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Multiple tilings of $\mathbb{Z}$ with long periods, and tiles with many-generated level semigroups

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Abstract. We consider multiple tilings of $\mathbb{Z}$ by translates of a finite multiset A of integers (called a *tile*). We say that a set of integers T is an *A-tiling of level d* if each integer can be written in exactly d ways as the sum of an element of T and an element of A. We find new exponential lower bounds on the longest period of A-tiling as a function of the diameter of A, which rejoin the exponential upper bounds given by Ruzsa (preprint, 2002) and Kolountzakis (2003). We also show the existence of tiles whose level semigroups have arbitrarily many generators (where the *level semigroup* of a tile A is the set of integers d such that A admits a tiling of level d).

Contents

Introduction

Let A be a finite multiset of integers (which we shall call a *tile*) and let d be a nonnegative integer. Another multiset T of integers is said to be an *A-tiling of level d* if every integer can be written in exactly d ways as the sum of an element of T and an element of A. If we encode A and T as power series:

$$A(x) = \sum_{i\in\mathbb{Z}} A[i]x^i, \qquad T(x) = \sum_{i\in\mathbb{Z}} T[i]x^i$$

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where $A[i]$, $T[i]$ stand for the number of times i appears in the multisets A and T respectively, then T is an A-tiling of level d if and only if

$$T(x)A(x) = d\sum_{i\in\mathbb{Z}} x^i. \tag{1}$$

One can thus view T as a set of translation positions of the set A such that the union of all the translates of A covers the integers exactly d times. Tilings of this kind have been discussed many places [2, 4, 5, 6, 8, 9, 13, 14, 15, 19, 20, 21], though most of the literature focuses on the case $d = 1$.

The purpose of this paper is to exhibit tiles and tilings with various interesting properties. More precisely we show how to construct tilings where the minimal period of T is exponentially long compared to the diameter of the tile A (it is known that T is always periodic; see discussion below) and how to find tiles A whose level semigroups have arbitrarily many generators.

We start by clarifying the latter statement. The "level semigroup" $\mathcal{L}(A)$ of a tile A is the set of integers d such that A admits a tiling of level d. It is obvious that $\mathcal{L}(A)$ is closed under addition, whence the "semigroup" terminology (in fact, since $0 \in \mathcal{L}(A)$ for any tile A (as $T = \emptyset$ is an A-tiling of level 0 regardless of A), $\mathcal{L}(A)$ is closed under arbitrary nonnegative integer linear combinations of its elements). Thus $\mathcal{L}(A)$ is always of the form $\mathbb{Z}^+(Q)$ for some set $Q \subseteq \mathbb{N}$, where "$\mathbb{Z}^+(Q)$" denotes the closure of Q under nonnegative integer linear combinations (for us $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$). If Q is chosen minimal then Q will be finite and we refer to the elements of Q as the "generators" of $\mathcal{L}(A)$. We will see that for any $k \in \mathbb{N}$ there is some tile A whose level semigroup has exactly k generators, and that this result still holds true if we restrict A to being a set rather than a multiset. (We had previously discussed [14] how to construct a tile with level semigroup $\mathbb{Z}^+(a, b)$ for any $a, b \in \mathbb{N}$. Kolountzakis and Lagarias [7] have also shown that when it comes to tilings of the real line by translates of a single real-valued function, there exist tiles whose level semigroups are not even finitely generated. Such a phenomenon cannot occur for tilings of the integers since any semigroup that is a subset of $\mathbb{Z}^+$ has finitely many generators.)

Our second result is a comment on the asymptotic relationship between the diameter $\operatorname{diam}(A) = \max\{j \in A\} - \min\{j \in A\}$ of a tile A and the longest period of an A-tiling. A simple pigeonhole argument [14] shows that all A-tilings of level d are periodic with period less than $(d+1)^{\operatorname{diam}(A)}$. Recently Kolountzakis [6] and Ruzsa [11] have shown there exists an upper bound $m(A)$ independent of the level d such that all A-tilings are periodic mod $m(A)$. More specifically, if we let $\mathcal{K}(A)$ stand for the longest minimal period of an A-tiling and let

$$D(n) = \max\{\mathcal{K}(A) : \operatorname{diam}(A) \le n\}$$

then Ruzsa shows there exists some $c > 0$ such that

$$D(n) < e^{c\sqrt{n\ln(n)}} \tag{2}$$

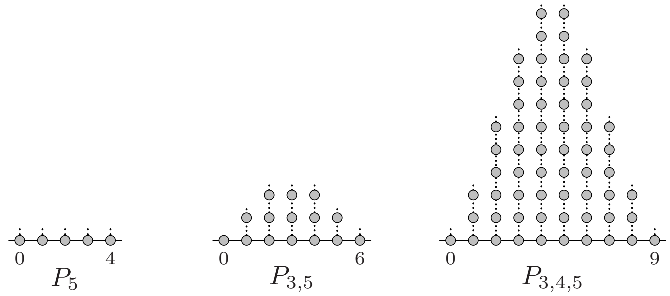

FIGURE 1. The tiles P_5, $P_{3,5}$ and $P_{3,4,5}$. The multiplicity with which an integer appears inside the tile is indicated by the number of points in the column above it.

for all n sufficiently large (Kolountzakis gives a similar though slightly weaker upper bound). We will conversely show that there exists a constant $b > 0$ such that

$$D(n) > e^{b\sqrt{n\ln(n)}} \tag{3}$$

for all n sufficiently large.

The lower bound (3) is achieved using tiles that contain integers with very high multiplicity and also using tilings of very high level. The question of finding similar upper and lower bounds for level 1 tilings is still open. Biró [1] has in fact shown that for level 1 tilings the upper bound (2) can be improved to

$$D(n) < e^{n^{(\frac{1}{3}+\epsilon)}} \tag{4}$$

for any $\epsilon > 0$, which implies that a lower bound of type (3) cannot hold for level 1 tilings. We have recently found, however, that the period of a level 1 A-tiling can still grow superpolynomially compared to $\mathrm{diam}(A)$ [18]. A few more remarks on (3) can be found at the end of the next section.

Notation. As already stated, $\mathbb{Z}^+(Q)$ stands for the closure under nonnegative integer linear combinations of the set Q. We write $\mathbb{Z}_n$ for the set $\{0, 1, \dots, n-1\}$. We use the standard notation $[x^k]P(x)$ for the coefficient of x^k in the formal power series (or polynomial) $P(x)$, where $[x^k]P(x)Q(x) = [x^k](P(x)Q(x))$.

Results I

In this section we introduce some of the main tiles of interest to us and use them to prove the lower bound (3). We define a tile $P_{n_1,\dots,n_k}$ for all $n_1, \dots, n_k \in \mathbb{N}$ and all $k \in \mathbb{N}$ by

$$P_{n_1,\dots,n_k}(x) = \prod_{j=1}^{k}(1 + x + \cdots + x^{n_j-1}).$$

Some sample tiles $P_{n_1,\dots,n_k}$ are sketched in Figure 1.

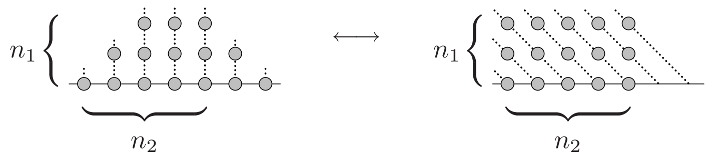

FIGURE 2. P_{n_1,n_2} as a coordinate-sum projection ($n_1 = 3, n_2 = 5$).

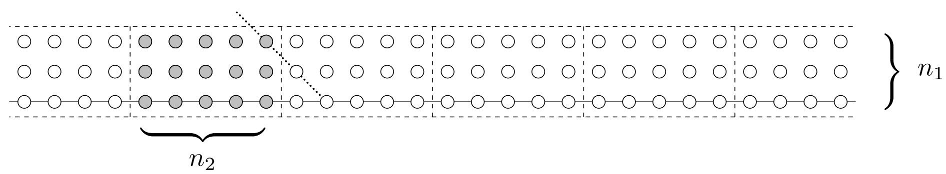

FIGURE 3. A coordinate-sum representation of a P_{n_1,n_2}-tiling of level n_1.

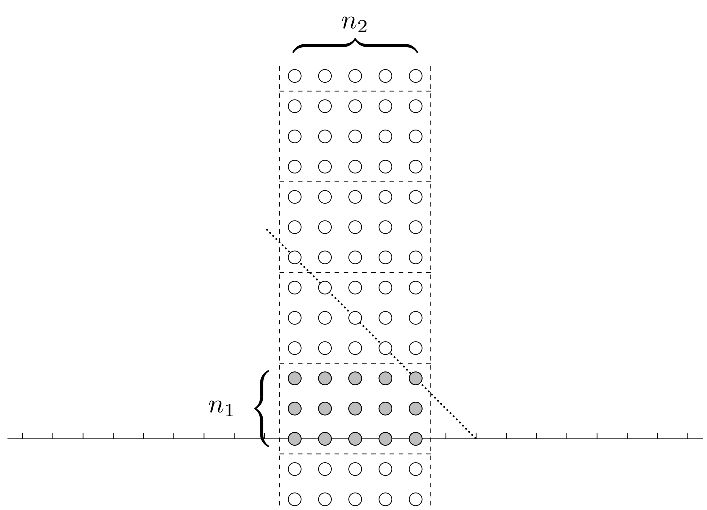

FIGURE 4. A coordinate-sum representation of a P_{n_1,n_2}-tiling of level n_2.

One can understand the tile $P_{n_1,\ldots,n_k}$ as the coordinate-sum projection of the array of points $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ in $\mathbb{Z}^k$ onto $\mathbb{Z}$. This idea is shown in Figure 2 for the tile $P_{3,5}$. This way of visualizing $P_{n_1,\ldots,n_k}$ is useful since it allows us to see right away that $P_{n_1,\ldots,n_k}$ admits tilings of levels $N/n_1, \ldots, N/n_k$ where $N = n_1 \cdots n_k$: simply pile translates of the array $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ end-to-end along one of the coordinate directions, and then take the coordinate-sum projection of the column obtained (see Figures 3 and 4, where this process is illustrated for the tile $P_{3,5}$).

Alternately, we may verify that $P_{n_1,\dots,n_k}$ admits a tiling of level N/n_i for $i = 1,\dots,k$ via the formal computation:

$$\begin{aligned}(n_i\mathbb{Z})(x)P_{n_1,\dots,n_k}(x) &= \sum_{t=-\infty}^{\infty} x^{n_i t} \prod_{1\le j\le k}(1+x+\cdots+x^{n_j-1})\\ &= \prod_{\substack{1\le j\le k\\ j\ne i}}(1+x+\cdots+x^{n_j-1}) \sum_{m=-\infty}^{\infty} x^m\\ &= \frac{N}{n_i}\sum_{m=-\infty}^{\infty} x^m,\end{aligned}$$

so that, by Equation (1), $n_i\mathbb{Z}$ is $P_{n_1,\dots,n_k}$-tiling of level N/n_i.

By taking the disjoint union of the tilings $n_1\mathbb{Z}, \dots, n_k\mathbb{Z}$ we obtain a $P_{n_1,\dots,n_k}$-tiling of least period $\mathrm{lcm}(n_1,\dots,n_k)$. Since $\mathrm{diam}(P_{n_1,\dots,n_k}) = (\sum n_i)-k+1$, we thus see that $P_{n_1,\dots,n_k}$-tilings can have quite long periods compared to $\mathrm{diam}(P_{n_1,\dots,n_k})$. This observation, coupled with a simple asymptotic computation, yields the lower bound (3):

Theorem 1. *There exists a constant $c > 0$ such that for all n sufficiently large there is a tile A of diameter less than n admitting an A-tiling T of least period greater than $e^{c\sqrt{n\ln(n)}}$.*

Proof. Recall that $\mathcal{K}(A)$ denotes the longest minimal period an A-tiling, and that $D(n) = \max\{\mathcal{K}(A) : \mathrm{diam}(A) \le n\}$. We thus need to show $D(n) > e^{c\sqrt{n\ln(n)}}$ for some $c > 0$ for all n sufficiently large.

Let $\pi(r)$ denote the number of primes at most r and let $p_1,\dots,p_{\pi(r)}$ be the first $\pi(r)$ primes. Let $\sigma_r = p_1+\cdots+p_{\pi(r)}$. By the prime number theorem

$$\sigma_r \le \frac{r^2}{\ln(r)}$$

for all r sufficiently large. If $r < \sqrt{\sigma_r \ln(\sqrt{\sigma(r)})}$ then

$$\sigma_r < \frac{\sigma_r \ln(\sqrt{\sigma(r)})}{\ln\left(\sqrt{\sigma_r\ln(\sqrt{\sigma(r)})}\right)} < \sigma_r$$

a contradiction, so

$$r \ge \sqrt{\sigma_r\ln(\sqrt{\sigma(r)})} = \frac{1}{\sqrt{2}}\sqrt{\sigma_r\ln(\sigma_r)}$$

for all r sufficiently large. But since

$$\mathrm{diam}(P_{p_1\dots p_{\pi(r)}}) = \sigma_r - r < \sigma_r \quad\text{and}\quad \mathcal{K}(P_{p_1\dots p_{\pi(r)}}) \ge p_1\cdots p_{\pi(r)},$$

we have

$$\begin{aligned} D(\sigma_r) &\geq p_1 \cdots p_{\pi(r)} \\ &\geq \epsilon e^r \\ &\geq \epsilon e^{\frac{1}{\sqrt{2}}\sqrt{\sigma_r \ln(\sigma_r)}} \end{aligned}$$

for all $\epsilon < 1$, for all r sufficiently large. Thus, since for any $n \in \mathbb{N}$ there is some $r \in \mathbb{N}$ such that $n \leq \sigma_r \leq 2n$, there is a constant $c > 0$ such that

$$D(n) > e^{c\sqrt{n \ln(n)}}$$

for all n sufficiently large. □

Theorem 1 thus depends on the idea that tilings with long periods can be obtained by "sandwiching" together many tilings of small but different period lengths. Some may find this sandwich method somewhat unappealing, and would rather have a construction where the tilings with long period do not decompose as the disjoint union of tilings of smaller period. We say that an A-tiling is *indecomposable* if it is not the disjoint union of two other nonempty A-tilings. As a matter of fact, indecomposable tilings with exponentially long periods do exist, as we show in [15]. In that paper we give the slightly weaker lower bound

$$D(n) > e^{c\sqrt[3]{n \ln(n)}}$$

for indecomposable tilings. The construction of that paper also involves tiles of the type $P_{n_1,\ldots,n_k}$.

Results II

In this section we show the existence of tiles whose level semigroups have arbitrarily many generators. We have already seen that $P_{n_1,\ldots,n_k}$ admits tilings $n_1\mathbb{Z}$, $\ldots$, $n_k\mathbb{Z}$ of levels N/n_1, $\ldots$, N/n_k where $N = n_1 \cdots n_k$. As a judicious choice of the numbers $n_1, \ldots, n_k$ turns $\mathbb{Z}^+(N/n_1, \ldots, N/n_k)$ into a k-generated semigroup (proved later) this would be sufficient to establish the above result *if* it were true that the tilings $n_1\mathbb{Z}, \ldots, n_k\mathbb{Z}$ and their translates were always the *only* indecomposable $P_{n_1,\ldots,n_k}$-tilings[1] — but this is not always the case! For example $P_{6,10,15}$

[1]For full disclosure, we note that the tilings $n_1\mathbb{Z}$, $\ldots$, $n_k\mathbb{Z}$ are not necessarily indecomposable themselves; for example if $n_i | n_j$, $n_i \neq n_j$ for some i, j.

FIGURE 5. Two representations of $P_{3,2}^{3,1}$.

admits an indecomposable tiling T given[2] by

$$\begin{aligned} T(x) &= (6\mathbb{Z})(x) + x^5(10\mathbb{Z})(x) - (15\mathbb{Z})(x) \\ &= \sum_{j=-\infty}^{\infty} x^{30j}(1 + x^6 + x^{12} + x^{18} + x^{24}) + (x^5 + x^{15} + x^{25}) - (1 + x^{15}) \\ &= \sum_{j=-\infty}^{\infty} x^{30j}(x^5 + x^6 + x^{12} + x^{18} + x^{24} + x^{25}) \end{aligned}$$

that is neither a translate of $6\mathbb{Z}$, $10\mathbb{Z}$ or $15\mathbb{Z}$ (one sees that T is a $P_{6,10,15}$-tiling precisely because $T(x)$ is a linear combination of $(6\mathbb{Z})(x)$, $(10\mathbb{Z})(x)$ and $(15\mathbb{Z})(x)$). As a general matter, indecomposable $P_{n_1,\dots,n_k}$-tilings can become very wild once $k \geq 3$ [15, 16], making it hard to characterize $\mathcal{L}(P_{n_1,\dots,n_k})$ in the general case.[3]

What we shall prove instead, and which will be sufficient for our purposes, is that $n_1\mathbb{Z}, \dots, n_k\mathbb{Z}$ and their translates are the only indecomposable $P_{n_1,\dots,n_k}$-tilings when $n_1, \dots, n_k$ are pairwise coprime. This is a sufficient but not a necessary condition for $n_1\mathbb{Z}$, $\dots$, $n_k\mathbb{Z}$ to be the only indecomposable $P_{n_1,\dots,n_k}$-tilings. For example all P_{n_1,n_2}-tilings are disjoint unions of $n_1\mathbb{Z}$, $n_2\mathbb{Z}$ regardless of whether n_1 and n_2 are coprime or not [14], and we will see further down that $n\mathbb{Z}$ is the only indecomposable $P_{n,\dots,n}$-tiling. We leave it as an open problem to determine exactly which tuples $(n_1, \dots, n_k)$ have the property that all $P_{n_1,\dots,n_k}$-tilings decompose as a disjoint union of $n_1\mathbb{Z}$, $\dots$, $n_k\mathbb{Z}$.

We will also prove analogous results on level semigroups for tiles that are sets instead of multisets. For this we need a more general class of tiles than the tiles $P_{n_1,\dots,n_k}$. We define a tile $P_{n_1,\dots,n_k}^{q_1,\dots,q_k}$ using k additional indices $q_1, \dots, q_k \in \mathbb{N}$ given by the polynomial:

$$P_{n_1,\dots,n_k}^{q_1,\dots,q_k}(x) = \prod_{j=1}^{k}(1 + x^{q_j} + x^{2q_j} + \dots + x^{(n_j-1)q_j}).$$

Note $P_{n_1,\dots,n_k} = P_{n_1,\dots,n_k}^{1,\dots,1}$. One can think of $P_{n_1,\dots,n_k}^{q_1,\dots,q_k}$ as the coordinate-sum projection of a k-dimensional array with spaced rows and columns, where the spacing is specified by the variables $q_1, \dots, q_k$, as shown in Figure 5.

Like for $P_{n_1,\dots,n_k}$, the coordinate-sum representation of $P_{n_1,\dots,n_k}^{q_1,\dots,q_k}$ allows us to see right away that $P_{n_1,\dots,n_k}^{q_1,\dots,q_k}$ admits tilings of levels N/n_1, $\dots$, N/n_k where $N =$

[2]This example is due independently to Rédei [10] and to Schoenberg [12], who both originally gave it in the context of vanishing sums of roots of unity. See [15, 16] for a connection between our work and vanishing sums of roots of unity.

[3]It is noteworthy that $\mathcal{L}(P_{n_1,\dots,n_k})$ can in fact be characterized without knowing the indecomposable $P_{n_1,\dots,n_k}$-tilings in certain special cases; compare [15] and [3].

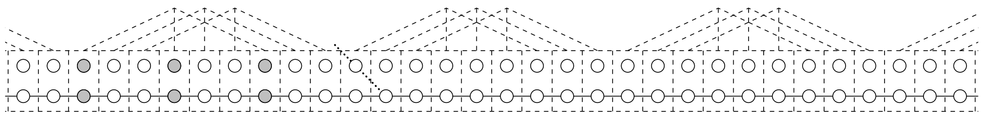

FIGURE 6. A $P^{3,1}_{3,2}$-tiling of level 2.

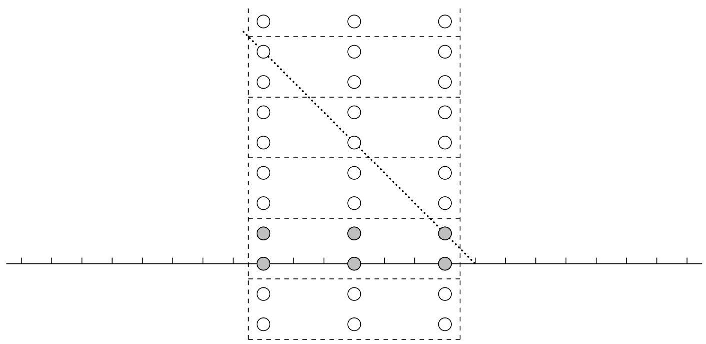

FIGURE 7. A $P^{3,1}_{3,2}$-tiling of level 3.

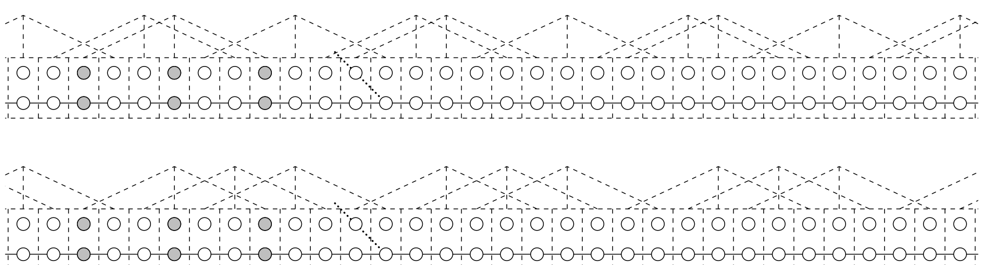

FIGURE 8. More $P^{3,1}_{3,2}$-tilings of level 2.

$n_1 \cdots n_k$ (regardless of the values of $q_1, \ldots, q_k$). The drawings corresponding to Figures 3, 4 are shown in Figures 6, 7. Note that if $q_i > 1$ then there will generally be several different $P^{q_1,\ldots,q_k}_{n_1,\ldots,n_k}$-tilings of level N/n_i that are inequivalent under translation, as shown in Figure 8, where we show some $P^{3,1}_{3,2}$-tilings of level two that are translation-inequivalent from the tiling of Figure 6.

Formally, one notes that if T is *any* $P^{q_i}_{n_i}$-tiling of level 1 then T is also a $P^{q_1,\ldots,q_k}_{n_1,\ldots,n_k}$-tiling of level N/n_i, since

$$\begin{aligned} T(x)P^{q_1,\dots,q_k}_{n_1,\dots,n_k}(x) &= T(x)P^{q_1}_{n_1}(x)\prod_{\substack{l=1\\l\neq i}}^{n} P^{q_l}_{n_l}(x) \\ &= \sum_{j=-\infty}^{\infty} x^j \prod_{\substack{l=1\\l\neq i}}^{n} P^{q_l}_{n_l}(x) \\ &= \frac{N}{n_i}\sum_{j=-\infty}^{\infty} x^j. \end{aligned}$$

Our main reason for considering the larger family of tiles $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$ instead of $P_{n_1,\dots,n_k}$ is that $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$ is a set instead of a multiset for certain choices of the variables $q_1,\dots,q_k$; for example if $q_1,\dots,q_k$ are taken rapidly enough increasing, e.g., if $q_{j+1} > (n_1-1)q_1 + \cdots + (n_j-1)q_j$ for $1 \le j < k$, then it is easy to check that $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$ is a set. This will allow us to prove level semigroup results for set-like tiles instead of multiset-like tiles.

The main result of this section is the following theorem on $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$-tilings:

Theorem 2. *If the products $n_1q_1,\dots,n_kq_k \in \mathbb{N}$ are pairwise coprime then the only indecomposable $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$-tilings are level 1 $P^{q_i}_{n_i}$-tiling for $1 \le i \le k$.*

The aforementioned result that the only indecomposable $P_{n_1,\dots,n_k}$-tilings are the translates of $n_1\mathbb{Z}$, $\dots$, $n_k\mathbb{Z}$ when $n_1,\dots,n_k$ are pairwise coprime is a corollary of Theorem 2 (applied with $q_1 = \cdots = q_k = 1$). Likewise, Theorem 2 is not best possible and does not constitute a characterization of which pairs of k-tuplets $(n_1,\dots,n_k)$, $(q_1,\dots,q_k)$ have the property that all indecomposable $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$-tilings are level 1 $P^{q_i}_{n_i}$-tiling for $1 \le i \le k$. The existence of set-like tiles with many-generated level semigroups is easy to deduce from Theorem 2; see Corollary 2 at the end of the paper.

We jot down without proof the elementary structure of indecomposable P^q_n-tilings before beginning the proof of Theorem 2:

Proposition 1. *If T is an indecomposable P^q_n-tiling then T has level 1 and there are numbers $b_0,\dots,b_{q-1}$ where $b_s \equiv s \bmod q$ such that $T[a] = 1 \iff a \equiv b_s \bmod nq$ for some $0 \le s < q$.*

Before starting the proof of Theorem 2 we note that all roots of $P^q_n(x)$ are nq-th roots of unity since $(1-x^q)P^q_n(x) = 1 - x^{nq}$, and that by extension all roots of $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}(x) = P^{q_1}_{n_1}(x)\cdots P^{q_k}_{n_k}(x)$ are m-th roots of unity where $m = \operatorname{lcm}(n_1q_1,\dots,n_kq_k)$.

Proof of Theorem 2. Assume by contradiction that $n_1q_1,\dots,n_kq_k$ are pairwise relatively prime and that T is an indecomposable $P^{q_1,\dots,q_k}_{n_1,\dots,n_k}$-tiling that is not a $P^{q_i}_{n_i}$-tiling for any $i \in \{1,\dots,k\}$ (this latter assumption obviously implies that $k \ge 2$). In particular T does not contain any $P^{q_1}_{n_1}$-tilings, so by Propopsition 1 there must be some $0 \le s < q_1$ for which there are numbers $a_0,\dots,a_{n_1-1}$, $a_i \equiv s - iq_1 \bmod n_1q_1$, such that $T[a_i] = 0$ for $0 \le i < n_1$ (we put $a_i \equiv s - iq_1$ instead of $a_i \equiv s + iq_1$ as this turns out more convenient later on). We can assume WLOG via a translation that $s = 0$, so that $a_i \equiv -iq_1 \bmod n_1q_1$ for $0 \le i < n_1$.

Let $l = \mathrm{lcm}(n_2q_2, \ldots, n_kq_k)$. Since $(1-x)P^{q_2,\ldots,q_k}_{n_2,\ldots,n_k}(x)$ divides $1 - x^l$ (as all the roots of the former polynomial are distinct l-th roots of unity) we have

$$\begin{aligned} 0 &= (1-x)T(x)P^{q_1,\ldots,q_k}_{n_1,\ldots,n_k}(x) \\ &= T(x)P^{q_1}_{n_1}(x)((1-x)P^{q_2,\ldots,q_k}_{n_2,\ldots,n_k}(x)) \\ &= T(x)P^{q_1}_{n_1}(x)(1-x^l) \\ &= T(x)(1-x^{n_1q_1})(1-x^l). \end{aligned} \tag{5}$$

In other words, $T(x)(1-x^l)$ is periodic mod n_1q_1 and therefore

$$T(x)(1-x^{zn_1q_1})(1-x^l) = 0$$

for any integer z. Let $z_0, \ldots, z_{n_1-1} \in \mathbb{Z}$ be chosen such that $iq_1 + z_in_1q_1 = -a_i$. We have

$$\begin{aligned} 0 &= T(x)(P^{q_1}_{n_1}(x) + (x^{z_0n_1q_1} - 1) + x^{q_1}(x^{z_1n_1q_1} - 1) + \cdots \\ &\qquad + x^{(n_1-1)q_1}(x^{z_{n_1-1}n_1q_1} - 1))(1-x^l) \\ &= T(x)(x^{-a_0} + \cdots + x^{-a_{n-1}})(1-x^l) \end{aligned}$$

so that, for any $u \in \mathbb{Z}$,

$$\begin{aligned} 0 &= [x^0]T(x)(x^{-a_0} + \cdots + x^{-a_{n-1}})(x^{-ul} - 1) \\ &= [x^0]T(x)(x^{-a_0-ul} + \cdots + x^{-a_{n-1}-ul} - x^{-a_0} - \cdots - x^{-a_{n-1}}) \\ &= T[a_0 + ul] + \cdots + T[a_{n-1} + ul] - T[a_0] - \cdots - T[a_{n-1}] \\ &= T[a_0 + ul] + \cdots + T[a_{n-1} + ul]. \end{aligned}$$

But $T[\cdot] \ge 0$, so we conclude that $T[a_0 + ul] = \cdots = T[a_{n-1} + ul] = 0$ for all $u \in \mathbb{Z}$.

From (5) we get more generally that

$$T(x)(1-x^{vn_1q_1})(1-x^{ul}) = 0$$

for all $u, v \in \mathbb{Z}$. Taking the coefficient of x^t in this expression yields

$$T[t] - T[t-ul] = T[t - vn_1q_1] - T[t - vn_1q_1 - ul] \tag{6}$$

for all $t, u, v \in \mathbb{Z}$. Since $\gcd(n_1q_1, l) = 1$ for all $t \in \mathbb{Z}$ there are $u_t, v_t \in \mathbb{Z}$ such that $t - v_tn_1q_1 = a_0 + u_tl$. By (6) we have that

$$\begin{aligned} T[t] - T[t-l] &= T[t - v_tn_1q_1] - T[t - v_tn_1q_1 - l] \\ &= T[a_0 + u_tl] - T[a_0 + (u_t - 1)l] \\ &= 0 - 0 = 0 \end{aligned}$$

for any $t \in \mathbb{Z}$, so T is periodic mod $l = \mathrm{lcm}(n_2q_2, \ldots, n_kq_k)$. By symmetry T is also periodic mod $\mathrm{lcm}(n_1q_1, \ldots, n_{i-1}q_{i-1}, n_{i+1}q_{i+1}, \ldots, n_kq_k)$ for any $1 \le i \le k$, but the gcd of all those periods is 1 so T is periodic mod 1. Therefore, since $T[a_0] = 0$, $T[j] = 0$ for all $j \in \mathbb{Z}$, as sought. $\square$

Corollary 1. *If $n_1q_1, \ldots, n_kq_k$ are pairwise coprime then*

$$\mathcal{L}(P^{q_1,\ldots,q_k}_{n_1,\ldots,n_k}) = \mathbb{Z}^+(N/n_1, \ldots, N/n_k)$$

where $N = n_1 \cdots n_k$.

Corollary 2. *For any $k \in \mathbb{N}$ there exists a set-like tile whose level semigroup has exactly k generators.*

Proof. By the prime number theorem there is some number $K > 0$ for which there exist k primes $p_1, \ldots, p_k$ greater than K and less than $2K$. Put $N = p_1 \cdots p_k$. Note that since

$$\max_{1\le i,j\le k} \frac{N/p_i}{N/p_j} = \max_{1\le i,j\le k} \frac{p_i}{p_j} < 2$$

none of the elements in $\{N/p_1, \ldots, N/p_k\}$ can be obtained as a nonnegative integer linear combination of the others. Therefore $\mathbb{Z}^+(N/p_1, \ldots, N/p_k)$ is a semigroup with exactly k generators.

Now we choose $q_1, \ldots, q_k$ such that:

(i) $q_{j+1} > (p_1 - 1)q_1 + \cdots + (p_j - 1)q_j$ for $1 \le j < k$.

(ii) The numbers $p_1q_1, \ldots, p_kq_k$ are pairwise coprime.

(Such q_j's obviously exist.) By the choice of the q_j's $P^{q_1,\ldots,q_k}_{p_1,\ldots,p_k}$ is a set and $P^{q_1,\ldots,q_k}_{p_1,\ldots,p_k}$ has level semigroup $\mathbb{Z}^+(N/p_1, \ldots, N/p_k)$ by Theorem 2, where $\mathbb{Z}^+(N/p_1, \ldots, N/p_k)$ is a semigroup with k generators. □

Theorem 2 shows that all $P^{q_1,\ldots,q_k}_{n_1,\ldots,n_k}$-tilings are periodic mod $\mathrm{lcm}(n_1q_1, \ldots, n_kq_k)$ when $n_1q_1, \ldots, n_kq_k$ are pairwise coprime. This is true no matter what the values of $n_1, \ldots, n_k$, $q_1, \ldots, q_k$; it is a consequence of the fact that the roots of $P^{q_1,\ldots,q_k}_{n_1,\ldots,n_k}(x)$ are all $\mathrm{lcm}(n_1q_1, \ldots, n_kq_k)$-th roots of unity and of the following result of Kolountzakis [6] and of Ruzsa [11]:

Theorem 3 (Kolountzakis [6], Ruzsa [11])**.** *Let A be a tile. If $A(x)$ has no roots that are roots of unity, then all A-tilings are periodic mod* 1. *Otherwise all A-tilings are periodic mod* $\mathrm{lcm}(m_1, \ldots, m_k)$ *where $m_1, \ldots, m_k$ are the orders of the roots of unity that are roots of $A(x)$.*

It follows in particular that all $P_{n,\ldots,n}$-tilings are periodic mod n, and therefore that $n\mathbb{Z}$ (and its translates) are the only indecomposable $P_{n,\ldots,n}$-tilings, as was mentioned above.

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