working in neighborhood i's coordinates, we choose T' so that the pointwise sup of the operator norm of each of its nonzero coefficient matrices is ≤ 1 and so that at least one of these sup's is at least $1/2$. We use the same conventions to relate Q and Q'. The growth factor $a_i(T,Q)$ is then determined (up to a uniformly bounded factor) by the equation

$$\langle T\rho_j w, Q\rho_\ell w\rangle = a_i(T,Q)\langle T'\rho_j w, Q'\rho_\ell w\rangle.$$

We proceed similarly in j and ℓ coordinates to define $a_j(T,Q)$ and $a_\ell(T,Q)$. (If any operator has all coefficients zero in a coordinate system, we simply ignore that term.) Finally we let $a_{i,j,\ell}(T,Q)$ be the largest of the three growth factors so defined.

Assume that we are far enough out the manifold's end that the terms of the form $\langle R(D_A)D^{2k-1}\rho_i w, D^{2k-1}\rho_i w\rangle$ and $\langle A^{2k-1}R(D_A)\rho_i w, A^{2k-1}R(D_A)\rho_i w\rangle$ are nonnegative. The former involves the operator D^{2k-1} that is uniformly elliptic of order $2k-1$ and a growth factor that grows as A^2. The latter involves the square of the L^2 norm of $\rho_i w$ and a growth factor that grows as A^{4k+2}. In every other inner product: every differential operator that appears is of order no greater than $2k-1$; if both differential operators in an inner product are of order $2k-1$, the growth factor is $o(A^2)$ in the strong sense; and regardless of the orders of the operators, there is a uniform constant α so that each a_i is no greater than the product of α with the infimum of the spectrum of A^2 on the ith neighborhood. The proofs of these assertions rely on standard results about commutators of the functions ρ_i with differential operators and on two other observations. One is that although $[D^{2k-1}, DA+AD]$ may have order $2k-1$, $DA+AD$ is $o(A^2)$ in the strong sense. The other is that for r large enough that A^2 represents multiplication by a scalar-valued function of r, $[D^{2k-1}, A^2]$ has order $2k-2$ with coefficients that are $o(A^2)$ in the strong sense.

We need to show that with one-third of each term of the sum of nonnegative terms

$$\sum_i (2\langle R(D_A)D^{2k-1}\rho_i w, D^{2k-1}\rho_i w\rangle + (1/3)\langle A^{2k-1}R(D_A)\rho_i w, A^{2k-1}R(D_A)\rho_i w\rangle)$$

set aside, what remains of these terms exceeds the sum of the absolute values of all the other terms once we are in neighborhoods far enough out the manifold's end. Consider first the terms of the form $\langle T\rho_j w, Q\rho_\ell w\rangle$, where each of T and Q is a differential operator of order $2k-1$. Up to a uniform constant possibly arising from ambiguity in the convention used to define the Sobolev norm, for a term $\langle T\rho_j w, Q\rho_\ell w\rangle$ originally associated with index i,

$$|\langle T\rho_j w, Q\rho_\ell w\rangle| \leq a_{i,j,\ell}(T,Q)\, \|\rho_j w\|_{0,2k-1}\, \|\rho_\ell w\|_{0,2k-1},$$

where the last two factors are calculated in j and ℓ coordinates respectively. Because of the way our neighborhoods intersect, there is a bound, independent of w, on the number of such nonzero terms in which any ρ_j or ρ_ℓ appears. Because the terms in which both differential operators are of order $2k-1$ have growth factors that are $o(A^2)$ in the strong sense, by Gårding's inequality the sum of one-third of our nonnegative terms exceeds the sum of the absolute values of terms of this type.

Finally consider terms of the form $\langle T\rho_j w, Q\rho_\ell w\rangle$, where T is of order $\leq 2k-2$ and Q is of order $\leq 2k-1$ (as well as terms where the orders of the operators are the reverse, but our notation will reflect only the first case explicitly). Proceeding as before, we get

$$|\langle T\rho_j w, Q\rho_\ell w\rangle| \le a_{i,j,\ell}(T,Q)\,\|\rho_j w\|_{0,2k-2}\,\|\rho_\ell w\|_{0,2k-1}.$$

Supressing reference to the growth factor, which is bounded by a constant multiple of A^2, we note that for each $\delta > 0$

$$\|\rho_j w\|_{0,2k-2}\,\|\rho_\ell w\|_{0,2k-1} \le \delta\|\rho_\ell w\|^2_{0,2k-1} + (1/4\delta)\|\rho_j w\|^2_{0,2k-2}.$$

By the relationships among Sobolev norms, for any $\epsilon > 0$ there is a C_ϵ for which

$$\delta\|\rho_\ell w\|^2_{0,2k-1} + (1/4\delta)\|\rho_j w\|^2_{0,2k-2} \\ \le \delta\|\rho_\ell w\|^2_{0,2k-1} + (1/4\delta)(\epsilon\|\rho_j w\|^2_{0,2k-1} + C_\epsilon\|\rho_j w\|^2).$$

Using the bound on the number of intersecting neighborhoods and the observation that the terms we have just been analyzing have growth factors that are bounded by a constant multiple of A^2, we see that we can choose δ small enough and, taking $1/4\delta$ into account, ϵ small enough that by Gårding's inequality,

$$(1/3)\sum_i 2\langle R(D_A)D^{2k-1}\rho_i w, D^{2k-1}\rho_i w\rangle$$

plus some constant multiple of $\sum_i\langle A^2\rho_i w, \rho_i w\rangle$ dominates the terms giving rise to the product of their growth factors with

$$\delta\|\rho_\ell w\|^2_{0,2k-1} + (1/4\delta)(\epsilon\|\rho_j w\|^2_{0,2k-1} + C_\epsilon\|\rho_j w\|^2)$$

eventually in r. The needed constant multiple of $\sum_i\langle A^2\rho_i w, \rho_i w\rangle$ is provided, eventually in r, by the more rapidly growing

$$(1/3)\sum_i (1/3)\langle A^{2k-1}R(D_A)\rho_i w, A^{2k-1}R(D_A)\rho_i w\rangle.$$

The proof that the other nonnegativity hypothesis of Proposition 5.3 is satisfied is analogous. This time the third nonnegative term on the left side of the inequality plays no role. The first term provides $\langle D\rho_i w, D\rho_i w\rangle$ with a growth factor of the order of A^{4k-2}, the second term has growth of the order of A^{4k}, and the operator in the final term has all coefficients bounded by a multiple of A^{4k-2} and has its second-order coefficients strongly $o(A^{4k-2})$.

6. Normal neighborhoods

To provide a setting for estimates in Section 8 we use the bounded geometry of M^* and of $E \otimes S$ to make some uniform choices of normal neighborhoods of points in M^*. (Recall that M^* is the double of the manifold M with boundary that is the setting for the Atiyah–Patodi–Singer index theorem of Section 9.) A reference for background material is Section 2 of [19]. Our discussion is largely taken from Section 5 of [9], where we also provide details of the extension of our operators and other structures from these normal neighborhoods to Euclidean space.

We start our construction by choosing for each $x \in M^*$ a normal neighborhood U_x. This is done in such a way that there is a radius R_0 so that for each x the open subset of $\mathbb{R}^m$ associated to U_x by the normal coordinate chart contains $\overline{B}(\vec{0}, R_0)$ and has x associated with $\vec{0}$. (Here $\overline{B}(\vec{0}, R_0)$ is the closure of the open ball $B(\vec{0}, R_0)$ with center the origin and radius R_0.) We also assume that $R_0 \le \delta_b/4$, where δ_b is the uniform width of N's collar neighborhood in M. The normal coordinate chart at x arises from the exponential mapping at x. We assume all of this has been done so that the distortion of the normal coordinate system satisfies a bound that is independent of x and so that the Christoffel symbols of M^* and of $E \otimes S$, as x ranges over M^*, form a bounded subset of the Fréchet space $C^\infty(B(\vec{0}, R_0))$.

We make a single choice of a smooth function $\phi\colon B(\vec{0}, R_0) \to \mathbb{R}$ satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B(\vec{0}, R_0/4)$, and support$(\phi) \subset B(\vec{0}, R_0/2)$.

Notation 6.1. For each x we give the name ϕ_x to the associated function on U_x, as well as to its extension by 0 to all of M^*. For each section w of $E \otimes S$ and each $x \in M^*$, we use the notation w_x for $\phi_x \cdot w$.

Remark 6.2. As discussed in [9] we can use normal coordinates for each U_x and a uniformly chosen trivialization of $E \otimes S$ over each U_x to make sense of extension by zero of w_x to $\mathbb{R}^m$ and thus to make sense of the (componentwise) Fourier transform of w_x. We will denote this Fourier transform by $\widehat{w}_x$.

Notation 6.3. For $x \in M^*$

$$a_x = \inf_{y \in U_x} |A(y)|.$$

Some of the calculations later in the paper apply to sets U_x satisfying the following assumption.

Assumption 6.4. On U_x, $\mathcal{A}$ is the product of a scalar-valued function and a unitary-valued vector-bundle map that intertwines the connections on E_0 and E_1. The number a_x is positive.

Suppose that we are in a normal neighborhood U_x where Assumption 6.4 is satisfied. In Section 5 of [9], we describe how to use the normal coordinates to transfer D_{A,M^*} and Δ_{A,M^*} from U_x to a subset of Euclidean space and then to use a partition of unity to extend these transferred operators to operators $D_{A,x}$ and $\Delta_{A,x}$ on the entire Euclidean space. The operator $\Delta_{A,x}$ takes the form $D_x^2 + a_x^2 + \nu_x$, where D_x is a Dirac operator and ν_x a smooth compactly supported vector-bundle map. (Conjugation by a unitary operator made from A is suppressed in the current discussion but treated explicitly in [9].) Later, to identify the local expressions that arise from using the heat equation to prove an index formula, we will use the following assumption.

Assumption 6.5. For each n and x let $B_{n,x}$ denote the supremum of the pointwise norms of entries of ν_x and of all of these entries' partial derivatives of order no greater than $4n$. Then for each n, as $x \to \infty$ in M^*, $B_{n,x}/a_x^2 \to 0$.

Remark 6.6. Our discussions of manifolds with cylindrical and linearly expanding ends provide examples that satisfy Assumptions 6.4 and 6.5.

7. Operators with boundary conditions

The operator whose index theory we study (under assumptions that make this index theory accessible) is the operator $\widetilde{D}_A$ on the manifold M with boundary. At this point we must specify the domain of $\widetilde{D}_A$ more carefully. The operator $\widetilde{D}_A$ that we use is the closure of the operator $\widetilde{D}_A$ defined on a domain consisting of sections satisfying two conditions. First these sections w must be smooth and in L^2 with

$\widetilde{D}_A w \in L^2$ as well. Second these sections must satisfy the Atiyah–Patodi–Singer boundary condition, which is defined as follows. Let D_A be the operator defined on N and described in Section 3. (Because [6] tells us that, defined on smooth compactly supported sections, D_A is essentially self-adjoint, there is no ambiguity about the self-adjoint extension to which we refer.) We assume that we are in a setting in which the operator D_A on N has empty essential spectrum. Let P be the projection onto the subspace of $L^2(E_0 \otimes S \oplus E_1 \otimes S)$ spanned by the eigenvectors of D_A with nonnegative eigenvalues. The Atiyah–Patodi–Singer boundary condition states that w is in the domain of $\widetilde{D}_A$ if w's restriction to N is in the kernel of P.

Having taken this care in defining the operator $\widetilde{D}_A$ on M, we wish to take corresponding care in the description of its adjoint $(\widetilde{D}_A)^*$, used in the construction of

$$D_A = \begin{pmatrix} 0 & (\widetilde{D}_A)^* \\ \widetilde{D}_A & 0 \end{pmatrix}.$$

The following proposition will identify the adjoint $(\widetilde{D}_A)^*$ and hence the operators D_A and Δ_A on M. First we introduce some notation.

Notation 7.1. Recall from Remark 3.6 the construction of the double M^* of M and of the extension of the operator $\widetilde{D}_A$ (without boundary conditions) to M^*. Because M^* is complete without boundary, the passage from $\widetilde{D}_A$ on M^* to D_A, and hence to Δ_A, on M^* is without ambiguity. When our discussion makes clear the underlying manifold, we will omit it from the notation, but when there is a chance of confusion, we will include the underlying manifold in the notation, as in $\widetilde{D}_{A,M}$, $D_{A,N}$, or Δ_{A,M^*}.

Proposition 7.2. *Let $\widetilde{T}$ denote the differential operator on M whose expression in local coordinates (including at boundary points) agrees with that of $(\widetilde{D}_{A,M^*})^*$. Apply $\widetilde{T}$ first to the set of sections w that are smooth and L^2 on M, whose images under $\widetilde{T}$ are L^2, and that satisfy the boundary condition $(1-P)(w|_N) = 0$ adjoint to the Atiyah–Patodi–Singer boundary condition. Let T be the closure, as a map from L^2 sections on M to L^2 sections on M, of this operator. Then $T = (\widetilde{D}_{A,M})^*$.*

Proof. Suppose that w has support in $N \times [0, 5\delta_b/6]$, "within the boundary's collar neighborhood." Because any inner product involving w or $T(w)$ may be calculated on $N \times [0, \delta_b)$, it is a consequence of the reasoning used to prove Proposition 2.12 of Part I of [1] that w is in the domain of T if and only if w is in the domain of $(\widetilde{D}_{A,M})^*$. Proposition 2.12 of Part I of [1] relies on parts of Proposition 2.5 of the same paper. The calculations that use separation of variables on the cylinder extend to our setting, although in doing them we must replace the smooth compactly supported sections of [1] by smooth sections with supports that are compact in the direction orthogonal to N. Note that the reasoning outlined here shows that for w supported within the boundary's collar neighborhood and in the domain of T, $T(w) = (\widetilde{D}_{A,M})^*(w)$.

Let h_1 be a smooth nonnegative function of u satisfying $0 \le h_1 \le 1$, $h_1(u) = 1$ for $u \in [0, \delta_b/3]$, and support$(h_1) \subset [0, \delta_b/2]$. We use h_1 to define a function, also called h_1, on $N \times [0, \delta_b]$, and, still using the name h_1, we extend this function by 0 to the rest of M and by 1 to what remains of M^*. We proceed similarly with a function h_2 that shares the properties of h_1 except that $h_2(u) = 1$ for $u \in [0, 2\delta_b/3]$ and support$(h_2) \subset [0, 5\delta_b/6]$.

Suppose that w is in the original domain of $\widetilde{T}$. Because $h_2 w$ is also in this domain, and $h_2 w$ has support in the boundary's collar neighborhood of width $5\delta_b/6$, $h_2 w \in \text{domain}((\widetilde{D}_{A,M})^*)$ with $(\widetilde{D}_{A,M})^*(h_2 w) = T(h_2 w)$. To show that $w \in \text{domain}((\widetilde{D}_{A,M})^*)$, we show that $(1-h_2)w \in \text{domain}((\widetilde{D}_{A,M})^*)$. (In the process we show that $(\widetilde{D}_{A,M})^*((1-h_2)w) = T((1-h_2)w)$.) If v is an arbitrary smooth element of domain$((\widetilde{D}_{A,M}))$,

$$\begin{aligned}\langle \widetilde{D}_{A,M} v, (1-h_2)w\rangle = \langle \widetilde{D}_{A,M}((1-h_1)v), (1-h_2)w\rangle &= \langle (1-h_1)v, T((1-h_2)w)\rangle \\ &= \langle v, T((1-h_2)w)\rangle.\end{aligned}$$

Here the first and last equalities arise from the supports of the sections that appear in the expressions. The middle equality follows from the definition of T and the observation that $(1-h_1)v$ and $(1-h_2)w$ extend smoothly by zero to M^*. Because $(\widetilde{D}_{A,M})^*$ is a closed operator, showing that any w in the original domain of $\widetilde{T}$ is in domain$((\widetilde{D}_{A,M})^*)$, with T and $(\widetilde{D}_{A,M})^*$ agreeing on w, shows that any $w \in$ domain(T) is in domain$((\widetilde{D}_{A,M})^*)$, with the operators agreeing on such w.

Suppose $w \in \text{domain}((\widetilde{D}_{A,M})^*)$. To show that $h_2 w$ is in domain$((\widetilde{D}_{A,M})^*)$, note that h_2 and $\partial h_2/\partial u$ are bounded real-valued functions and that multiplication by h_2 takes domain$(\widetilde{D}_{A,M})$ to itself, and calculate that, for $v \in \text{domain}(\widetilde{D}_{A,M})$,

$$\begin{aligned}\langle h_2 w, \widetilde{D}_{A,M} v\rangle = \langle w, h_2 \widetilde{D}_{A,M} v\rangle &= \langle w, (-\partial h_2/\partial u)v + \widetilde{D}_{A,M}(h_2 v)\rangle \\ &= \langle (-\partial h_2/\partial u + h_2(\widetilde{D}_{A,M})^*)w, v\rangle.\end{aligned}$$

By showing that $h_2 w \in \text{domain}((\widetilde{D}_{A,M})^*)$, we have shown also that $(1-h_2)w \in$ domain$((\widetilde{D}_{A,M})^*)$ and, by the first paragraph of this proof, that $h_2 w \in \text{domain}(T)$.

Let v be an arbitrary smooth element of domain$(\widetilde{D}_{A,M^*})$. The section $(1-h_1)v$ is in both domain$(\widetilde{D}_{A,M^*})$ and domain$(\widetilde{D}_{A,M})$. By support considerations

$$\begin{aligned}\langle \widetilde{D}_{A,M^*} v, (1-h_2)w\rangle &= \langle \widetilde{D}_{A,M^*}((1-h_1)v), (1-h_2)w\rangle \\ &= \langle \widetilde{D}_{A,M}((1-h_1)v), (1-h_2)w\rangle \\ &= \langle (1-h_1)v, (\widetilde{D}_{A,M})^*((1-h_2)w)\rangle \\ &= \langle v, (\widetilde{D}_{A,M})^*((1-h_2)w)\rangle.\end{aligned}$$

It follows that $(1-h_2)w \in \text{domain}((\widetilde{D}_{A,M^*})^*)$. By [6] there is a sequence of smooth sections w_j converging to $(1-h_2)w$ in the norm

$$(\langle w_j, w_j\rangle + \langle (\widetilde{D}_{A,M^*})^* w_j, (\widetilde{D}_{A,M^*})^* w_j\rangle)^{1/2},$$

where the inner products are on M^*. The sequence of sections $(1-h_1)w_j$ also converges to $(1-h_2)w$ in this norm, and the sections in this sequence satisfy the conditions defining domain(T). Hence for $w \in \text{domain}((\widetilde{D}_{A,M})^*)$, $(1-h_2)w \in$ domain(T). Combining $(1-h_2)w \in \text{domain}(T)$ with $h_2 w \in \text{domain}(T)$, we see that $w \in \text{domain}(T)$. □

Estimates in the next section will depend on the following proposition.

Proposition 7.3. *Assume that $D_{A,N}$ has empty essential spectrum. Then for each positive integer k there is a positive constant Γ_k that makes the following true: for every $w \in \mathrm{domain}(D^k_{A,M})$ having support contained in some proper subset $N\times[0,\delta']$ of the boundary's open collar neighborhood $N \times [0, \delta_b)$, there is a section $w^* \in \mathrm{domain}(D^k_{A,M^*})$ having support in $N \times [-\delta', \delta']$ and satisfying the inequality*

$$\sum_{j=0}^{k}\langle D^j_{A,M^*}w^*, D^j_{A,M^*}w^*\rangle \le \Gamma_k \sum_{j=0}^{k}\langle D^j_{A,M}w, D^j_{A,M}w\rangle.$$

Proof. Use an orthonormal L^2-basis of eigenvectors $\{\Phi_\lambda\}$ for $D_{A,N}$ (with λ representing the associated eigenvalues, and with the notational ambiguity caused by multiplicity introducing no confusion) to write $w = \sum_\lambda f_\lambda\Phi_\lambda$, where each f_λ is a function defined on $[0,\infty)$ with support contained in $[0, \delta']$. Assume for the moment that each f_λ is in the Sobolev space $H^k([0, \delta_b))$. As discussed, e.g., in [7], there are numbers c_t for $t = 1, \dots , k{+}1$ that satisfy the system of equations $\sum_{t=1}^{k+1}(-t)^m c_t = 1$ indexed by $m = 0, \dots , k$. With these numbers we can extend each f_λ to $u < 0$ by defining $f^*_\lambda(u) = \sum_{t=1}^{k+1} c_t f_\lambda(-tu)$ for $u \le 0$. We denote the extended function by f^*_λ. One can calculate directly from the construction that f^*_λ is well-defined at 0, as are all derivatives of f^*_λ through order k. One can also calculate that there is a constant α_0 for which the L^2 norms satisfy $\|f^*_\lambda\| \le \alpha_0\|f_\lambda\|$, and for each $\ell \le k$ there is a constant α_ℓ for which

$$\left\|\frac{d^\ell}{du^\ell}f^*_\lambda\right\| \le \alpha_\ell\left\|\frac{d^\ell}{du^\ell}f_\lambda\right\|.$$

In these inequalities the derivatives and L^2 norms are calculated in $(-\delta_b, \delta_b)$ for f^*_λ and in $[0, \delta_b)$ for f_λ. All of the constants are independent of f_λ.

For $w = \sum f_\lambda\Phi_\lambda$, define $w^* = \sum f^*_\lambda\Phi_\lambda$. To prove the inequality in the proposition, it suffices to prove that for each $j \le k$ there are constants β_j, γ_j, and ϵ_j (independent of λ, f_λ, and Φ_λ) for which

$$\|D^j_{A,M^*}f^*_\lambda\Phi_\lambda\| \le \beta_j\sum_{\ell=0}^{j}|\lambda|^{j-\ell}\left\|\frac{d^\ell}{du^\ell}f^*_\lambda\right\| \le \gamma_j\sum_{\ell=0}^{j}|\lambda|^{j-\ell}\left\|\frac{d^\ell}{du^\ell}f_\lambda\right\| \le \epsilon_j\|D^j_{A,M}f_\lambda\Phi_\lambda\|.$$

The first inequality in the expression above follows from the observation that $D_{A,M^*}(f^*_\lambda\Phi_\lambda) = ((df^*_\lambda/du) + \lambda)\Phi_\lambda$ and from the triangle inequality. The second inequality follows from the relationship between f_λ and f^*_λ. The third inequality follows from the observation that $D_{A,M}(f_\lambda\Phi_\lambda) = ((df_\lambda/du) + \lambda)\Phi_\lambda$ and from the following lemma. Note that this lemma also establishes that for $w = \sum_\lambda f_\lambda\Phi_\lambda \in \mathrm{domain}(D^k_{A,M})$, satisfying the support condition we are assuming, each $f_\lambda \in H^k([0, \delta_b))$. □

Lemma 7.4. *For each natural number k and each $j \in \{0, \dots , k\}$, there exists a constant $\zeta_{j,k}$ for which $D^k_{A,M}(f_\lambda\Phi_\lambda) = g_\lambda\Phi_\lambda$ implies that*

$$|\lambda|^{k-j}\left\|\frac{d^j f_\lambda}{du^j}\right\| \le \zeta_{j,k}\|g_\lambda\| \quad \textit{and} \quad \left\|\frac{d^k f_\lambda}{du^k}\right\| \le \zeta_{k,k}\|g_\lambda\|.$$

Here $\|\ \|$ denotes the norm on $L^2([0,\infty))$.

Proof. The case $\lambda = 0$ follows directly from the observation that $D_{A,M}(f_0\Phi_0) = df_0/du$. For other values of λ the proof is by induction on k. For $k = 1$ the arguments in the proof of Proposition 2.5 of Part I of [1] extend to show that $|\lambda|\|f_\lambda\| \leq 2\|g_\lambda\|$ and $\|df_\lambda/du\| \leq 3\|g_\lambda\|$. Suppose that the result is true for $k = n$. To prove it for $k = n+1$, note that for $D_{A,M}^{n+1}(f_\lambda\Phi_\lambda) = g_\lambda\Phi_\lambda$, we can set $D_{A,M}(f_\lambda\Phi_\lambda) = F_\lambda\Phi_\lambda$ and use the inductive hypothesis in the form $|\lambda|\|f_\lambda\| \leq 2\|F_\lambda\|$ and $|\lambda|^n\|F_\lambda\| \leq \zeta_{0,n}\|g_\lambda\|$ to conclude that $|\lambda|^{n+1}\|f_\lambda\| \leq 2\zeta_{0,n}\|g_\lambda\|$. It then follows from $(df_\lambda/du)+\lambda f_\lambda = F_\lambda$ that $\|df_\lambda/du\| \leq |\lambda|\|f_\lambda\|+\|F_\lambda\|$ and hence by the preceding conclusion that $|\lambda|^n\|df_\lambda/du\| \leq |\lambda|^{n+1}\|f_\lambda\|+|\lambda|^n\|F_\lambda\| \leq (2\zeta_{0,n}+\zeta_{0,n})\|g_\lambda\|$. The proof concludes by induction on the order of f_λ's derivative. We illustrate this with the next step: because $(d^2f_\lambda/du^2) + \lambda(df_\lambda/du) = dF_\lambda/du$ (and previous reasoning puts the second and third terms in L^2), $\|d^2f_\lambda/du^2\| \leq |\lambda|\|df_\lambda/du\| + \|dF_\lambda/du\|$ and so $|\lambda|^{n-1}\|d^2f_\lambda/du^2\| \leq |\lambda|^n\|df_\lambda/du\| + |\lambda|^{n-1}\|dF_\lambda/du\| \leq (3\zeta_{0,n} + \zeta_{1,n})\|g_\lambda\|$. □

8. Heat kernels

At the heart of a proof of an Atiyah–Patodi–Singer index theorem for a perturbed Dirac operator $\widetilde{D}_{A,M}$ is analysis of the associated heat operator $\exp(-t\Delta_{A,M})$. By identifying conditions under which this heat operator is trace class, we provide a way to prove that $\widetilde{D}_{A,M}$ is Fredholm. These conditions let us represent the heat operator's supertrace, for $t > 0$, by the integral along the diagonal of the pointwise supertrace of the heat kernel. One key to this representation is the decay of the pointwise trace as $(x,x) \to \infty$ on the diagonal of $M \times M$. To extract from this representation an index formula, we need, for the operator's supertrace, an asymptotic expansion in which the dominant terms are determined by local data at points away from the boundary of M and by data from the boundary's collar neighborhood at points in some collar neighborhood of the boundary. A standard tool for getting an asymptotic expansion with such properties is the use of the Fourier transform and finite propagation speed for solutions of the wave equation to express the heat operator as the sum of two terms, one of which is determined by local data. Bounded geometry permits one to make the associated estimates uniformly in $(x,y) \in M \times M$ [17]. To extract from these estimates asymptotics that persist after integration over the base manifold, we need to show that the constants in these estimates decay sufficiently rapidly as x and/or y go to infinity on M. Both kinds of decay required by this discussion will arise from the observation that the heat operator is continuous between the domains of any two powers of Δ_A, from the characterization of domains of powers of Δ_A given by Assumption 5.1, and from quantitative versions of Assumption 4.1. This section provides the details behind this assertion by extending the discussion given in Section 7 of [9].

Throughout this section we use uniform choices of normal neighborhoods U_x of points $x \in M^*$ and we use the notation introduced in Section 6, in particular in Notation 6.1, Remark 6.2, and Notation 6.3. We will also need the following notation.

Notation 8.1. Let $\widetilde{a}_x$ denote the larger of 1 and a_x.

Throughout this section we assume that Assumption 5.1 holds on M^*.

Notation 8.2. We use the terminology Dirac delta distribution and notation δ_x for an element dual to continuous sections of $E \otimes S$ and defined on a section by the application of a linear functional of norm one to the section's value at x. For our arguments we do not need to specify the functional and hence the distribution more precisely. We use this terminology and notation on both M and M^*.

Notation 8.3. For positive integers n let W^{2n} denote the completion of the set of smooth sections of $E \otimes S$ that are in the domains of $\Delta_A, \ldots, \Delta_A^n$ in the norm defined by

$$\|w\|_{2n}^2 = \langle w, w\rangle + \sum_{k=1}^{n} \langle (\Delta_A)^k w, (\Delta_A)^k w\rangle.$$

Here the inner products are the L^2 inner products. The notation $\| \quad \|$ without subscripts refers to the L^2 norm. Let W^{-2n} be the dual of W^{2n}, with norm $\| \quad \|_{-2n}$ the standard one associated to linear functionals. We use the notation introduced here either with sections over M, in which case the operator involved is $\Delta_{A,M}$ and the inner products are L^2 inner products over M, or with sections over M^*, in which case the operator involved is Δ_{A,M^*} and the inner products are L^2 inner products over M^*. We let context or explicit comment identify the sense in which to interpret each use of this notation.

Remark 8.4. We use the expression *uniform constant* to refer to a constant that may depend on our given data (M, E, S, D, and A), on general choices (such as U_x and ϕ_x chosen as in Section 6), and on algebraic identities, but that is independent of other data (such as a vector-bundle section or point in M) appearing in the expression in which the constant appears.

Lemma 8.5. *Assume that Assumption 5.1 holds on M^*. Choose an arbitary natural number n and $x \in M^*$. If $w \in W^{2n}$, then $w_x \in W^{2n}$. Moreover there is a uniform constant K such that*

$$\|(D^{2n} + \widetilde{a}_x^{2n}) w_x\| \le K \|w\|_{2n}.$$

Here the sections, norms, and domains are over M^*.

Proof. The assertion that $w_x \in W^{2n}$ is purely local. By Assumption 5.1 it depends only on the degree of differentiability of w and of w_x. This is unaffected by multiplication of w by the smooth ϕ_x.

To prove the inequality we first apply Assumption 5.1 to the terms in the definition of $\|w\|_{2n}$. Then we note that, by the product rule, $\|D^{2n} w_x\|^2$ is bounded by a sum of terms, each of which is bounded by a product of two sup norms of partial derivatives (of degree $\le 2n$) of ϕ_x times a product of factors of total degree 2 in expressions of the form $\|D^{2k} w\|$ for nonnegative integers $k \le n$. This last assertion follows from repeated application of the Cauchy–Schwarz inequality and self-adjointness. For example in the hardest case

$$|\langle D^{2r+1} w, D^{2s} w\rangle| \le \|D^{2r+1} w\| \cdot \|D^{2s} w\|$$

with

$$\langle D^{2r+1} w, D^{2r+1} w\rangle^{1/2} = \langle D^{2r+2} w, D^{2r} w\rangle^{1/2} \le \|D^{2r+2} w\|^{1/2} \|D^{2r} w\|^{1/2}. \qquad \square$$

Proposition 8.6. *Assume that Δ_{A,M^*} satisfies Assumption 5.1. If $2n > m/2$ and, over M^*, $w \in W^{2n}$, then w is a continuous section of $E \otimes S$. For each n such that*

$2n > m/2$ there is a uniform constant C such that for every $x \in M^*$,

$$\sup_{y \in U_x} |w_x(y)| \le C \cdot \widetilde{a}_x^{(m/2)-2n} \cdot \|w\|_{2n}.$$

The same conclusions, but involving norms over M, apply to sections w over M.

Proof. The continuity statement follows from $W^{2n} \subset \mathrm{domain}(D^{2n})$, the ellipticity of D, and the standard Sobolev imbedding theorem. The bound on the pointwise norm of w_x follows from a bound on the L^1 norm of $\widehat{w}_x$. By the Hölder inequality

$$\left(\int_{\mathbb{R}^m} |\widehat{w}_x(\xi)|\, d\xi\right)^2 \le \left(\int_{\mathbb{R}^m} |\widehat{w}_x(\xi)|^2 (\widetilde{a}_x^{2n} + |\xi|^{2n})^2\, d\xi\right)\left(\int_{\mathbb{R}^m} (\widetilde{a}_x^{2n} + |\xi|^{2n})^{-2}\, d\xi\right).$$

The left side of the inequality is the square of the L^1 norm of $\widehat{w}_x$. By Lemma 8.5 the first integral on the right side is bounded by a uniform constant multiple of $\|w\|_{2n}^2$. Switching to spherical coordinates, we can replace the second integral on the right by a constant (volume of a unit sphere) multiple of $\int_0^\infty (\widetilde{a}_x^{2n} + \rho^{2n})^{-2} \rho^{m-1}\, d\rho$. Factoring out $\widetilde{a}_x^{-4n}$ and changing variables by $v = \rho/\widetilde{a}_x$, we get for this integral $\widetilde{a}_x^{m-4n} \int_0^\infty (1 + v^{2n})^{-2} v^{m-1}\, dv$, which is finite for $2n > m/2$. The transition from M^* to M follows from Proposition 7.3, which permits the extension of sections from M to M^* and provides an accompanying norm inequality. □

Corollary 8.7. *Let δ_x be a Dirac delta distribution at $x \in M^*$. For $2n > m/2$, $\delta_x \in W^{-2n}$. If Assumption 5.1 holds, then for n such that $2n > m/2$, there is a uniform constant C' such that*

$$\|\delta_x\|_{-2n} \le C' \cdot \widetilde{a}_x^{(m/2)-2n}.$$

The same statements are true on M.

Proof. Because all elements of W^{2n} are continuous, $\delta_x \in W^{-2n}$. The norm estimate on δ_x follows from the bound on the sup norm given by Proposition 8.6. □

Lemma 8.8. *Let f be a Schwartz function on $\mathbb{R}$ and choose an integer n for which $2n > m/2$. Then $f(D_A)$ is a continuous map $W^{-2n} \to W^{2n}$, and the norm of this linear map is bounded above by $\sup\{(1 + |x|^{4n})|f(x)| : x \in \mathbb{R}\}$ times a constant that depends only on n. These statements are true for either D_{A,M^*} on M^* or $D_{A,M}$ on M.*

Proof. This is a consequence of the definitions of W^{2n} and W^{-2n} and of algebraic identities. □

Notation 8.9. For f as in the preceding lemma, let $\widehat{f}$ denote its Fourier transform. The L^1_{4n} norm of $\widehat{f}$ is $\sum_{i=0}^{4n} \int_{\mathbb{R}} |(d/d\xi)^i \widehat{f}(\xi)|\, d\xi$.

Lemma 8.10. *In the setting of the preceding lemma, the norm of $f(D_A)\colon W^{-2n} \to W^{2n}$ is bounded by a uniform constant times the L^1_{4n} norm of $\widehat{f}$. Again the assertion does not depend on whether the spaces of sections are defined over M^* or M.*

Proof. This proof follows from the preceding lemma, the Fourier transform's treatment of multiplication and differentiation, and algebraic identities. □

Lemma 8.11. *Choose a positive integer n. Let g be a Schwartz function on $\mathbb{R}$, and let θ be a smooth compactly supported function on $\mathbb{R}$ that is identically one in some neighborhood of 0. For real τ, let $G(\lambda) = g(\lambda)(1 - \theta(\tau\lambda))$. Then for τ in some neighborhood of 0, for each N there is a constant K_N such that the L^1_{4n} norm of $G(\lambda)$ is bounded above by $K_N \tau^N$.*

Proof. This lemma follows from the product rule and the rapid decay of g and its derivatives. □

Lemma 8.12. *Let f be a Schwartz function on $\mathbb{R}$. Then the kernel $k[f]$ of $f(D_A)$ is smooth. Again the assertion is true for both $f(D_{A,M^*})$ and $f(D_{A,M})$.*

Proof. By Lemma 8.5, the ellipticity of D, and the standard Sobolev imbedding theorem, a section over M^* that is in the domain of D^k_{A,M^*} for all k is smooth. By Proposition 7.3 the same is true over M. By Lemma 8.8 $f(D_A)$ is a continuous map from the space of L^2 sections to the domain of any power of D_A. Hence as in, e.g., Proposition 5.8 of [18], $k[f]$ is smooth. □

Lemma 8.13. *Let f and $k[f]$ be as in Lemma 8.12. Assume that Assumption 5.1 holds. Suppose that $2n > m/2$ and that the L^1_{4n} norm of $\widehat{f}$ is bounded above by a constant B. Then there is a uniform constant c such that*

$$|k[f](x, y)| \leq c \cdot B \cdot \widetilde{a}_x^{(m/2)-2n} \cdot \widetilde{a}_y^{(m/2)-2n}.$$

Again the assertion is true for both $f(D_{A,M^*})$ and $f(D_{A,M})$.

Proof. The linear map $k[f](x, y)$ is defined by a sum of terms of the form $f(D_A)(\delta_y)$ with the linear functionals defining δ_y ranging over a basis of the dual space of $(E \otimes S)_y$. By Corollary 8.7, $\|\delta_y\|_{-2n}$ is bounded by a uniform constant times $\widetilde{a}_y^{(m/2)-2n}$. By Lemma 8.10, $\|f(D_A)(\delta_y)\|_{2n} \leq B \cdot \widetilde{a}_y^{(m/2)-2n}$ times a uniform constant. By Proposition 8.6, $|f(D_A)(\delta_y)(x)| \leq \widetilde{a}_x^{(m/2)-2n} \cdot B \cdot \widetilde{a}_y^{(m/2)-2n}$ times a uniform constant. □

Proposition 8.14. *Assume that Assumption 5.1 holds. Suppose that on M there is a nonnegative smooth function $F(x)$ whose integral over M is finite and for which $\widetilde{a}_x^{m-4n} \leq F(x)$. Then for a Schwartz function f, $f(D_A)$ is a Hilbert–Schmidt operator.*

Proof. By the characterization of $k[f]$ given in Lemma 8.13, our assumption makes the kernel square integrable on $M \times M$. □

Theorem 8.15. *Under the assumptions of Proposition 8.14, for $t > 0$, the operator $\exp(-t\Delta_{A,M})$ is trace class with a smooth kernel, the integral of whose pointwise trace on the diagonal gives the operator's trace. Also the integral of the kernel's pointwise supertrace along the diagonal gives the operator's supertrace.*

Proof. By Lemma 8.12 the operator $\exp(-(t/2)\Delta_{A,M})$ is represented by a smooth kernel. By Proposition 8.14 $\exp(-(t/2)\Delta_{A,M})$ is Hilbert–Schmidt. As the composition of a pair of Hilbert–Schmidt operators, $\exp(-t\Delta_{A,M})$ is trace class with trace given by integrating the pointwise trace of its kernel along the diagonal. (See, e.g., the proof of Theorem 6.10 of [18].) The assertion about the supertrace merely involves introducing the appropriate signs. □

Corollary 8.16. *Under the assumptions of Proposition 8.14, the operator $\widetilde{D}_{A,M}$ is Fredholm. Its Fredholm index equals the integral along the diagonal of the pointwise supertrace of the operator $\exp(-t\Delta_{A,M})$'s kernel for any $t > 0$.*

Proof. This result follows directly from the preceding theorem. □

Example 8.17. On a cylindrical end let the perturbation be formed from a power r^p of the radial variable times a unitary-valued vector-bundle map that is independent of r. Then the integrability condition in Proposition 8.14 is satisfied for choices of n and p positive and large enough to ensure $p(m-4n) < -1$. On a linearly expanding end the considerations are the same except that the cross-section contributes a factor of r^{m-1} to the volume. Now the integrability condition requires of positive n and p that $p(m-4n)+m-1 < -1$, and thus that $p(m-4n) < -m$.

Lemma 8.18. *Suppose that $\theta \in C_c^\infty(\mathbb{R})$ with $\theta \equiv 1$ in some neighborhood of 0. Then*

$$\exp(-t\Delta_A) = (4\pi t)^{-1/2}\int_{\mathbb{R}} \exp(-\xi^2/4t)\exp(i\xi D_A)\theta(\xi)\,d\xi + (4\pi t)^{-1/2}\int_{\mathbb{R}} \exp(-\xi^2/4t)\exp(i\xi D_A)(1-\theta(\xi))\,d\xi$$

for either Δ_{A,M^*} or $\Delta_{A,M}$.

Proof. $\Delta_A = D_A^2$. The integrals arise from the Fourier transform of a Gaussian. □

Lemma 8.19. *Choose a positive integer n. There is a deleted neighborhood of $t = 0$ in $\{t \in \mathbb{R} \colon t \geq 0\}$ such that for t in this neighborhood the following estimates hold: for any N there is a constant C_N for which the L^1_{4n} norm of*

$$\xi \mapsto \exp(-\xi^2/4t)(1-\theta(\xi))$$

is bounded above by $C_N t^N$.

Proof. In the integrals defining the L^1_{4n} norm, use the change of variable $\lambda = \xi/(2t^{1/2})$. Apply Lemma 8.11. □

Lemma 8.20. *Suppose that Assumption 5.1 holds for Δ_A and that n is such that $2n > m/2$. Choose $\theta \in C_c^\infty(\mathbb{R})$ with $\theta \equiv 1$ in some neighborhood of 0. Let $k_t^{1-\theta}$ be the kernel of the operator defined by*

$$(4\pi t)^{-1/2}\int_{\mathbb{R}} \exp(-\xi^2/4t)\exp(i\xi D_A)(1-\theta(\xi))\,d\xi.$$

Then there is a deleted neighborhood of $t = 0$ in $\{t \in \mathbb{R} \colon t \geq 0\}$ such that for t in this neighborhood the following estimates hold: for any N there is a constant C_N for which $|k_t^{1-\theta}(x,y)| \leq C_N \cdot t^N \cdot \widetilde{a}_x^{(m/2)-2n} \cdot \widetilde{a}_y^{(m/2)-2n}$ times a uniform constant. Again the assertion is true for both D_{A,M^*} and $D_{A,M}$.

Proof. Apply Lemma 8.19 to replace the B in Lemma 8.13. □

Remark 8.21. As will be discussed in detail in the next section, this lemma provides the foundation for identifying conditions under which the significant parts of an asymptotic expansion for the heat kernel depend only on local data.

9. The index theorem

In this section we prove an Atiyah–Patodi–Singer index theorem for the perturbed Dirac operators studied in this paper. Our proof follows the outline of the proof in [1] but uses estimates proven in the current paper as well as some results in [9]. Throughout this section we make the following assumptions.

Assumption 9.1. The manifolds, vector bundles, and operators are as described in Sections 2 and 3. In short we consider perturbed Dirac operators between sections of vector bundles having bounded geometry over complete oriented Riemannian manifolds with bounded geometry. Recall that we denote the m–dimensional manifold with boundary by M, its boundary by N, and its double by M^*. Also we denote by $\widetilde{D}_{A,M}$ the perturbed Dirac operator on M whose Fredholm index we study.

We assume that the perturbation's growth is sufficiently rapid and regular. In particular we require that Assumption 4.1 be satisfied because $i(A)(x) \to \infty$ as $x \to \infty$ on M (or on M^*) and because $DA + AD$ is $o(A^2)$. This assumption about the perturbation on M implies the same assumptions about the perturbation's restriction to N and about the perturbation's natural extension to M^*.

Our more detailed assumptions about the perturbation's regularity are Assumptions 6.4 (applied to x outside some compact subset of M^*) and 6.5. A decay condition on elements in the domains of powers of Δ_{A,M^*} is expressed by Assumption 5.1. We assume that the number of values no greater than L in the spectrum of $\Delta_{A,N}$ grows at a rate bounded by some power of L. Finally we assume that there is an $n > m/2$ for which the integrability hypothesis of Proposition 8.14 is satisfied.

Throughout the paper we have discussed examples satisfying these assumptions on manifolds with cylindrical or linearly expanding ends.

Notation 9.2. We follow [1] in letting $\eta(s)$ denote the eta function associated with the operator $D_{A,N}$.

$$\eta(s) = \sum_{\lambda} \operatorname{sign}(\lambda)|\lambda|^{-s},$$

where λ ranges over the nonzero elements of the spectrum of $D_{A,N}$ and these elements are counted according to multiplicity. By our assumption on the spectrum of $\Delta_{A,N}$ (and hence on the spectrum of $D_{A,N}$), this function is holomorphic when the real part of s is sufficiently large. As in [1] the proof of the index theorem establishes that $\eta(s)$ has an analytic continuation to the plane that is holomorphic in a neighborhood of $s = 0$. We let $\eta(0)$ denote the value at 0 of this continuation.

Notation 9.3. We let $\mathcal{I}(M)$ denote the index differential form associated with the operator $\mathcal{D}$. For example if $\mathcal{D}$ is the spin Dirac operator, $\mathcal{I}(M) = \widehat{A}(M)$, the $\widehat{A}$ differential form expressed in terms of the curvature of M. (There is more than one convention regarding the differential form. One should choose a convention consistent with the convention used for the Chern character form introduced below. See, e.g., [18].)

Notation 9.4. We let $\operatorname{ch}(E_0, E_1, \mathcal{A})$ denote the Chern character differential form equal to the difference of the Chern character form arising from the connection on E_0 and the Chern character form arising from the connection on E_1. Our assumption that, off some compact set, these connections are intertwined by $\mathcal{A}$'s unitary part guarantees that $\mathrm{ch}(E_0, E_1, \mathcal{A})$ is compactly supported.

Notation 9.5. Let h denote the dimension of the kernel of $D_{A,N}$.

Theorem 9.6. *Under the assumptions collected in Assumption 9.1, the perturbed Dirac operator $\widetilde{D}_{A,M}$, with the Atiyah–Patodi–Singer boundary conditions described at the beginning of Section 7, is Fredholm. Its Fredholm index is given by the formula*

$$\mathrm{index}(\widetilde{D}_{A,M}) = \int_M \mathcal{I}(M)\mathrm{ch}(E_0, E_1, \mathcal{A}) \; - \; \frac{h + \eta(0)}{2}.$$

Proof. The proof follows the strategy of [1]. The estimates of the previous sections replace the estimates arising from compactness of the manifold in [1].

Corollary 8.16 establishes that $\widetilde{D}_{A,M}$ is Fredholm, with index equal to the integral along the diagonal of the pointwise supertrace of $\exp(-t\Delta_{A,M})$'s kernel for any $t > 0$. Let $\{\theta, 1-\theta\}$ be a smooth partition of unity on $\mathbb{R}$, with θ an even function identically equal to 1 on some neighborhood of 0 and with support contained in $(-R_0/8, R_0/8)$, where R_0 is as in Section 6. Suppose that x is an element of M with distance at least $\delta_b/4$ from the boundary of M. Our analysis of the pointwise supertrace of the kernel of $\exp(-t\Delta_A)$ follows the pattern established in Section 8 of [9]. Lemma 8.18 of the present paper decomposes $\exp(-t\Delta_A)$ into a sum of two operators, with the value of the first operator's kernel at (x, x) completely determined by local data. We give the name $k_t^\theta(x, x)$ to the value at (x, x) of this locally determined operator's kernel. We use the notation $k_t(x, x)$ for the value at (x, x) of the kernel of $\exp(-t\Delta_A)$. The pointwise supertrace of $k_t^\theta(x, x)$ admits an asymptotic expansion via the standard heat equation methods. In particular because the bundles E_0 and E_1 have connections that, off some compact set, are intertwined by the unitary part of $\mathcal{A}$, off some compact set of values of x, $\mathrm{tr}_s k_t^\theta(x, x) = 0$. (See, in particular, Lemmas 8.7 and 8.8 of [9].) By Lemma 8.20 of the present paper and our integrability assumption, the integral over the set of x in M with distance at least $\delta_b/4$ from M's boundary of the pointwise supertrace of $k_t(x, x)$ has an asymptotic expansion agreeing with the asymptotic expansion of the integral over the same set of the pointwise supertrace of $k_t^\theta(x, x)$.

It follows that

$$\mathrm{index}(\widetilde{D}_{A,M}) = \int_{CN} \mathrm{tr}_s k_t(z, z) \; + \; F(t),$$

where: $\mathrm{tr}_s k_t(z, z)$ is the pointwise supertrace of $\exp(-t\Delta_{A,M})$'s kernel; the integral is over the collar neighborhood CN of N having width $\delta_b/4$; and $F(t)$ is the asymptotic expansion in powers of t arising from applying standard local heat equation techniques to $\exp(-t\Delta_{A,M})$ over a compact subset of M.

In an asymptotic expansion for $\int_{CN} \mathrm{tr}_s k_t(z, z)$, our integrability assumption and Lemma 8.20 imply we can ignore $\int_{CN} \mathrm{tr}_s k_t^{1-\theta}(z, z)$ and focus on $\int_{CN} \mathrm{tr}_s k_t^\theta(z, z)$. Because all of our structures are product structures over the collar neighborhood $N \times [0, \delta_b]$, we can extend all of the structures to product structures over $N \times [0, \infty)$, which we will call IN. On IN we have a perturbed Dirac operator. This perturbed Dirac operator differs from our standard examples in that its perturbation may fail to be invertible over a noncompact set, the product of $[0, \infty)$ with the compact subset of N where invertibility fails. However because our operator has boundary operator $D_{A,N}$, we may impose Atiyah–Patodi–Singer boundary conditions, as on M. Also, because IN is complete, as on M we may use these boundary conditions and their adjoint boundary conditions to form on IN the self-adjoint first-order operator $D_{A,IN}$ and its square $\Delta_{A,IN}$. As in Notation 8.3 there are norms associated with domains of powers of $\Delta_{A,IN}$. The extension of elements of these domains that are supported in the interior of $N \times [0, \delta_b)$ to sections supported in the interior of $N \times (-\delta_b, \delta_b)$ can be achieved just as in Section 7. For sections with support in $N \times (-\delta_b, \delta_b)$, calculations with positive integer powers of Δ_A on the double of IN are exactly the same as calculations with positive integer powers of Δ_A on M^*. In fact both Proposition 7.3 and Lemma 7.4 can be applied to sections with arbitrary support in IN. It follows that we can use multiplication by a smooth, compactly supported, nonnegative function of u (the variable normal to N) to proceed from a section in the domain of some power of $\Delta_{A,IN}$ to a section in the domain of the same power of Δ_{A,M^*} in such a way that the sections agree on CN. Furthermore we can choose a constant, independent of the sections, so that the latter section's norm, defined as in Notation 8.3 using the operator Δ_{A,M^*}, is bounded by that constant times the former section's norm, defined as in Notation 8.3 using the operator $\Delta_{A,IN}$.

Let $\exp(-t\Delta_{A,IN})$ be the heat operator associated with our operator on IN. It has a kernel that we will denote $k_{IN,t}(x, y)$. Section 8's analysis (Lemmas 8.8, 8.10, and 8.11) of functions of D_A on M and on M's double applies equally well on IN and on IN's double. By the comments on $N \times [0, \delta_b)$ in the preceding paragraph, Section 8's analysis of sections' sup norms and of delta distributions' norms extends without change to sections compactly supported in $N \times [0, \delta_b)$. It follows from the comments at the end of the preceding paragraph that for $z \in N \times [0, \delta_b/4]$, when we use our partition of unity $\{\theta, 1 - \theta\}$ to write $k_{IN,t}(z, z) = k^{\theta}_{IN,t}(z, z) + k^{1-\theta}_{IN,t}(z, z)$, our integrability assumption implies that in an asymptotic expansion we can replace $\int_{CN} \mathrm{tr}_s k^{\theta}_{IN,t}(z, z)$ by $\int_{CN} \mathrm{tr}_s k_{IN,t}(z, z)$. By the finite propagation speed of solutions of the wave equation, for $z \in N \times [0, \delta_b/4]$, $k^{\theta}_t(z, z) = k^{\theta}_{IN,t}(z, z)$.

At this point we can rewrite our index formula as

$$\mathrm{index}(\widetilde{D}_{A,M}) = \int_{CN} \mathrm{tr}_s k_{IN,t}(z, z) \; + \; F(t).$$

Because the estimate on pages 53–54 of Part I of [1] depends only on the rate of growth of the boundary operator's spectrum, it applies in our case as well. Thus as an asymptotic expansion our index formula can be written

$$\mathrm{index}(\widetilde{D}_{A,M}) = \int_{IN} \mathrm{tr}_s k_{IN,t}(z, z) \; + \; F(t).$$

As in Part I of [1], this index formula provides the asymptotic expansion for the first term on the right side of the equation, and this asymptotic expansion justifies replacing $\int_{IN} \mathrm{tr}_s k_{IN,t}(z, z)$ by $-(h + \eta(0))/2$ and $F(t)$ by $\int_M \mathcal{I}(M)\mathrm{ch}(E_0, E_1, \mathcal{A})$. (The discrepancy between $\int_M$ and $\int_{M \setminus CN}$ in this last expression is explained by observing that the collar's product structure forces $\int_{CN} \mathcal{I}(M)\mathrm{ch}(E_0, E_1, \mathcal{A})$ to equal 0.) □

10. Comments on the eta invariant

In this section we make two comments about the eta invariants of self-adjoint perturbed Dirac operators, which appeared in our index theorem for perturbed Dirac operators satisfying Atiyah–Patodi–Singer boundary conditions. First we note that the eta invariant is nontrivial, even when restricted to the class of operators we consider. Second we observe that any eta invariant of a self-adjoint perturbed Dirac operator that appears in the index theorem is equal, modulo $\mathbb{Z}$, to an eta invariant of a related elliptic operator on the boundary of a related compact manifold with boundary.

The nontriviality of the eta invariant follows from the observation that our self-adjoint perturbed Dirac operators can have nontrivial spectral flow and hence nontrivial continuous variation of their eta invariants. To build an example of a family of perturbed Dirac operators with nontrivial spectral flow, we start with the standard example of nontrivial spectral flow for a family of elliptic operators on a compact manifold: the convex combination $X_u = (1-u)T + u\exp(-2\pi it)\circ T\circ\exp(2\pi it)$ where u is the parameter for the spectral flow and T is the Dirac operator on the circle parametrized by $t \in [0,1]$. (See, e.g., [11].) As is discussed in Part III of [1], this spectral flow equals the index of an elliptic operator $\frac{\partial}{\partial u} + X_u$ on the Cartesian product of the circle parametrized by u with the circle parametrized by t. A reinterpretation of this assertion is that the spectral flow is the Kasparov product $[X] \otimes_{C(S^1)} [i\frac{\partial}{\partial u}]$ of the class in $KK^1(\mathbb{C}, C(S^1))$ represented by the family of operators X_u and of the class in $KK^1(C(S^1), \mathbb{C})$ represented by the operator $i\frac{\partial}{\partial u}$. The identification of an integer with this Kasparov product is justified by the observation that the Kasparov product is an element of $KK(\mathbb{C}, \mathbb{C})$, which is isomorphic to $\mathbb{Z}$ under the index map. (See, e.g., [3] for a discussion of G.G. Kasparov's KK theory.)

The Kasparov product $[X] \otimes_{C(S^1)} [i\frac{\partial}{\partial u}]$ is equal to $[\mathcal{H}] \otimes_{\mathbb{C}} ([X] \otimes_{C(S^1)} [i\frac{\partial}{\partial u}])$ for any class $[\mathcal{H}] \in KK(\mathbb{C}, \mathbb{C})$ represented by an operator of index one. The operator of index one that we choose is a modified version of the operator

$$d + d^* + (xdx\wedge + ydy\wedge) + (xdx\wedge + ydy\wedge)^*$$

from L^2 differential forms of even degree to L^2 differential forms of odd degree on $\mathbb{R}^2$. This operator is taken from [12] and is discussed as a perturbed Dirac operator in [9]. The modification of this operator (and of the spaces of L^2 forms that it acts on) that we make is based on changing the metric on the plane to one having a cylindrical end. (This will put analysis of the spectrum of an operator we consider next directly in the framework discussed in Section 4.) We give the name $\mathcal{H}$ to the operator thus constructed on the modified Euclidean space.

By associativity of the Kasparov product, we can also view $[\mathcal{H}] \otimes_{\mathbb{C}} ([X] \otimes_{C(S^1)} [i\frac{\partial}{\partial u}])$ as $([\mathcal{H}] \otimes_{\mathbb{C}} [X]) \otimes_{C(S^1)} [i\frac{\partial}{\partial u}]$. The observation that $[\mathcal{H}] \otimes_{\mathbb{C}} [X]$ is represented by a family of self-adjoint Dirac operators on the Cartesian product of the modified $\mathbb{R}^2$ with the circle identifies the Kasparov product $([\mathcal{H}] \otimes_{\mathbb{C}} [X]) \otimes_{C(S^1)} [i\frac{\partial}{\partial u}]$ as the spectral flow of that family and hence shows that the spectral flow of that family equals the nontrivial spectral flow of the family X_u.

The identification, modulo $\mathbb{Z}$, of the eta invariant of a self-adjoint perturbed Dirac operator on a boundary, as it appeared in our index theorem, with the eta invariant of a related operator on a compact boundary arises from observing that the two eta invariants appear in boundary contributions of Atiyah–Patodi–Singer index formulas in which the integrals of the index forms are equal. As in the rest of the paper, let $\widetilde{D}_A$ be a perturbed Dirac operator satisfying our usual conditions on a manifold M with boundary N. Choose a compact oriented hypersurface H in M that has boundary equal to its intersection with M's boundary. Choose the hypersurface so that its boundary has a collar neighborhood equal to its intersection with N's collar neighborhood. Choose the hypersurface so that it is in the interior of the set U of points in M at which $\mathcal{A}$ is the product of a scalar-valued function and a unitary-valued vector-bundle map that intertwines the connections on E_0 and E_1. In addition choose the hypersurface so that it separates M into a compact subset Y and a subset contained in the interior of U. Using deformations supported in the interior of U, smoothly deform all structures so that the hypersurface H has a collar neighborhood (extending on both sides of H) over which all structures are products.

We now make a compact manifold $M(Y) = Y \bigcup_H Y$ with boundary $N(Y)$ by gluing two copies of Y via the identity map on the hypersurface H. One copy of Y takes all structures from the structures on M, after these structures have been deformed to give H a collar neighborhood. With the exception of the treatment of the auxiliary vector bundle E, the gluing of structures follows the pattern used when we doubled M by gluing along N. In particular the second copy of Y receives a reversed orientation; under the new orientation what was $S^{\pm}$ becomes $S^{\mp}$; and the Clifford action of a unit vector field normal to H is used at each point of H to identify the fiber of the spinor bundle over one copy of Y with the fiber of the spinor bundle over the other copy. The auxiliary vector bundle over the second copy of Y is $E_0 \oplus E_0$. Over each point on H, $E_0 \oplus E_1$ is attached to $E_0 \oplus E_0$ via the map that is the identity on the first summand and is the inverse of $\mathcal{A}$'s unitary part on the second summand. The vector-bundle map $\mathcal{A}\colon E_0 \to E_1$ over the first copy of Y is extended to a map that, over the second copy of Y, can be identified with multiplication by a nonvanishing scalar-valued function. We may make this map the identity map $E_0 \to E_0$ over the complement, in the second copy of Y, of H's collar neighborhood.

With Atiyah–Patodi–Singer boundary conditions, the perturbed Dirac operator $\widetilde{D}_{A,M(Y)}$ on the compact $M(Y)$ defines a Fredholm operator whose index is given by the formula in [1]. While the indices of $\widetilde{D}_{A,M(Y)}$ and $\widetilde{D}_{A,M}$ may differ, as may the dimensions of the kernels of $D_{A,N(Y)}$ and $D_{A,N}$, the integrals of the differential forms appearing in the index formulas for $\widetilde{D}_{A,M(Y)}$ and $\widetilde{D}_{A,M}$ are exactly the same. It follows from these observations and the index formulas that the eta invariants of $D_{A,N(Y)}$ and $D_{A,N}$ agree modulo $\mathbb{Z}$.

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