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Regular AF subalgebras of some crossed products

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Abstract. Let $C(X) \rtimes_T \mathsf{Z}$ be the crossed product associated to a dynamical system (X, T). We characterize the regular AF subalgebras of $C(X) \rtimes_T \mathsf{Z}$ that can arise as the algebra $A_Y = \langle C(X), uC_0(X \backslash Y)\rangle$ for some closed subset Y of X. We also characterize the minimal homeomorphisms in A_Y terms.

Contents

1. Regular AF subalgebras

Let X be the Cantor set and let $T : X \to X$ be a minimal homeomorphism. Denote by u the canonical unitary implementing the action $\mathsf{Z} \ni n \to \alpha_T^n \in \mathrm{Aut}(C(X))$, where $\alpha_T : C(X) \to C(X)$ is given by $\alpha_T(f) = f \circ T^{-1}$ for every $f \in C(X)$ and by $C(X) \rtimes_T \mathsf{Z}$ the crossed product $C(X) \rtimes_{\alpha_T} \mathsf{Z}$. Set $C_0(X \backslash Y) = \{f \in C(X) | f|_Y = 0\}$. Recall the following theorem of Putnam ([2]):

Theorem 1.1. *For any nonempty closed subset Y of X, the C*-subalgebra of $C(X) \rtimes_T \mathsf{Z}$ generated by $C(X)$ and $uC_0(X \backslash Y)$ is an AF algebra.*

Set $A_Y = \langle C(X), uC_0(X \backslash Y)\rangle$. For any C*-subalgebra A of $C(X) \rtimes_T \mathsf{Z}$ containing $C(X)$, the normalizer of $C(X)$ in $U(A)$ is given by $N(C(X), A) = \{v \in U(A) | vC(X)v^* = C(X)\}$. (Here $U(A)$ is the set of unitary elements in A.) The algebra A is called *regular* if $N(C(X), A)$ generates A. It is

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well-known that A_Y is regular. Another property of A_Y is contained in the following result of Poon (see [1]):

Theorem 1.2. *Let Y be a nonempty subset of X. For every $n \geqslant 1$ and any clopen subset W of X one has that $u^n\chi_W (= u\chi_{T^{n-1}(W)}u\chi_{T^{n-2}(W)}\dots u\chi_W) \in A_Y$ if and only if $u\chi_{T^{n-1}(W)}, u\chi_{T^{n-2}(W)}, \dots, u\chi_W \in A_Y$. (Here χ_W denotes the characteristic function of W.)*

We say that a C*-subalgebra A of the crossed product $C(X) \rtimes_T \mathsf{Z}$ has property $\star$ if for any $n \geqslant 1$ and any clopen subset W of X,

$$u\chi_{T^{n-1}(W)}, u\chi_{T^{n-2}(W)}, \dots, u\chi_W \in A$$

whenever $u^n\chi_W \in A$.

Theorem 1.3. *For any regular AF subalgebra A of $C(X) \rtimes_T \mathsf{Z}$ that contains $C(X)$ and has property $\star$, there is a closed nonempty set Y of X such that $A_Y = A$.*

Let us find the natural candidate for Y. Consider the set

$$I = \{f \in C(X) \mid uf \in A\} \subseteq C(X).$$

Lemma 1.4. *The subset I is a closed ideal of $C(X)$.*

Proof. If $f, g \in I$ then $uf, ug \in A$, and since A is an algebra $u(f+g) = uf + ug \in A$, so $f + g \in I$. If $f \in I$ and $g \in C(X)$ then $uf \in A$ and since $g \in C(X) \subseteq A$ we get $ufg \in A$, hence $fg \in I$. Remark that since $g \circ T^{-1} \in C(X) \subseteq A$ and $uf \in A$ we get that $ugf = (g \circ T^{-1})uf \in A$, hence $gf \in I$. Suppose that $f_n \to f$ and $f_n \in I$. Then $uf_n \to uf$, and since $uf_n \in A$ it follows that $uf \in A$, hence $f \in I$. □

Since I is an ideal in $C(X)$ we know that $I = C_0(X \backslash Y)$ for some closed subset Y of X. Since A is AF we see that u is not in A, so Y is not empty.

Lemma 1.5. *The inclusion $A_Y \subseteq A$ is always valid.*

Proof. It is sufficient to show that the generators of A_Y are in A. By the definition of A we know that $C(X) \subseteq A$. If $f \in C_0(X \backslash Y) = I$ then $uf \in A$, hence $uC_0(X \backslash Y) \subseteq A$. Therefore $A_Y \subseteq A$. □

Lemma 1.6. *The inclusion $A \subseteq A_Y$ holds.*

Proof. Since A is a regular algebra it is enough to show that each unitary in $N(C(X), A)$ is also in A_Y. Letting v be a unitary in $N(C(X), A)$, it is known from Lemma 5.1 of [2] that $v = f \sum_{n \in \mathsf{Z}} p_n u^n$, where f is a unitary in $C(X)$, every p_n is a projection in $C(X)$, only finitely many p_n are different than 0, $p_n p_k = 0$ for $n \neq k$ and $\sum_{n \in \mathsf{Z}} p_n = 1$. Since the projections p_n are orthogonal, we obtain that $p_n u^n \in A$ for every $n \in \mathsf{Z}$.

We show that $p_n u^n \in A_Y$. Suppose that $n \geqslant 1$. If $p_n = \chi_W$ for a clopen subset W of X, from the chain of equalities $p_n u^n = u^n(p_n \circ T^n) = u^n \chi_{T^{-n}(W)} = u\chi_{T^{-1}(W)} u\chi_{T^{-2}(W)} \dots u\chi_{T^{-n}(W)}$ and the $\star$ condition,

$$u\chi_{T^{-1}(W)}, u\chi_{T^{-2}(W)}, \dots, u\chi_{T^{-n}(W)} \in A.$$

Hence we see that $\chi_{T^{-1}(W)}, \chi_{T^{-2}(W)}, \dots, \chi_{T^{-n}(W)} \in I = C_0(X \backslash Y)$. From the decomposition $p_n u^n = u\chi_{T^{-1}(W)} u\chi_{T^{-2}(W)} \dots u\chi_{T^{-n}(W)}$ conclude that $p_n u^n \in A_Y$. If $p_{-n} u^{-n} \in A$ for some $n \geqslant 1$, we note that $u^n p_{-n} = (p_{-n} u^{-n})^* \in A$, and as above one gets that $u^n p_{-n} \in A_Y$, and therefore $p_{-n} u^{-n} \in A_Y$. □

Remark 1.7. (1) Condition $\star$ can not be removed. To this end, let T be a minimal homeomorphism of X such that T^2 is also minimal. We embed $C(X) \rtimes_{T^2} \mathsf{Z}$ into $C(X) \rtimes_T \mathsf{Z}$ via $C(X) \ni f \to f \in C(X)$ and $v \to u^2$, where v is the canonical unitary in $C(X) \rtimes_{T^2} \mathsf{Z}$ implementing the action generated by T^2. Let Y be a closed, nonvoid subset of X, with $Y \neq X$, and consider the regular AF subalgebra $\langle C(X), vC_0(X \backslash Y)\rangle$ of $C(X) \rtimes_{T^2} \mathsf{Z}$. We can see this subalgebra as $\langle C(X), u^2 C_0(X \backslash Y)\rangle$ in the crossed product $C(X) \rtimes_T \mathsf{Z}$. The last algebra is then again regular and AF, not satisfying the condition $\star$ and not of the form $\langle C(X), uC_0(X \backslash Z)\rangle$ for any nonempty closed subset Z of X.

(2) If $\star$ is removed and Y is a singleton it was noted by Poon (see Corollary 4.4 of [1]) that Lemma 1.6 still holds.

2. Minimality

In this section we give a characterization of the minimal homeomorphisms in terms of approximately finitely-dimensionality of A_Ys. If X is a zero-dimensional compact space and T a self-homeomorphism of X, not necessarily minimal, then we can still define the C*-subalgebra $\langle C(X), uC_0(X \backslash Y)\rangle$. It was proved in [1] that A_Y is an AF algebra iff $X = \bigcup_{n \in \mathsf{Z}} T^n(W)$ for every clopen subset W containing Y.

Proposition 2.1. *Let X be a zero-dimensional compact space and $T : X \to X$ a self-homeomorphism. Then T is minimal iff A_Y is AF for any closed nonempty subset Y of X.*

Proof. Suppose that A_Y is AF for any nonempty closed subset Y of X. If there is a proper closed subset Y of X such that $T(Y) = Y$ (i.e., $Y \neq \emptyset, Y \neq X$) then Y is not clopen. Pick an element $x_0 \in X \backslash Y$ and note again that $Y \bigcup \{x_0\} \neq X$ (otherwise Y were clopen). Let W be a clopen subset of X such that $Y \cup \{x_0\} \subseteq W \subseteq X$ and $Y \cup \{x_0\} \neq W$, $W \neq X$.

Since x_0 is not in Y and $x_0 \in W$ there is a clopen subset U such that $x_0 \in U \subseteq W$ and $U \cap Y = \emptyset$. But $W^c \bigcup \{x_0\}$ is a closed subset of X that is included in the clopen subset $W^c \bigcup U$. By Poon's result (cited above) one gets that

$X = \bigcup_{n\in\mathbb{Z}} T^n(W^c \bigcup U)$. Since Y is not empty let $y_0 \in Y$, we get that $y_0 \in T^{n_0}(W^c \bigcup U)$ for some $n_0 \in \mathbb{Z}$. We distinguish two cases. If $y_0 \in T^{n_0}(W^c)$, taking into account that also $y_0 \in Y$ implies that $y_0 \in T^{n_0}(Y)$, one gets that $y_0 \in T^{n_0}(W^c) \bigcap T^{n_0}(Y) = T^{n_0}(W^c \bigcap Y) = T^{n_0}(\emptyset) = \emptyset$, a contradiction. If $y_0 \in T^{n_0}(U)$ then again $y_0 \in T^{n_0}(U) \bigcap T^{n_0}(Y) = T^{n_0}(U \bigcap Y) = T^{n_0}(\emptyset) = \emptyset$, contradiction.

The other implication is always true by [2]. □

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