A discrete Lyapunov theorem for the exponential stability of evolution families

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ABSTRACT. We propose a discrete time approach for the exponential stability of evolution families on a Hilbert space by proposing a Liapunov-type equation which involves only discrete time arguments.

The result of A.M. Lyapunov has come into widespread usage in many topics of mathematics. In particular, it continues to be of great importance in modern treatments of the asymptotic behaviour of the solutions of differential systems.

Let us recall that the theorem of Lyapunov states that if A is an $n \times n$ complex matrix then A has all its characteristic roots with real parts negative if and only if for any positive definite Hermitian H there exists an unique positive definite Hermitian matrix B satisfying the equation

$$A^*B + BA = -H$$

(where * denotes the conjugate transpose of a matrix) (see [1]).

This very familiar result was extended in a natural way to strongly continuous semigroups of operators on a complex Hilbert space, by R. Datko [7]. The result of Datko requires the mathematical sophistication of the modern functional analysis tools. A similar result is given by Krein and Daleckij in [6] in the case of the semigroup $T(t) = e^{tA}$ where A is a bounded linear operator, first for exponential stability and then for exponential dichotomy.

Also, in this context results related to the passing from the bounded linear operator A to the case of an unbounded one, can be found in the papers due to C. Chicone [3], J. Goldstein [8], Y. Latushkin [3, 11], S. Montgomery-Smith [11], L. Pandolfi [12], A. Pazy [13] and Vu Quoc Phong [16, 17].

Let $B(X)$ be the Banach algebra of all bounded linear operators acting on the Hilbert space X. The $B(X)$-valued function $T = \{T(t)\}_{t \geq 0}$ is a semigroup of linear operators if:

- $T(0)$ is the identity on X.
- $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$.

If in addition T is strongly continuous (i.e., $\lim_{t \to 0_+} T(t)x = x$, for all $x \in X$) then we will call T a C_0-semigroup.

Also, the generator of a C_0-semigroup T is the operator defined by

$$D(A) = \left\{ x \in X : \exists \lim_{t \to 0_+} \frac{T(t)x - x}{t} \text{ in } X \right\}, \quad Ax = \lim_{t \to 0_+} \frac{T(t)x - x}{t}.$$

The C_0-semigroup T is claimed to be exponentially stable if there exist $N, \nu > 0$ such that $\|T(t)\| \le Ne^{-\nu t}$, for all $t \ge 0$.

The so-called "Lyapunov-type" result for the semigroups of linear operators establishes that in the case when the generator A of a C_0 semigroup T is bounded, then the C_0-semigroup is exponentially stable if and only if there exists a positive, self-adjoint, bounded operator W on X such that

(L) $$A^*W + WA = -I.$$

As we already noticed, this result was extended also for an unbounded operator A. More precise if A is an unbounded closed operator which generates a C_0-semigroup, then the equation (L) becomes

(L1) $$\langle Ax, Wx \rangle + \langle Wx, Ax \rangle = -\|x\|^2 \quad \text{for all } x \in D(A).$$

For further details, we refer the reader to [7].

We will now easily derive an equivalent form of the above equation, a form that does not contain the generator A. We will then attempt to propose a "Lyapunov-type" equation for the general case of evolution families.

Assume that (L1) holds and let $f_x : \mathbb{R} \to \mathbb{C}$, the function defined by

$$f_x(t) = \langle WT(t)x, T(t)x \rangle,$$

$x \in D(A)$. One can easily see that f_x is differentiable and

$$\begin{aligned} f_x'(t) &= \langle WAT(t)x, T(t)x \rangle + \langle WT(t)x, AT(t)x \rangle \\ &= \langle AT(t)x, WT(t)x \rangle + \langle WT(t)x, AT(t)x \rangle \\ &= -\|T(t)x\|^2 \end{aligned}$$

since we are assuming that (L1) holds and we have $T(t)x \in D(A)$ whenever $x \in D(A)$ (see for instance [14]). Integrating with respect to s on $[0, t]$ we have

$$\langle WT(t)x, T(t)x \rangle - \langle Wx, x \rangle = -\int_0^t \|T(s)x\|^2 ds$$

which is equivalent to

$$\left\langle T^*(t)WT(t)x + \int_0^t T^*(s)T(s)x \, ds, x \right\rangle = \langle Wx, x \rangle,$$

for all $x \in D(A)$. Using the fact that the generator has dense domain we obtain that

$$\left\langle T^*(t)WT(t)x + \int_0^t T^*(s)T(s)x \, ds - Wx, x \right\rangle = 0$$

for all $x \in X$, which implies

$$\text{(L2)} \qquad T^*(t)WT(t)x + \int_0^t T^*(s)T(s)x\,ds = Wx, \quad \text{for all } t \geq 0, \ x \in X.$$

(See, for instance, [9, Exercise 16.46(b), page 253].) Following this idea it can be seen also that it is not required that $WD(A) \subset D(A^*)$.

It is easy to check that if (L2) holds then (L1) is also true. Now we recall:

Definition 1. A family of bounded linear operators $\mathcal{U} = \{U(t,s)\}_{t \geq s > 0}$ is called an evolution family if the following statements hold:

(e_1) $U(t,t) = I$, for all $t \geq 0$.
(e_2) $U(t,s)U(s,r) = U(t,r)$, for all $t \geq s \geq r \geq 0$.
(e_3) There are $M, \omega > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \quad \text{for all} \quad t \geq s \geq 0.$$

We note that many authors impose some strong continuity hypotheses on evolution families. Here we do not need any continuity assumptions, so we can extend the area of application of our results to a larger class of evolution families.

Remark 1. If $T = \{T(t)\}_{t \geq 0}$ is a C_0-semigroup then the family $U(t,s) = T(t-s)$ is an evolution family.

Definition 2. An evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ is called uniformly exponentially stable if there exist $N, \nu > 0$ such that

$$\|U(t,s)\| \leq Ne^{-\nu(t-s)}, \quad \text{for all} \quad t \geq s \geq 0.$$

Definition 3. A map $H : \mathbb{N} \to B(X)$ is called uniformly positive if there exists $a > 0$ such that

$$\langle H(m)x, x \rangle \geq a\|x\|^2, \quad \text{for all} \quad m \in \mathbb{N}, \ x \in X.$$

For such a map H, motivated by (L2) and Remark 1 we consider

$$\text{(L}_H) \qquad 0 = U^*(n+m,m)W(m+n)U(n+m,m)$$
$$+ \sum_{k=1}^{n-1} U^*(k+m,m)H(m)U(k+m,m)$$
$$- W(m)$$

for all $m \in \mathbb{N}$, $n \in \mathbb{N}^*$. We claim this equation to be a discrete time variant of (L) for the case of general evolution families. In many mathematical situations it is desirable to find discrete time formulations of the problems in order to find numerical algorithms or to make it easier to verify results. Thus, we are willing to give here a discrete time form for our results, establishing in fact a discrete characterization for the uniform exponential stability of the general case of continuous time evolution families.

Remark 2. If $\mathcal{U}$ is an uniformly exponentially stable evolution family then the equation (L$_H$) has at most one solution. In fact we have that if $W_1, W_2 : \mathbb{N} \to B(X)$ with $\sup_{n \in \mathbb{N}} \|W_i(n)\| < \infty$, $i \in \{1,2\}$ are solutions of (L$_H$) then

$$\|W_1(m) - W_2(m)\| \leq \|U^*(n+m,m)\| \ \|W_1(m+n) - W_2(m+n)\| \ \|U(n+m,m)\|$$
$$\leq \sup_{k \in \mathbb{N}} \|W_1(k) - W_2(k)\|N^2e^{-2\nu n}, \quad \text{for all} \quad m, n \in \mathbb{N}.$$

Hence $W_1 = W_2$.

Lemma 1. *If $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is an evolution family and $(a_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers that tends towards 0 and $\|U(m+n, m)\| \leq a_n$ for all $m, n \in \mathbb{N}$, then $\mathcal{U}$ is exponentially stable.*

Proof. Let

$$n_0 = \inf\left\{n \in \mathbb{N}^* : a_n \leq \frac{1}{e}\right\}, \quad s \geq 0, \ t \geq 2n_0.$$

Let m, n be two natural numbers such that $s \in [m, m+1)$, $\frac{t}{n_0} \in [n, n+1)$. Then $t + s \geq nn_0 + m \geq 1 + m \geq s$ and $U(t+s, s) = U(t+s, nn_0 + m)U(nn_0 + m, s)$ which implies that

$$\|U(t+s, s\| \leq \|U(t+s, nn_0 + m)\| \ \|U(nn_0 + m, s\|$$
$$\leq Me^{\omega(t - nn_0 + s - m)}\|U(nn_0 + m, m+1)\| \ \|U(m+1, s)\|$$
$$\leq M^2 e^{3\omega} \prod_{k=0}^{n-2} \|U(m+1+(k+1)n_0, m+1+kn_0)\|$$
$$\cdot \|U(nn_0 + m, (n-1)n_0 + m + 1)\|$$
$$\leq M^3 e^{3\omega + n_0 - 1} e^{-(n-1)}$$
$$\leq M^3 e^{3\omega + n_0 - \frac{t}{n_0} + 1}.$$

It is also easy to check that

$$\|U(t+s, s)\| \leq Me^{\omega t} \leq Me^{2n_0\omega} \leq Me^{2n_0\omega + 2 - \frac{t}{n_0}}$$

for all $s \geq 0$ and all $t \in [0, 2n_0]$. It follows that

$$\|U(t+s, s)\| \leq Ne^{-\nu t}, \quad \text{for all} \quad t, s \geq 0$$

where $N = \max\{M^3 e^{3\omega + n_0 + 1}, Me^{2n_0\omega + 2}\}$, $\nu = \frac{1}{n_0}$. $\qquad \square$

Theorem 1. *An evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is uniformly exponentially stable if and only if there exists $H : \mathbb{N} \to B(X)$ uniformly positive such that the equation (L_H) has a positive solution $W : \mathbb{N} \to B(X)$ with $\sup_{m \in \mathbb{N}} \|W(m)\| < \infty$.*

Proof. *Necessity.* Let $H, W : \mathbb{N} \to B(X)$ given by

$$H(m) = I, \ W(m) = \sum_{k=0}^{\infty} U^*(k+m, m)U(k+m, m).$$

It is clear that

$$\langle H(m)x, x \rangle \geq \|x\|^2, \quad \text{for all} \quad m \in \mathbb{N} \quad \text{and all} \quad x \in X$$

and

$$\|W(m)\| \leq \sum_{k=0}^{\infty} \|U(k+m, m)\|^2 \leq \sum_{k=0}^{\infty} N^2 e^{-2\nu k} = \frac{N^2}{1 - e^{-2\nu}} \quad \text{for all} \quad m \in \mathbb{N}.$$

On the other hand

$$
U^*(m+n,m)W(m+n)U(m+n,m)
$$
$$
= \sum_{k=0}^{\infty} U^*(m+n,m)U^*(k+m+n,m+n)U(k+m+n,m+n)U(m+n,m)
$$
$$
= \sum_{k=0}^{\infty} U^*(k+m+n,m)U(k+m+n,m)
$$
$$
= \sum_{j=n}^{\infty} U^*(j+m,m)U(j+m,m)
$$

for all $m,n \in \mathbb{N}$ and hence

$$
U^*(m+n,m)W(m+n)U(m+n,m) + \sum_{k=0}^{n-1} U^*(k+m,m)H(m)U(k+m,m)
$$
$$
= \sum_{k=0}^{\infty} U^*(k+m,m)U(k+m,m)
$$
$$
= W(m)
$$

for all $m \in \mathbb{N}$, $n \in \mathbb{N}^*$, which shows that (L_H) has a positive solution with $\sup_{m \in \mathbb{N}} \|W(m)\| < \infty$. Using Remark 2 we obtain what is required.

Sufficiency. Let $K = \sup_{m \in \mathbb{N}} \|W(m)\|$.

$$
\sum_{k=0}^{n-1} \|U(k+m,m)x\|^2
$$
$$
\leq \sum_{k=0}^{n-1} \frac{1}{a}\langle H(m)U(k+m,m)x, U(k+m,m)x\rangle
$$
$$
= \left\langle \sum_{k=0}^{n-1} \frac{1}{a}U^*(k+m,m)H(m)U(k+m,m)x, x \right\rangle
$$
$$
= \frac{1}{a}\langle W(m)x, x\rangle - \frac{1}{a}\langle U^*(m+n,m)W(m+n)U(m+n,m)x, x\rangle
$$
$$
\leq \frac{K}{a}\|x\|^2 - \frac{1}{a}\langle W(m+n)U(m+n,m)x, U(m+n,m)x\rangle
$$
$$
\leq \frac{K}{a}\|x\|^2
$$

for all $m \in \mathbb{N}$, $n \in \mathbb{N}^*$, $x \in X$. It follows that

$$
\sum_{k=0}^{\infty} \|U(k+m,m)x\|^2 \leq \frac{K}{a}\|x\|^2, \quad \text{for all} \quad m \in \mathbb{N}, \ x \in X,
$$

and hence $\|U(k+m,m)\|^2 \leq \dfrac{K}{a}$, for all $m, k \in \mathbb{N}$. Next, we have that

$$
\begin{aligned}
(n+1)\|U(n+m,m)x\|^2 &= \sum_{k=0}^{n} \|U(n+m,m)x\|^2 \\
&\leq \sum_{k=0}^{n} \|U(n+m,m+k)\|^2 \ \|U(k+m,m)x\|^2 \\
&\leq \frac{K}{a} \sum_{k=0}^{n} \|U(k+m,m)x\|^2 \\
&\leq \frac{K}{a} \sum_{k=0}^{\infty} \|U(k+m,m)x\|^2 \\
&\leq \frac{K^2}{a^2} \|x\|^2 \qquad \text{for all } m, n \in N, \ x \in X,
\end{aligned}
$$

and so

$$
\|U(m+n,m)\| \leq \frac{K}{a\sqrt{n+1}}
$$

for all $m, n \in \mathbb{N}$. By Lemma 1, U is uniformly exponentially stable. $\square$

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