

# On certain conformally invariant systems of differential equations

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ABSTRACT. Several systems of differential operators are constructed and their study is commenced. These systems are generalizations, in a reasonable sense, of the Heisenberg Laplacian operators introduced by Folland and Stein. In particular, they admit large groups of conformal symmetries; various real form of the special linear groups, even special orthogonal groups, and the exceptional group of type  $E_6$  appear in this capacity.

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## 1. Introduction

This work is concerned with certain conformally invariant systems of differential equations and so we begin by defining this notion and making a few remarks about its history and significance. Let  $G$  be a Lie group that acts on a manifold  $M$  and  $\mathcal{E} \rightarrow M$  a vector bundle with a compatible  $G$  action.

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A system  $D_1, \dots, D_n$  of differential operators on sections of  $\mathcal{E}$  is conformally invariant with respect to  $G$  if there are  $c_{ij} \in C^\infty(G)$  such that

$$(1.1) \quad g \circ D_i \circ g^{-1} = \sum_{j=1}^n c_{ji}(g) D_j$$

for  $1 \leq i \leq n$  and all  $g \in G$ . It is often convenient to consider instead the infinitesimal version of (1.1), which requires that there are  $C_{ij} \in C^\infty(\mathfrak{g})$  such that

$$(1.2) \quad [X, D_i] = \sum_{j=1}^n C_{ji}(X) D_j$$

for  $1 \leq i \leq n$  and all  $X$  in the Lie algebra  $\mathfrak{g}$  of  $G$ . In fact, (1.2) is often the primary definition of conformal invariance, and makes sense even when there is no group action associated with the Lie algebra  $\mathfrak{g}$  of vector fields on  $M$ , but we shall not need to belabor the distinction here. Conformal invariance is one possible answer to the question of what to require of the system  $D_1, \dots, D_n$  so that the condition  $D_i \bullet \varphi = 0$  for  $1 \leq i \leq n$  has  $G$ -invariant significance. Note that it encompasses, but is more permissive than, the requirement that the system itself be  $G$ -invariant.

In the case  $n = 1$ , at least, conformal invariance has a long history in mathematics and mathematical physics. An outstanding example is the Laplacian  $\Delta$ , which is conformally invariant with respect to the group generated by the Euclidean motions and the Kelvin transform. At first, this example was considered on the manifold  $M = \mathbb{R}^n$ , with  $\Delta$  acting on functions on  $M$ , but this has the disadvantage that  $G$  acts by maps that are generally not everywhere defined. It was later realized that it was preferable to take  $M$  to be a suitable compactification of  $\mathbb{R}^n$ , with  $\Delta$  acting on sections of a line bundle over  $M$ . With the advantage of hindsight, we may say that  $M$  is a real generalized flag manifold for  $G$  and the line bundle is  $G$ -homogeneous. Many of the characteristic features of the theory of conformally invariant systems are prefigured in various chapters of the voluminous classical theory of the Laplacian. In addition to the discovery of the Kelvin transform and the associated fundamental solution, Maxwell's theorem giving the structure of the space of harmonic polynomials, and the discovery of symmetry reductions leading to other important equations of mathematical physics are particularly significant from the point of view of conformal invariance.

An important step forward in the study of conformally invariant operators was taken by Kostant [15]. In his contribution to [1], Kostant explains that he was motivated to undertake this work by Segal's observation that the space of solutions of the wave equation on spacetime is invariant under the action of the conformal group  $SU(2, 2)$ , even though the equation itself is not invariant. The explanation is the conformal invariance of the wave equation, which Kostant instead called quasi-invariance. Note that this is exactly the

signature-(3, 1) analogue of the conformal invariance of the Laplacian under the group generated by the Euclidean motions and the Kelvin transform, slightly disguised by the low-dimensional isomorphism  $SU(2, 2) \cong Spin(4, 2)$ . In this work, Kostant considered the situation in which  $M$  is a real generalized flag manifold and  $\mathcal{E}$  is a homogeneous line bundle. He showed that, in this setting, there is an intimate connection between conformally invariant operators and homomorphisms from one scalar generalized Verma module to another. With the benefit of hindsight once again, one may see traces of this relationship in Maxwell's work on harmonic polynomials, the module of harmonic polynomials being a quotient of a scalar generalized Verma module by the image of the homomorphism associated to the Laplacian by Kostant's theorem. Kostant's work reduces the discovery of conformally invariant operators on real generalized flag manifolds to the algebraic problem of understanding homomorphisms between scalar generalized Verma modules, or equivalently to the problem of constructing submodules of these modules when they exist.

The referee has suggested that Kostant's result be illustrated in the simplest possible case. Accordingly, in this paragraph and the next we shall take  $G = SL(2, \mathbb{R})$  and  $\bar{Q}$  to be the subgroup of  $G$  consisting of lower-triangular matrices. For  $u \in \mathbb{R}^\times$  and  $s \in \mathbb{C}$  we let  $|u|_+^s = |u|^s$  and  $|u|_-^s = \text{sgn}(u)|u|^s$ . For  $s \in \mathbb{C}$  and  $\varepsilon \in \{\pm\}$ , define  $\chi_{s,\varepsilon} : \bar{Q} \rightarrow \mathbb{C}^\times$  by

$$\chi_{s,\varepsilon} \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} = |a|_\varepsilon^s.$$

Let  $\mathbb{C}_{s,\varepsilon}$  denote  $\mathbb{C}$  as a  $\bar{Q}$ -module on which  $\bar{Q}$  acts by  $\chi_{s,\varepsilon}$ . Then

$$\mathcal{L}_{s,\varepsilon} = G \times_{\bar{Q}} \mathbb{C}_{-s,\varepsilon}$$

is a homogeneous line bundle over  $G/\bar{Q}$ . Let  $N$  denote the subgroup of  $G$  consisting of upper-triangular matrices with ones on the diagonal. The set  $N\bar{Q}/\bar{Q}$  is open and dense in  $G\bar{Q}$  and is diffeomorphic to  $\mathbb{R}$ . A smooth section of  $\mathcal{L}_{s,\varepsilon}$  restricted to this set may be regarded as a smooth function on  $\mathbb{R}$ . The (partially defined) action of  $G$  on such functions is generated by the translation operators  $(T_p f)(u) = f(u + p)$  for  $p \in \mathbb{R}$  and the Kelvin transform  $\mathbb{K}_{s,\varepsilon}$  given by

$$(\mathbb{K}_{s,\varepsilon} f)(u) = |u|_\varepsilon^{-s} f\left(-\frac{1}{u}\right).$$

A smooth function  $f$  extends to a smooth section of  $\mathcal{L}_{s,\varepsilon}$  over  $G/\bar{Q}$  precisely when  $\mathbb{K}_{s,\varepsilon} f$  extends smoothly across zero. Note that  $\mathbb{K}_{s,\varepsilon}^{-1} = \varepsilon \mathbb{K}_{s,\varepsilon}$ .

The Lie algebra  $\mathfrak{g}$  of  $G$  has a standard basis  $H, X, Y$ , with commutation relations  $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$ . The Lie algebra of  $\bar{\mathfrak{q}}$  is spanned by  $H$  and  $Y$ . The Verma module associated to the bundle  $\mathcal{L}_{s,\varepsilon}$  is  $\mathcal{M}(s) = \mathcal{U}(\bar{\mathfrak{q}}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_s$ , where  $\mathbb{C}_s$  is  $\mathbb{C}$  with the  $\bar{\mathfrak{q}}$ -action  $H z = s z$  and  $Y z = 0$ . We write  $1_s$  for 1 when regarded as an element of  $\mathbb{C}_s$ . One verifies

the identities

$$\begin{aligned} H(X^m \otimes 1_s) &= (s + 2m)X^m \otimes 1_s, \\ Y(X^m \otimes 1_s) &= -m(s + m - 1)X^{m-1} \otimes 1_s, \end{aligned}$$

in  $\mathcal{M}(s)$  for  $m \geq 0$ . It follows that  $\mathcal{M}(1 - m)$  is reducible for  $m \geq 1$ , the cyclic submodule generated by  $X^m \otimes 1_s$  being proper. Equivalently, there is a nonzero homomorphism

$$\mathcal{M}(m + 1) \rightarrow \mathcal{M}(1 - m)$$

that maps  $1 \otimes 1_{m+1}$  to  $X^m \otimes 1_{1-m}$ . Kostant's theory assigns the operator  $\frac{\partial^m}{\partial u^m}$  to this homomorphism and predicts that it will be conformally invariant with respect to  $G$ . Indeed, this operator commutes with  $T_p$  for all  $p$  and satisfies the identity

$$\mathbb{K}_{1-m,\varepsilon} \circ \frac{\partial^m}{\partial u^m} \circ \mathbb{K}_{1-m,\varepsilon}^{-1} = u^{2m} \frac{\partial^m}{\partial u^m},$$

which confirms the conformal invariance. Note that  $\varepsilon$  plays no role in the Verma module nor in the conformal-invariance identities. However, it does reappear if one examines the solution space of the equation  $\frac{\partial^m f}{\partial u^m} = 0$ . Of course, this solution space is spanned by the set  $\{1, u, \dots, u^{m-1}\}$ . One confirms using the Kelvin transform that the elements of this set extend to smooth sections of  $\mathcal{L}_{1-m,+}$  if  $m$  is odd and of  $\mathcal{L}_{1-m,-}$  if  $m$  is even, but not vice versa. By conformal invariance, these solution spaces are  $G$ -invariant and hence afford representations of  $G$ . The possibility of constructing representations in this manner is one of the motivations for the study of conformally invariant operators.

In [3], the result of Kostant that was just illustrated was generalized in two ways: first, by allowing the manifold  $M$  to be more general than a real generalized flag manifold and, secondly, by considering systems in place of single operators. At least in currently published work, no use has yet been made of the first generalization, although there are in fact a number of interesting situations in which it can be applied. With the focus remaining for the moment on real generalized flag manifolds, the second generalization, to systems, has proved to be fruitful. The paper [2] is, in a sense, a companion to [3]. In it, conformally invariant systems on the generalized flag manifolds associated with the Heisenberg parabolic subgroups are studied. One of the main results is the explicit construction of a sufficient number of conformally invariant systems on these manifolds to account for all first reducibility points in the associated families of scalar generalized Verma modules for all simple Lie algebras (excluding  $\mathfrak{sl}(2)$ , which has no Heisenberg parabolic). The majority of these conformally invariant systems consist of more than one operator, and so such a result would not be possible if one restricted oneself to considering conformally invariant operators.

One of the technical innovations of [2] was the introduction of a systematic method for constructing candidate conformally invariant systems. The

method went unnamed in [2], and has remained so in several subsequent works that made use of it. We now propose to call it the  $\tau$  method, in reference to the notation that is usually used to describe the central objects in the method. Following a suggestion of the author, Kubo [17, 18] generalized the  $\tau$  method and applied it to the construction of conformally invariant systems on a large class of generalized flag manifolds beyond those of Heisenberg type. The systems constructed by Kubo in this work consist of first-order or of second-order operators, and do not account for all first reducibility points in the relevant family of scalar generalized Verma modules. This is unsurprising, since the systems constructed in [2] include systems consisting of third-order or of fourth-order operators. The  $\tau$  method naturally produces candidates for such systems in the setting of Kubo's work also, but they have not yet been investigated. More recently still, the author has found a further generalization of the  $\tau$  method, which is sufficient to bring all known conformally invariant systems on generalized flag manifolds within its purview, and to produce many new candidate systems as well. This work will be reported elsewhere.

It is now time to explain how the present work connects to the general program some aspects of which have been described above. In the course of using the  $\tau$  method to construct conformally invariant systems in [2], a second-order conformally invariant operator associated to the algebras of type A was found which had no analogue in the other types. This is explained in Theorem 5.3 of [2] and the surrounding discussion. Here it is important to understand that going from the algebraic setting of [2] to a concrete conformally invariant system of differential operators on a real generalized flag manifold involves one additional choice, that of a suitable real form for the complex algebra in which the algebraic construction has been carried out. Thus a single abstract conformally invariant system typically gives rise to many concrete examples, each associated with a certain real form. In this way, the anomalous abstract conformally invariant operator identified in Theorem 5.3 of [2] had to give rise to a second-order conformally invariant operator on the Heisenberg real generalized flag manifolds associated with the groups  $\mathrm{SL}(n, \mathbb{R})$  and  $\mathrm{SU}(p, q)$ . These manifolds have the Heisenberg group as a dense open subset and restricting to this subset places us in a situation that has received a lot of attention. Based on existing knowledge, it was natural to guess that, in the case of  $\mathrm{SU}(n+1, 1)$ , the associated operator was the Heisenberg Laplacian

$$L_s^{n,0} = - \sum_{j=1}^n \left( \frac{\partial^2}{\partial w_j \partial \bar{w}_j} - i w_j \frac{\partial^2}{\partial w_j \partial t} + i \bar{w}_j \frac{\partial^2}{\partial \bar{w}_j \partial t} \right) - |w|^2 \frac{\partial^2}{\partial t^2} + i s \frac{\partial}{\partial t}$$

on the Heisenberg group  $H_n = \mathbb{C}^n \oplus \mathbb{R}$  for some value of the complex parameter  $s$ , and a computation confirmed that this was so. The Heisenberg Laplacians have received a great deal of attention since their introduction

by Folland and Stein [7] in connection with a problem of complex analysis. The conformal invariance of these operators with respect to  $SU(n+1, 1)$  was already well known and a number of different arguments had been given for it. The notes of Greiner and Koornwinder [8] are a particularly good source for this perspective on the Heisenberg Laplacians. It is also possible to consider signed Heisenberg Laplacians  $L_s^{p,q}$  with  $p+q=n$ , obtained by modifying  $L_s^{n,0}$  in the usual way. These operators are conformally invariant with respect to the group  $SU(p+1, q+1)$  and, for a suitable value of  $s$ , are precisely the operators obtained from the abstract conformally invariant operator when it is interpreted in the appropriate real form.

At this point, two questions naturally presented themselves. First, could one obtain the entire family  $L_s^{p,q}$  from the  $\tau$  method instead of being restricted to a single value of  $s$ ? Secondly, when viewed from the perspective of the  $\tau$  method, what was the essential ingredient that made the construction of these conformally invariant operators possible and could it be generalized to yield interesting new conformally invariant operators or systems? It turned out that the key to both questions was the fact that the Heisenberg parabolic in type A is not maximal, whereas it is so in all other types. A careful analysis of the  $\tau$  method applied to nonmaximal parabolic subalgebras revealed that it always yields candidate systems that depend on one or more parameters, and that among them there are always anomalous systems whose construction generalizes that of the Heisenberg Laplacians. It is not possible to be more precise here without an extended discussion of the  $\tau$  method, but three facts about the candidate systems can be stated. Let  $S$  be the set of nodes in the Dynkin diagram that must be deleted to obtain the diagram associated with the nonmaximal parabolic subalgebra. Then the number of parameters on which the anomalous systems depend is  $|S| - 1$ , the degree of the operators in the systems is the sum of the coefficients of the simple roots in  $S$  in the highest root, and the number of operators in the system is equal to the dimension of the center of the nilradical of the parabolic subalgebra. The purpose of the current work is to begin the analysis of these anomalous systems in the simplest possible case, that where  $S$  has two elements (so that the systems depend on a single parameter) both occurring with coefficient 1 in the highest root (so that the systems consist of second order operators).

To describe the contents of the present work in more specific terms, we must assume some familiarity with Lie theory. As just indicated, the essential data for the construction of the systems considered here are a complex simple Lie algebra  $\mathfrak{g}$  and two simple roots  $\alpha$  and  $\beta$  that occur in the highest root of  $\mathfrak{g}$  with coefficient 1. The list of all possible choices of such data may be found in Section 2. To this data there is associated a parabolic subalgebra  $\mathfrak{q}$  of  $\mathfrak{g}$ , which we may write in the customary notation as  $\mathfrak{l} \oplus \mathfrak{n}$ . Here  $\mathfrak{l}$  is a reductive Lie algebra with two-dimensional center and  $\mathfrak{n}$  is a two-step nilpotent Lie algebra that is also an  $\mathfrak{l}$ -module. As such, it decomposes into a direct sum  $\mathfrak{n} = V \oplus \mathfrak{z}(\mathfrak{n})$  of  $\mathfrak{l}$ -submodules. The main purpose of Section 2

is to construct a family  $d\chi_z$  of characters of  $\mathfrak{l}$ , depending on a single complex parameter  $z$ , and a proper submodule of the generalized Verma module  $\mathcal{M}(d\chi_z) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\chi_z}$ . This proper submodule may be thought of as corresponding to a nonzero homomorphism

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathfrak{z}(\mathfrak{n}) \rightarrow \mathcal{M}(d\chi_z)$$

of  $\mathcal{U}(\mathfrak{g})$ -modules. Although the  $\tau$  method was used to find the elements that generate this submodule, we do not explain how this was done here. The elements are simply written down and their essential properties are verified. The author intends to return to this point later in the context of explaining a generalization of the  $\tau$  method beyond what may be found in [2] and [17, 18].

In Section 3, we explain how the construction made in Section 2 leads to a conformally invariant system with conformal group a suitable real Lie group  $G$  with complex Lie algebra  $\mathfrak{g}$ . Initially, the system is constructed on  $N$ , the nilpotent Lie group in  $G$  corresponding to the Lie algebra  $\mathfrak{n}$ . To this point, the construction can be carried out for any value of the parameter  $z$ . However, for later purposes, we are interested in extending the system to the generalized flag manifold  $G/\bar{Q}$  as a system of operators on the sections of a homogeneous line bundle  $\mathcal{L} \rightarrow G/\bar{Q}$ . Note that  $G/\bar{Q}$  is a compactification of  $N$ . For this to succeed we need to construct a character  $\chi_z$  of  $L$  whose differential is  $d\chi_z$  and, for some real forms, the existence of such a character restricts the possible values of  $z$ . The precise list of suitable real forms and the corresponding restrictions on  $z$ , if any, are given in Section 3.

At this point, the conformally invariant systems are, in some sense, in hand. However, nothing has been said about their specific shape nor about their properties. In Section 4, we focus on the systems that have been constructed in the split real forms and make them more explicit. For some purposes, such as the study of polynomial solutions, it is completely adequate to study only one real form. For other purposes, such as the study of  $K$ -finite solutions, there are substantial differences between the behavior of the various real forms of the same system, but it is still possible to transfer some information from one to another. This may partially justify our focusing on the split real form at this point. By a careful choice of coordinates on  $N$  (which is less symmetric than the usual choice) we are able to arrange that the central derivatives in the system (the analogues of the  $t$ -derivatives in  $L_s^{n,0}$ ) are all first order. That is, we are able to arrange for the terms analogous to the second  $t$ -derivative in  $L_s^{n,0}$  to disappear. This is extremely helpful in the analysis of the system. Let  $m$  be the dimension of  $\mathfrak{z}(\mathfrak{n})$ . Then the conformally invariant system we are considering consists of  $m$  operators and takes the shape

$$(1.3) \quad \overrightarrow{\Delta} + ((z - z_0)I - F) \overrightarrow{\partial}$$

when written in vector form. Here  $\overrightarrow{\Delta}$  is a column vector of second-order operators on  $V$ , each resembling a Euclidean ultrahyperbolic operator,  $\overrightarrow{\partial}$  is the column vector of central derivatives,  $F$  is an  $m$ -by- $m$  matrix of first order operators on  $V$ , and  $z_0$  is a constant depending on the original data that was used to construct the system.

Since the present construction encompasses the operator  $L_s^{n,0}$  and related operators, we may look to the theory of these operators for guidance about what questions to ask and what answers to expect for the systems in general. A prominent feature that is encountered early on in the theory of  $L_s^{n,0}$  is the existence of isolated undesirable values for the parameter  $s$ . What makes these values undesirable is that when  $s$  takes one of these values, the operator  $L_s^{n,0}$  can exhibit unwanted anomalous behavior. Although the precise set of bad values varies from property to property, it appears always to be a subset of  $n + 2\mathbb{Z}$ . For example, Folland and Stein [7] constructed a fundamental solution for  $L_s^{n,0}$  when  $s \notin n + 2\mathbb{Z}$ , but showed that  $L_s^{n,0}$  may fail to have a fundamental solution when  $s \in n + 2\mathbb{Z}$ . A similar phenomenon occurred for the Heisenberg ultrahyperbolic operator (which corresponds to the same abstract conformally invariant operator as does  $L_s^{n,0}$ , but in the split real form) in [13]. To compare this work with the results of Folland and Stein, it is necessary to notice that a difference in normalization means that  $s = 2z$  and  $z_0 = n/2$ , so that the condition  $s \in n + 2\mathbb{Z}$  corresponds to the condition  $z \in z_0 + \mathbb{Z}$ . In [14], it was shown that a certain  $\mathcal{U}(\mathfrak{g})$ -module naturally associated to the Heisenberg ultrahyperbolic operator is irreducible when  $z \notin z_0 + \mathbb{Z}$  but may become reducible when  $z \in z_0 + \mathbb{Z}$ . For the Heisenberg ultrahyperbolic equation, this dichotomy turns out to be closely related to the question of whether solutions to the equation are determined by their initial conditions, where we regard the central coordinate  $t$  in the equation as being analogous to time. The form of (1.3) suggests that a similar analogy may be fruitful here, since (1.3) can be rearranged to resemble an evolution equation with multiple times.

The observations made in the previous paragraph lead us to consider the restriction map  $p \mapsto p|_V$  on formal power series solutions to the system (1.3) and ask whether it is injective. Many of the classes of solutions that we shall eventually be interested in are analytic and so results on formal power series solutions are applicable to them. Also note that this is one instance where results on the split real form can be transferred to other real forms, since the formal power series for different real forms will be related by a complex change of variables. The form of the system given above implies that the restriction map is injective if and only if  $z \notin z_0 + \sigma(F)$ , where  $\sigma(F)$  denotes the spectrum of the operator  $F$  on  $\mathbb{C}[[V]]^m$ . Thus the set  $z_0 + \sigma(F)$  plays the role of the undesirable set for the restriction problem. This, in turn, leads us to seek to determine  $\sigma(F)$  and also to hope that  $\sigma(F) \subset \mathbb{Z}$ , as the examples described above would suggest.



In the last section, Section 5, we turn our attention to the problem of determining the spectrum  $\sigma(F)$ . In most cases, we are able to do so exactly. In one case, we presently have to settle for a bound on  $\sigma(F)$  that may be too large by a finite number of values. We use three different techniques to study  $\sigma(F)$ , which is why Section 5 involves much more case-by-case argument than the other sections. The first technique is based upon Umeda’s Capelli adjoint identity [19]. This is a noncommutative analogue of the method of finding the eigenvalues of an operator by considering its characteristic polynomial and so is very natural. Unfortunately, it only gives bounds on  $\sigma(F)$  in general, because in the noncommutative case the characteristic polynomial can have roots that are not eigenvalues. The second method is based upon finding links between the equation  $F\varphi = \lambda\varphi$  and the de Rham complex on  $V$ . It gives the precise value of  $\sigma(F)$  when it succeeds, but the author does not know how to push it through in all cases. The third method is based on invariant theory and the fact that the operator  $F$  commutes with the action of  $L$  on  $\mathbb{C}[V]^m$  that arises from identifying this module with  $\mathbb{C}[V] \otimes \mathfrak{z}(\mathfrak{n})^*$ . Again, it gives the precise value of  $\sigma(F)$  when it succeeds, but it requires explicit information about highest weight vectors in  $\mathbb{C}[V] \otimes \mathfrak{z}(\mathfrak{n})^*$  that is awkward to obtain in some degenerate cases. In a sense, the last method is the most principled and it is likely that it can be used to determine  $\sigma(F)$  in the cases where it isn’t precisely determined here, but we leave this as a project for the future. In all cases,  $\sigma(F)$  contains  $\mathbb{N}$  and  $\sigma(F) - \mathbb{N}$  is a finite set of negative integers. In particular,  $\sigma(F) \subset \mathbb{Z}$  as had been hoped.

The referee has suggested that we give an explicit example of one of the conformally invariant systems that is constructed in Sections 2 and 3. We shall do so for the system in type  $A_4$  that corresponds to the choice  $\alpha = e_2 - e_3$  and  $\beta = e_3 - e_4$  in the standard model for this root system. This system is the second in a family of systems that the author believes will prove particularly interesting from the point of view of the theory of special functions. As in our previous example, we write down the system on the group  $N$ . In this case,  $N$  consists of all 5-by-5 matrices of the form

$$n(x, y, t) = \begin{pmatrix} 1 & 0 & x_1 & t_1 & t_2 \\ 0 & 1 & x_2 & t_3 & t_4 \\ 0 & 0 & 1 & y_1 & y_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here  $x$  denotes the column vector  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $y$  denotes the row vector  $(y_1 \ y_2)$ , and  $t$  denotes the 2-by-2 matrix  $\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$ . In these coordinates, the associated system consists of the following four operators depending on the parameter  $z$ :

$$D_1 = \frac{\partial^2}{\partial x_1 \partial y_1} + \left( z - \frac{1}{2} \right) \frac{\partial}{\partial t_1} - x_1 \frac{\partial^2}{\partial x_1 \partial t_1} - x_2 \frac{\partial^2}{\partial x_1 \partial t_3},$$

$$\begin{aligned}
D_2 &= \frac{\partial^2}{\partial x_1 \partial y_2} + \left(z - \frac{1}{2}\right) \frac{\partial}{\partial t_2} - x_1 \frac{\partial^2}{\partial x_1 \partial t_2} - x_2 \frac{\partial^2}{\partial x_1 \partial t_4}, \\
D_3 &= \frac{\partial^2}{\partial x_2 \partial y_1} + \left(z - \frac{1}{2}\right) \frac{\partial}{\partial t_3} - x_1 \frac{\partial^2}{\partial x_2 \partial t_1} - x_2 \frac{\partial^2}{\partial x_2 \partial t_3}, \\
D_4 &= \frac{\partial^2}{\partial x_2 \partial y_2} + \left(z - \frac{1}{2}\right) \frac{\partial}{\partial t_4} - x_1 \frac{\partial^2}{\partial x_2 \partial t_2} - x_2 \frac{\partial^2}{\partial x_2 \partial t_4}.
\end{aligned}$$

Each of these operators is invariant under left translation on  $N$ . There is also a Kelvin transform  $\mathbb{K}_{z,\varepsilon_1,\varepsilon_2}$  that depends on the parameter  $z$  and two sign parameters  $\varepsilon_1, \varepsilon_2 \in \{\pm\}$ . It acts on functions on  $N$  by

$$\begin{aligned}
(\mathbb{K}_{z,\varepsilon_1,\varepsilon_2} f)(n(x, y, t)) &= |\det(t) - y \operatorname{adj}(t)x|_{\varepsilon_1}^{-1/2+z} |\det(t)|_{\varepsilon_2}^{-1/2-z} \\
&\quad \cdot f\left(n\left(-\frac{\operatorname{adj}(t)x}{\det(t) - y \operatorname{adj}(t)x}, \frac{y \operatorname{adj}(t)}{\det(t)}, \frac{\operatorname{adj}(t)}{\det(t)}\right)\right),
\end{aligned}$$

where  $\operatorname{adj}(t) = \det(t)t^{-1}$  is the classical adjoint of  $t$ . It has been written this way to emphasize the role of the polynomials  $\det(t)$  and  $\det(t) - y \operatorname{adj}(t)x$ . Note that we have  $\mathbb{K}_{z,\varepsilon_1,\varepsilon_2}^{-1} = \mathbb{K}_{z,\varepsilon_1,\varepsilon_2}$ . The group generated by the left translation operators and the Kelvin transform is isomorphic to  $G = \mathrm{SL}(5, \mathbb{R})$  and the system  $D_1, D_2, D_3, D_4$  is conformally invariant with respect to this group. The identities that express  $\mathbb{K}_{z,\varepsilon_1,\varepsilon_2} \circ D_j \circ \mathbb{K}_{z,\varepsilon_1,\varepsilon_2}^{-1}$  in terms of  $D_1, D_2, D_3, D_4$  are a little complicated in this case and so we refrain from writing them explicitly. They can be deduced from Theorem 3.6 if desired. Accompanying the Kelvin transforms are the basic solutions

$$\varphi_{z,\varepsilon_1,\varepsilon_2}(n(x, y, t)) = |\det(t) - y \operatorname{adj}(t)x|_{\varepsilon_1}^{-1/2+z} |\det(t)|_{\varepsilon_2}^{-1/2-z}$$

to the system. These are the analogues for this system of the solutions to the Heisenberg ultrahyperbolic equation that were studied in [13]. We remark also that there is a second real form of the system  $D_1, D_2, D_3, D_4$ , which is conformal with respect to the group  $G = \mathrm{SU}(3, 2)$  instead of  $G = \mathrm{SL}(5, \mathbb{R})$ .

## 2. Construction of the systems. I

The purpose of this section is to identify a suitable family of scalar generalized Verma modules and to construct proper submodules in each of these modules explicitly. The conformally invariant systems of differential equations that we wish to study will eventually derive from these submodules by application of the general theory developed in [3].

Let  $\mathfrak{g}$  be a finite-dimensional, complex, simple, simply-laced Lie algebra. Choose a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and let  $R = R(\mathfrak{g}, \mathfrak{h})$  be the root system of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Choose a positive system  $R^+$  in  $R$  and let  $R^s$  be the corresponding set of simple roots and  $R^- = -R^+$ . Let  $\prec$  be any total order on  $R^s$ . Let  $\mathbb{B}$  be a multiple of the Killing form of  $\mathfrak{g}$ . Then  $\mathbb{B}$  induces an inner product  $(\cdot, \cdot)$  on  $\mathfrak{h}^*$  and we normalize  $\mathbb{B}$  so that  $(\gamma, \gamma) = 2$  for all  $\gamma \in R$ .

Let  $\mathbb{Z}R^s$  denote the root lattice of  $R$  and define  $f : \mathbb{Z}R^s \times \mathbb{Z}R^s \rightarrow \mathbb{Z}$  to be the unique  $\mathbb{Z}$ -bilinear form that satisfies

$$f(\delta, \varepsilon) = \begin{cases} (\delta, \varepsilon) & \text{if } \delta \prec \varepsilon, \\ 1 & \text{if } \delta = \varepsilon, \\ 0 & \text{if } \delta \succ \varepsilon, \end{cases}$$

for  $\delta, \varepsilon \in R^s$ . Let  $e : \mathbb{Z}R^s \rightarrow \mathbb{Z}$  be given by

$$e(\xi) = \begin{cases} 1 & \text{if } \xi \in R^+, \\ -1 & \text{if } \xi \in R^-, \\ 0 & \text{if } \xi \notin R. \end{cases}$$

In terms of these functions, we define

$$(2.1) \quad N_{\mu, \nu} = e(\mu)e(\nu)e(\mu + \nu)(-1)^{f(\mu, \nu)}$$

for  $\mu, \nu \in R$ .

For  $\gamma \in R$ , denote by  $\mathfrak{g}_\gamma$  the root space associated to  $\gamma$ . It is known that we may choose  $X_\gamma \in \mathfrak{g}_\gamma$  for each  $\gamma \in R$  in such a way that  $[X_\mu, X_\nu] = N_{\mu, \nu}X_{\mu+\nu}$  whenever  $\mu + \nu \in R$ . Since  $N_{\mu, \nu} = 0$  when  $\mu + \nu \notin R$ , it does no harm to write the same equation even when  $\mu + \nu \notin R$ . For each  $\gamma \in R$ , we let  $H_\gamma = [X_\gamma, X_{-\gamma}]$ . Then  $X_\gamma, H_\gamma, X_{-\gamma}$  is an  $\mathfrak{sl}(2)$ -triple for all  $\gamma \in R$ . Moreover,  $\mathbb{B}(X_\gamma, X_{-\gamma}) = 1$  for all  $\gamma \in R$  and  $\mu(H_\nu) = (\mu, \nu)$  for all  $\mu, \nu \in R$ . It follows from this that  $\mathbb{B}(H_\mu, H_\nu) = (\mu, \nu)$  for all  $\mu, \nu \in R$ .

The following well-known fact will be needed below; note that the word “exactly” can be placed in the statement because  $R$  is simply laced.

**Lemma 2.1.** *Suppose that  $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in R$  are such that  $\gamma_1 + \gamma_2 \in R$ , no two of the  $\gamma_j$  are opposite, and  $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 0$ . Then exactly one of  $\gamma_1 + \gamma_3$  and  $\gamma_1 + \gamma_4$  lies in  $R$ .*

**Proof.** That at least one of  $\gamma_1 + \gamma_3$  and  $\gamma_1 + \gamma_4$  lies in  $R$  follows from the identity

$$N_{\gamma_1, \gamma_2}N_{\gamma_3, \gamma_4} - N_{\gamma_1, \gamma_3}N_{\gamma_2, \gamma_4} + N_{\gamma_1, \gamma_4}N_{\gamma_2, \gamma_3} = 0,$$

for which see Theorem 4.1.2 (iv) in [6], for example. Suppose that both  $\gamma_1 + \gamma_3$  and  $\gamma_1 + \gamma_4$  are roots. Then  $\gamma_2 + \gamma_3 = -(\gamma_1 + \gamma_4)$  is also a root. It follows that the angle between  $\gamma_1$  and  $\gamma_2$  is  $120^\circ$ , as are the angles between  $\gamma_1$  and  $\gamma_3$  and between  $\gamma_2$  and  $\gamma_3$ . From this we conclude that  $\gamma_1, \gamma_2, \gamma_3$  are coplanar and that they sum to 0. This is impossible, since it implies that  $\gamma_4 = 0$ , and the second claim is verified.  $\square$

Let  $\alpha$  and  $\beta$  be distinct simple roots and suppose that the coefficients of  $\alpha$  and  $\beta$  in the expression for the highest root as a combination of simple roots are both 1. We shall associate a one-parameter family of conformally invariant systems to this data. Before describing the construction of this family, we wish to delineate its scope by listing all the possibilities for  $R$ ,  $\alpha$ , and  $\beta$ . First,  $R$  must be of type  $A_l$  ( $l \geq 2$ ),  $D_l$  ( $l \geq 4$ ), or  $E_6$ . We number the simple roots in these systems using the conventions of Bourbaki [4]. For the

reader's convenience, the numbering is displayed in Figure 1. Secondly, we list the set from which  $\alpha$  and  $\beta$  may be drawn. For type  $A_l$ , it is  $\{\alpha_1, \dots, \alpha_l\}$ , for type  $D_l$  it is  $\{\alpha_1, \alpha_{l-1}, \alpha_l\}$ , and for type  $E_6$  it is  $\{\alpha_1, \alpha_6\}$ . The order of  $\alpha$  and  $\beta$  is irrelevant, so that we shall construct  $\binom{l}{2}$  families in type  $A_l$  (some of which are conjugate under an outer automorphism), three families in type  $D_l$  (two of which are conjugate under an outer automorphism), and a single family in type  $E_6$ .

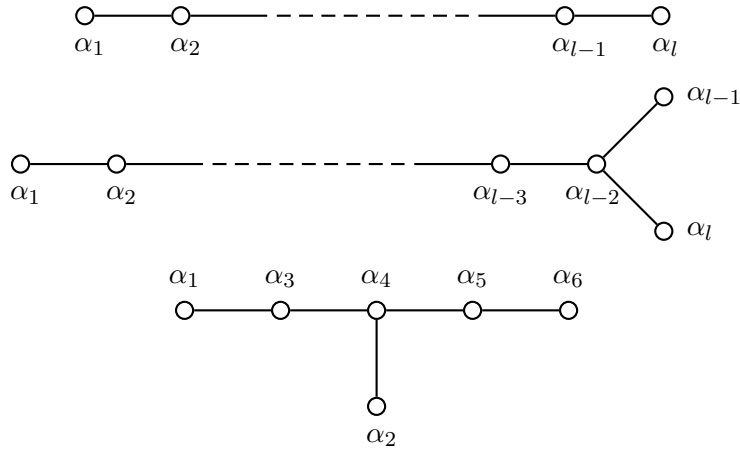
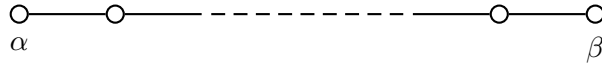


FIGURE 1. Numbering of the simple roots.

We now return to the construction. Let  $C_{\alpha\beta} \subset R^s$  be the minimal connected set of vertices in the Dynkin diagram that contains both  $\alpha$  and  $\beta$ . By inspection, the induced graph on  $C_{\alpha\beta}$  always has the form



of a Dynkin diagram of type  $A$  with  $\alpha$  and  $\beta$  at its ends. We assume that the total order  $\prec$  on  $R^s$  is chosen so that

$$C_{\alpha\beta} = \{\delta \in R^s \mid \alpha \preceq \delta \preceq \beta\}$$

is the interval from  $\alpha$  to  $\beta$  and every subinterval of  $C_{\alpha\beta}$  is a connected set of vertices. Note that we may then enumerate  $C_{\alpha\beta}$  (from left to right in the above diagram) as

$$(2.2) \quad \alpha = \delta_0 \prec \delta_1 \prec \dots \prec \delta_{d-1} \prec \delta_d = \beta,$$

where  $d = d(\alpha, \beta)$  is the length of the path from  $\alpha$  to  $\beta$ . If  $C \subset R^s$  is any nonempty connected set of vertices in the Dynkin diagram then we write

$$\langle C \rangle = \sum_{\delta \in C} \delta.$$

Recall that  $\langle C \rangle \in R$  for all such  $C$  (see Corollary VI.1.6.3 in [4], for example).

There are unique elements  $H_0, Z_0 \in \mathfrak{h}$  such that  $\alpha(H_0) = 1$ ,  $\beta(H_0) = 1$ ,  $\alpha(Z_0) = 1$ ,  $\beta(Z_0) = -1$ , and  $\gamma(H_0) = \gamma(Z_0) = 0$  for all  $\gamma \in R^s - \{\alpha, \beta\}$ . The semisimple endomorphisms  $\text{ad}(H_0)$  and  $\text{ad}(Z_0)$  of  $\mathfrak{g}$  commute with one another and have integral eigenvalues. Thus they induce a  $\mathbb{Z}^2$ -grading of  $\mathfrak{g}$ . We write  $\mathfrak{g}(j, k)$  for the eigenspace on which  $\text{ad}(H_0)$  acts by  $j$  and  $\text{ad}(Z_0)$  by  $k$ . The assumption that the coefficients of  $\alpha$  and  $\beta$  in the expression for the highest root are both 1 implies that this  $\mathbb{Z}^2$ -grading reduces to

$$(2.3) \quad \mathfrak{g} = \mathfrak{g}(-2, 0) \oplus \mathfrak{g}(-1, -1) \oplus \mathfrak{g}(-1, 1) \oplus \mathfrak{g}(0, 0) \oplus \mathfrak{g}(1, -1) \oplus \mathfrak{g}(1, 1) \oplus \mathfrak{g}(2, 0).$$

There is a corresponding partition

$$R = R(-2, 0) \cup R(-1, -1) \cup R(-1, 1) \cup R(0, 0) \cup R(1, -1) \cup R(1, 1) \cup R(2, 0)$$

of  $R$ , where

$$R(j, k) = \{\gamma \in R \mid \mathfrak{g}_\gamma \subset \mathfrak{g}(j, k)\}.$$

The summand  $\mathfrak{g}(0, 0)$  in (2.3) is a subalgebra of  $\mathfrak{g}$  and the remaining summands are  $\mathfrak{g}(0, 0)$ -modules. For brevity and consistency with earlier work, we shall write  $\mathfrak{l} = \mathfrak{g}(0, 0)$ . The other summands in (2.3) are orthogonal to  $\mathfrak{l}$  with respect to the Killing form of  $\mathfrak{g}$  and hence with respect to  $\mathbb{B}$ . Consequently, the restriction of  $\mathbb{B}$  to  $\mathfrak{l}$  is nondegenerate and hence  $\mathfrak{l}$  is reductive. It follows from the construction that the center of  $\mathfrak{l}$  is spanned by  $H_0$  and  $Z_0$  and hence that

$$(2.4) \quad \mathfrak{l} = \mathbb{C}H_0 \oplus \mathbb{C}Z_0 \oplus \mathfrak{l}^{ss}$$

with  $\mathfrak{l}^{ss}$  semisimple. Moreover, the decomposition (2.4) is orthogonal with respect to  $\mathbb{B}$ .

For each  $\delta \in R^s$  let  $\varpi_\delta$  be the fundamental weight dual to  $\delta$ . Both  $\varpi_\alpha$  and  $\varpi_\beta$  vanish on  $\mathfrak{h} \cap \mathfrak{l}^{ss}$  and so we may extend them uniquely to elements of  $\mathfrak{l}^*$  that satisfy  $\varpi_\alpha(\mathfrak{l}^{ss}) = \varpi_\beta(\mathfrak{l}^{ss}) = \{0\}$ . We denote these extensions by the same symbols. Then  $\varpi_\alpha$  and  $\varpi_\beta$  span the space of characters of  $\mathfrak{l}$  and we write  $d\chi = z_\alpha \varpi_\alpha + z_\beta \varpi_\beta$  with  $z_\alpha, z_\beta \in \mathbb{C}$ . (The compound notation  $d\chi$  is chosen here because  $d\chi$  will eventually appear as the differential of a character  $\chi$  of a group.)

**Proposition 2.2.** *The  $\mathfrak{l}$ -modules  $\mathfrak{g}(\pm 2, 0)$  and  $\mathfrak{g}(\pm 1, \pm 1)$  are simple. Let  $\lambda = \langle C_{\alpha\beta} \rangle$ . The root vector  $X_\lambda$  is a lowest weight vector in  $\mathfrak{g}(2, 0)$ ,  $X_\alpha$  is a lowest weight vector in  $\mathfrak{g}(1, 1)$ , and  $X_\beta$  is a lowest weight vector in  $\mathfrak{g}(1, -1)$ . The root vector  $X_{-\lambda}$  is a highest weight vector in  $\mathfrak{g}(-2, 0)$ ,  $X_{-\alpha}$  is a highest weight vector in  $\mathfrak{g}(-1, -1)$ , and  $X_{-\beta}$  is a highest weight vector in  $\mathfrak{g}(-1, 1)$ .*

**Proof.** We verify that  $\mathfrak{g}(1, 1)$  is a simple  $\mathfrak{l}$ -module, since the others are similar. Consider the algebra

$$\mathfrak{r} = \mathfrak{g}(-1, -1) \oplus \mathfrak{l} \oplus \mathfrak{g}(1, 1).$$

(Note that it is an algebra, because  $\mathfrak{g}(2, 2) = \mathfrak{g}(-2, -2) = \{0\}$ .) The restriction of  $\mathbb{B}$  to  $\mathfrak{r}$  is nondegenerate and so  $\mathfrak{r}$  is reductive. The center of  $\mathfrak{r}$  is contained in the center of  $\mathfrak{l}$  and it follows from this observation that the

center of  $\mathfrak{r}$  is spanned by  $H_0 - Z_0$ . Thus  $\mathfrak{r}^{ss}$  has rank  $\dim(\mathfrak{h}) - 1$ . The Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is also a Cartan subalgebra of  $\mathfrak{r}$  and  $\mathfrak{h} \cap \mathfrak{r}^{ss}$  is a Cartan subalgebra of  $\mathfrak{r}^{ss}$ . The  $\mathbb{B}$ -orthogonal projection  $\mathfrak{h} \rightarrow \mathfrak{h} \cap \mathfrak{r}^{ss}$  induces an inclusion of  $(\mathfrak{h} \cap \mathfrak{r}^{ss})^*$  into  $\mathfrak{h}^*$  and identifies  $R(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$  as a subsystem of  $R(\mathfrak{g}, \mathfrak{h})$ . We have chosen a positive system in  $R(\mathfrak{g}, \mathfrak{h})$  and this induces a positive system in  $R(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$ . We have  $R^s - \{\alpha, \beta\} \subset R^s(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$  and  $R^s(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$  has precisely one more element than  $R^s - \{\alpha, \beta\}$ .

Suppose that  $\mu$  is a lowest weight for an irreducible submodule of  $\mathfrak{g}(1, 1)$ . Then  $\mu \in R(1, 1)$  and the restriction of  $\mu$  to  $\mathfrak{h} \cap \mathfrak{r}^{ss}$  lies in  $R^+(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$ . Suppose that  $\mu$  is decomposable as  $\mu = \mu_1 + \mu_2$  with  $\mu_1, \mu_2 \in R^+(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$ . Then the same equation holds in  $R(\mathfrak{g}, \mathfrak{h})$ . After switching the order of  $\mu_1$  and  $\mu_2$  if necessary, we may conclude that  $\mu_1 \in R(1, 1)$  and  $\mu_2 \in R(0, 0) \cap R^+$ . This contradicts the assumption that  $\mu$  is a lowest weight. Thus no such decomposition is possible and  $\mu \in R^s(\mathfrak{r}^{ss}, \mathfrak{h} \cap \mathfrak{r}^{ss})$ . The conclusion of the previous paragraph now implies that  $\mu$  is uniquely determined and hence that  $\mathfrak{g}(1, 1)$  is simple.

It is clear from construction that  $X_\alpha$  is a lowest weight vector in  $\mathfrak{g}(1, 1)$  and that  $X_\beta$  is a lowest weight vector in  $\mathfrak{g}(1, -1)$ . Recall that the support of a root in  $R^s$  is always a connected set of vertices in the Dynkin diagram (see Corollary VI.1.6.3 in [4]). If  $\delta \in R(2, 0)$  then  $\text{supp}(\delta)$  contains both  $\alpha$  and  $\beta$  and hence contains  $C_{\alpha\beta}$ . This makes the fact that  $X_\lambda$  is a lowest weight vector in  $\mathfrak{g}(2, 0)$  equally clear. The invariant form  $\mathbb{B}$  identifies  $\mathfrak{g}(-1, \pm 1)$  with the contragredient of  $\mathfrak{g}(1, \mp 1)$  and  $\mathfrak{g}(-2, 0)$  with the contragredient of  $\mathfrak{g}(2, 0)$ . This observation allows us to deduce the claims about  $\mathfrak{g}(-1, \pm 1)$  and  $\mathfrak{g}(-2, 0)$  from the information already obtained about  $\mathfrak{g}(1, \pm 1)$  and  $\mathfrak{g}(2, 0)$ .  $\square$

For  $\zeta \in R(2, 0)$  we define

$$S(\zeta) = \{(\mu, \nu) \in R(1, 1) \times R(1, -1) \mid \mu + \nu = \zeta\}.$$

**Lemma 2.3.** *For each  $\zeta \in R(2, 0)$ , the cardinality of the set  $S(\zeta)$  is  $d(\alpha, \beta)$ .*

**Proof.** Let  $\lambda = \langle C_{\alpha\beta} \rangle$ . The elements of  $S(\lambda)$  have the form  $(\langle C_\alpha \rangle, \langle C_\beta \rangle)$  where  $C_\alpha$  is a connected subset of  $R^s$  that contains  $\alpha$ ,  $C_\beta$  is a connected subset of  $R^s$  that contains  $\beta$ ,  $C_\alpha \cap C_\beta = \emptyset$ , and  $C_\alpha \cup C_\beta = C_{\alpha\beta}$ . Any such pair is determined by choosing the first element in the interval  $C_{\alpha\beta}$  that is not contained in  $C_\alpha$  and there are  $d(\alpha, \beta)$  choices for this element. This verifies the claim for  $S(\lambda)$ .

Now take  $\zeta \in R(2, 0) - \{\lambda\}$ . Since  $\zeta$  is not a lowest weight, we may find  $\delta \in R^s - \{\alpha, \beta\}$  such that  $\zeta - \delta \in R(2, 0)$ . It will suffice to show that  $S(\zeta)$  and  $S(\zeta - \delta)$  have the same number of elements, for then we may continue to descend until we reach  $S(\lambda)$ . Suppose that  $(\mu, \nu) \in S(\zeta - \delta)$ . Then  $\mu + \nu + \delta + (-\zeta) = 0$  and  $\mu + \nu \in R$ . By Lemma 2.1, it follows that exactly one of  $\mu + \delta$  and  $\mu - \zeta$  is a root. Consequently, exactly one of  $\mu + \delta$  and  $\nu + \delta$  is a root. If  $\mu + \delta$  is a root then  $(\mu + \delta, \nu) \in S(\zeta)$  and if  $\nu + \delta$  is a root then  $(\mu, \nu + \delta) \in S(\zeta)$ . A similar argument with  $-\delta$  in place of  $\delta$

shows that if  $(\mu', \nu') \in S(\zeta)$  then exactly one of  $\mu' - \delta$  and  $\nu' - \delta$  is a root. If  $\mu' - \delta$  is a root then  $(\mu' - \delta, \nu') \in S(\zeta - \delta)$  and if  $\nu' - \delta$  is a root then  $(\mu', \nu' - \delta) \in S(\zeta - \delta)$ . The resulting maps from  $S(\zeta)$  to  $S(\zeta - \delta)$  and back are mutually inverse and this completes the proof.  $\square$

**Lemma 2.4.** *Let  $\zeta \in R(2, 0)$ . Then we have*

$$\begin{aligned} \{\nu \in R(1, -1) \mid (\mu, \nu) \in S(\zeta) \text{ for some } \mu \in R(1, 1)\} \\ = \{\nu \in R(1, -1) \mid (\nu, \zeta) = 1\}. \end{aligned}$$

*A similar equation holds with the roles of  $R(1, 1)$  and  $R(1, -1)$  reversed.*

**Proof.** First suppose that  $(\mu, \nu) \in S(\zeta)$  for some  $\mu \in R(1, 1)$ . Then  $\zeta = \mu + \nu$  and the fact that the squared lengths of  $\mu$ ,  $\nu$ , and  $\zeta$  are all 2 implies that  $(\mu, \nu) = -1$ . Thus  $(\nu, \zeta) = (\nu, \mu + \nu) = -1 + 2 = 1$ , as required. Conversely, suppose that  $(\nu, \zeta) = 1$ . Then  $s_\zeta(\nu) = \nu - \zeta \in R$  and so  $\mu = \zeta - \nu \in R$ . Since  $\zeta \in R(2, 0)$  and  $\nu \in R(1, -1)$ , we have  $\mu \in R(1, 1)$  and then  $(\mu, \nu) \in S(\zeta)$ , as required. A similar argument establishes the last claim.  $\square$

Let

$$\mathfrak{n} = \mathfrak{g}(1, -1) \oplus \mathfrak{g}(1, 1) \oplus \mathfrak{g}(2, 0).$$

Note that  $\mathfrak{n}$  is a two-step nilpotent subalgebra of  $\mathfrak{g}$  with center  $\mathfrak{g}(2, 0)$ . Choose a basis  $\{X_i\}$  for  $\mathfrak{g}(1, 1)$  and let  $\{\bar{X}_i\}$  be the basis for  $\mathfrak{g}(-1, -1)$  that is  $\mathbb{B}$ -dual to  $\{X_i\}$ . Similarly, choose a basis  $\{Y_j\}$  for  $\mathfrak{g}(1, -1)$  and let  $\{\bar{Y}_j\}$  be the basis for  $\mathfrak{g}(-1, 1)$  that is  $\mathbb{B}$ -dual to  $\{Y_j\}$ . Let  $\mathcal{U}(\mathfrak{n})$  be the universal enveloping algebra of  $\mathfrak{n}$  and define a map  $\omega : \mathfrak{g}(2, 0) \rightarrow \mathcal{U}(\mathfrak{n})$  by

$$(2.5) \quad \omega(Z) = \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) X_i Y_j.$$

One checks that  $\omega$  is independent of the choice of the bases  $\{X_i\}$  and  $\{Y_j\}$ . Essentially, this is because of the appearance of the basis-dual-basis pairs in the sum.

Let  $\mathbf{G} = \text{Aut}(\mathfrak{g})^\circ$  be the connected component of the identity in the automorphism group of  $\mathfrak{g}$  and  $\mathbf{L} \subset \mathbf{G}$  be the connected subgroup with Lie algebra  $\mathfrak{l}$ . The group  $\mathbf{L}$  acts adjointly on both  $\mathfrak{g}(2, 0)$  and  $\mathcal{U}(\mathfrak{n})$ .

**Lemma 2.5.** *If  $l \in \mathbf{L}$  and  $Z \in \mathfrak{g}(2, 0)$  then  $\omega(\text{Ad}(l)Z) = \text{Ad}(l)\omega(Z)$ .*

**Proof.** Let  $\{X_i\}$ ,  $\{\bar{X}_i\}$ ,  $\{Y_j\}$ , and  $\{\bar{Y}_j\}$  be chosen as above. The form  $\mathbb{B}$  is  $\text{Ad}(\mathbf{G})$ -invariant and this implies that  $\{\text{Ad}(l)X_i\}$ ,  $\{\text{Ad}(l)\bar{X}_i\}$ ,  $\{\text{Ad}(l)Y_j\}$ , and  $\{\text{Ad}(l)\bar{Y}_j\}$  are also basis-dual-basis pairs. We have observed that  $\omega$  is independent of the choice of bases and so

$$\begin{aligned} \omega(\text{Ad}(l)Z) &= \sum_{i,j} \mathbb{B}(\text{Ad}(l)Z, [\text{Ad}(l)\bar{X}_i, \text{Ad}(l)\bar{Y}_j]) \text{Ad}(l)X_i \text{Ad}(l)Y_j \\ &= \sum_{i,j} \mathbb{B}(\text{Ad}(l)Z, \text{Ad}(l)[\bar{X}_i, \bar{Y}_j]) \text{Ad}(l)(X_i Y_j) \end{aligned}$$

$$\begin{aligned}
&= \text{Ad}(l) \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) X_i Y_j \\
&= \text{Ad}(l) \omega(Z),
\end{aligned}$$

as required.  $\square$

Let

$$\bar{\mathfrak{n}} = \mathfrak{g}(-2, 0) \oplus \mathfrak{g}(-1, -1) \oplus \mathfrak{g}(-1, 1)$$

and  $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{n}}$ . Then  $\bar{\mathfrak{q}}$  is a parabolic subalgebra of  $\mathfrak{g}$  with nilradical  $\bar{\mathfrak{n}}$ . We extend  $d\chi$  to be a character of  $\bar{\mathfrak{q}}$  by making it trivial on  $\bar{\mathfrak{n}}$  and let  $\mathbb{C}_{d\chi}$  be the one-dimensional  $\bar{\mathfrak{q}}$ -module on which  $\bar{\mathfrak{q}}$  acts by  $d\chi$ . Let  $\mathcal{M}(d\chi) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\chi}$  be the scalar generalized Verma module that corresponds to  $\mathbb{C}_{d\chi}$ . Recall that the map  $\mathcal{U}(\mathfrak{n}) \rightarrow \mathcal{M}(d\chi)$  that is given by  $u \mapsto u \otimes 1$  is a vector space isomorphism. We shall use this map to identify  $\mathcal{U}(\mathfrak{n})$  and  $\mathcal{M}(d\chi)$  when convenient. There is an action of  $\mathbf{L}$  on  $\mathcal{M}(d\chi)$  that satisfies

$$(2.6) \quad l(u \otimes 1) = \text{Ad}(l)u \otimes 1$$

for all  $l \in \mathbf{L}$  and  $u \in \mathcal{U}(\mathfrak{g})$ . It is well defined because the character  $d\chi$  factors through the projection map  $\mathfrak{l} \rightarrow \mathfrak{l}/\mathfrak{l}^{ss}$  and so  $d\chi(\text{Ad}(l)W) = d\chi(W)$  for all  $l \in \mathbf{L}$  and  $W \in \mathfrak{l}$ . The action satisfying (2.6) cannot be regarded as natural and is only available because we happen to be considering a scalar generalized Verma module. Nevertheless, it will be convenient to make use of it here, particularly because it makes the above-mentioned vector space isomorphism  $\mathcal{U}(\mathfrak{n}) \rightarrow \mathcal{M}(d\chi)$   $\mathbf{L}$ -equivariant. It is worth observing that the differential of the action satisfying (2.6) is the action of  $\mathfrak{l}$  that satisfies

$$(2.7) \quad W(u \otimes 1) = Wu \otimes 1 - d\chi(W)u \otimes 1$$

for all  $W \in \mathfrak{l}$  and  $u \in \mathcal{U}(\mathfrak{g})$ . Despite the additional term in (2.7), an  $\mathbf{L}$ -invariant subspace of  $\mathcal{M}(d\chi)$  is also  $\mathfrak{l}$ -invariant, a fact that will be used below.

**Proposition 2.6.** *There are (possibly inhomogeneous) linear forms*

$$c_{\pm} : \mathbb{C}^2 \rightarrow \mathbb{C}$$

such that

$$\begin{aligned}
\bar{Y}\omega(Z) \otimes 1 &= c_+(z_{\alpha}, z_{\beta})[\bar{Y}, Z] \otimes 1, \\
\bar{X}\omega(Z) \otimes 1 &= c_-(z_{\alpha}, z_{\beta})[\bar{X}, Z] \otimes 1
\end{aligned}$$

in  $\mathcal{M}(d\chi)$  for  $\bar{X} \in \mathfrak{g}(-1, -1)$ ,  $\bar{Y} \in \mathfrak{g}(-1, 1)$ ,  $Z \in \mathfrak{g}(2, 0)$ , and  $(z_{\alpha}, z_{\beta}) \in \mathbb{C}^2$ .

**Proof.** Suppose that  $\{X_i\}$ ,  $\{\bar{X}_i\}$ ,  $\{Y_j\}$ , and  $\{\bar{Y}_j\}$  are chosen as in the definition of  $\omega$ . Let  $\bar{Y} \in \mathfrak{g}(-1, 1)$  and  $Z \in \mathfrak{g}(2, 0)$ . Then  $[\bar{Y}, X_i] = 0$ ,  $[\bar{Y}, Y_j] \in \mathfrak{l}$ , and  $\bar{Y} \otimes 1 = 0$  in  $\mathcal{M}(d\chi)$ . A brief calculation using these facts reveals that

$$(2.8) \quad \bar{Y}\omega(Z) \otimes 1 = \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) d\chi([\bar{Y}, Y_j]) X_i \otimes 1.$$



Let  $\bar{X} \in \mathfrak{g}(-1, -1)$ . Then  $[\bar{X}, X_i] \in \mathfrak{l}$ ,  $[\bar{X}, Y_j] = 0$ , and  $\bar{X} \otimes 1 = 0$  in  $\mathcal{M}(d\chi)$ . A slightly longer calculation using these facts reveals that

$$(2.9) \quad \begin{aligned} \bar{X}\omega(Z) \otimes 1 &= \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) [[\bar{X}, X_i], Y_j] \otimes 1 \\ &\quad + \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) d\chi([\bar{X}, X_i]) Y_j \otimes 1. \end{aligned}$$

The first conclusion that we draw from (2.8) and (2.9) is that, under the identification of  $\mathcal{M}(d\chi)$  with  $\mathcal{U}(\mathfrak{n})$ ,  $\bar{Y}\omega(Z) \otimes 1$  is identified with an element of  $\mathfrak{g}(1, 1)$  and  $\bar{X}\omega(Z) \otimes 1$  is identified with an element of  $\mathfrak{g}(1, -1)$ . Thus, with slight abuse of notation, there is a map from  $\mathfrak{g}(-1, 1) \otimes \mathfrak{g}(2, 0)$  to  $\mathfrak{g}(1, 1)$  such that

$$(2.10) \quad \bar{Y} \otimes Z \mapsto \bar{Y}\omega(Z) \otimes 1.$$

Similarly, with the same abuse of notation, there is a map

$$\mathfrak{g}(-1, -1) \otimes \mathfrak{g}(2, 0) \rightarrow \mathfrak{g}(1, -1)$$

given by

$$(2.11) \quad \bar{X} \otimes Z \mapsto \bar{X}\omega(Z) \otimes 1.$$

By Lemma 2.5 and the observations made above, these maps are homomorphisms of  $\mathbf{L}$ -modules and hence of  $\mathfrak{l}$ -modules. Since all root spaces are one dimensional, all the weights of  $\mathfrak{g}(\pm 1, \pm 1)$  and  $\mathfrak{g}(\pm 2, 0)$  have multiplicity one. It is well-known that this implies that the tensor products  $\mathfrak{g}(-1, 1) \otimes \mathfrak{g}(2, 0)$  and  $\mathfrak{g}(-1, -1) \otimes \mathfrak{g}(2, 0)$  are multiplicity free. From this we conclude that the space of homomorphisms from  $\mathfrak{g}(-1, 1) \otimes \mathfrak{g}(2, 0)$  to  $\mathfrak{g}(1, 1)$  is at most one dimensional, as is the space of homomorphisms from  $\mathfrak{g}(-1, -1) \otimes \mathfrak{g}(2, 0)$  to  $\mathfrak{g}(1, -1)$ . The maps that satisfy

$$(2.12) \quad \bar{Y} \otimes Z \mapsto [\bar{Y}, Z]$$

and

$$(2.13) \quad \bar{X} \otimes Z \mapsto [\bar{X}, Z]$$

are nonzero and lie in the spaces of homomorphisms just referred to. Thus the spaces are precisely one dimensional, the map satisfying (2.10) is a multiple of the map satisfying (2.12), and the map satisfying (2.11) is a multiple of the map satisfying (2.13). In light of this conclusion, we may return to (2.8) and (2.9) to conclude that the constant of proportionality in the first case is a linear function of  $z_\alpha$  and  $z_\beta$  and the constant of proportionality in the second case is a possibly inhomogeneous linear form in  $z_\alpha$  and  $z_\beta$ .  $\square$

**Proposition 2.7.** *With notation as in Proposition 2.6, we have*

$$\begin{aligned} c_+(z_\alpha, z_\beta) &= -z_\beta, \\ c_-(z_\alpha, z_\beta) &= z_\alpha - d(\alpha, \beta). \end{aligned}$$

**Proof.** It suffices to compute both sides of the identities displayed in Proposition 2.6 for some  $\bar{X}$ ,  $\bar{Y}$ , and  $Z$  such that  $[\bar{X}, Z] \neq 0$  and  $[\bar{Y}, Z] \neq 0$ . Let  $\lambda = \langle C_{\alpha\beta} \rangle$ . We shall choose  $\bar{X} = X_{-\alpha}$ ,  $\bar{Y} = X_{-\beta}$ , and  $Z = X_\lambda$ . Throughout the proof, we shall enumerate the set  $C_{\alpha\beta}$  as in (2.2) and write  $d = d(\alpha, \beta)$ .

The sets  $\{X_\mu\}_{\mu \in R(1,1)}$  and  $\{X_{-\mu}\}_{\mu \in R(1,1)}$  are  $\mathbb{B}$ -dual bases of  $\mathfrak{g}(1, 1)$  and  $\mathfrak{g}(-1, -1)$ , respectively. Similarly,  $\{X_\nu\}_{\nu \in R(1,-1)}$  and  $\{X_{-\nu}\}_{\nu \in R(1,-1)}$  are  $\mathbb{B}$ -dual bases of  $\mathfrak{g}(1, -1)$  and  $\mathfrak{g}(-1, 1)$ , respectively. Thus we have

$$(2.14) \quad \omega(X_\lambda) = \sum_{\substack{\mu \in R(1,1) \\ \nu \in R(1,-1)}} \mathbb{B}(X_\lambda, [X_{-\mu}, X_{-\nu}]) X_\mu X_\nu.$$

The commutator  $[X_{-\mu}, X_{-\nu}]$  is  $\mathbb{B}$ -orthogonal to  $X_\lambda$  unless  $\mu + \nu = \lambda$ . Thus (2.14) may be rewritten as

$$(2.15) \quad \omega(X_\lambda) = \sum_{(\mu, \nu) \in S(\lambda)} N_{-\mu, -\nu} X_\mu X_\nu.$$

If  $(\mu, \nu) \in S(\lambda)$  then  $\mu = \delta_0 + \cdots + \delta_{j-1}$  and  $\nu = \delta_j + \cdots + \delta_d$  for some  $1 \leq j \leq d$ . Thus

$$\begin{aligned} f(-\mu, -\nu) &= \sum_{a=0}^{j-1} \sum_{b=j}^d f(\delta_a, \delta_b) \\ &= \sum_{a=0}^{j-1} \sum_{b=j}^d (\delta_a, \delta_b) \\ &= (\delta_{j-1}, \delta_j) \\ &= -1 \end{aligned}$$

and so

$$N_{-\mu, -\nu} = (-1)(-1)(-1)(-1)^{-1} = 1$$

by (2.1). It follows that

$$(2.16) \quad \omega(X_\lambda) = \sum_{(\mu, \nu) \in S(\lambda)} X_\mu X_\nu.$$

We have

$$\begin{aligned} X_{-\beta} \omega(X_\lambda) \otimes 1 &= \sum_{(\mu, \nu) \in S(\lambda)} X_{-\beta} X_\mu X_\nu \otimes 1 \\ &= \sum_{(\mu, \nu) \in S(\lambda)} X_\mu X_{-\beta} X_\nu \otimes 1 \\ &= \sum_{(\mu, \nu) \in S(\lambda)} X_\mu [X_{-\beta}, X_\nu] \otimes 1 \\ &= \sum_{(\mu, \nu) \in S(\lambda)} d\chi([X_{-\beta}, X_\nu]) X_\mu \otimes 1 \end{aligned}$$

$$\begin{aligned}
&= -d\chi(H_\beta)X_{\lambda-\beta} \otimes 1 \\
&= -z_\beta X_{\lambda-\beta} \otimes 1.
\end{aligned}$$

On the other hand,

$$f(-\beta, \lambda) = -\sum_{j=0}^d f(\delta_d, \delta_j) = -1$$

and so

$$N_{-\beta, \lambda} = (-1)(1)(1)(-1)^{-1} = 1.$$

This gives  $[X_{-\beta}, X_\lambda] = X_{\lambda-\beta}$  and by comparing this with the value of  $X_{-\beta}\omega(X_\lambda) \otimes 1$  given above we get  $c_+(z_\alpha, z_\beta) = -z_\beta$ .

We also have

$$\begin{aligned}
&X_{-\alpha}\omega(X_\lambda) \otimes 1 \\
&= \sum_{(\mu, \nu) \in S(\lambda)} X_{-\alpha}X_\mu X_\nu \otimes 1 \\
&= \sum_{(\mu, \nu) \in S(\lambda)} [X_{-\alpha}, X_\mu]X_\nu \otimes 1 \\
&= -H_\alpha X_{\lambda-\alpha} \otimes 1 + \sum_{(\mu, \nu) \in S(\lambda) - \{(\alpha, \lambda-\alpha)\}} N_{-\alpha, \mu} X_{\mu-\alpha} X_\nu \otimes 1 \\
&= -[H_\alpha, X_{\lambda-\alpha}] \otimes 1 - X_{\lambda-\alpha} H_\alpha \otimes 1 \\
&\quad + \sum_{(\mu, \nu) \in S(\lambda) - \{(\alpha, \lambda-\alpha)\}} N_{-\alpha, \mu} N_{\mu-\alpha, \nu} X_{\lambda-\alpha} \otimes 1 \\
&= -(\lambda - \alpha, \alpha) X_{\lambda-\alpha} \otimes 1 - z_\alpha X_{\lambda-\alpha} \otimes 1 \\
&\quad + \sum_{(\mu, \nu) \in S(\lambda) - \{(\alpha, \lambda-\alpha)\}} N_{-\alpha, \mu} N_{\mu-\alpha, \nu} X_{\lambda-\alpha} \otimes 1 \\
&= X_{\lambda-\alpha} \otimes 1 - z_\alpha X_{\lambda-\alpha} \otimes 1 + \sum_{(\mu, \nu) \in S(\lambda) - \{(\alpha, \lambda-\alpha)\}} N_{-\alpha, \mu} N_{\mu-\alpha, \nu} X_{\lambda-\alpha} \otimes 1.
\end{aligned}$$

Now suppose that  $\mu = \delta_0 + \dots + \delta_{j-1}$  and  $\nu = \delta_j + \dots + \delta_d$  in the last sum, where  $2 \leq j \leq d$ . Then

$$f(-\alpha, \mu) = -\sum_{a=0}^{j-1} f(\delta_0, \delta_a) = -1 + 1 = 0$$

and so

$$N_{-\alpha, \mu} = (-1)(1)(1)(-1)^0 = -1.$$

Furthermore,

$$f(\mu - \alpha, \nu) = \sum_{a=1}^{j-1} \sum_{b=j}^d f(\delta_a, \delta_b) = (\delta_{j-1}, \delta_j) = -1$$

and so

$$N_{\mu-\alpha, \nu} = (1)(1)(1)(-1)^{-1} = -1.$$

Introducing these values above, we obtain

$$\begin{aligned} X_{-\alpha}\omega(X_\lambda) \otimes 1 &= (1 - z_\alpha)X_{\lambda-\alpha} \otimes 1 + \sum_{(\mu, \nu) \in S(\lambda) - \{(\alpha, \lambda-\alpha)\}} X_{\lambda-\alpha} \otimes 1 \\ &= (1 - z_\alpha)X_{\lambda-\alpha} \otimes 1 + (d - 1)X_{\lambda-\alpha} \otimes 1 \\ &= (d - z_\alpha)X_{\lambda-\alpha} \otimes 1. \end{aligned}$$

Now

$$f(-\alpha, \lambda) = - \sum_{j=0}^d f(\delta_0, \delta_j) = -1 + 1 = 0$$

and so

$$N_{-\alpha, \lambda} = (-1)(1)(1)(-1)^0 = -1.$$

This gives  $[X_{-\alpha}, X_\lambda] = -X_{\lambda-\alpha}$  and by comparing this with the value of  $X_{-\alpha}\omega(X_\lambda) \otimes 1$  obtained above we get  $c_-(z_\alpha, z_\beta) = z_\alpha - d$ .  $\square$

Let  $z \in \mathbb{C}$ . Set  $z_0 = d(\alpha, \beta)/2$  and define

$$d\chi_z = (z_0 - z)\varpi_\alpha + (z_0 + z)\varpi_\beta.$$

Define  $\omega_z : \mathfrak{g}(2, 0) \rightarrow \mathcal{U}(\mathfrak{n})$  by

$$\omega_z(Z) = \omega(Z) + (z_0 + z)Z.$$

**Theorem 2.8.** *For all  $z \in \mathbb{C}$ ,*

$$\{\omega_z(Z) \otimes 1 \mid Z \in \mathfrak{g}(2, 0)\}$$

*is a  $\bar{\mathfrak{q}}$ -submodule of  $\mathcal{M}(d\chi_z)$ .*

**Proof.** Let  $E$  be the set referred to in the statement. Then  $E$  is a subspace of  $\mathcal{M}(d\chi_z)$  and it follows from Lemma 2.5 that  $\mathfrak{l}E \subset E$ . To complete the proof it suffices to show that  $\bar{\mathfrak{n}}E = \{0\}$ . Since  $\bar{\mathfrak{n}}$  is generated by  $\mathfrak{g}(-1, 1)$  and  $\mathfrak{g}(-1, -1)$ , it suffices to show that  $\bar{Y}\omega_z(Z) \otimes 1 = 0$  and  $\bar{X}\omega_z(Z) \otimes 1 = 0$  for all  $\bar{X} \in \mathfrak{g}(-1, -1)$ ,  $\bar{Y} \in \mathfrak{g}(-1, 1)$ , and  $Z \in \mathfrak{g}(2, 0)$ . By Propositions 2.6 and 2.7, we have

$$\begin{aligned} \bar{Y}\omega_z(Z) \otimes 1 &= \bar{Y}\omega(Z) \otimes 1 + (z_0 + z)\bar{Y}Z \otimes 1 \\ &= -(z_0 + z)[\bar{Y}, Z] \otimes 1 + (z_0 + z)[\bar{Y}, Z] \otimes 1 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \bar{X}\omega_z(Z) \otimes 1 &= \bar{X}\omega(Z) \otimes 1 + (z_0 + z)\bar{X}Z \otimes 1 \\ &= (z_0 - z - 2z_0)[\bar{X}, Z] \otimes 1 + (z_0 + z)[\bar{X}, Z] \otimes 1 \\ &= -(z_0 + z)[\bar{X}, Z] \otimes 1 + (z_0 + z)[\bar{X}, Z] \otimes 1 \\ &= 0. \end{aligned}$$

These two equations imply the required conclusion.  $\square$

Note that Theorem 2.8 implies that

$$\mathcal{N}(d\chi_z) = \mathcal{U}(\mathfrak{g})\{\omega_z(Z) \otimes 1 \mid Z \in \mathfrak{g}(2,0)\}$$

is a proper submodule of  $\mathcal{M}(d\chi_z)$ . This completes the first part of the construction.

In the definition (2.5) of the map  $\omega$ , the spaces  $\mathfrak{g}(1,1)$  and  $\mathfrak{g}(1,-1)$  do not play symmetric roles. This seems to be unavoidable, but it will be convenient to have an alternate expression for  $\omega$  which is as symmetric as possible between the two spaces. For  $X, Y \in \mathfrak{g}$  we define

$$\lrcorner XY \lrcorner = \frac{1}{2}(XY + YX)$$

in  $\mathcal{U}(\mathfrak{g})$ . With the notation as in (2.5), define

$$(2.17) \quad \lrcorner \omega \lrcorner(Z) = \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) \lrcorner X_i Y_j \lrcorner.$$

**Lemma 2.9.** *For all  $Z \in \mathfrak{g}(2,0)$  we have  $\lrcorner \omega \lrcorner(Z) = \omega(Z) + z_0 Z$ .*

**Proof.** It suffices to verify this for  $Z = X_\zeta$  with  $\zeta \in R(2,0)$ . We have

$$\begin{aligned} \omega(X_\zeta) &= \sum_{(\mu,\nu) \in S(\zeta)} N_{-\mu,-\nu} X_\mu X_\nu \\ &= \sum_{(\mu,\nu) \in S(\zeta)} N_{-\mu,-\nu} \left( \lrcorner X_\mu X_\nu \lrcorner + \frac{1}{2}[X_\mu, X_\nu] \right) \\ &= \lrcorner \omega \lrcorner(X_\zeta) + \frac{1}{2} \sum_{(\mu,\nu) \in S(\zeta)} N_{-\mu,-\nu} N_{\mu,\nu} X_\zeta. \end{aligned}$$

For  $(\mu, \nu) \in S(\zeta)$  we have

$$N_{-\mu,-\nu} = -(-1)^{f(-\mu,-\nu)} = -(-1)^{f(\mu,\nu)} = -N_{\mu,\nu}$$

and so  $N_{-\mu,-\nu} N_{\mu,\nu} = -N_{\mu,\nu}^2 = -1$ . Introducing this evaluation above gives

$$\begin{aligned} \omega(X_\zeta) &= \lrcorner \omega \lrcorner(X_\zeta) - \frac{1}{2} \sum_{(\mu,\nu) \in S(\zeta)} X_\zeta \\ &= \lrcorner \omega \lrcorner(X_\zeta) - \frac{d(\alpha, \beta)}{2} X_\zeta \\ &= \lrcorner \omega \lrcorner(X_\zeta) - z_0 X_\zeta, \end{aligned}$$

since, by Lemma 2.3,  $S(\zeta)$  has  $d(\alpha, \beta)$  elements. This is equivalent to the asserted identity.  $\square$

Lemma 2.9 suggests a reason for our normalization of the parameter  $z$ . Indeed  $\lrcorner \omega \lrcorner(Z) = \omega_0(Z)$ , so that the point  $z = 0$  is made to correspond to the most symmetric member of the family  $\omega_z$ .

### 3. Construction of the systems. II

In this section we explain how the construction made in Section 2 gives rise to conformally invariant systems of differential operators via the theory developed in [3]. In fact, each of the families constructed in Section 2 gives rise to several families of conformally invariant systems indexed by suitable real forms of  $\mathfrak{g}$ . We begin by identifying which real forms are the suitable ones.

Let  $\mathfrak{g}_0$  be a real form of  $\mathfrak{g}$  and consider the Satake diagram of this real form. (Various choices are required to obtain this diagram, but the eventual result is independent of the choices and they need not be specified at present.) We require that the nodes corresponding to the simple roots  $\alpha$  and  $\beta$  be unpainted in the Satake diagram and that the set  $\{\alpha, \beta\}$  be stable under the Satake involution. More concretely, the second condition requires that if  $\alpha$  is joined to another simple root by an arrow in the Satake diagram then the other root is  $\beta$ . If these conditions are satisfied then the real form is suitable for the construction of a family of conformally invariant systems of differential operators. The split real form is always suitable for any possible choice of  $\alpha$  and  $\beta$ . All noncompact real forms in type A are suitable for at least one possible choice of  $\alpha$  and  $\beta$ . All noncompact real forms in type  $E_6$  are suitable for the only possible choice of  $\alpha$  and  $\beta$ . In addition to the split real form, there are two real forms in type D that are suitable for exactly one possible choice of  $\alpha$  and  $\beta$ . The Satake diagrams and possible choices of  $\alpha$  and  $\beta$  for suitable nonsplit real forms are shown in Figures 2 and 3 for types A and D, respectively. The corresponding data for type  $E_6$  is not displayed since no noncompact real form is excluded in that case.

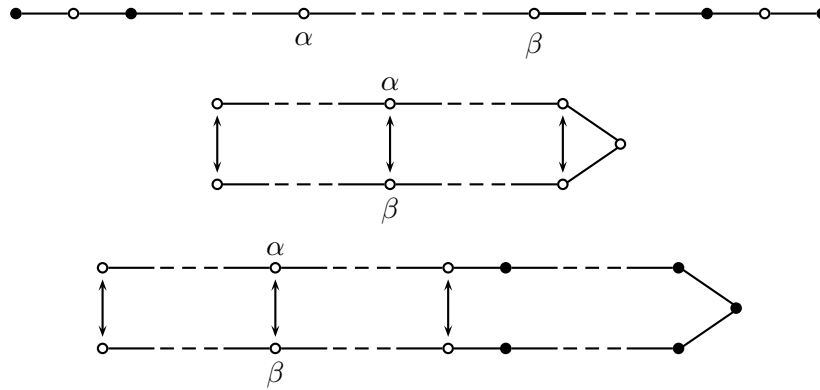


FIGURE 2. Satake diagrams of suitable nonsplit real forms in Type A.

Let  $\mathfrak{g}_0$  be a suitable real form of  $\mathfrak{g}$  and let  $G$  be a real, connected, linear Lie group with Lie algebra  $\mathfrak{g}_0$ . (We systematically employ the convention by which the name of a real object includes a zero subscript which is removed

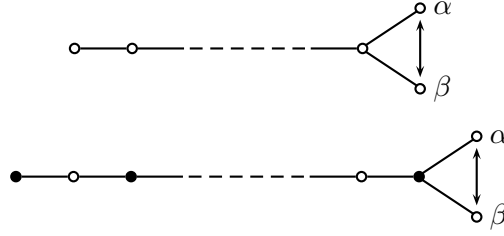


FIGURE 3. Satake diagrams of suitable nonsplit real forms in Type D.

to name the complexification.) Let  $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$  be a Cartan decomposition of  $\mathfrak{g}_0$  with Cartan involution  $\theta$ . Choose a maximally noncompact  $\theta$ -stable Cartan subalgebra  $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$  with  $\mathfrak{a}_0 \subset \mathfrak{p}_0$  and  $\mathfrak{t}_0 \subset \mathfrak{k}_0$  as usual. The complexification  $\mathfrak{h}$  of  $\mathfrak{h}_0$  is a Cartan subalgebra of  $\mathfrak{g}$  with  $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a}_0 \oplus i\mathfrak{t}_0$ . Choose an ordered basis of  $\mathfrak{a}_0$ , extend it by an ordered basis of  $i\mathfrak{t}_0$  to obtain an ordered basis of  $\mathfrak{h}_{\mathbb{R}}$ , and introduce the corresponding lexicographic order on  $\mathfrak{h}_{\mathbb{R}}^*$ . Also order  $\mathfrak{a}_0^*$  lexicographically using the chosen ordered basis of  $\mathfrak{a}_0$ . We obtain positive systems  $R^+(\mathfrak{g}, \mathfrak{h})$  and  $R^+(\mathfrak{g}_0, \mathfrak{a}_0)$  in the root system and restricted root system, respectively, from these choices. With these choices, if  $\gamma \in R^+(\mathfrak{g}, \mathfrak{h})$  and  $\text{res}_{\mathfrak{a}_0}(\gamma) \neq 0$  then  $-\theta(\gamma) \in R^+(\mathfrak{g}, \mathfrak{h})$  and  $\text{res}_{\mathfrak{a}_0}(\gamma) \in R^+(\mathfrak{g}_0, \mathfrak{a}_0)$ .

**Lemma 3.1.** *We have  $H_0 \in \mathfrak{a}_0$ . If  $\alpha$  and  $\beta$  are not connected by an arrow in the Satake diagram then  $Z_0 \in \mathfrak{a}_0$  also; otherwise,  $Z_0 \in i\mathfrak{t}_0$ .*

**Proof.** Note first that  $H_0, Z_0 \in \mathfrak{h}_{\mathbb{R}} = \mathfrak{a}_0 \oplus i\mathfrak{t}_0$ . Let us write  $\gamma \mapsto \gamma'$  for the Satake involution on the set

$$S = \{\gamma \in R^s(\mathfrak{g}, \mathfrak{h}) \mid \text{res}_{\mathfrak{a}_0}(\gamma) \neq 0\}.$$

By hypothesis,  $\alpha, \beta \in S$  and the set  $\{\alpha, \beta\}$  is stable under the Satake involution. If  $\gamma \in S$  then

$$(3.1) \quad -\theta(\gamma) = \gamma' + \sum_{\delta \in R^s(\mathfrak{g}, \mathfrak{h}) - S} c_{\delta, \gamma} \delta$$

with  $c_{\delta, \gamma} \in \mathbb{N}$ . It follows that if  $\gamma \in R^s(\mathfrak{g}, \mathfrak{h}) - \{\alpha, \beta\}$  then  $-\theta(\gamma)(H_0) = 0$  and so  $\gamma(\theta(H_0)) = 0$ . On the other hand, if we apply (3.1) to  $H_0$  with  $\gamma \in \{\alpha, \beta\}$  then we obtain  $-\theta(\gamma)(H_0) = 1$  and so  $\alpha(\theta(H_0)) = \beta(\theta(H_0)) = -1$ . Consequently,  $\theta(H_0) = -H_0$  and so  $H_0 \in \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{p} = \mathfrak{a}_0$ . Similarly, by applying (3.1) to  $Z_0$  we obtain  $\gamma(\theta(Z_0)) = 0$  for all  $\gamma \in R^s(\mathfrak{g}, \mathfrak{h}) - \{\alpha, \beta\}$ . If  $\alpha' = \beta$  then we also obtain  $\alpha(\theta(Z_0)) = 1$  and  $\beta(\theta(Z_0)) = -1$ . It follows that  $\theta(Z_0) = Z_0$  in this case and so  $Z_0 \in \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k} = i\mathfrak{t}_0$ . Finally, if  $\alpha' = \alpha$  then a similar argument gives  $\theta(Z_0) = -Z_0$  and hence  $Z_0 \in \mathfrak{a}_0$ .  $\square$

It follows from Lemma 3.1 and the definition of  $\mathfrak{g}(j, k)$  that  $\mathfrak{l}$  and  $\mathfrak{g}(\pm 2, 0)$  are defined over  $\mathbb{R}$ . The spaces  $\mathfrak{g}(1, 1) \oplus \mathfrak{g}(1, -1)$  and  $\mathfrak{g}(-1, 1) \oplus \mathfrak{g}(-1, -1)$  are also defined over  $\mathbb{R}$ , but the individual summands are only defined over

$\mathbb{R}$  when  $\alpha$  and  $\beta$  are not connected by an arrow in the Satake diagram. It will be convenient to write  $s(\alpha, \beta)$  for the sign of the restriction of the Satake involution to the set  $\{\alpha, \beta\}$ . This is 1 if the Satake involution fixes  $\alpha$  and  $\beta$  and  $-1$  if it does not.

We shall need a notation for complex conjugation. Since bar is already being used for opposition and is somewhat unwieldy over long expressions in any case, we shall instead write  $z^c$  for the complex conjugate of  $z \in \mathbb{C}$ . Let  $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$  be conjugation with respect to the real form  $\mathfrak{g}_0$ . This map satisfies  $\sigma([X, Y]) = [\sigma(X), \sigma(Y)]$  for all  $X, Y \in \mathfrak{g}$  and so induces a conjugate-linear automorphism  $\sigma : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$ . For all  $X, Y \in \mathfrak{g}$ , we have

$$(3.2) \quad \mathbb{B}(\sigma(X), \sigma(Y)) = \mathbb{B}(X, Y)^c.$$

**Proposition 3.2.** *For  $Z \in \mathfrak{g}(2, 0)$  and  $z \in \mathbb{C}$  we have*

$$\sigma(\omega_z(Z)) = s(\alpha, \beta)\omega_{s(\alpha, \beta)z^c}(\sigma(Z)).$$

**Proof.** Let  $\{X_i\}$  and  $\{Y_j\}$  be bases for  $\mathfrak{g}(1, 1)$  and  $\mathfrak{g}(1, -1)$ , respectively, and  $\{\bar{X}_i\}$  and  $\{\bar{Y}_j\}$  be the  $\mathbb{B}$ -dual bases of  $\mathfrak{g}(-1, -1)$  and  $\mathfrak{g}(-1, 1)$ , respectively. Then

$$\ulcorner \omega^\urcorner(Z) = \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j]) \ulcorner X_i Y_j \urcorner$$

and so

$$\begin{aligned} \sigma(\ulcorner \omega^\urcorner(Z)) &= \sum_{i,j} \mathbb{B}(Z, [\bar{X}_i, \bar{Y}_j])^c \sigma(\ulcorner X_i Y_j \urcorner) \\ &= \sum_{i,j} \mathbb{B}(\sigma(Z), \sigma([\bar{X}_i, \bar{Y}_j])) \ulcorner \sigma(X_i) \sigma(Y_j) \urcorner \\ &= \sum_{i,j} \mathbb{B}(\sigma(Z), [\sigma(\bar{X}_i), \sigma(\bar{Y}_j)]) \ulcorner \sigma(X_i) \sigma(Y_j) \urcorner, \end{aligned}$$

where we used (3.2) from the first line to the second. From (3.2) it follows that  $\{\sigma(X_i)\}$  and  $\{\sigma(\bar{X}_i)\}$  are  $\mathbb{B}$ -dual, as are  $\{\sigma(Y_j)\}$  and  $\{\sigma(\bar{Y}_j)\}$ . If  $s(\alpha, \beta) = 1$  then  $\{\sigma(X_i)\}$  is a basis for  $\mathfrak{g}(1, 1)$  and  $\{\sigma(Y_j)\}$  is a basis for  $\mathfrak{g}(1, -1)$ . The above expression for  $\sigma(\ulcorner \omega^\urcorner(Z))$  then implies that  $\sigma(\ulcorner \omega^\urcorner(Z)) = \ulcorner \omega^\urcorner(\sigma(Z))$ . On the other hand, if  $s(\alpha, \beta) = -1$  then  $\{\sigma(X_i)\}$  is a basis for  $\mathfrak{g}(1, -1)$  and  $\{\sigma(Y_j)\}$  is a basis for  $\mathfrak{g}(1, 1)$  and so we must rewrite the above expression as

$$\sigma(\ulcorner \omega^\urcorner(Z)) = - \sum_{i,j} \mathbb{B}(\sigma(Z), [\sigma(\bar{Y}_j), \sigma(\bar{X}_i)]) \ulcorner \sigma(Y_j) \sigma(X_i) \urcorner$$

to see that  $\sigma(\ulcorner \omega^\urcorner(Z)) = -\ulcorner \omega^\urcorner(\sigma(Z))$ . These two conclusions may be expressed uniformly as

$$\sigma(\ulcorner \omega^\urcorner(Z)) = s(\alpha, \beta) \ulcorner \omega^\urcorner(\sigma(Z)).$$

By using this equation, we obtain

$$\sigma(\omega_z(Z)) = \sigma(\ulcorner \omega^\urcorner(Z) + zZ)$$



$$\begin{aligned}
 &= s(\alpha, \beta) \Gamma \omega^\Gamma(\sigma(Z)) + z^c \sigma(Z) \\
 &= s(\alpha, \beta) (\Gamma \omega^\Gamma(\sigma(Z)) + s(\alpha, \beta) z^c \sigma(Z)) \\
 &= s(\alpha, \beta) \omega_{s(\alpha, \beta) z^c}(\sigma(Z)),
 \end{aligned}$$

as claimed. □

It follows from the discussion above that  $\bar{\mathfrak{q}}$  is defined over  $\mathbb{R}$ , so that  $\bar{\mathfrak{q}}_0 = \bar{\mathfrak{q}} \cap \mathfrak{g}_0$  is a real parabolic subalgebra of  $\mathfrak{g}_0$ . Let  $\bar{Q} = N_G(\bar{\mathfrak{q}}_0)$ . Then  $\bar{Q}$  is a parabolic subgroup of  $G$ . If we let  $\mathfrak{c}_0 \subset \mathfrak{a}_0$  be the intersection of the kernels of all the simple restricted roots other than  $\text{res}_{\mathfrak{a}_0}(\alpha)$  and  $\text{res}_{\mathfrak{a}_0}(\beta)$  and define  $L = Z_G(\mathfrak{c}_0)$  then we have a factorization  $\bar{Q} = L\bar{N}$ , where  $\bar{N}$  is the unipotent radical of  $\bar{Q}$ . The next task is to describe as far as possible the characters  $\chi_z : L \rightarrow \mathbb{C}^\times$  whose differential  $d\chi_z$  is equal to the character of  $\mathfrak{l}$  that we have already given this name above. We shall call such a character a *suitable* character of  $L$ . There are two issues to be addressed: first, for which values of  $z$  is there any suitable character and, secondly, when such a character exists how many are there with the given value of  $z$ ?

We begin to answer these questions by recalling some standard facts about the structure of  $L$ . Let  $L^\circ$  be the connected component of the identity in  $L$ . Then  $[L^\circ, L^\circ]$  is the connected subgroup of  $L^\circ$  with Lie algebra  $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}^{ss}$ . This subgroup is usually denoted by  $(L^\circ)^{ss}$  and is closed in  $L^\circ$ . The quotient  $L^\circ / (L^\circ)^{ss}$  is therefore a connected abelian Lie group with Lie algebra isomorphic to  $\mathbb{C}H_0 \oplus \mathbb{C}Z_0$ . Note that  $(L^\circ)^\circ$  is a characteristic subgroup of  $L^\circ$  and so normal in  $L$ . Thus the component group  $L/L^\circ$  acts by conjugation on  $L^\circ / (L^\circ)^{ss}$ . In fact, this action is trivial. To see this, recall that there is a finite elementary abelian 2-group  $F$  such that  $L = FL^\circ$ . If we define

$$\kappa_\delta = \exp(\pi i H_\delta)$$

for each real root  $\delta$  then  $\kappa_\delta \in L$  and the group  $F$  is generated by the set  $\{\kappa_\delta\}$ . One sees directly that  $\text{Ad}(\kappa_\delta)$  induces the trivial map on the Lie algebra of  $L^\circ / (L^\circ)^{ss}$ , from which the claim follows. The intersection of  $F$  with  $L^\circ$  may be nontrivial, but we can choose a complement  $F_L$  for  $F \cap L^\circ$  in  $F$ . Then  $L \cong F_L \ltimes L^\circ$  and the order of  $F_L$  is equal to the number of connected components of  $L$ . Since  $F_L$  acts trivially on  $L^\circ / (L^\circ)^{ss}$ , it stabilizes every character of  $L^\circ$ . Thus every character of  $L^\circ$  has precisely  $|F_L|$  extensions to a character of  $L$ . This answers the second question first: if there is any suitable character for a particular value of  $z$  then the number of suitable characters for that value of  $z$  is equal to the number of connected components of  $L$ .

**Proposition 3.3.** *Suppose that  $\alpha$  and  $\beta$  are not joined by an arrow in the Satake diagram. Then there is a suitable character for every  $z \in \mathbb{C}$ .*

**Proof.** By Lemma 3.1, we have  $H_0, Z_0 \in \mathfrak{a}_0$ . The map

$$(u_1, u_2) \mapsto \exp(u_1 H_0 + u_2 Z_0) (L^\circ)^{ss}$$

is an isomorphism from  $\mathbb{R}^2$  to  $L^\circ/(L^\circ)^{ss}$ . In terms of this isomorphism,

$$\chi_z(u_1, u_2) = \exp(u_1 d\chi_z(H_0) + u_2 d\chi_z(Z_0))$$

is a suitable character for all  $z$ .  $\square$

**Lemma 3.4.** *Suppose that  $\alpha$  and  $\beta$  are joined by an arrow in the Satake diagram. Then  $\exp(2\pi i Z_0) \in Z(G)$ . In addition, we have*

$$d\chi_z(Z_0) = -z\|\varpi_\alpha - \varpi_\beta\|^2.$$

**Proof.** By Lemma 3.1,  $iZ_0 \in \mathfrak{g}_0$  and so  $g = \exp(2\pi i Z_0) \in G$ . The map  $\text{ad}(Z_0)$  on  $\mathfrak{g}$  is semisimple and its eigenvalues lie in  $\{0, \pm 1\}$ . Thus the map  $\text{Ad}(g)$  on  $\mathfrak{g}$  is semisimple and its eigenvalues are all 1. It follows that  $\text{Ad}(g)$  is the identity map and so  $g \in Z(G)$ .

Since  $\alpha$  and  $\beta$  are joined by an arrow in the Satake diagram, there is a diagram automorphism that interchanges  $\alpha$  to  $\beta$ . This automorphism also interchanges  $\varpi_\alpha$  and  $\varpi_\beta$  and so  $\|\varpi_\alpha\| = \|\varpi_\beta\|$ . Under the identification of  $\mathfrak{h}_\mathbb{R}$  and  $\mathfrak{h}_\mathbb{R}^*$  induced by the inner product,  $H_0$  corresponds to  $\varpi_\alpha + \varpi_\beta$  and  $Z_0$  corresponds to  $\varpi_\alpha - \varpi_\beta$ . Thus

$$\begin{aligned} d\chi_z(Z_0) &= ((z_0 - z)\varpi_\alpha + (z_0 + z)\varpi_\beta, \varpi_\alpha - \varpi_\beta) \\ &= z_0(\varpi_\alpha + \varpi_\beta, \varpi_\alpha - \varpi_\beta) - z(\varpi_\alpha - \varpi_\beta, \varpi_\alpha - \varpi_\beta) \\ &= z_0(\|\varpi_\alpha\|^2 - \|\varpi_\beta\|^2) - z\|\varpi_\alpha - \varpi_\beta\|^2 \\ &= -z\|\varpi_\alpha - \varpi_\beta\|^2, \end{aligned}$$

as claimed.  $\square$

Note that our assumptions imply that  $Z(G)$  is finite, so that  $\exp(2\pi i Z_0)$  has finite order.

**Proposition 3.5.** *Suppose that  $\alpha$  and  $\beta$  are joined by an arrow in the Satake diagram. Let  $a$  be the order of the element  $\exp(2\pi i Z_0)$ . Then there is a suitable character for  $z$  if and only if*

$$z \in \frac{1}{a\|\varpi_\alpha - \varpi_\beta\|^2} \mathbb{Z}.$$

**Proof.** By Lemma 3.1, we have  $H_0 \in \mathfrak{a}_0$  and  $iZ_0 \in \mathfrak{t}_0$ . It follows from the definition of  $a$  that the map

$$(u_1, u_2 + \mathbb{Z}) \mapsto \exp(u_1 H_0 + 2\pi i a u_2 Z_0)(L^\circ)^{ss}$$

is an isomorphism from  $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})$  to  $L^\circ/(L^\circ)^{ss}$ . In terms of this isomorphism, there is a suitable character associated to  $z$  if and only if

$$d\chi_z(\partial/\partial u_2) \in 2\pi i \mathbb{Z}.$$

This is equivalent to the condition that  $d\chi_z(2\pi i a Z_0) \in 2\pi i \mathbb{Z}$  or  $d\chi_z(a Z_0) \in \mathbb{Z}$ . From Lemma 3.4, this is equivalent to the condition

$$az\|\varpi_\alpha - \varpi_\beta\|^2 \in \mathbb{Z},$$

as stated.  $\square$

There is a homogeneous line bundle over the generalized flag manifold  $G/\bar{Q}$  associated to any suitable character. The  $\bar{q}$ -submodule of  $\mathcal{M}(d\chi_z)$  that was identified in Theorem 2.8 is associated to a conformally invariant system of differential operators that act on sections of this line bundle over an appropriate dense open subset of  $G/\bar{Q}$ . The theoretical background for this construction was developed in [3]. The specific application of that theory in situations such as the current one was further described in [12]. In [12] it was assumed that the character  $\chi_z$  is real-valued, which will not always be the case here, but the modifications necessary to allow complex-valued characters are minor. In [14] the constructions were described explicitly in the specific case that was studied there. For the reader's convenience, and to prepare the ground for future work on these systems, we shall summarize the rest of the construction here.

Let  $N$  be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{n}$ . The set  $NL\bar{N}$  is open and dense in  $G$  and if  $g \in NL\bar{N}$  then  $g$  has a unique factorization in the form

$$(3.3) \quad g = \zeta(g)a(g)\bar{\zeta}(g)$$

with  $\zeta(g) \in N$ ,  $a(g) \in L$ , and  $\bar{\zeta}(g) \in \bar{N}$ . For  $g \in G$ , we define  $U_g \subset G/\bar{Q}$  by

$$U_g = N\bar{Q}/\bar{Q} \cap gN\bar{Q}/\bar{Q}.$$

Note that each  $U_g$  is a dense open set in  $G/\bar{Q}$  and that  $N$  may be identified with  $U_e$  via  $n \mapsto n\bar{Q}$ . If  $n \in N$  and  $n\bar{Q} \in U_g$  then  $g^{-1}n \in NL\bar{N}$ .

We have seen above that suitable characters are indexed by a suitable value of  $z$ , as identified in Propositions 3.3 and 3.5, together with a character of the component group  $L/L^\circ$  (which is necessarily a sign character). We shall often simplify the notation by suppressing the dependence on  $z$  and the sign character of the component group. Let  $\chi : L \rightarrow \mathbb{C}^\times$  be a suitable character and extend  $\chi$  to a character of  $\bar{Q}$  by making it trivial on  $\bar{N}$ . There is a homogeneous line bundle  $\mathcal{L} \rightarrow G/\bar{Q}$  associated to  $\chi$ . If  $U \subset G/\bar{Q}$  is open and  $W \subset G$  is its preimage then we may identify the space  $\Gamma(U, \mathcal{L})$  of smooth sections of  $\mathcal{L}$  over  $U$  with the space of smooth functions  $\varphi : W \rightarrow \mathbb{C}$  that satisfy  $\varphi(g\bar{q}) = \chi(\bar{q})\varphi(g)$  for all  $g \in W$  and  $\bar{q} \in \bar{Q}$ . The space  $\Gamma(\mathcal{L})$  with the left-translation action of  $G$  affords a model of the smooth induced representation  $\text{Ind}(G, \bar{Q}, \chi^{-1})$ . We write  $\pi$  for this representation of  $G$  and  $\Pi$  for the derived representation of  $\mathfrak{g}$ . Note that while  $\pi$  depends both on  $z$  and on the sign character of  $L/L^\circ$ ,  $\Pi$  depends only on  $z$ . The representation  $\Pi$  realizes  $\mathfrak{g}$  as an algebra of first-order differential operators on  $G/\bar{Q}$ . In particular, the action of  $\mathfrak{g}$  via  $\Pi$  is local on  $G/\bar{Q}$  and so it may be extended to an action on smooth sections of  $\mathcal{L}$  over any open subset of  $G/\bar{Q}$ . The restriction map  $\Gamma(U_e, \mathcal{L}) \rightarrow C^\infty(N)$  is injective and the action of  $\mathfrak{g}$  on  $\Gamma(U_e, \mathcal{L})$  via  $\Pi$  extends to an action on  $C^\infty(N)$ . We shall make similar extensions silently from now on.

Let  $\mathbb{D}(N)$  be the algebra of smooth linear differential operators on  $N$  and define

$$\mathbb{D}(N)^{\mathfrak{n}} = \{D \in \mathbb{D}(N) \mid [\Pi(X), D] = 0 \text{ for all } X \in \mathfrak{n}\}.$$

This is precisely the subalgebra of  $\mathbb{D}(N)$  consisting of operators that are invariant under left translation by elements of  $N$ . Let  $R : \mathfrak{n}_0 \rightarrow \mathbb{D}(N)$  be the derived map associated to the right-translation action of  $N$  on  $C^\infty(N)$ . We may extend  $R$  to a complex-linear map  $R : \mathfrak{n} \rightarrow \mathbb{D}(N)$  and then to an algebra homomorphism  $R : \mathcal{U}(\mathfrak{n}) \rightarrow \mathbb{D}(N)$ . The image of this map is precisely  $\mathbb{D}(N)^{\mathfrak{n}}$  and  $R$  corestricts to an isomorphism between the algebras  $\mathcal{U}(\mathfrak{n})$  and  $\mathbb{D}(N)^{\mathfrak{n}}$ . We define an action of  $L$  on  $\mathbb{D}(N)$  by  $g * D = \pi(g) \circ D \circ \pi(g^{-1})$ . This action preserves the subalgebra  $\mathbb{D}(N)^{\mathfrak{n}}$  and the map  $R$  becomes  $L$ -intertwining when  $\mathcal{U}(\mathfrak{n})$  is given the adjoint action. The derived action of  $*$  is  $Y * D = [\Pi(Y), D]$  for  $Y \in \mathfrak{l}$  and, in particular, we have

$$(3.4) \quad [\Pi(Y), R(u)] = R(\text{ad}(Y)u)$$

for  $Y \in \mathfrak{l}$  and  $u \in \mathcal{U}(\mathfrak{n})$ .

The map  $R$  and several other maps are displayed in the following diagram in which the square commutes:

$$(3.5) \quad \begin{array}{ccccc} \mathcal{M}(d\chi) & \xrightarrow{x \mapsto x \otimes 1} & \mathcal{M}(d\chi) \otimes_{\mathbb{C}} \mathbb{C}_{-d\chi} & \xrightarrow{\Lambda \mapsto D_\Lambda} & \mathbb{D}(N)^{\mathfrak{n}} \\ & & \downarrow & & \uparrow R \\ & & \mathcal{U}(\mathfrak{n}) \otimes_{\mathbb{C}} \mathbb{C}_{-d\chi} & \xrightarrow{x \otimes w \mapsto wx} & \mathcal{U}(\mathfrak{n}). \end{array}$$

The left-hand vertical map is induced by the identification between  $\mathcal{M}(d\chi)$  and  $\mathcal{U}(\mathfrak{n})$  that was introduced above. The map  $\Lambda \mapsto D_\Lambda$  is defined directly in [3], but can equally well be defined as the composition of the other three arrows in the square. The equivalence of the two definitions follows from Proposition 16 in [3]. The remaining maps are self-explanatory.

Recall that in Section 2 we defined a map  $\omega_z : \mathfrak{g}(2, 0) \rightarrow \mathcal{U}(\mathfrak{n})$  by

$$\omega_z(Z) = \omega(Z) + (z_0 + z)Z,$$

where  $\omega$  is defined in (2.5). We now define a map  $\Omega_z : \mathfrak{g}(2, 0) \rightarrow \mathbb{D}(N)^{\mathfrak{n}}$  by

$$(3.6) \quad \Omega_z(Z) = R(\omega_z(Z))$$

for  $Z \in \mathfrak{g}(2, 0)$ . In light of Theorem 2.8, it follows by applying Theorem 15 of [3] that the system  $\{\Omega_z(Z) \mid Z \in \mathfrak{g}(2, 0)\}$  is conformally invariant on the restriction of the bundle  $\mathcal{L}$  to  $U_e$ . We shall make the content of this assertion more explicit momentarily, but first we make a few other observations about the system  $\Omega_z$ . By construction,  $\Omega_z(Z) \in \mathbb{D}(N)^{\mathfrak{n}}$  for all  $Z \in \mathfrak{g}(2, 0)$ . Thus the system  $\Omega_z$  is *straight* in the sense of [3]. If  $g \in L$  and  $Z \in \mathfrak{g}(2, 0)$  then

$$(3.7) \quad \begin{aligned} g * \Omega_z(Z) &= g * R(\omega_z(Z)) \\ &= R(\text{Ad}(g)\omega_z(Z)) \\ &= R(\omega_z(\text{Ad}(g)Z)) \end{aligned}$$

$$= \Omega_z(\text{Ad}(g)Z),$$

where we have used Lemma 2.5 and the definition of  $\omega_z(Z)$  from the second line to the third. In particular,  $\Omega_z$  is an *L-stable* system in the sense of [3]. A similar calculation using the derived action of  $\mathfrak{l}$  reveals that

$$[\Pi(H_0), \Omega_z(Z)] = 2\Omega_z(Z)$$

for all  $Z \in \mathfrak{g}(2, 0)$ . In particular, the system is *homogeneous* in the sense of [3]. These are the hypotheses, in addition to conformal invariance, that are required in order to be able to apply the results of [12] to the system  $\Omega_z$ .

The following result makes explicit the conformal invariance of the system  $\Omega_z$ . In expressing the result, we encounter the common notational annoyance of a function that is most easily named by its values. To circumvent this, suppose that  $U$  is an open set in  $N$ ,  $T : C^\infty(U) \rightarrow C^\infty(U)$  is a linear map, and  $n \in U$ . In these circumstances, define  $T_n : C^\infty(U) \rightarrow \mathbb{C}$  by  $T_n(\varphi) = T(\varphi)(n)$ . We also require the projection map  $\text{pr}_\mathfrak{l} : \mathfrak{g} \rightarrow \mathfrak{l}$  onto the second summand in the decomposition  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l} \oplus \bar{\mathfrak{n}}$ . In keeping with the usual convention in the algebraic treatment of differential operators, we write  $D\bullet\varphi$  for the result of applying  $D$  to  $\varphi$ .

**Theorem 3.6.** *Let  $g \in G$  and  $Y \in \mathfrak{g}$ . Then we have*

$$(3.8) \quad (\pi(g) \circ \Omega_z(Z) \circ \pi(g^{-1}))_n = \Omega_z(\text{Ad}(a(g^{-1}n)^{-1})Z)_n$$

for all  $Z \in \mathfrak{g}(2, 0)$  and all  $n \in N$  such that  $n\bar{Q} \in U_g$ . In addition, we have

$$(3.9) \quad [\Pi(Y), \Omega_z(Z)]_n = \Omega_z([\text{pr}_\mathfrak{l}(\text{Ad}(n^{-1})Y), Z])_n$$

for all  $Z \in \mathfrak{g}(2, 0)$  and all  $n \in N$ . The subspace  $\Gamma(\mathcal{L})^{\Omega_z}$  of  $\Gamma(\mathcal{L})$  consisting of those  $\varphi$  such that  $\Omega_z(Z)\bullet\varphi = 0$  on  $U_e$  for all  $Z \in \mathfrak{g}(2, 0)$  is invariant under  $G$ .

**Proof.** The first identity is a basis-free form of the first identity in Proposition 2.3 in [12] specialized to the conformally invariant system being considered here. The last statement also follows from that proposition. The second identity follows by applying Theorem 15 in [3] to the current situation.  $\square$

We wish to make a few technical remarks about Theorem 3.6 and its proof. First, although the common solution space  $\Gamma(\mathcal{L})^{\Omega_z}$  is  $G$ -invariant, the reader should not conclude that the operators  $\Omega_z(Z)$  extend to smooth differential operators on sections of the bundle  $\mathcal{L}$  over  $G/\bar{Q}$ . Indeed this is false, since the fact that the operators  $\Omega_z(Z)$  are invariant under left translation by elements of  $N$  forces them to have singularities at infinity in the generalized flag manifold. There is a conformally invariant system that is equivalent to  $\Omega_z$  (in the sense defined in [3]) and whose members do extend to be operators on sections of  $\mathcal{L}$  over  $G/\bar{Q}$ . However, this system is hard to write down explicitly and so is less convenient to work with. Since equivalent systems have the same common solution spaces, we may have the best of both worlds.

The reader might feel that (3.9) should derive from (3.8) by differentiation, and formally this is so. Note however that (3.9) is an instance of the definition of conformal invariance that was used in [3] and that conformal invariance was used to derive Proposition 2.3 in [12]. Thus (3.9) is logically prior to (3.8) and cannot be derived from it without circularity. It would be enough to note that Theorem 15 in [3] gives the required conformal invariance and then to derive the specific form of (3.9) by differentiation of (3.8) when  $Y \in \mathfrak{g}_0$ ; it then follows in general by linearity. This route presents a technical difficulty, however, since (3.8) is only valid on  $\{n \in N \mid n\bar{Q} \in U_g\}$  and we now wish to vary  $g$  whilst keeping  $n$  fixed. Fortunately, this difficulty may be avoided by a technical ruse. To wit, instead of dealing with the derivative directly it suffices to take a sequence  $\{\tau_j\}$  of positive real numbers that approach zero and consider the limit of the relevant difference quotients only over this sequence. If we let  $g_j = e^{\tau_j Y}$  then the set

$$V = \bigcap_{j=1}^{\infty} \{n \in N \mid n\bar{Q} \in U_{g_j}\}$$

is dense in  $N$  by the Baire Category Theorem. The differentiation argument goes through for  $n \in V$  and then (3.9) follows in general by continuity.

The last observation we wish to make is that (3.9) is independent of the real form  $\mathfrak{g}_0$  of  $\mathfrak{g}$  that has been chosen. Thus (3.9) holds for all  $z \in \mathbb{C}$  even when we are considering a real form such that  $\alpha$  and  $\beta$  are joined by an arrow in the Satake diagram. That is, in these cases we still obtain a conformally invariant system of differential operators on  $N$ . The difference is that there is no line bundle  $\mathcal{L}$  over  $G/\bar{Q}$  to consider unless  $z$  happens to satisfy the condition identified in Proposition 3.5.

#### 4. The system $\Omega_z$ on the split real form. I

In this section, we study the conformally invariant system  $\Omega_z$  in the setting of the split real form of  $\mathfrak{g}$ . The embodiments of  $\Omega_z$  on other real forms are related to this one by a complex-linear change of variables. This means that for some purposes it is sufficient to study the system on the split real form. For example, this is so if one wishes to understand the polynomial solutions to the system. For other purposes, such as studying the space of  $K$ -finite solutions, each real form must be considered separately.

The system  $\Omega_z$  is comprised of differential operators on the connected group  $N \subset G$  whose Lie algebra is  $\mathfrak{n}$ . Thus we must begin by establishing a model for this group. The underlying manifold of  $N$  is  $\mathfrak{n}_0$ , which decomposes as  $\mathfrak{n}_0 = V_0 \oplus \mathfrak{g}_0(2, 0)$ , where  $V_0 = \mathfrak{g}_0(1, 1) \oplus \mathfrak{g}_0(1, -1)$ . It will be important to distinguish between elements of  $\mathfrak{n}_0$  and elements of  $N$  and so we shall write  $n(X, Y, T)$  for  $X + Y + T$  regarded as an element of  $N$ , where  $X \in \mathfrak{g}_0(1, 1)$ ,  $Y \in \mathfrak{g}_0(1, -1)$ , and  $T \in \mathfrak{g}_0(2, 0)$ . We shall take the operation on  $N$  to be

$$(4.1) \quad n(X_1, Y_1, T_1)n(X_2, Y_2, T_2) = n(X_1 + X_2, Y_1 + Y_2, T_1 + T_2 + [X_1, Y_2]).$$

The reader may be surprised to see (4.1) instead of the more symmetrical and more usual operation

$$(4.2) \quad \tilde{n}(X_1, Y_1, T_1)\tilde{n}(X_2, Y_2, T_2) \\ = \tilde{n}\left(X_1 + X_2, Y_1 + Y_2, T_1 + T_2 + \frac{1}{2}[X_1, Y_2] - \frac{1}{2}[X_2, Y_1]\right).$$

In fact, the map

$$n(X, Y, T) \mapsto \tilde{n}\left(X, Y, T - \frac{1}{2}[X, Y]\right)$$

is easily seen to be an isomorphism between these structures. The reason for choosing (4.1) in place of (4.2) is that this choice leads to a system of coordinates in which  $\Omega_z$  takes a relatively simple form. Note that with this model the inverse is given by

$$(4.3) \quad n(X, Y, T)^{-1} = n(-X, -Y, -T + [X, Y]),$$

the exponential map  $\exp : \mathfrak{n}_0 \rightarrow N$  is given by

$$(4.4) \quad \exp(X + Y + T) = n\left(X, Y, T + \frac{1}{2}[X, Y]\right),$$

and the conjugation by elements of  $L$  is given by

$$(4.5) \quad gn(X, Y, T)g^{-1} = n(\text{Ad}(g)X, \text{Ad}(g)Y, \text{Ad}(g)T)$$

for  $g \in L$ .

We shall need notation for specific differential operators on  $N$ . To this end, we introduce the directional derivatives  $\partial_\mu$  for  $\mu \in R(1, 1) \cup R(1, -1) \cup R(2, 0)$ . If  $\mu \in R(1, 1)$  and  $\varphi \in C^\infty(N)$  then

$$(\partial_\mu \bullet \varphi)(n(X, Y, T)) = \left. \frac{d}{d\tau} \right|_{\tau=0} \varphi(n(X + \tau X_\mu, Y, T))$$

and similarly for  $\mu \in R(1, -1) \cup R(2, 0)$ . We also introduce the coordinates  $x_\mu, y_\nu, t_\zeta$ , for  $\mu \in R(1, 1)$ ,  $\nu \in R(1, -1)$ , and  $\zeta \in R(2, 0)$ , respectively. By definition, these coordinates are dual to the root vectors  $X_\mu, X_\nu$ , and  $X_\zeta$ , respectively, so that we have

$$\partial_\mu = \frac{\partial}{\partial x_\mu}$$

for  $\mu \in R(1, 1)$ , and similarly for  $y_\nu$  and  $t_\zeta$ .

**Lemma 4.1.** *For  $\mu \in R(1, 1)$ ,  $\nu \in R(1, -1)$ , and  $\xi \in R(2, 0)$ , we have*

$$R(X_\mu) = \partial_\mu, \\ R(X_\nu) = \partial_\nu + \sum_{\gamma \in R(1, 1)} N_{\gamma, \nu} x_\gamma \partial_{\gamma + \nu}, \\ R(X_\xi) = \partial_\xi.$$

**Proof.** We prove the second identity, since the other two are almost immediate. For  $\varphi \in C^\infty(N)$  we have

$$\begin{aligned} (R(X_\nu) \bullet \varphi)(n(X, Y, T)) &= \left. \frac{d}{d\tau} \right|_{\tau=0} \varphi(n(X, Y, T)n(0, \tau X_\nu, 0)) \\ &= \left. \frac{d}{d\tau} \right|_{\tau=0} \varphi(n(X, Y + \tau X_\nu, T + \tau[X, X_\nu])). \end{aligned}$$

Now

$$[X, X_\nu] = \sum_{\gamma \in R(1,1)} x_\gamma [X_\gamma, X_\nu] = \sum_{\gamma \in R(1,1)} N_{\gamma, \nu} x_\gamma X_{\gamma+\nu}$$

and so

$$\begin{aligned} &(R(X_\nu) \bullet \varphi)(n(X, Y, T)) \\ &= \left. \frac{d}{d\tau} \right|_{\tau=0} \varphi \left( n \left( X, Y + \tau X_\nu, T + \sum_{\gamma \in R(1,1)} N_{\gamma, \nu} x_\gamma \tau X_{\gamma+\nu} \right) \right) \\ &= \left( \left( \partial_\nu + \sum_{\gamma \in R(1,1)} N_{\gamma, \nu} x_\gamma \partial_{\gamma+\nu} \right) \bullet \varphi \right) (n(X, Y, T)), \end{aligned}$$

as required.  $\square$

We are now ready to determine the explicit form of the operators in the system  $\Omega_z$ , but to express the result concisely it will be convenient to introduce some further notation. For  $\zeta \in R(2, 0)$ , let

$$(4.6) \quad \Delta_\zeta = \sum_{(\mu, \nu) \in S(\zeta)} N_{-\mu, -\nu} \partial_\mu \partial_\nu$$

and, for  $\zeta, \xi \in R(2, 0)$ , let

$$(4.7) \quad F_{\zeta, \xi} = \sum_{\substack{\nu \in R(1, -1) \\ (\nu, \zeta) = (\nu, \xi) = 1}} N_{\zeta-\nu, \nu} N_{\xi-\nu, \nu} x_{\xi-\nu} \partial_{\zeta-\nu}.$$

There is an obvious notion of a vector or matrix indexed by arbitrary finite sets rather than only by sets of the form  $\{1, \dots, m\}$  and we shall use this notion here. Thus we may consider the  $R(2, 0)$ -column vectors  $\vec{\partial} = [\partial_\zeta]$ ,  $\vec{\Delta} = [\Delta_\zeta]$  and  $\vec{\Omega}_z = [\Omega_z(X_\zeta)]$  and the  $R(2, 0)$ -by- $R(2, 0)$  matrix  $F = [F_{\zeta, \xi}]$ . We use the usual symbol  $I$  for the identity matrix (with the indexing set to be determined from context).

**Proposition 4.2.** *We have*

$$\vec{\Omega}_z = \vec{\Delta} + ((z - z_0)I - F) \vec{\partial}.$$



**Proof.** Let  $\zeta \in R(2, 0)$ . By choosing bases consisting of roots vectors in the definition of  $\omega$  we obtain

$$\omega(X_\zeta) = \sum_{(\mu, \nu) \in S(\zeta)} N_{-\mu, -\nu} X_\mu X_\nu.$$

Thus

$$\begin{aligned} \Omega_z(X_\zeta) &= R(\omega_z(X_\zeta)) \\ &= R(\omega(X_\zeta) + (z + z_0)X_\zeta) \\ &= (z + z_0)R(X_\zeta) + \sum_{(\mu, \nu) \in S(\zeta)} N_{-\mu, -\nu} R(X_\mu) R(X_\nu) \\ &= (z + z_0)\partial_\zeta + \sum_{(\mu, \nu) \in S(\zeta)} N_{-\mu, -\nu} \partial_\mu \left( \partial_\nu + \sum_{\gamma \in R(1, 1)} N_{\gamma, \nu} x_\gamma \partial_{\gamma + \nu} \right) \\ &= (z + z_0)\partial_\zeta + \sum_{(\mu, \nu) \in S(\zeta)} N_{-\mu, -\nu} \partial_\mu \partial_\nu + \sum_{\substack{(\mu, \nu) \in S(\zeta) \\ \gamma \in R(1, 1)}} N_{-\mu, -\nu} N_{\gamma, \nu} \partial_\mu x_\gamma \partial_{\gamma + \nu} \\ &= (z + z_0)\partial_\zeta + \Delta_\zeta + \sum_{\substack{(\mu, \nu) \in S(\zeta) \\ \gamma \in R(1, 1)}} N_{-\mu, -\nu} N_{\gamma, \nu} (\delta_{\mu, \gamma} + x_\gamma \partial_\mu) \partial_{\gamma + \nu} \\ &= (z + z_0)\partial_\zeta + \Delta_\zeta + \sum_{(\mu, \nu) \in S(\zeta)} N_{-\mu, -\nu} N_{\mu, \nu} \partial_\zeta \\ &\quad + \sum_{\substack{(\mu, \nu) \in S(\zeta) \\ \gamma \in R(1, 1)}} N_{-\mu, -\nu} N_{\gamma, \nu} x_\gamma \partial_\mu \partial_{\gamma + \nu}. \end{aligned}$$

It follows from (2.1) that  $N_{-\mu, -\nu} = -N_{\mu, \nu}$ . Moreover, if  $(\mu, \nu) \in S(\zeta)$  then  $N_{\mu, \nu} \in \{\pm 1\}$  and so  $N_{-\mu, -\nu} N_{\mu, \nu} = -1$ . By Lemma 2.3, the third term in the last line above evaluates to  $-d(\alpha, \beta)\partial_\zeta = -2z_0\partial_\zeta$ . This gives

$$\Omega_z(X_\zeta) = \Delta_\zeta + (z - z_0)\partial_\zeta - \sum_{\substack{(\mu, \nu) \in S(\zeta) \\ \gamma \in R(1, 1)}} N_{\mu, \nu} N_{\gamma, \nu} x_\gamma \partial_\mu \partial_{\gamma + \nu}.$$

It remains to rewrite the last term in this expression. We begin by collecting the sum according to the value of  $\gamma + \nu$  when  $N_{\gamma, \nu} \neq 0$ . This gives

$$\sum_{\xi \in R(2, 0)} \left( \sum_{\substack{(\mu, \nu) \in S(\zeta) \\ (\gamma, \nu) \in S(\xi)}} N_{\mu, \nu} N_{\gamma, \nu} x_\gamma \partial_\mu \right) \partial_\xi.$$

By Lemma 2.4 this may be reexpressed as

$$\sum_{\xi \in R(2,0)} \left( \sum_{\substack{\nu \in R(1,-1) \\ (\nu, \zeta) = (\nu, \xi) = 1}} N_{\zeta - \nu, \nu} N_{\xi - \nu, \nu} x_{\xi - \nu} \partial_{\zeta - \nu} \right) \partial_{\xi},$$

which is

$$\sum_{\xi \in R(2,0)} F_{\zeta, \xi} \partial_{\xi},$$

by definition. It follows that

$$\Omega_z(X_{\zeta}) = \Delta_{\zeta} + (z - z_0) \partial_{\zeta} - \sum_{\xi \in R(2,0)} F_{\zeta, \xi} \partial_{\xi},$$

for all  $\zeta \in R(2,0)$ . This is the component-by-component expression of the claimed identity.  $\square$

The entries of the  $R(2,0)$ -by- $R(2,0)$  matrix  $F$  lie in  $D(\mathfrak{g}(1,1))$ . We wish to consider the vector space endomorphism of the  $\mathbb{C}[\mathfrak{g}(1,1)]$ -module  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  induced by  $F$ . If  $\zeta \in R(2,0)$  then let  $e_{\zeta}$  be the  $\zeta^{\text{th}}$  standard basis vector for  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$ . If  $\varphi \in \mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  then

$$F \bullet \varphi = \sum_{\zeta, \xi \in R(2,0)} F_{\zeta, \xi} \bullet \varphi_{\xi} e_{\zeta}.$$

Note that every entry of  $F$  is a homogeneous operator of degree zero with respect to the standard grading of  $\mathbb{C}[\mathfrak{g}(1,1)]$ . Thus  $F \bullet$  preserves each graded component of  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  if this module is given the componentwise grading arising from the standard grading of  $\mathbb{C}[\mathfrak{g}(1,1)]$ . In particular, the map  $F \bullet$  is locally finite on  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$ . Thus we may consider the spectrum  $\sigma(F) \subset \mathbb{C}$  of  $F \bullet$  on  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  and if  $w \notin \sigma(F)$  then the operator  $F \bullet - wI$  on  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  is invertible. The homogeneity of  $F$  also implies that the action of  $F$  on  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  extends to an action on the module  $\mathbb{C}[[x]]^{R(2,0)}$  of  $R(2,0)$ -tuples of formal power series centered at 0 in  $\mathfrak{g}(1,1)$ . Of course, this action is no longer locally finite, but it remains true that if  $w \notin \sigma(F)$  then the operator  $F \bullet - wI$  on  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  is invertible, since this may be checked one homogeneous component at a time. Finally, the operators  $F_{\zeta, \xi}$  are independent of the variables  $y$  and so the action of  $F$  on  $\mathbb{C}[[x]]^{R(2,0)}$  extends further to an action on  $\mathbb{C}[[x, y]]^{R(2,0)}$ . This extended action once again has the invertibility property just described.

**Theorem 4.3.** *Let  $t_0$  be the coordinate of a point in  $\mathfrak{g}(2,0)$  and suppose that  $z \notin z_0 + \sigma(F)$ . If  $p \in \mathbb{C}[[x, y, t - t_0]]$  is a formal power series solution to the system  $\Omega_z$  and  $p(x, y, t_0) = 0$  then  $p$  is identically zero.*

**Proof.** Let us express  $p$  in the form

$$p = \sum_k q_k (t - t_0)^k,$$

where  $k \in \mathbb{N}^{R(2,0)}$  and  $q_k \in \mathbb{C}[x, y]$  for each  $k$ . Since  $\Omega_z(X_\zeta) \bullet p = 0$  for all  $\zeta \in R(2, 0)$ , we have the equation

$$[\Delta_\zeta \bullet p] + ((z - z_0)I - F) \bullet [\partial_\zeta \bullet p] = [0]$$

of  $R(2, 0)$ -tuples. Thus

$$\begin{aligned} [0] &= \sum_k [\Delta_\zeta \bullet q_k] (t - t_0)^k + ((z - z_0)I - F) \bullet \left[ \sum_k k_\zeta q_k (t - t_0)^{k - e_\zeta} \right] \\ &= \sum_k [\Delta_\zeta \bullet q_k] (t - t_0)^k + ((z - z_0)I - F) \bullet \left[ \sum_k (k_\zeta + 1) q_{k + e_\zeta} (t - t_0)^k \right] \\ &= \sum_k \left( [\Delta_\zeta \bullet q_k] + ((z - z_0)I - F) \bullet [(k_\zeta + 1) q_{k + e_\zeta}] \right) (t - t_0)^k \end{aligned}$$

and so we must have

$$[\Delta_\zeta \bullet q_k] + ((z - z_0)I - F) \bullet [(k_\zeta + 1) q_{k + e_\zeta}] = [0]$$

for all  $k$ . The assumption on  $z$  and the discussion preceding the statement of the theorem allow us to rewrite this expression as

$$[(k_\zeta + 1) q_{k + e_\zeta}] = -((z - z_0)I - F)^{-1} \bullet [\Delta_\zeta \bullet q_k]$$

for all  $k$ . Since  $k_\zeta + 1 \geq 1$ , this identity allows us to determine all  $q_k$  inductively from  $q_0$ . We complete the proof by noting that  $p(x, y, t_0) = q_0(x, y)$ .  $\square$

Note that Theorem 4.3 is sharp. Indeed, if  $z \in z_0 + \sigma(F)$  and  $\varphi = [\varphi_\xi] \in \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  is an eigenvector of  $F \bullet$  with eigenvalue  $z - z_0$  then

$$p = \sum_{\xi \in R(2,0)} t_\xi \varphi_\xi$$

is a nonzero solution to the system  $\Omega_z$ , but  $p(x, y, 0)$  is identically zero.

The last order of business in this section is to establish the equivariance of the map  $F \bullet$  with respect to a suitable action of  $L$ . Recall from Section 3 that if  $\varphi$  is a function on  $N$  and  $g \in L$  then  $\pi(g)\varphi$  is the function defined by  $(\pi(g)\varphi)(n) = \chi(g)^{-1} \varphi(g^{-1}ng)$ . The  $*$ -action of  $L$  on  $\mathbb{D}(N)$  is then defined by  $g * D = \pi(g) \circ D \circ \pi(g^{-1})$ . In fact, the factors of  $\chi$  in this expression cancel one another. Thus it will be convenient to introduce the action of  $L$  on functions on  $N$  that is given by  $(g \cdot \varphi)(n) = \varphi(g^{-1}ng)$  and to note that, with this action,  $g * D = g \circ D \circ g^{-1}$ . In particular, by (3.7), we have

$$(4.8) \quad g^{-1} \circ \Omega_z(\text{Ad}(g)Z) = \Omega_z(Z) \circ g^{-1}$$

for all  $g \in L$ ,  $Z \in \mathfrak{g}(2, 0)$ , and  $z \in \mathbb{C}$ .

Let us identify the vector  $\varphi = [\varphi_\xi] \in \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  with the element

$$\sum_{\xi \in R(2,0)} \varphi_\xi \otimes t_\xi \in \mathbb{C}[\mathfrak{g}(1, 1)] \otimes \mathfrak{g}(2, 0)^*.$$

In this way, we obtain an action of  $L$  on  $\mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  via transport of structure from the tensor product action of  $L$  on  $\mathbb{C}[\mathfrak{g}(1, 1)] \otimes \mathfrak{g}(2, 0)^*$ .

**Proposition 4.4.** *With the action of  $L$  on  $\mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  that was defined above, the map  $F\bullet : \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)} \rightarrow \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  is  $L$ -intertwining.*

**Proof.** Let  $\varphi = [\varphi_\xi] \in \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  and define

$$\psi(\varphi) = \sum_{\xi \in R(2,0)} t_\xi \varphi_\xi.$$

This function is independent of  $y_\nu$  for all  $\nu \in R(1, -1)$  and so  $\Delta_\zeta \bullet \psi(\varphi) = 0$  for all  $\zeta \in R(2, 0)$ . Thus, by Proposition 4.2, we have

$$\begin{aligned} \Omega_z(X_\zeta) \bullet \psi(\varphi) &= (z - z_0) \partial_\zeta \bullet \psi(\varphi) - \sum_{\xi \in R(2,0)} F_{\zeta, \xi} \partial_\xi \bullet \psi(\varphi) \\ &= (z - z_0) \varphi_\zeta - \sum_{\xi \in R(2,0)} F_{\zeta, \xi} \varphi_\xi \end{aligned}$$

and, in particular,

$$(4.9) \quad \Omega_{z_0}(X_\zeta) \bullet \psi(\varphi) = -(F\bullet \varphi)_\zeta$$

for all  $\zeta \in R(2, 0)$ .

Let us introduce matrix coefficients  $m_{\tau, \zeta}$  for the adjoint action of  $L$  on  $\mathfrak{g}(2, 0)$  by

$$(4.10) \quad \text{Ad}(g)X_\zeta = \sum_{\tau \in R(2,0)} m_{\tau, \zeta}(g)X_\tau.$$

It follows as usual from (4.10) that

$$(4.11) \quad \text{Ad}^*(g)t_\zeta = \sum_{\tau \in R(2,0)} m_{\zeta, \tau}(g^{-1})t_\tau.$$

The action of  $L$  on  $\mathbb{C}[\mathfrak{g}(2, 0)]$  restricts to the coadjoint action on the space of linear polynomials and so (4.11) may also be written as

$$(4.12) \quad g \cdot t_\zeta = \sum_{\tau \in R(2,0)} m_{\zeta, \tau}(g^{-1})t_\tau$$

if we regard  $t_\zeta$  as a function on  $N$ . For  $g \in L$  and  $\varphi \in \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  we have

$$\begin{aligned} g \cdot \psi(\varphi) &= \sum_{\xi \in R(2,0)} g \cdot (t_\xi \varphi_\xi) \\ &= \sum_{\xi \in R(2,0)} (g \cdot t_\xi)(g \cdot \varphi_\xi) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\xi, \tau \in R(2,0)} m_{\xi, \tau}(g^{-1}) t_{\tau}(g \cdot \varphi_{\xi}) \\
&= \sum_{\tau \in R(2,0)} t_{\tau} \left( \sum_{\xi \in R(2,0)} m_{\xi, \tau}(g^{-1})(g \cdot \varphi_{\xi}) \right) \\
&= \sum_{\tau \in R(2,0)} t_{\tau}(g \cdot \varphi)_{\tau} \\
&= \psi(g \cdot \varphi).
\end{aligned}$$

(Of course, it was this computation that governed the choice of action of  $L$  on  $\mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$ .) By this identity, (4.9), (4.10) and (4.8), we have

$$\begin{aligned}
(F \bullet (g^{-1} \cdot \varphi))_{\zeta} &= -\Omega_{z_0}(X_{\zeta}) \bullet \psi(g^{-1} \bullet \varphi) \\
&= -\Omega_{z_0}(X_{\zeta}) \bullet (g^{-1} \cdot \psi(\varphi)) \\
&= -g^{-1} \cdot \Omega_{z_0}(\text{Ad}(g)X_{\zeta}) \bullet \psi(\varphi) \\
&= -g^{-1} \cdot \sum_{\tau \in R(2,0)} m_{\tau, \zeta}(g) \Omega_{z_0}(X_{\tau}) \bullet \psi(\varphi) \\
&= g^{-1} \cdot \sum_{\tau \in R(2,0)} m_{\tau, \zeta}(g) (F \bullet \varphi)_{\tau} \\
&= \sum_{\tau \in R(2,0)} m_{\tau, \zeta}(g) g^{-1} \cdot (F \bullet \varphi)_{\tau} \\
&= (g^{-1} \cdot (F \bullet \varphi))_{\zeta}
\end{aligned}$$

for all  $\zeta \in R(2, 0)$  and  $g \in L$ . It follows that

$$F \bullet (g^{-1} \cdot \varphi) = g^{-1} \cdot (F \bullet \varphi)$$

for all  $g \in L$ , as required.  $\square$

As noted above, the map  $F \bullet : \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)} \rightarrow \mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$  is homogeneous of degree zero. Since  $L$  is reductive and its action on  $\mathbb{C}[\mathfrak{g}(1, 1)]$  is degree preserving, the  $L$ -module  $\mathbb{C}[\mathfrak{g}(1, 1)] \otimes \mathfrak{g}(2, 0)^*$  decomposes as the direct sum of various irreducible components. It follows from Proposition 4.4 that  $F \bullet$  leaves each  $L$ -isotype in this decomposition stable. The decomposition may not be multiplicity free and so we cannot necessarily invoke Schur's Lemma to conclude that  $F \bullet$  is actually a scalar on each isotype. We can, however, always determine the spectrum  $\sigma(F)$  if we can compute the action of  $F \bullet$  on all highest-weight vectors in  $\mathbb{C}[\mathfrak{g}(1, 1)]^{R(2,0)}$ .

## 5. The system $\Omega_z$ on the split real form. II

In this section, we make the structure of the system  $\Omega_z$  more explicit in each of the cases in which it has been constructed. This allows us to estimate

and, in most cases, determine the spectrum of the operator  $F^\bullet$  that appears in Theorem 4.3.

**5.1. The systems in Type  $A_l$ .** Let  $l \geq 2$ , choose  $a, b \geq 1$  such that  $a + b \leq l$ , and let  $c = l - a - b + 1$ . We take the standard model for the root system  $R$  of type  $A_l$ , in which the positive roots are  $\{e_i - e_j \mid 1 \leq i < j \leq l + 1\}$ . With the labels introduced in Figure 1, we have  $\alpha_i = e_i - e_{i+1}$ . Let  $\alpha = \alpha_a$  and  $\beta = \alpha_{a+b}$ . Note that  $d(\alpha, \beta) = b$ . We use the total order on  $R^s$  that is specified by

$$\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{l-1} \prec \alpha_l,$$

which satisfies the requirements introduced in Section 2. Let

$$\mu(i, j) = e_i - e_{a+j}$$

for  $1 \leq i \leq a$  and  $1 \leq j \leq b$ ,

$$\nu(i, j) = e_{a+i} - e_{a+b+j}$$

for  $1 \leq i \leq b$  and  $1 \leq j \leq c$ , and

$$\zeta(i, j) = e_i - e_{a+b+j}$$

for  $1 \leq i \leq a$  and  $1 \leq j \leq c$ . With this notation, we have

$$\begin{aligned} R(1, 1) &= \{\mu(i, j) \mid 1 \leq i \leq a \text{ and } 1 \leq j \leq b\}, \\ R(1, -1) &= \{\nu(i, j) \mid 1 \leq i \leq b \text{ and } 1 \leq j \leq c\}, \\ R(2, 0) &= \{\zeta(i, j) \mid 1 \leq i \leq a \text{ and } 1 \leq j \leq c\}. \end{aligned}$$

With the total order that has been specified above, one finds that

$$(5.1) \quad N_{\mu(i,j), \nu(m,n)} = -\delta_{j,m}.$$

Note that these are not the structure constants that would be obtained by considering the standard basis for the algebra  $\mathfrak{sl}(l + 1)$ , but rather their negatives. We continue with the nonstandard choice for consistency's sake. It follows from (5.1) that

$$N_{-\mu(i,j), -\nu(m,n)} = \delta_{j,m}$$

and by using this in (4.6) we get

$$(5.2) \quad \Delta_{\zeta(i,j)} = \sum_{k=1}^b \partial_{\mu(i,k)} \partial_{\nu(k,j)}.$$

By using (5.1) in (4.7), we get

$$(5.3) \quad F_{\zeta(i,j), \zeta(m,n)} = \delta_{j,n} \sum_{k=1}^b x_{\mu(m,k)} \partial_{\mu(i,k)}.$$

The first conclusion we draw from this evaluation is that the  $R(2, 0)$ -by- $R(2, 0)$  matrix  $F$  may be arranged so as to have a block-diagonal structure. In fact, if we order the roots  $\zeta(i, j)$  in reverse lexicographic order then the resulting  $ac$ -by- $ac$  matrix will become block diagonal with  $c$   $a$ -by- $a$  blocks

down the diagonal. Moreover, these blocks are actually identical, since the cofactor of  $\delta_{j,n}$  in (5.3) depends only on  $i$  and  $m$ . For brevity, let us introduce the notation  $x_{m,k} = x_{\mu(m,k)}$  and  $\partial_{i,k} = \partial_{\mu(i,k)}$ . Then we have

$$F = P \boxplus P \boxplus \cdots \boxplus P,$$

where  $\boxplus$  denotes the block sum of matrices and  $P$  is the  $a$ -by- $a$  matrix with entries

$$P_{i,m} = \sum_{k=1}^b x_{m,k} \partial_{i,k}.$$

It follows from this that in order to determine the spectrum of  $F\bullet$  on the module  $\mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  it suffices to determine the spectrum of  $P\bullet$  on the module  $\mathbb{C}[\mathfrak{g}(1,1)]^a$ .

**Lemma 5.1.** *For  $1 \leq p, q, r, s \leq a$  we have*

$$[P_{p,q}, P_{r,s}] = \delta_{p,s} P_{r,q} - \delta_{q,r} P_{p,s}.$$

**Proof.** The required identity follows from a calculation that is based upon the commutator formula  $[\partial_{p,k}, x_{s,m}] = \delta_{p,s} \delta_{k,m}$ .  $\square$

It follows from Lemma 5.1 that the operators  $P_{p,q}$  span a copy of  $\mathfrak{gl}(a)$ . In fact, the map  $E_{q,p} \mapsto P_{p,q}$  is an isomorphism from  $\mathfrak{gl}(a)$  to their span. Note the interchange of indices that is involved in this isomorphism.

In order to estimate the spectrum of  $P\bullet$  on  $\mathbb{C}[\mathfrak{g}(1,1)]^a$ , we shall have to make a brief incursion into the marches of Capelli's identity. Let  $\tilde{P}$  be the  $a$ -by- $a$  matrix

$$\tilde{P}(u) = [P_{m,n} - \delta_{m,n}(u + a - m)],$$

whose entries we regard as elements of the tensor product algebra  $\mathbb{C}[u] \otimes \mathcal{U}(\mathfrak{gl}(a))$ . Let  $\psi_1, \dots, \psi_a$  be the standard basis for  $\mathbb{C}^a$ . In the tensor product algebra  $\mathbb{C}[u] \otimes \mathcal{U}(\mathfrak{gl}(a)) \otimes \wedge^\bullet \mathbb{C}^a$ , we define elements

$$\tau_q(u) = \sum_{p=1}^a (P_{p,q} - \delta_{p,q}u)\psi_p.$$

Note that

$$(5.4) \quad \tau_1(u + a - 1)\tau_2(u + a - 2) \cdots \tau_a(u) = \text{cdet}(\tilde{P}(u))\psi_1 \cdots \psi_a,$$

where  $\text{cdet}$  stands for the column determinant

$$\text{cdet}(A) = \sum_{\pi} \text{sgn}(\pi) A_{\pi(1)1} \cdots A_{\pi(a)a}.$$

As usual,  $\text{cdet}(\tilde{P}(u)) \in \mathbb{C}[u] \otimes \mathcal{Z}(\mathfrak{gl}(a))$ , where  $\mathcal{Z}(\mathfrak{gl}(a))$  denotes the center of the universal enveloping algebra. Several methods are available for proving this assertion. One of the clearest and most efficient is the method outlined by Itoh in Sections 1 and 2 of [11] (see also Section 1 of [10]). This method involves equating  $\text{cdet}(\tilde{P}(u))$  with a symmetrized determinant of

$\tilde{P}(u)$  whose invariance under  $\mathrm{GL}(a)$  can easily be established. Since the fact is familiar, we shall not provide the details.

We choose the standard diagonal Cartan subalgebra of  $\mathfrak{gl}(a)$ , but use the opposite of the usual positive system of roots. With this choice (but not with the standard one), the entries above the diagonal in  $\tilde{P}(u)$  are weight raising. Let  $(\rho_{\varpi}, W_{\varpi})$  be the irreducible representation of  $\mathfrak{gl}(a)$  with highest weight  $\varpi = (\varpi_1, \dots, \varpi_a)$ . Note that we will have  $\varpi_1 \leq \varpi_2 \leq \dots \leq \varpi_a$  because of the choice of positive system that has been made. The fact recalled in the previous paragraph implies that  $\mathrm{cdet}(\tilde{P}(u))$  acts on  $\mathbb{C}[u] \otimes W_{\varpi}$  by a scalar. This scalar may be calculated by considering the action of  $\mathrm{cdet}(\tilde{P}(u))$  on a highest weight vector in  $W_{\varpi}$ . The result of this calculation is that

$$(5.5) \quad \rho_{\varpi}(\mathrm{cdet}(\tilde{P}(u))) = (\varpi_1 - (u + a - 1))(\varpi_2 - (u + a - 2)) \cdots (\varpi_a - u).$$

Lemma 5.2 and Lemma 5.3 are due to Umeda (Lemma 1 and Proposition 2 in [19]). However, due to the interchange of indices and a difference of normalization, our versions look slightly different, so it seems safest to sketch the proof of the more substantial of the two.

**Lemma 5.2.** *For all  $1 \leq q, r \leq a$  we have*

$$\tau_q(u + 1)\tau_r(u) = -\tau_r(u + 1)\tau_q(u).$$

*In particular,*

$$\tau_q(u + 1)\tau_q(u) = 0$$

*for all  $1 \leq q \leq a$ .*

**Proof.** This is a consequence of a calculation using Lemma 5.1.  $\square$

**Lemma 5.3.** *There is an  $a$ -by- $a$  matrix  $Q(u)$  with entries in  $\mathbb{C}[u] \otimes \mathcal{U}(\mathfrak{gl}(a))$  such that*

$$Q(u)(P - uI_a) = \mathrm{cdet}(\tilde{P}(u))I_a.$$

**Proof.** We first define elements  $\eta_1(u), \dots, \eta_a(u) \in \mathbb{C}[u] \otimes \mathcal{U}(\mathfrak{gl}(a)) \otimes \wedge^{\bullet} \mathbb{C}^a$  by

$$\begin{aligned} \eta_1(u) &= (-1)^{a-1} \tau_2(u + a - 1) \tau_3(u + a - 2) \cdots \tau_a(u + 1), \\ \eta_a(u) &= \tau_1(u + a - 1) \cdots \tau_{a-1}(u + 1), \end{aligned}$$

and

$$\begin{aligned} \eta_m(u) &= (-1)^{a-m} \tau_1(u + a - 1) \tau_2(u + a - 2) \cdots \tau_{m-1}(u + a - m + 1) \\ &\quad \cdot \tau_{m+1}(u + a - m) \cdots \tau_a(u + 1) \end{aligned}$$

for  $2 \leq m \leq a - 1$ . It follows from (5.4) and repeated use of Lemma 5.2 that

$$\eta_m(u)\tau_n(u) = \delta_{m,n} \mathrm{cdet}(\tilde{P}(u))\psi_1 \cdots \psi_a$$

for all  $1 \leq m, n \leq a$ . Next we define the matrix  $Q(u)$  by

$$\eta_m(u) = \sum_{j=1}^a (-1)^{a-j} Q_{m,j}(u) \psi_1 \cdots \hat{\psi}_j \cdots \psi_a,$$



where the hat denotes omission. Then

$$\begin{aligned} & \delta_{m,n} \text{cdet}(\tilde{P}(u)) \psi_1 \cdots \psi_a \\ &= \eta_m(u) \tau_n(u) \\ &= \sum_{j,k=1}^a (-1)^{a-j} Q_{m,j}(u) (P_{k,n} - \delta_{k,n}u) \psi_1 \cdots \widehat{\psi}_j \cdots \psi_a \psi_k \\ &= \sum_{j=1}^a Q_{m,j}(u) (P_{j,n} - \delta_{j,n}u) \psi_1 \cdots \psi_a \end{aligned}$$

and so

$$\sum_{j=1}^a Q_{m,j}(u) (P - uI_a)_{j,n} = \delta_{m,n} \text{cdet}(\tilde{P}(u)),$$

as required.  $\square$

**Theorem 5.4.** *We have  $\sigma(F) \subset \mathbb{Z} \cap [-(a-1), \infty)$ . If  $b \geq a$  then this inclusion is an equality.*

**Proof.** As we remarked above,  $\sigma(F)$  is equal to  $\sigma(P)$ , the spectrum of  $P_\bullet$  on  $\mathbb{C}[\mathfrak{g}(1,1)]^a$ . We begin by establishing that  $\sigma(P) \subset \mathbb{Z} \cap [-(a-1), \infty)$ . Suppose that  $\varphi = [\varphi_j]$  is an eigenvector of  $P_\bullet$  with eigenvalue  $\lambda$ . It follows from Lemma 5.3 that  $\text{cdet}(\tilde{P}(\lambda))_\bullet \varphi_j = 0$  for all  $1 \leq j \leq a$ . Let us fix some  $j$  such that  $\varphi_j \neq 0$ .

From the explicit formula for  $P_{m,n}$ , we see that the space of linear polynomials on  $\mathfrak{g}(1,1)$  is isomorphic to the direct sum of  $b$  copies of the standard representation of  $\mathfrak{gl}(a)$ . Indeed, each of the variables  $x_{a,1}, \dots, x_{a,b}$  is a highest weight vector for the action with weight  $(0, \dots, 0, 1)$ . (Naturally, we continue to use the nonstandard positive system for  $\mathfrak{gl}(a)$  that was fixed above.) It follows that, as a  $\mathfrak{gl}(a)$ -module,  $\mathbb{C}[\mathfrak{g}(1,1)]$  decomposes as the direct sum of various polynomial representations. These could be made quite explicit, but all we require here is that their highest weights have the form  $\varpi = (\varpi_1, \dots, \varpi_a)$  with  $\varpi_1 \leq \dots \leq \varpi_a$  and  $\varpi_m \in \mathbb{N}$  for all  $1 \leq m \leq a$ . The polynomial  $\varphi_j$  fixed in the previous paragraph may be decomposed into isotypic components as

$$\varphi_j = \sum_{\varpi} \varphi_j^{\varpi}$$

and we have

$$\begin{aligned} 0 &= \text{cdet}(\tilde{P}(\lambda))_\bullet \varphi_j \\ &= \sum_{\varpi} (\varpi_1 - (\lambda + a - 1)) (\varpi_2 - (\lambda + a - 2)) \cdots (\varpi_a - \lambda) \varphi_j^{\varpi}. \end{aligned}$$

Distinct isotypes are independent and it follows that if we choose a highest weight  $\varpi$  such that  $\varphi_j^{\varpi} \neq 0$  then

$$(\varpi_1 - (\lambda + a - 1)) (\varpi_2 - (\lambda + a - 2)) \cdots (\varpi_a - \lambda) = 0.$$

This equation implies that  $\lambda = \varpi_m - a + m$  for some  $1 \leq m \leq a$ . Consequently,  $\lambda \in \mathbb{Z}$  and  $\lambda \geq -a + m \geq -(a - 1)$ . This establishes the first claim.

Henceforth, we suppose that  $b \geq a$ . The next claim is that the spectrum  $\sigma(P)$  is invariant under the map  $\lambda \mapsto \lambda + 1$ . Let

$$X = [x_{p,q}]_{1 \leq p, q \leq a},$$

which is possible by our assumption on  $b$ , and define  $D = \det(X)$ . One verifies from the explicit formula for  $P_{i,m}$  that  $P_{i,m} \bullet x_{r,s} = \delta_{i,r} x_{m,s}$ . From this it follows that  $P_{i,m} \bullet D$  is the determinant of the matrix obtained from  $X$  by replacing the  $i^{\text{th}}$  row by the  $m^{\text{th}}$  row. That is,  $P_{i,m} \bullet D = \delta_{i,m} D$ . Suppose that  $\varphi = [\varphi_j]$  is an eigenvector of  $P \bullet$  with eigenvalue  $\lambda$  and let  $\varphi^+ = [D\varphi_j]$ . We have

$$\begin{aligned} \sum_{m=1}^a P_{i,m} \bullet \varphi_m^+ &= \sum_{m=1}^a P_{i,m} \bullet (D\varphi_m) \\ &= \sum_{m=1}^a (DP_{i,m} \bullet \varphi_m + \varphi_m P_{i,m} \bullet D) \\ &= \lambda D\varphi_i + \varphi_i D \\ &= (\lambda + 1)D\varphi_i \\ &= (\lambda + 1)\varphi_i^+. \end{aligned}$$

Since  $\varphi^+$  is certainly nonzero, it follows that  $\lambda + 1 \in \sigma(P)$ , as claimed.

All that remains in order to complete the proof is to show that  $-(a - 1) \in \sigma(P)$ . Let  $D_1, \dots, D_a$  be the (unsigned) cofactors of the first column in  $X$ . We wish to calculate  $P_{i,m} \bullet D_m$ . First note that if  $i = m$  then the entries of the cofactor  $D_m$  are all constant with respect to  $P_{i,m}$  and so  $P_{i,m} \bullet D_m = 0$ . Now suppose that  $i \neq m$ . As above,  $P_{i,m} \bullet D_m$  will be the determinant of the matrix that results from the comatrix of  $x_{m,1}$  by deleting the  $i^{\text{th}}$  row and replacing it with the  $m^{\text{th}}$  row. Apart from the order of the rows, this is  $D_i$ . Since the  $m^{\text{th}}$  row has been inserted immediately before the  $(i + 1)^{\text{st}}$  row and must be moved to be immediately after the  $(m - 1)^{\text{st}}$  row in order to obtain  $D_i$ , the precise expression is

$$P_{i,m} \bullet D_m = (-1)^{(i+1)-(m-1)+1} D_i = (-1)^{i+m+1} D_i.$$

Let  $\varphi = [(-1)^{j+1} D_j]$ . Then

$$\begin{aligned} \sum_{m=1}^a P_{i,m} \bullet \varphi_m &= \sum_{m=1}^a (-1)^{m+1} P_{i,m} \bullet D_m \\ &= \sum_{m \neq i} (-1)^{m+1} (-1)^{i+m+1} D_i \\ &= (-1)^i (a - 1) D_i \\ &= -(a - 1) \varphi_i \end{aligned}$$

and it follows that  $\varphi$  is an eigenvector of  $P_\bullet$  with eigenvalue  $-(a - 1)$ , as required.  $\square$

The inclusion in Theorem 5.4 is not generally an equality when  $b < a$ . The following result gives the precise situation in the extreme case in which  $b = 1$ .

**Proposition 5.5.** *Suppose that  $a \geq 2$  and  $b = 1$ . Then  $\sigma(F) = \mathbb{Z} \cap [-1, \infty)$ .*

**Proof.** Since  $b = 1$ , we shall drop the second index on  $x_{m,1}$  and  $\partial_{m,1}$  to write  $x_m$  and  $\partial_m$ . The nonzero vector  $\varphi = [\varphi_j]$  is an eigenvector of  $P_\bullet$  with eigenvalue  $\lambda$  if and only if

$$\begin{aligned} x_1 \partial_1 \bullet \varphi_1 + x_2 \partial_1 \bullet \varphi_2 + \cdots + x_a \partial_1 \bullet \varphi_a &= \lambda \varphi_1, \\ x_1 \partial_2 \bullet \varphi_1 + x_2 \partial_2 \bullet \varphi_2 + \cdots + x_a \partial_2 \bullet \varphi_a &= \lambda \varphi_2, \\ &\vdots \\ x_1 \partial_a \bullet \varphi_1 + x_2 \partial_a \bullet \varphi_2 + \cdots + x_a \partial_a \bullet \varphi_a &= \lambda \varphi_a. \end{aligned}$$

We first confirm that  $-1 \in \sigma(P)$ . In fact,  $\varphi_1 = x_2$ ,  $\varphi_2 = -x_1$ , and  $\varphi_j = 0$  for  $3 \leq j \leq a$  is a solution to the system with  $\lambda = -1$ . We now assume that we have a solution to the system with  $\lambda \neq -1$ . Let  $i \neq j$ . By differentiating the  $i^{\text{th}}$  equation with respect to  $x_j$ , the  $j^{\text{th}}$  with respect to  $x_i$ , and subtracting the results, we obtain

$$\partial_i \bullet \varphi_j - \partial_j \bullet \varphi_i = \lambda \partial_j \bullet \varphi_i - \lambda \partial_i \bullet \varphi_j.$$

This is equivalent to  $(\lambda + 1) \partial_j \bullet \varphi_i = (\lambda + 1) \partial_i \bullet \varphi_j$  and, since we have assumed that  $\lambda \neq -1$ , we conclude that  $\partial_j \bullet \varphi_i = \partial_i \bullet \varphi_j$  for all  $i \neq j$ . It follows that there is a polynomial  $g \in \mathbb{C}[x_1, \dots, x_a]$  such that  $\varphi_i = \partial_i \bullet g$  for  $1 \leq i \leq a$ . When we substitute the identities  $\varphi_j = \partial_j \bullet g$  into the  $i^{\text{th}}$  equation we obtain

$$x_1 \partial_i \partial_1 \bullet g + x_2 \partial_i \partial_2 \bullet g + \cdots + x_a \partial_i \partial_a \bullet g = \lambda \partial_i \bullet g,$$

which may be rewritten first as

$$\partial_i \bullet (x_1 \partial_1 \bullet g + x_2 \partial_2 \bullet g + \cdots + x_a \partial_a \bullet g) = (\lambda + 1) \partial_i \bullet g$$

and then as

$$\partial_i \bullet (\mathbb{E} \bullet g - (\lambda + 1)g) = 0,$$

where  $\mathbb{E}$  denotes the Euler operator with respect to the variables  $x_1, \dots, x_a$ . We conclude that there is a constant  $C$  such that

$$\mathbb{E} \bullet g - (\lambda + 1)g = C.$$

It follows that the polynomial  $h = g + C/(\lambda + 1)$  satisfies the equation  $\mathbb{E} \bullet h = (\lambda + 1)h$ . This equation implies that  $h$  is homogeneous of degree  $\lambda + 1$  and, in particular, that  $\lambda + 1 \in \mathbb{N}$ . Since we have assumed that  $\lambda \neq -1$ , it follows that  $\lambda \in \mathbb{N}$ . Overall, this implies that  $\sigma(P) \subset \mathbb{Z} \cap [-1, \infty)$ . We have already seen that  $-1$  lies in  $\sigma(P)$ . The last step is to show that  $\mathbb{N} \subset \sigma(P)$  also. To do so, take  $\lambda \in \mathbb{N}$  and choose a nonzero homogeneous polynomial  $g$  of degree  $\lambda + 1$ . We may then reverse the above calculation to see that

$\varphi = [\partial_i \bullet g]$  is a nonzero solution to the system for the given value of  $\lambda$ , which completes the proof.  $\square$

Note that, as a byproduct of the proof of Proposition 5.5, we determined the structure of all eigenvectors of  $P_\bullet$  with eigenvalues  $\lambda \in \mathbb{N}$ . They are simply the gradients of homogeneous polynomials of degree  $\lambda + 1$ . Note also that eigenvectors of  $P_\bullet$  with eigenvalue  $-1$  definitely do not share the same structure; the example given in the proof does not have this structure, for example.

**5.2. The first system in Type  $D_l$ .** Let  $l \geq 4$ . We take the standard model of the root system  $R$  of type  $D_l$ , in which the positive roots are  $\{e_i \pm e_j \mid 1 \leq i < j \leq l\}$ . With the labels introduced in Figure 1 we have  $\alpha_i = e_i - e_{i+1}$  for  $1 \leq i \leq l-1$  and  $\alpha_l = e_{l-1} + e_l$ . Let  $\alpha = \alpha_{l-1}$  and  $\beta = \alpha_l$ . Note that  $d(\alpha, \beta) = 2$ . We use the total order on  $R^s$  that is specified by

$$\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{l-3} \prec \alpha_{l-1} \prec \alpha_{l-2} \prec \alpha_l,$$

which satisfies the requirements introduced in Section 2. Let

$$\mu(i) = e_i - e_l$$

for  $1 \leq i \leq l-1$ ,

$$\nu(i) = e_i + e_l$$

for  $1 \leq i \leq l-1$ , and

$$\zeta(i, j) = e_i + e_j$$

for  $1 \leq i < j \leq l-1$ . With this notation, we have

$$R(1, 1) = \{\mu(i) \mid 1 \leq i \leq l-1\},$$

$$R(1, -1) = \{\nu(i) \mid 1 \leq i \leq l-1\},$$

$$R(2, 0) = \{\zeta(i, j) \mid 1 \leq i < j \leq l-1\}.$$

**Lemma 5.6.** *We have*

$$N_{\mu(i), \nu(j)} = \begin{cases} 1 & \text{if } i < j \leq l-2, \\ -1 & \text{if } i < j = l-1, \\ 0 & \text{if } i = j, \\ -1 & \text{if } i > j. \end{cases}$$

**Proof.** First,  $\mu(i) + \nu(j) \in R$  if and only if  $i \neq j$ , which verifies the third value. Suppose now that  $i \neq j$  and  $1 \leq i, j \leq l-2$ . We have

$$\mu(i) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2} + \alpha_{l-1}$$

and

$$\nu(j) = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_l$$

and so

$$\begin{aligned}
 & f(\mu(i), \nu(j)) \\
 &= f(\alpha_{l-1}, \alpha_l) + \sum_{j \leq q \leq l-2} f(\alpha_{l-1}, \alpha_q) + \sum_{i \leq p \leq l-2} f(\alpha_p, \alpha_l) + \sum_{\substack{i \leq p \leq l-2 \\ j \leq q \leq l-2}} f(\alpha_p, \alpha_q) \\
 &= (\alpha_{l-1}, \alpha_l) + (\alpha_{l-1}, \alpha_{l-2}) + (\alpha_{l-2}, \alpha_l) + \sum_{\substack{i \leq p \leq l-2 \\ j \leq q \leq l-2}} \delta_{p,q} + \sum_{\substack{i \leq p \leq l-2 \\ j \leq q \leq l-2}} (\alpha_p, \alpha_q) \mathbf{1}_{p < q} \\
 &= -2 + (l-2) - \max(i, j) + 1 - \sum_{\substack{i \leq p \leq l-2 \\ j-1 \leq p \leq l-3}} 1 \\
 &= (l-3) - \max(i, j) - (l-3) + \max(i, j-1) - 1 \\
 &= \max(i, j-1) - \max(i, j) - 1,
 \end{aligned}$$

where  $\mathbf{1}_{p < q}$  is the indicator function of the set where  $p < q$ . If  $i < j$  then  $i \leq j-1$  and  $f(\mu(i), \nu(j)) = (j-1) - j - 1 = -2$ , which gives  $N_{\mu(i), \nu(j)} = 1$ . If  $i > j$  then  $f(\mu(i), \nu(j)) = i - i - 1 = -1$ , which gives  $N_{\mu(i), \nu(j)} = -1$ . It remains to evaluate  $f(\mu(i), \nu(l-1))$  when  $i < l-1$  and  $f(\mu(l-1), \nu(j))$  when  $j < l-1$ . For the first, we have

$$\begin{aligned}
 f(\mu(i), \nu(l-1)) &= f(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2} + \alpha_{l-1}, \alpha_l) \\
 &= f(\alpha_{l-2}, \alpha_l) \\
 &= -1
 \end{aligned}$$

and so  $N_{\mu(i), \nu(l-1)} = -1$ . For the second, we have

$$\begin{aligned}
 f(\mu(l-1), \nu(j)) &= f(\alpha_{l-1}, \alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_l) \\
 &= f(\alpha_{l-1}, \alpha_{l-2}) \\
 &= -1
 \end{aligned}$$

and so  $N_{\mu(l-1), \nu(j)} = -1$ . □

A calculation based on Lemma 5.6 and (4.6) shows that

$$\Delta_{\zeta(i,j)} = \begin{cases} \partial_{\mu(j)} \partial_{\nu(i)} - \partial_{\mu(i)} \partial_{\nu(j)} & \text{if } i < j \leq l-2, \\ \partial_{\mu(l-1)} \partial_{\nu(i)} + \partial_{\mu(i)} \partial_{\nu(l-1)} & \text{if } i < j = l-1. \end{cases}$$

Note that this expression could be uniformized simply by changing the signs of  $X_{\pm \nu(l-1)}$ .

In order to simplify the notation for  $F$ , we shall write  $x_{\mu(i)}$  as  $x_i$  and  $\partial_{\mu(i)}$  as  $\partial_i$  in what follows.

**Lemma 5.7.** *For  $1 \leq i < j \leq l-1$  and  $1 \leq m < n \leq l-1$  we have*

$$F_{\zeta(i,j), \zeta(m,n)} = \delta_{i,m} x_n \partial_j + \delta_{j,n} x_m \partial_i - \delta_{i,n} x_m \partial_j - \delta_{j,m} x_n \partial_i.$$

**Proof.** The quantity  $F_{\zeta(i,j),\zeta(m,n)}$  is expressed as a sum over those  $\nu(k)$  such that  $(\zeta(i,j),\nu(k)) = 1$  and  $(\zeta(m,n),\nu(k)) = 1$ . The first equation holds exactly when  $k \in \{i,j\}$  and the second exactly when  $k \in \{m,n\}$ . Thus  $F_{\zeta(i,j),\zeta(m,n)} = 0$  unless  $\{i,j\} \cap \{m,n\} \neq \emptyset$ . If  $i = m$  and  $j = n$  then (4.7) implies that

$$F_{\zeta(i,j),\zeta(i,j)} = N_{\mu(j),\nu(i)}N_{\mu(j),\nu(i)}x_j\partial_j + N_{\mu(i),\nu(j)}N_{\mu(i),\nu(j)}x_i\partial_i.$$

Since the structure constants are  $\pm 1$ , this evaluates to

$$F_{\zeta(i,j),\zeta(i,j)} = x_j\partial_j + x_i\partial_i,$$

which agrees with the formula in the statement in this case. Now suppose that  $i = m$  and  $j \neq n$ . Then

$$F_{\zeta(i,j),\zeta(i,n)} = N_{\mu(j),\nu(i)}N_{\mu(n),\nu(i)}x_n\partial_j$$

and  $N_{\mu(j),\nu(i)} = N_{\mu(n),\nu(i)} = -1$  by Lemma 5.6. Thus  $F_{\zeta(i,j),\zeta(i,n)} = x_n\partial_j$ , as claimed. Similarly, if  $j = n$  and  $i \neq m$  then

$$F_{\zeta(i,j),\zeta(m,j)} = N_{\mu(i),\nu(j)}N_{\mu(m),\nu(j)}x_m\partial_i.$$

If  $j \leq l - 2$  then  $N_{\mu(i),\nu(j)} = N_{\mu(m),\nu(j)} = 1$ . If  $j = l - 1$  then  $N_{\mu(i),\nu(j)} = N_{\mu(m),\nu(j)} = -1$ . In either case, the product of the structure constants is 1 and so  $F_{\zeta(i,j),\zeta(m,j)} = x_m\partial_i$ . This once again agrees with the formula given in the statement. If  $i = n$  then  $m < i < j$  and

$$F_{\zeta(i,j),\zeta(m,i)} = N_{\mu(j),\nu(i)}N_{\mu(m),\nu(i)}x_m\partial_j.$$

The inequality  $m < i < j$  implies that  $i \neq l - 1$  and so Lemma 5.6 gives  $N_{\mu(j),\nu(i)} = -1$  and  $N_{\mu(m),\nu(i)} = 1$ . Thus  $F_{\zeta(i,j),\zeta(m,i)} = -x_m\partial_j$ , as claimed. Finally, suppose that  $j = m$ . Then  $i < j < n$  and

$$F_{\zeta(i,j),\zeta(j,n)} = N_{\mu(i),\nu(j)}N_{\mu(n),\nu(j)}x_n\partial_i.$$

The inequality  $i < j < n$  implies that  $j \neq l - 1$  and so Lemma 5.6 gives  $N_{\mu(i),\nu(j)} = 1$  and  $N_{\mu(n),\nu(j)} = -1$ . Thus  $F_{\zeta(i,j),\zeta(j,n)} = -x_n\partial_i$ , as claimed.  $\square$

**Theorem 5.8.** *We have  $\sigma(F) = \{-2\} \cup \mathbb{N}$ .*

**Proof.** Suppose that  $\varphi = [\varphi_{\zeta(i,j)}]$  lies in the kernel of  $F\bullet - \lambda$ . To simplify the notation, we abbreviate  $\varphi_{\zeta(i,j)}$  to  $\varphi_{i,j}$ . It follows from Lemma 5.7 that  $\varphi$  satisfies the equation

$$(5.6) \quad \sum_{1 \leq m < n \leq l-1} (\delta_{i,m}x_n\partial_j + \delta_{j,n}x_m\partial_i - \delta_{i,n}x_m\partial_j - \delta_{j,m}x_n\partial_i)\bullet\varphi_{m,n} = \lambda\varphi_{i,j}$$

for all  $1 \leq i < j \leq l - 1$ . We refer to (5.6) as the  $(i,j)$ -equation. Let us choose indices  $1 \leq i < j < k \leq l - 1$ . By applying the operator  $\partial_k$  to the  $(i,j)$ -equation we obtain

$$\begin{aligned} & \sum_{1 \leq m < n \leq l-1} (\delta_{i,m} x_n \partial_j \partial_k + \delta_{j,n} x_m \partial_i \partial_k - \delta_{i,n} x_m \partial_j \partial_k - \delta_{j,m} x_n \partial_i \partial_k) \bullet \varphi_{m,n} \\ & + \sum_{1 \leq m < n \leq l-1} (\delta_{i,m} \delta_{n,k} \partial_j + \delta_{j,n} \delta_{k,m} \partial_i - \delta_{i,n} \delta_{m,k} \partial_j - \delta_{j,m} \delta_{n,k} \partial_i) \bullet \varphi_{m,n} \\ & = \lambda \partial_k \bullet \varphi_{i,j} \end{aligned}$$

and, by taking into account the inequalities  $m < n$  and  $i < j < k$  in the second sum, this simplifies to

$$(5.7) \quad \sum_{1 \leq m < n \leq l-1} (\delta_{i,m} x_n \partial_j \partial_k + \delta_{j,n} x_m \partial_i \partial_k - \delta_{i,n} x_m \partial_j \partial_k - \delta_{j,m} x_n \partial_i \partial_k) \bullet \varphi_{m,n} + \partial_j \bullet \varphi_{i,k} - \partial_i \bullet \varphi_{j,k} = \lambda \partial_k \bullet \varphi_{i,j}.$$

Similarly, we apply the operator  $\partial_j$  to the  $(i, k)$ -equation to obtain

$$(5.8) \quad \sum_{1 \leq m < n \leq l-1} (\delta_{i,m} x_n \partial_j \partial_k + \delta_{k,n} x_m \partial_i \partial_j - \delta_{i,n} x_m \partial_j \partial_k - \delta_{k,m} x_n \partial_i \partial_j) \bullet \varphi_{m,n} + \partial_k \bullet \varphi_{i,j} + \partial_i \bullet \varphi_{j,k} = \lambda \partial_j \bullet \varphi_{i,k}.$$

We also apply the operator  $\partial_i$  to the  $(j, k)$ -equation to obtain

$$(5.9) \quad \sum_{1 \leq m < n \leq l-1} (\delta_{j,m} x_n \partial_i \partial_k + \delta_{k,n} x_m \partial_i \partial_j - \delta_{j,n} x_m \partial_i \partial_k - \delta_{k,m} x_n \partial_i \partial_j) \bullet \varphi_{m,n} + \partial_j \bullet \varphi_{i,k} - \partial_k \bullet \varphi_{i,j} = \lambda \partial_i \bullet \varphi_{j,k}.$$

We now take equation (5.9) minus equation (5.8) plus equation (5.7). In this combination, all the terms in the sums over  $1 \leq m < n \leq l-1$  cancel in pairs, and we obtain

$$2\partial_j \bullet \varphi_{i,k} - 2\partial_i \bullet \varphi_{j,k} - 2\partial_k \bullet \varphi_{i,j} = \lambda(\partial_i \bullet \varphi_{j,k} - \partial_j \bullet \varphi_{i,k} + \partial_k \bullet \varphi_{i,j}),$$

which is equivalent to

$$(5.10) \quad (\lambda + 2)(\partial_i \bullet \varphi_{j,k} - \partial_j \bullet \varphi_{i,k} + \partial_k \bullet \varphi_{i,j}) = 0.$$

It follows from this equation that either  $\lambda = -2$  or

$$(5.11) \quad d\left(\sum_{1 \leq i < j \leq l-1} \varphi_{i,j} dx_i \wedge dx_j\right) = 0.$$

Let us assume for the moment that  $\lambda \neq -2$ . Then (5.11) holds and it follows that there are polynomials  $g_1, \dots, g_{l-1}$  such that

$$d\left(\sum_{m=1}^{l-1} g_m dx_m\right) = \sum_{1 \leq i < j \leq l-1} \varphi_{i,j} dx_i \wedge dx_j,$$

which is equivalent to the equations

$$(5.12) \quad \varphi_{i,j} = \partial_i \bullet g_j - \partial_j \bullet g_i$$

for all  $1 \leq i < j \leq l-1$ .

The next step is to substitute (5.12) into (5.6). As a preliminary to this substitution, note that the left-hand side of (5.6) may be reexpressed as

$$\begin{aligned} & \sum_{1 \leq m < n \leq l-1} (\delta_{j,n} x_m - \delta_{j,m} x_n) \partial_i \bullet \varphi_{m,n} + \sum_{1 \leq m < n \leq l-1} (\delta_{i,m} x_n - \delta_{i,n} x_m) \partial_j \bullet \varphi_{m,n} \\ &= \sum_{1 \leq m < n \leq l-1} (\partial_i (\delta_{j,n} x_m - \delta_{j,m} x_n) - \delta_{i,m} \delta_{j,n}) \bullet \varphi_{m,n} \\ & \quad + \sum_{1 \leq m < n \leq l-1} (\partial_j (\delta_{i,m} x_n - \delta_{i,n} x_m) - \delta_{i,m} \delta_{j,n}) \bullet \varphi_{m,n} \\ &= -2\varphi_{i,j} + \partial_i \bullet \left( \sum_{1 \leq m < n \leq l-1} (\delta_{j,n} x_m - \delta_{j,m} x_n) \varphi_{m,n} \right) \\ & \quad + \partial_j \bullet \left( \sum_{1 \leq m < n \leq l-1} (\delta_{i,m} x_n - \delta_{i,n} x_m) \varphi_{m,n} \right). \end{aligned}$$

Now, by (5.12),

$$\begin{aligned} & \sum_{1 \leq m < n \leq l-1} (\delta_{j,n} x_m - \delta_{j,m} x_n) \varphi_{m,n} \\ &= \sum_{1 \leq m < n \leq l-1} (\delta_{j,n} x_m - \delta_{j,m} x_n) (\partial_m \bullet g_n - \partial_n \bullet g_m) \\ &= \sum_{1 \leq m < j} x_m \partial_m \bullet g_j + \sum_{j < n \leq l-1} x_n \partial_n \bullet g_j - \sum_{1 \leq m < j} x_m \partial_j \bullet g_m - \sum_{j < n \leq l-1} x_n \partial_j \bullet g_n \\ &= \mathbb{E} \bullet g_j - x_j \partial_j \bullet g_j - \sum_{1 \leq m \leq l-1} x_m \partial_j \bullet g_m + x_j \partial_j \bullet g_j \\ &= \mathbb{E} \bullet g_j - \sum_{1 \leq m \leq l-1} x_m \partial_j \bullet g_m \\ &= \mathbb{E} \bullet g_j - \sum_{1 \leq m \leq l-1} (\partial_j x_m - \delta_{j,m}) \bullet g_m \\ &= \mathbb{E} \bullet g_j - \partial_j \bullet P + g_j \\ &= (\mathbb{E} + 1) \bullet g_j - \partial_j \bullet P, \end{aligned}$$

where we have written  $\mathbb{E}$  for the Euler operator with respect to the variables  $x_1, \dots, x_{l-1}$  and

$$P = \sum_{1 \leq m \leq l-1} x_m g_m.$$



A similar calculation shows that, by (5.12),

$$\sum_{1 \leq m < n \leq l-1} (\delta_{i,m}x_n - \delta_{i,n}x_m)\varphi_{m,n} = -(\mathbb{E} + 1)\bullet g_i + \partial_i\bullet P.$$

In light of these evaluations, the result of substituting (5.12) into (5.6) is equivalent to the equation

$$\partial_i\bullet((\mathbb{E} + 1)\bullet g_j - \partial_j\bullet P) - \partial_j\bullet((\mathbb{E} + 1)\bullet g_i - \partial_i\bullet P) = (\lambda + 2)(\partial_i\bullet g_j - \partial_j\bullet g_i).$$

By rearranging this equation and using the fact that  $\partial_i\partial_j\bullet P = \partial_j\partial_i\bullet P$  we obtain

$$(5.13) \quad \partial_i\bullet((\mathbb{E} - (\lambda + 1))\bullet g_j) = \partial_j\bullet((\mathbb{E} - (\lambda + 1))\bullet g_i).$$

Since (5.13) holds for all  $i < j$ , we conclude that there is a polynomial  $h$  such that

$$(5.14) \quad (\mathbb{E} - (\lambda + 1))\bullet g_i = \partial_i\bullet h$$

for  $1 \leq i \leq l - 1$ . This completes the second step.

For all  $1 \leq k \leq l - 1$  we have  $[\mathbb{E}, \partial_k] = -\partial_k$  and so  $\mathbb{E}\partial_k = \partial_k(\mathbb{E} - 1)$ . Thus

$$\begin{aligned} (\mathbb{E} - \lambda)\bullet\varphi_{i,j} &= (\mathbb{E} - \lambda)\bullet(\partial_i\bullet g_j - \partial_j\bullet g_i) \\ &= \partial_i\bullet((\mathbb{E} - (\lambda + 1))\bullet g_j) - \partial_j\bullet((\mathbb{E} - (\lambda + 1))\bullet g_i) \\ &= \partial_i\bullet(\partial_j\bullet h) - \partial_j\bullet(\partial_i\bullet h) \\ &= 0, \end{aligned}$$

where we have used both (5.12) and (5.14) in the course of the calculation. Since the spectrum of  $\mathbb{E}\bullet$  on the polynomial algebra in  $x_1, \dots, x_{l-1}$  is  $\mathbb{N}$ , it follows from this argument that if  $\lambda \neq -2$  and  $\lambda \in \sigma(F)$  then  $\lambda \in \mathbb{N}$ . Thus we have shown that  $\sigma(F) \subset \{-2\} \cup \mathbb{N}$ . To complete the proof, we must also establish the reverse inclusion. To show that  $-2 \in \sigma(F)$ , one verifies that if  $\varphi = [\varphi_{i,j}]$  with  $\varphi_{1,2} = x_3$ ,  $\varphi_{1,3} = -x_2$ ,  $\varphi_{2,3} = x_1$ , and  $\varphi_{i,j} = 0$  for all other values of  $(i, j)$  then  $F\bullet\varphi = -2\varphi$ . Let  $s \in \mathbb{N}$ . To show that  $s \in \sigma(F)$ , one verifies that if  $\varphi = [\varphi_{i,j}]$  with  $\varphi_{1,2} = x_1^s$  and  $\varphi_{i,j} = 0$  for all other values of  $(i, j)$  then  $F\bullet\varphi = s\varphi$ .  $\square$

As with the remark after the proof of Theorem 5.4, we may note that the proof of Theorem 5.8 yields quite a bit of extra information about the structure of solutions to the equation  $F\bullet\varphi = \lambda\varphi$  provided that  $\lambda \neq -2$ . However, the solutions to  $F\bullet\varphi = -2\varphi$  do not share this structure; in particular, the 2-form  $x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3$  is not closed.

**5.3. The second and third systems in Type  $D_l$ .** We use the same model of the root system of type  $D_l$  ( $l \geq 4$ ) as was used in the previous subsection. For the second system, we take  $\alpha = \alpha_1$  and  $\beta = \alpha_l$ . Note that  $d(\alpha, \beta) = l - 2$ . We use the total order on  $R^s$  that is specified by

$$\alpha_1 \prec \alpha_2 \prec \dots \prec \alpha_{l-2} \prec \alpha_l \prec \alpha_{l-1},$$

which satisfies the requirements introduced in Section 2. Let

$$\mu(i) = e_1 - e_{i+1}$$

for  $1 \leq i \leq l-1$ ,

$$\nu(i, j) = e_{i+1} + e_{j+1}$$

for  $1 \leq i < j \leq l-1$ , and

$$\zeta(i) = e_1 + e_{i+1}$$

for  $1 \leq i \leq l-1$ . With this notation, we have

$$\begin{aligned} R(1, 1) &= \{\mu(i) \mid 1 \leq i \leq l-1\}, \\ R(1, -1) &= \{\nu(i, j) \mid 1 \leq i < j \leq l-1\}, \\ R(2, 0) &= \{\zeta(i) \mid 1 \leq i \leq l-1\}. \end{aligned}$$

**Lemma 5.9.** *We have*

$$N_{\mu(i), \nu(j, k)} = \begin{cases} 0 & \text{if } i \notin \{j, k\}, \\ -1 & \text{if } i = j, \\ 1 & \text{if } i = k \leq l-2, \\ -1 & \text{if } i = k = l-1. \end{cases}$$

**Proof.** Note first that  $\mu(i) + \nu(j, k) \in R$  if and only if  $i \in \{j, k\}$ . This verifies the first value. We have

$$\mu(i) = \alpha_1 + \cdots + \alpha_i$$

for  $1 \leq i \leq l-1$ . Also

$$\begin{aligned} &\nu(j, k) \\ &= \begin{cases} \alpha_{j+1} + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l & \text{if } k \leq l-3, \\ \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_{l-1} + \alpha_l & \text{if } k = l-2, \\ \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_l & \text{if } k = l-1. \end{cases} \end{aligned}$$

We can simplify our work a little if we notice that we only care about  $f(\mu, \nu)$  modulo 2 and this value only depends on  $\mu$  and  $\nu$  modulo the lattice  $2\mathbb{Z}R^s$ . Let us write  $\equiv$  for congruence modulo  $2\mathbb{Z}R^s$  or modulo 2, as context dictates. Then

$$\nu(j, k) \equiv \begin{cases} \alpha_{j+1} + \cdots + \alpha_k + \alpha_{l-1} + \alpha_l & \text{if } k \leq l-2, \\ \alpha_{j+1} + \cdots + \alpha_{l-2} + \alpha_l & \text{if } k = l-1. \end{cases}$$

Suppose that  $k \leq l-2$ . Then

$$\begin{aligned} &f(\mu(i), \nu(j, k)) \\ &\equiv f\left(\sum_{1 \leq m \leq i} \alpha_m, \sum_{j+1 \leq n \leq k} \alpha_n + \alpha_{l-1} + \alpha_l\right) \end{aligned}$$

$$\begin{aligned}
 &\equiv \sum_{1 \leq m \leq i} \sum_{j+1 \leq n \leq k} f(\alpha_m, \alpha_n) + \sum_{1 \leq m \leq i} f(\alpha_m, \alpha_{l-1}) + \sum_{1 \leq m \leq i} f(\alpha_m, \alpha_l) \\
 &\equiv \sum_{1 \leq m \leq i} \sum_{j+1 \leq n \leq k} \delta_{m,n} + \sum_{\substack{1 \leq m \leq i \\ j \leq m \leq k-1}} 1 + \delta_{i,l-2} + (\delta_{i,l-2} + \delta_{i,l-1}) \\
 &\equiv \min(i, k) - (j + 1) + 1 + \min(i, k - 1) - j + 1 + \delta_{i,l-1} \\
 &\equiv \min(i, k) + \min(i, k - 1) + 1 + \delta_{i,l-1} \\
 &\equiv \begin{cases} 1 & \text{if } i < k, \\ \delta_{i,l-1} & \text{if } i \geq k. \end{cases}
 \end{aligned}$$

This evaluation confirms the second and third values displayed in the statement. It remains to treat the cases where  $k = l - 1$ . We have

$$\begin{aligned}
 &f(\mu(i), \nu(j, l - 1)) \\
 &= f\left(\sum_{1 \leq m \leq i} \alpha_m, \sum_{j+1 \leq n \leq l-2} \alpha_n + \alpha_l\right) \\
 &\equiv \sum_{1 \leq m \leq i} \sum_{j+1 \leq n \leq l-2} f(\alpha_m, \alpha_n) + \sum_{1 \leq m \leq i} f(\alpha_m, \alpha_l) \\
 &\equiv \sum_{\substack{1 \leq m \leq i \\ j+1 \leq m \leq l-2}} 1 + \sum_{\substack{1 \leq m \leq i \\ j \leq m \leq l-3}} 1 + (\delta_{i,l-2} + \delta_{i,l-1}) \\
 &\equiv \min(i, l - 2) - (j + 1) + 1 + \min(i, l - 3) - j + 1 + \delta_{i,l-2} + \delta_{i,l-1} \\
 &\equiv \min(i, l - 2) + \min(i, l - 3) + 1 + \delta_{i,l-2} + \delta_{i,l-1}.
 \end{aligned}$$

By considering  $i \leq l - 3$ ,  $i = l - 2$ , and  $i = l - 1$  in turn, one verifies that this expresses is always congruent to 1 modulo 2. This confirms the fourth value given in the statement.  $\square$

A calculation based on Lemma 5.9 and (4.6) implies that

$$\Delta_{\zeta(p)} = \sum_{1 \leq i < p} \partial_{\mu(i)} \partial_{\nu(i,p)} - \sum_{p < i \leq l-2} \partial_{\mu(i)} \partial_{\nu(p,i)} + \partial_{\mu(l-1)} \partial_{\nu(p,l-1)}$$

if  $1 \leq p \leq l - 2$  and

$$\Delta_{\zeta(l-1)} = \sum_{1 \leq i < l-1} \partial_{\mu(i)} \partial_{\nu(i,l-1)}.$$

More symmetric expressions would be obtained by changing the signs of  $X_{\pm \nu(p,l-1)}$  for  $1 \leq p < l - 1$ .

In order to simplify the notation, we abbreviate  $x_{\mu(i)}$  as  $x_i$  and  $\partial_{\mu(i)}$  as  $\partial_i$  in what follows.

**Lemma 5.10.** *For  $1 \leq p \leq l - 1$  we have*

$$F_{\zeta(p), \zeta(p)} = \mathbb{E} - x_p \partial_p,$$

where  $\mathbb{E}$  is the Euler operator  $\mathbb{E} = x_1\partial_1 + \cdots + x_{l-1}\partial_{l-1}$ . If  $1 \leq p, q \leq l-1$  and  $p \neq q$  then

$$F_{\zeta(p), \zeta(q)} = \pm x_p \partial_q,$$

where the sign is positive if either  $p$  or  $q$  is equal to  $l-1$  and negative otherwise.

**Proof.** The elements  $\nu \in R(1, -1)$  that satisfy  $(\zeta(p), \nu) = 1$  are  $\nu = \nu(i, p)$  with  $1 \leq i < p$  and  $\nu = \nu(p, i)$  with  $p < i \leq l-1$ . We have  $\zeta(p) - \nu(p, i) = \mu(i)$  and  $\zeta(p) - \nu(i, p) = \mu(i)$ . Thus

$$\begin{aligned} F_{\zeta(p), \zeta(p)} &= \sum_{1 \leq i < p} N_{\mu(i), \nu(i, p)}^2 x_i \partial_i + \sum_{p < i \leq l-1} N_{\mu(i), \nu(p, i)}^2 x_i \partial_i \\ &= \mathbb{E} - x_p \partial_p, \end{aligned}$$

since  $N_{\mu(i), \nu(i, p)}$  and  $N_{\mu(i), \nu(p, i)}$  are both  $\pm 1$ . If  $p < q$  then the only  $\nu \in R(1, -1)$  such that  $(\zeta(p), \nu) = 1$  and  $(\zeta(q), \nu) = 1$  is  $\nu = \nu(p, q)$ . We have  $\zeta(p) - \nu(p, q) = \mu(q)$  and  $\zeta(q) - \nu(p, q) = \mu(p)$ , and so

$$F_{\zeta(p), \zeta(q)} = N_{\mu(p), \nu(p, q)} N_{\mu(q), \nu(p, q)} x_p \partial_q.$$

By Lemma 5.9, the coefficient in this expression is  $-1$  unless  $q = l-1$ , in which case it is  $1$ . The evaluation of  $F_{\zeta(p), \zeta(q)}$  when  $p > q$  is similar.  $\square$

**Theorem 5.11.** *We have  $\sigma(F) = \{2-l\} \cup \mathbb{N}$ .*

**Proof.** We begin by applying the automorphism of  $\mathbb{C}[x_1, \dots, x_{l-1}]$  that changes the sign of  $x_{l-1}$ . This does not affect the spectrum of  $F^\bullet$ , but does allow us to write  $F$  in the more convenient form

$$F = \mathbb{E}I - [x_p \partial_q].$$

Suppose that  $\varphi = [\varphi_i] \neq 0$  satisfies  $F^\bullet \varphi = \lambda \varphi$ . Each entry in  $F$  is a homogeneous operator of degree zero and it follows that  $F^\bullet$  commutes with projection onto the terms in a fixed degree. Thus we may assume that  $\varphi_i$  is homogeneous of degree  $n$ , say, for all  $i$ . Then the equation  $F^\bullet \varphi = \lambda \varphi$  is equivalent to the system of equations

$$(5.15) \quad \sum_{q=1}^{l-1} x_p \partial_q \bullet \varphi_q = (n - \lambda) \varphi_p$$

for  $1 \leq p \leq l-1$ . If we define  $\psi = \sum_{q=1}^{l-1} \partial_q \bullet \varphi_q$  then this system of equations is equivalent to  $x_p \psi = (n - \lambda) \varphi_p$  for  $1 \leq p \leq l-1$ . If  $\lambda = n$  then this system is equivalent to the single equation  $\psi = 0$ . Since  $\partial_{l-1}$  is a surjective map from the space of homogeneous polynomials of degree  $n$  to the space of homogeneous polynomials of degree  $n-1$ , given homogeneous polynomials  $\varphi_1, \dots, \varphi_{l-2}$  of degree  $n$  we may find a homogeneous polynomial  $\varphi_{l-1}$  of degree  $n$  such that  $(\varphi_1, \dots, \varphi_{l-1})$  solves the equation  $\psi = 0$ . It follows that every natural number lies in  $\sigma(F)$ . Now suppose that  $\lambda \notin \mathbb{N}$ , so that  $n - \lambda \neq 0$  above, and let  $g = \psi / (n - \lambda)$ . Note that  $g$  is a homogeneous polynomial of degree  $n-1$ . We have  $\varphi_p = x_p g$  for all  $1 \leq p \leq l-1$ ,

and substituting this into (5.15) and simplifying the result we find that the system (5.15) is equivalent to the equation

$$(5.16) \quad (l-1)g + \mathbb{E}\bullet g = (n-\lambda)g.$$

However,  $\mathbb{E}\bullet g = (n-1)g$  and  $g \neq 0$  because  $\varphi \neq 0$ . It follows from (5.16) and these facts that  $\lambda = 2-l$ . That is, the only possible eigenvalue of  $F\bullet$  that is not a natural number is  $2-l$ . Finally, let  $\varphi = [x_i]$ . One verifies that  $F\bullet\varphi = (2-l)\varphi$ , so that  $2-l$  does indeed lie in  $\sigma(F)$ .  $\square$

For the third system, we would take  $\alpha = \alpha_1$  and  $\beta = \alpha_{l-1}$ . This choice is conjugate to the choice for the second system under the nontrivial diagram automorphism of the Dynkin diagram. Consequently, we obtain essentially the same system as was considered above, with the roles of the simple roots  $\alpha_{l-1}$  and  $\alpha_l$  interchanged. The spectrum  $\sigma(F)$  is the same as for the second system. Thus we do not consider this case further.

**5.4. The system in Type  $E_6$ .** We use the labels for the simple roots that were introduced in Figure 1. Let  $\alpha = \alpha_1$  and  $\beta = \alpha_6$ . Note that  $d(\alpha, \beta) = 4$ . We use the total order on  $R^s$  that is specified by

$$\alpha_1 \prec \alpha_3 \prec \alpha_4 \prec \alpha_5 \prec \alpha_6 \prec \alpha_2,$$

which satisfies the requirements introduced in Section 2. Next we enumerate the sets  $R(1, 1)$ ,  $R(1, -1)$  and  $R(2, 0)$ . Each of these sets has eight elements. We have  $R(1, 1) = \{\mu_1, \dots, \mu_8\}$  where

$$\begin{aligned} \mu_1 &= \alpha_1, \\ \mu_2 &= \alpha_1 + \alpha_3, \\ \mu_3 &= \alpha_1 + \alpha_3 + \alpha_4, \\ \mu_4 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ \mu_5 &= \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \\ \mu_6 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \\ \mu_7 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\ \mu_8 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5. \end{aligned}$$

We have  $R(1, -1) = \{\nu_1, \dots, \nu_8\}$  where

$$\begin{aligned} \nu_1 &= \alpha_6, \\ \nu_2 &= \alpha_5 + \alpha_6, \\ \nu_3 &= \alpha_4 + \alpha_5 + \alpha_6, \\ \nu_4 &= \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \\ \nu_5 &= \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ \nu_6 &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ \nu_7 &= \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ \nu_8 &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6. \end{aligned}$$

We have  $R(2, 0) = \{\zeta_1, \dots, \zeta_8\}$  where

$$\begin{aligned}\zeta_1 &= \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ \zeta_2 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ \zeta_3 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ \zeta_4 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ \zeta_5 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\ \zeta_6 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\ \zeta_7 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\ \zeta_8 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.\end{aligned}$$

One advantage of enumerating the sets  $R(1, 1)$ ,  $R(1, -1)$ , and  $R(2, 0)$  in these orders is that the three 8-by-8 matrices  $[(\mu_i, \mu_j)]$ ,  $[(\nu_i, \nu_j)]$ , and  $[(\zeta_i, \zeta_j)]$  of inner products between roots drawn from the same set are equal to one another. The common value is

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Unfortunately, it is not possible to assign an order so that the two 8-by-8 matrices  $[(\mu_i, \zeta_j)]$  and  $[(\nu_i, \zeta_j)]$  are equal to one another and so the values of both must be recorded. They are

$$[(\mu_i, \zeta_j)] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$[(\nu_i, \zeta_j)] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We would like to remark on the structure of these two matrices. In the first matrix, the first four entries in the rows are an enumeration of all 4-bit strings with an odd number of 1s, and the last four entries are the mirror-image of the bit-complement of the first four. Moreover, the  $i^{\text{th}}$  row is the bit-complement of the  $(9 - i)^{\text{th}}$  row. The second matrix has the same structural features, except that the first four entries in the rows are an enumeration of all 4-bit strings with an even number of 1s. We also require the 8-by-8 matrix  $[(\mu_i, \nu_j)]$ . It is

$$[(\mu_i, \nu_j)] = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the structure of this matrix is similar to the structure of the matrix  $[(\nu_i, \zeta_j)]$ .

We shall also need certain structure constants. With the total order on  $R^s$  that was fixed above, the matrix of the bilinear form  $f$  with respect to the ordered basis  $\alpha_1, \dots, \alpha_6$  is

$$[f] = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For  $\mu \in R(1, 1)$  and  $\nu \in R(1, -1)$  we have

$$N_{\mu, \nu} = (\mu, \nu)(-1)^{f(\mu, \nu)+1},$$

since  $\mu + \nu \in R$  if and only if  $(\mu, \nu) = -1$ . By making use of this and the matrix  $[f]$ , we find that

$$[N_{\mu_i, \nu_j}] = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It will be useful to list the sets  $S(\zeta_1), \dots, S(\zeta_8)$  explicitly. They are

$$S(\zeta_1) = \{(\mu_1, \nu_5), (\mu_2, \nu_3), (\mu_3, \nu_2), (\mu_5, \nu_1)\},$$

$$\begin{aligned}
S(\zeta_2) &= \{(\mu_1, \nu_6), (\mu_2, \nu_4), (\mu_4, \nu_2), (\mu_6, \nu_1)\}, \\
S(\zeta_3) &= \{(\mu_1, \nu_7), (\mu_3, \nu_4), (\mu_4, \nu_3), (\mu_7, \nu_1)\}, \\
S(\zeta_4) &= \{(\mu_2, \nu_7), (\mu_3, \nu_6), (\mu_4, \nu_5), (\mu_8, \nu_1)\}, \\
S(\zeta_5) &= \{(\mu_1, \nu_8), (\mu_5, \nu_4), (\mu_6, \nu_3), (\mu_7, \nu_2)\}, \\
S(\zeta_6) &= \{(\mu_2, \nu_8), (\mu_5, \nu_6), (\mu_6, \nu_5), (\mu_8, \nu_2)\}, \\
S(\zeta_7) &= \{(\mu_3, \nu_8), (\mu_5, \nu_7), (\mu_7, \nu_5), (\mu_8, \nu_3)\}, \\
S(\zeta_8) &= \{(\mu_4, \nu_8), (\mu_6, \nu_7), (\mu_7, \nu_6), (\mu_8, \nu_4)\}.
\end{aligned}$$

With these sets and the structure constants  $N_{\mu, \nu}$  in hand, we can compute the operators  $\Delta_\zeta$ . We abbreviate  $\Delta_{\zeta_i}$  as  $\Delta_i$  and find that

$$\begin{aligned}
\Delta_1 &= \partial_{\mu_1} \partial_{\nu_5} + \partial_{\mu_2} \partial_{\nu_3} + \partial_{\mu_3} \partial_{\nu_2} + \partial_{\mu_5} \partial_{\nu_1}, \\
\Delta_2 &= \partial_{\mu_1} \partial_{\nu_6} + \partial_{\mu_2} \partial_{\nu_4} + \partial_{\mu_4} \partial_{\nu_2} + \partial_{\mu_6} \partial_{\nu_1}, \\
\Delta_3 &= \partial_{\mu_1} \partial_{\nu_7} - \partial_{\mu_3} \partial_{\nu_4} + \partial_{\mu_4} \partial_{\nu_3} + \partial_{\mu_7} \partial_{\nu_1}, \\
\Delta_4 &= -\partial_{\mu_2} \partial_{\nu_7} - \partial_{\mu_3} \partial_{\nu_6} + \partial_{\mu_4} \partial_{\nu_5} + \partial_{\mu_8} \partial_{\nu_1}, \\
\Delta_5 &= \partial_{\mu_1} \partial_{\nu_8} - \partial_{\mu_5} \partial_{\nu_4} + \partial_{\mu_6} \partial_{\nu_3} - \partial_{\mu_7} \partial_{\nu_2}, \\
\Delta_6 &= -\partial_{\mu_2} \partial_{\nu_8} - \partial_{\mu_5} \partial_{\nu_6} + \partial_{\mu_6} \partial_{\nu_5} - \partial_{\mu_8} \partial_{\nu_2}, \\
\Delta_7 &= \partial_{\mu_3} \partial_{\nu_8} - \partial_{\mu_5} \partial_{\nu_7} + \partial_{\mu_7} \partial_{\nu_5} - \partial_{\mu_8} \partial_{\nu_3}, \\
\Delta_8 &= -\partial_{\mu_4} \partial_{\nu_8} + \partial_{\mu_6} \partial_{\nu_7} - \partial_{\mu_7} \partial_{\nu_6} + \partial_{\mu_8} \partial_{\nu_4}.
\end{aligned}$$

The next thing that we must do is to identify  $\mathfrak{g}(1, 1)$  and  $\mathfrak{g}(2, 0)$  as representations of  $\mathfrak{l}^{ss}$ . From the Dynkin diagram, we see that  $\mathfrak{l}^{ss} \cong \mathfrak{so}(8)$  but, because  $\mathfrak{so}(8)$  has a number of outer automorphisms, there are a number of genuinely different isomorphisms between  $\mathfrak{l}^{ss}$  and  $\mathfrak{so}(8)$ . We can rigidify the situation sufficiently by deciding how we shall identify  $\alpha_2, \dots, \alpha_5$  in the standard model of the root system of type  $D_4$ . The identification we choose is

$$\begin{aligned}
\alpha_2 &\longleftrightarrow e_3 + e_4, \\
\alpha_3 &\longleftrightarrow e_1 - e_2, \\
\alpha_4 &\longleftrightarrow e_2 - e_3, \\
\alpha_5 &\longleftrightarrow e_3 - e_4.
\end{aligned}$$

We can then identify  $\mu_1, \dots, \mu_8$  as weights with respect to the Cartan subalgebra that is spanned by  $H_{\alpha_2}, \dots, H_{\alpha_5}$ . The correspondence of weights is as follows:

$$\begin{aligned}
\mu_1 &\longleftrightarrow -e_1, & \mu_5 &\longleftrightarrow -e_4, \\
\mu_2 &\longleftrightarrow -e_2, & \mu_6 &\longleftrightarrow e_3, \\
\mu_3 &\longleftrightarrow -e_3, & \mu_7 &\longleftrightarrow e_2, \\
\mu_4 &\longleftrightarrow e_4, & \mu_8 &\longleftrightarrow e_1.
\end{aligned}$$

In particular, with the given identification of  $\mathfrak{l}^{ss}$  with  $\mathfrak{so}(8)$ ,  $\mathfrak{g}(1, 1)$  is isomorphic to the standard module. We can also identify  $\zeta_1, \dots, \zeta_8$  as weights.



The correspondence of weights is as follows:

$$\begin{aligned} \zeta_1 &\longleftrightarrow -\frac{1}{2}(e_1 + e_2 + e_3 + e_4), & \zeta_5 &\longleftrightarrow -\frac{1}{2}(e_1 - e_2 - e_3 + e_4), \\ \zeta_2 &\longleftrightarrow -\frac{1}{2}(e_1 + e_2 - e_3 - e_4), & \zeta_6 &\longleftrightarrow \frac{1}{2}(e_1 - e_2 + e_3 - e_4), \\ \zeta_3 &\longleftrightarrow -\frac{1}{2}(e_1 - e_2 + e_3 - e_4), & \zeta_7 &\longleftrightarrow \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \\ \zeta_4 &\longleftrightarrow \frac{1}{2}(e_1 - e_2 - e_3 + e_4), & \zeta_8 &\longleftrightarrow \frac{1}{2}(e_1 + e_2 + e_3 + e_4). \end{aligned}$$

In particular, with the given identification of  $\mathfrak{f}^{ss}$  with  $\mathfrak{so}(8)$ ,  $\mathfrak{g}(2, 0)$  is isomorphic to the even half-spin module.

Let  $\Gamma(p_1, p_2, p_3, p_4)$  be the irreducible  $\mathfrak{so}(8)$ -module with highest weight  $p_1e_1 + p_2e_2 + p_3e_3 + p_4e_4$ . Then we have

$$\Gamma(m, 0, 0, 0) \otimes \Gamma(1/2, 1/2, 1/2, 1/2) \cong \begin{cases} \Gamma(1/2, 1/2, 1/2, 1/2) & \text{if } m = 0, \\ \Gamma(m + 1/2, 1/2, 1/2, 1/2) \oplus \Gamma(m - 1/2, 1/2, 1/2, -1/2) & \text{if } m \geq 1. \end{cases}$$

To confirm this, recall that the highest weight of a constituent of the tensor product necessarily has the form  $(m, 0, 0, 0) + \varpi$ , where  $\varpi$  is a weight of  $\Gamma(1/2, 1/2, 1/2, 1/2)$ . Moreover, the tensor product is multiplicity free since  $\Gamma(1/2, 1/2, 1/2, 1/2)$  is weight multiplicity free. As the Cartan constituent,  $\Gamma(m + 1/2, 1/2, 1/2, 1/2)$  must occur, and a check of the dimensions on both sides shows that the other possible constituent must also occur unless  $m = 0$ . In order to make the decomposition of the tensor product concrete, we need to locate a highest weight vector for the summand  $\Gamma(m - 1/2, 1/2, 1/2, -1/2)$ . As will become clear below, it suffices to do this for  $m = 1$ . For the purpose at hand, we shall use  $\mathfrak{g}(1, 1)$  and  $\mathfrak{g}(2, 0)$  as models for  $\Gamma(1, 0, 0, 0)$  and  $\Gamma(1/2, 1/2, 1/2, 1/2)$ , respectively.

**Lemma 5.12.** *The vector*

$$r = X_{\mu_5} \otimes X_{\zeta_8} + X_{\mu_6} \otimes X_{\zeta_7} - X_{\mu_7} \otimes X_{\zeta_6} + X_{\mu_8} \otimes X_{\zeta_5}$$

*is a highest weight vector for  $\Gamma(1/2, 1/2, 1/2, -1/2)$  in  $\mathfrak{g}(1, 1) \otimes \mathfrak{g}(2, 0)$ .*

**Proof.** By using the correspondence of weights given above, it is easy to check that each of the terms in  $r$  has weight  $\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$ . Thus it suffices to check that  $r$  is annihilated by  $X_{\alpha_2}$ ,  $X_{\alpha_3}$ ,  $X_{\alpha_4}$ , and  $X_{\alpha_5}$ . We have

$$X_{\alpha_2} \cdot (X_{\mu_5} \otimes X_{\zeta_8}) = N_{\alpha_2, \mu_5} X_{\mu_6} \otimes X_{\zeta_8} = X_{\mu_6} \otimes X_{\zeta_8},$$

since  $f(\alpha_2, \mu_5) = 0$  because  $\alpha_2$  is the largest simple root and doesn't occur in  $\mu_5$ . Similarly,

$$X_{\alpha_2} \cdot (X_{\mu_6} \otimes X_{\zeta_7}) = N_{\alpha_2, \zeta_7} X_{\mu_6} \otimes X_{\zeta_8} = -X_{\mu_6} \otimes X_{\zeta_8},$$

since  $f(\alpha_2, \zeta_7) = 1$  because  $\alpha_2$  is the largest simple root and occurs in  $\zeta_7$  with coefficient 1. The other two terms are annihilated by  $X_{\alpha_2}$  and so  $X_{\alpha_2} \cdot r = 0$ . We can confirm in the same way that

$$\begin{aligned} X_{\alpha_4} \cdot (X_{\mu_6} \otimes X_{\zeta_7}) &= -X_{\mu_7} \otimes X_{\zeta_7}, \\ X_{\alpha_4} \cdot (X_{\mu_7} \otimes X_{\zeta_6}) &= -X_{\mu_7} \otimes X_{\zeta_7}, \end{aligned}$$

and  $X_{\alpha_4}$  annihilates the other two terms in  $r$ , so that  $X_{\alpha_4} \cdot r = 0$ . Furthermore,

$$\begin{aligned} X_{\alpha_3} \cdot (X_{\mu_7} \otimes X_{\zeta_6}) &= -X_{\mu_8} \otimes X_{\zeta_6}, \\ X_{\alpha_3} \cdot (X_{\mu_8} \otimes X_{\zeta_5}) &= -X_{\mu_8} \otimes X_{\zeta_6}, \end{aligned}$$

and  $X_{\alpha_3}$  annihilates the other two terms in  $r$ , so that  $X_{\alpha_3} \cdot r = 0$ . Finally,  $X_{\alpha_5}$  annihilates every term in  $r$  separately, and so  $X_{\alpha_5} \cdot r = 0$ .  $\square$

We also need to identify the quadratic vectors in  $\mathfrak{g}(1, 1)$  and  $\mathfrak{g}(2, 0)$  that are preserved by  $\mathfrak{l}^{ss}$ .

**Lemma 5.13.** *The vector*

$$q = X_{\mu_1} X_{\mu_8} + X_{\mu_2} X_{\mu_7} + X_{\mu_3} X_{\mu_6} - X_{\mu_4} X_{\mu_5}$$

*spans a copy of the trivial representation in  $\text{sym}^2(\mathfrak{g}(1, 1))$ . The vector*

$$s = X_{\zeta_1} X_{\zeta_8} + X_{\zeta_2} X_{\zeta_7} - X_{\zeta_3} X_{\zeta_6} + X_{\zeta_4} X_{\zeta_5}$$

*spans a copy of the trivial representation in  $\text{sym}^2(\mathfrak{g}(2, 0))$ .*

**Proof.** These two facts can be verified by calculations similar to the ones used to prove Lemma 5.12.  $\square$

Next we require the values of the operators  $F_{\zeta_i, \zeta_j}$ , which we shall abbreviate as  $F_{i,j}$ . These will be expressed in terms of the variables  $x_{\mu_i}$  and partial derivatives  $\partial_{\mu_i}$ , which we shall abbreviate as  $x_i$  and  $\partial_i$ , respectively. The operators  $F_{i,i}$  each have four terms. They are as follows:

$$\begin{aligned} F_{1,1} &= x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_5 \partial_5, \\ F_{2,2} &= x_1 \partial_1 + x_2 \partial_2 + x_4 \partial_4 + x_6 \partial_6, \\ F_{3,3} &= x_1 \partial_1 + x_3 \partial_3 + x_4 \partial_4 + x_7 \partial_7, \\ F_{4,4} &= x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 + x_8 \partial_8, \\ F_{5,5} &= x_1 \partial_1 + x_5 \partial_5 + x_6 \partial_6 + x_7 \partial_7, \\ F_{6,6} &= x_2 \partial_2 + x_5 \partial_5 + x_6 \partial_6 + x_8 \partial_8, \\ F_{7,7} &= x_3 \partial_3 + x_5 \partial_5 + x_7 \partial_7 + x_8 \partial_8, \\ F_{8,8} &= x_4 \partial_4 + x_6 \partial_6 + x_7 \partial_7 + x_8 \partial_8. \end{aligned}$$

Note that  $F_{i,i} + F_{9-i,9-i} = \mathbb{E}$ , the Euler operator with respect to the variables  $x_1, \dots, x_8$ , for all  $1 \leq i \leq 8$ . The operators  $F_{i,j}$  with  $i \neq j$  each have two terms unless they are zero. Their values are displayed in Table 1. To reduce

the amount of calculation that is necessary, we note a symmetry property of the operators  $F_{i,j}$ . Recall that

$$F_{\zeta,\xi} = \sum_{\substack{\nu \in R(1,-1) \\ (\nu,\zeta)=(\nu,\xi)=1}} N_{\zeta-\nu,\nu} N_{\xi-\nu,\nu} x_{\xi-\nu} \partial_{\zeta-\nu}.$$

This expression makes it clear that  $F_{\xi,\zeta}$  may be obtained from  $F_{\zeta,\xi}$  simply by interchanging the subscripts on the  $x$  and  $\partial$  factor in each term. We also note, for future reference, that one may observe the symmetry  $F_{i,j} = \pm F_{9-j,9-i}$  for all  $i \neq j$  from the values given in Table 1.

It follows from Lemma 5.13 that  $L^{ss}$  preserves the quadratic form

$$Q = x_1x_8 + x_2x_7 + x_3x_6 - x_4x_5$$

on  $\mathfrak{g}(1,1)$ .

**Lemma 5.14.** *We have*

$$F_{i,j} \bullet Q = \delta_{i,j} Q$$

for all  $1 \leq i, j \leq 8$ . Consequently, if  $\varphi \in \mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  is an eigenvector of  $F \bullet$  with eigenvalue  $\lambda$  then  $Q\varphi$  is an eigenvector of  $F \bullet$  with eigenvalue  $\lambda + 1$ .

**Proof.** The first claim follows by inspection of the values of  $F_{i,j}$  that have been given above and in Table 1. The second claim follows from a formal calculation using the first.  $\square$

**Theorem 5.15.** *We have  $\sigma(F) = \mathbb{Z} \cap [-3, \infty)$ .*

**Proof.** Let

$$\nabla = \partial_1\partial_8 + \partial_2\partial_7 + \partial_3\partial_6 - \partial_4\partial_5.$$

Dually to Lemma 5.14, one verifies that

$$[F_{i,j}, \nabla] = -\delta_{i,j} \nabla.$$

This has the consequence that if  $\varphi \in \mathbb{C}[\mathfrak{g}(1,1)]^{R(2,0)}$  is an eigenvector of  $F \bullet$  with eigenvalue  $\lambda$  then  $\nabla \bullet \varphi$  is an eigenvector of  $F \bullet$  with eigenvalue  $\lambda - 1$  unless  $\nabla \bullet \varphi = 0$ .

We call an element of  $\mathbb{C}[\mathfrak{g}(1,1)]$  *harmonic* if it is annihilated by  $\nabla$  and denote by  $\mathcal{H}$  the space of harmonic polynomials in  $\mathbb{C}[\mathfrak{g}(1,1)]$ . It is well known that

$$\mathbb{C}[\mathfrak{g}(1,1)] \cong \mathbb{C}[Q] \otimes \mathcal{H}$$

as  $\mathfrak{l}^{ss}$ -modules, where the isomorphism from right to left satisfies  $Q^j \otimes h \mapsto Q^j h$ . Furthermore,

$$\mathcal{H} \cong \bigoplus_{m \geq 0} \Gamma(m, 0, 0, 0)$$

as  $\mathfrak{l}^{ss}$ -modules and  $x_1^m \in \mathcal{H}$  is a highest weight vector in the  $\Gamma(m, 0, 0, 0)$  summand. These facts may be derived from the discussion in Sections 3.5 and 3.6 of [9], for example. It follows from these observations and Lemma 5.14 that  $\sigma(F)$  is the union of the arithmetic progressions with common

	1	2	3	4	5	6	7	8
1	*	$x_4\partial_3 + x_6\partial_5$	$x_4\partial_2 + x_7\partial_5$	$x_4\partial_1 + x_8\partial_5$	$x_6\partial_2 - x_7\partial_3$	$x_6\partial_1 - x_8\partial_3$	$x_7\partial_1 - x_8\partial_2$	0
2	$x_3\partial_4 + x_5\partial_6$	*	$-x_3\partial_2 + x_7\partial_6$	$-x_3\partial_1 + x_8\partial_6$	$-x_5\partial_2 - x_7\partial_4$	$-x_5\partial_1 - x_8\partial_4$	0	$-x_7\partial_1 + x_8\partial_2$
3	$x_2\partial_4 + x_5\partial_7$	$-x_2\partial_3 + x_6\partial_7$	*	$-x_2\partial_1 + x_8\partial_7$	$x_5\partial_3 + x_6\partial_4$	0	$-x_5\partial_1 - x_8\partial_4$	$x_6\partial_1 - x_8\partial_3$
4	$x_1\partial_4 + x_5\partial_8$	$-x_1\partial_3 + x_6\partial_8$	$-x_1\partial_2 + x_7\partial_8$	*	0	$x_5\partial_3 + x_6\partial_4$	$x_5\partial_2 + x_7\partial_4$	$-x_6\partial_2 + x_7\partial_3$
5	$x_2\partial_6 - x_3\partial_7$	$-x_2\partial_5 - x_4\partial_7$	$x_3\partial_5 + x_4\partial_6$	0	*	$-x_2\partial_1 + x_8\partial_7$	$x_3\partial_1 - x_8\partial_6$	$-x_4\partial_1 - x_8\partial_5$
6	$x_1\partial_6 - x_3\partial_8$	$-x_1\partial_5 - x_4\partial_8$	0	$x_3\partial_5 + x_4\partial_6$	$-x_1\partial_2 + x_7\partial_8$	*	$-x_3\partial_2 + x_7\partial_6$	$x_4\partial_2 + x_7\partial_5$
7	$x_1\partial_7 - x_2\partial_8$	0	$-x_1\partial_5 - x_4\partial_8$	$x_2\partial_5 + x_4\partial_7$	$x_1\partial_3 - x_6\partial_8$	$-x_2\partial_3 + x_6\partial_7$	*	$-x_4\partial_3 - x_6\partial_5$
8	0	$-x_1\partial_7 + x_2\partial_8$	$x_1\partial_6 - x_3\partial_8$	$-x_2\partial_6 + x_3\partial_7$	$-x_1\partial_4 - x_5\partial_8$	$x_2\partial_4 + x_5\partial_7$	$-x_3\partial_4 - x_5\partial_6$	*

TABLE 1. The Operators  $F_{i,j}$  with  $i \neq j$

difference 1 and first term an eigenvalue of  $F\bullet$  on  $\mathcal{H}^{R(2,0)}$ . Thus we must compute the eigenvalues of  $F\bullet$  on  $\mathcal{H}^{R(2,0)}$ . As we noted above,

$$\begin{aligned} \mathcal{H}^{R(2,0)} &\cong \mathcal{H} \otimes \Gamma(1/2, 1/2, 1/2, 1/2) \\ &\cong \bigoplus_{m \geq 0} \Gamma(m + 1/2, 1/2, 1/2, 1/2) \oplus \bigoplus_{m \geq 1} \Gamma(m - 1/2, 1/2, 1/2, -1/2). \end{aligned}$$

In particular,  $\mathcal{H}^{R(2,0)}$  is multiplicity free. By Proposition 4.4 and Schur’s Lemma,  $F\bullet$  is necessarily a scalar on each summand in the above decomposition. In order to determine this scalar it suffices to compute the action of  $F\bullet$  on a highest weight vector in each summand.

A highest weight vector in the  $\Gamma(1/2, 1/2, 1/2, -1/2)$ -summand in the above decomposition was identified in Lemma 5.12. In the model used there,  $X_{\mu_s}^m \otimes X_{\zeta_s}$  is a highest weight vector in the  $\Gamma(m + 1/2, 1/2, 1/2, 1/2)$ -summand. By using Lemma 5.13 we may identify  $\mathfrak{g}(1, 1)$  and  $\mathfrak{g}(2, 0)$  with their contragredients in an  $L^{ss}$ -equivariant fashion and thus transfer the above highest weight vectors into the required model. On doing so, we find that

$$\varphi^{m,+} = [x_1^m, 0, 0, 0, 0, 0, 0, 0]^T$$

is a highest weight vector in the  $\Gamma(m+1/2, 1/2, 1/2, 1/2)$ -summand in  $\mathcal{H}^{R(2,0)}$  and

$$\varphi^{m,-} = [-x_1^{m-1}x_4, x_1^{m-1}x_3, x_1^{m-1}x_2, x_1^m, 0, 0, 0, 0]^T$$

is a highest weight vector in the  $\Gamma(m - 1/2, 1/2, 1/2, -1/2)$ -summand in  $\mathcal{H}^{R(2,0)}$ . By direct calculation, using the values of  $F_{i,j}$  that have been given above, we find that

$$F\bullet\varphi^{m,+} = m\varphi^{m,+}$$

for  $m \geq 0$  and

$$F\bullet\varphi^{m,-} = -3\varphi^{m,-}$$

for  $m \geq 1$ . It follows that the spectrum of  $F\bullet$  on  $\mathcal{H}^{R(2,0)}$  is  $\{-3\} \cup \mathbb{N}$ . In combination with the reductions that were made above, this completes the proof.  $\square$

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