

Density of orbits of semigroups of endomorphisms acting on the Adeles

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ABSTRACT. We investigate the question of whether or not the orbit of a point in \mathbb{A}/\mathbb{Q} , under the natural action of a subset $\Sigma \subseteq \mathbb{Q}$, is dense in \mathbb{A}/\mathbb{Q} . We prove that if the set Σ is a multiplicative semigroup of \mathbb{Q}^\times which contains at least two multiplicatively independent elements, one of which is an integer, then the orbit $\Sigma\alpha$ of any point α with irrational real coordinate is dense.

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1. Introduction

Let S be a multiplicative semigroup of positive integers which contains at least two multiplicatively independent elements (i.e., elements whose logarithms are \mathbb{Q} -linearly independent). A well known theorem of Furstenberg [4, Theorem IV.1] states that if $\alpha \in \mathbb{R}/\mathbb{Z}$ is irrational then the set

$$S\alpha := \{s\alpha : s \in S\} \subseteq \mathbb{R}/\mathbb{Z}$$

is dense in \mathbb{R}/\mathbb{Z} . This is a fundamental example of rigidity in dynamical systems, and it was extended by Berend in [1] to study the action of \mathbb{Q} on the Adelic quotient \mathbb{A}/\mathbb{Q} . Berend proved that under certain conditions on a multiplicative semigroup $\Sigma \subseteq \mathbb{Q}$, every point $\alpha \in \mathbb{A}/\mathbb{Q}$ with an infinite orbit under the action of Σ turns out to have a dense orbit. Berend's hypotheses on Σ require it to contain elements whose absolute values are greater than one, in all completions of \mathbb{Q} (Archimedean and non-Archimedean).

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In this paper we study semigroups of \mathbb{Q} which are generated by only two elements. In this case, in contrast with the results of Berend and Furstenberg, there are orbits of ‘nonlacunary’ semigroups which are neither finite nor dense (see Section 4). Our main result is the following theorem.

Theorem 1.1. *Suppose that $\gamma \in \mathbb{Q}$ and $\delta \in \mathbb{Z}$ are multiplicatively independent and let*

$$\Sigma = \Sigma(\gamma, \delta) := \{\gamma^a \delta^b : a, b \in \mathbb{N}\}.$$

Then for any $\alpha \in \mathbb{A}/\mathbb{Q}$ with $\alpha_\infty \notin \mathbb{Q}$ the set

$$\Sigma\alpha := \{\sigma\alpha : \sigma \in \Sigma\} \subseteq \mathbb{A}/\mathbb{Q},$$

where $\sigma\alpha$ denotes the natural action of \mathbb{Q} on \mathbb{A}/\mathbb{Q} , is dense in \mathbb{A}/\mathbb{Q} .

It is possible that the hypothesis that $\delta \in \mathbb{Z}$ could be weakened but, as the examples of Section 4 show, it cannot be removed completely. Our paper is organized as follows. In Section 2 we give a basic background about the Adeles and a metric which generates the topology on \mathbb{A}/\mathbb{Q} . In Section 3 we give the proof of our main theorem, and in Section 4 we give several examples of nondense orbits for the actions of semigroups of \mathbb{Q} on \mathbb{A}/\mathbb{Q} . The reader who is new to these ideas is encouraged to read the final section before the proof of the main result.

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2. Preliminaries

We use \mathbb{A} to denote the rational Adeles with the usual restricted product topology, under which it is a locally compact Abelian group (we refer the reader to [5, Chapter 5] for a basic treatment of the Adeles). The group \mathbb{Q} diagonally embeds in \mathbb{A} , and we also denote the image of this embedding as \mathbb{Q} . This slight ambiguity in notation should cause no confusion in what follows. It is well known that \mathbb{Q} is a discrete subgroup of \mathbb{A} and that the quotient \mathbb{A}/\mathbb{Q} is compact. Our canonical choice of fundamental domain for this quotient will be

$$\mathcal{F} := [0, 1) \times \prod_p \mathbb{Z}_p.$$

We will view \mathcal{F} as a topological space, with a base for its topology being given by the following two types of sets:

(i) Sets of the form

$$U_\infty \times \prod_p U_p,$$

where each $U_p \subseteq \mathbb{Z}_p$ is an open ball, $U_\infty \subseteq (0, 1)$ is an open interval, and $U_p = \mathbb{Z}_p$ for all but finitely many primes.

(ii) Sets of the form

$$\left((\alpha, 1) \times \prod_p U_p \right) \cup \left([0, \beta) \times \prod_p (U_p - 1) \right),$$

where $\alpha, \beta \in (0, 1)$, each $U_p \subseteq \mathbb{Z}_p$ is an open ball, and $U_p = \mathbb{Z}_p$ for all but finitely many primes.

The reason for this choice is made clear by the following proposition.

Proposition 2.1. *As a topological space \mathbb{A}/\mathbb{Q} is homeomorphic to \mathcal{F} .*

Proof. Call the two types of sets in the basis above sets of types (i) and (ii). First we show that both types of sets are open in the quotient topology on \mathbb{A}/\mathbb{Q} . Let $\phi : \mathbb{A} \rightarrow \mathbb{A}/\mathbb{Q}$ be the quotient map, and suppose that $X \subseteq \mathcal{F}$ is a set of type (i). Then

$$\phi^{-1}(X) = \bigcup_{\eta \in \mathbb{Q}} (X + \eta),$$

and for any $\eta \in \mathbb{Q}$ we have that $U_p + \eta = \mathbb{Z}_p$ for all but finitely many primes. Therefore each set $X + \eta$ is open in \mathbb{A} and we conclude that X is open in \mathbb{A}/\mathbb{Q} .

Similarly, if $X \subseteq \mathcal{F}$ is a set of type (ii) then write $X = A \cup B$ with

$$A = \left((\alpha, 1) \times \prod_p U_p \right) \quad \text{and} \quad B = \left([0, \beta) \times \prod_p (U_p - 1) \right),$$

and note that

$$\begin{aligned} \phi^{-1}(X) &= \bigcup_{\eta \in \mathbb{Q}} ((A + \eta) \cup (B + (\eta + 1))) \\ &= \bigcup_{\eta \in \mathbb{Q}} \left(\left((\alpha, 1 + \beta) \times \prod_p U_p \right) + \eta \right) \end{aligned}$$

is open in \mathbb{A} .

Next we show that any open set in \mathbb{A}/\mathbb{Q} can be written as a union of sets of types (i) and (ii). If $X \subseteq \mathbb{A}/\mathbb{Q}$ is open then we can write

$$\phi^{-1}(X) = \bigcup_{i \in I} Y_i,$$

where each $Y_i \subseteq \mathbb{A}$ is a set of the form

$$Y_i = V_{i,\infty} \times \prod_p V_{i,p},$$

with each $V_{i,p} \subseteq \mathbb{Q}_p$ an open ball, $V_{i,p} = \mathbb{Z}_p$ for all but finitely many primes, and $V_{i,\infty} \subseteq \mathbb{R}$ an open interval of length less than one. Then we have that

$$X = \bigcup_{i \in I} \phi(Y_i),$$

and each of the sets in this union, viewed as a subset of \mathcal{F} , is a set of type (i) or (ii). \square

As a word of warning we point out that, although their topological structure is similar, \mathbb{A}/\mathbb{Q} is *not* isomorphic as a group to the direct product

$$\mathbb{R}/\mathbb{Z} \times \prod_p \mathbb{Z}_p.$$

This fact has already been featured in the proof above, and indeed the group structure is reflected by the shapes of the sets of type (ii).

For convenience we will work with a metric on \mathbb{A}/\mathbb{Q} which induces its topology. We define $d : \mathcal{F} \times \mathcal{F} \rightarrow [0, \infty)$ by

$$d(\alpha, \beta) = \min_{\eta \in \{0, \pm 1\}} \max \left(|\alpha_\infty - \beta_\infty - \eta|_\infty, \max_p \left(\frac{|\alpha_p - \beta_p - \eta|_p}{p} \right) \right).$$

The function d defines a metric on \mathcal{F} and the metric topology is the same as the quotient topology on \mathbb{A}/\mathbb{Q} , since it is precisely that generated by the type (i) and (ii) sets above (see also [6, Proposition 3.3]). We will use the following proposition to quantify the density of the integers at the finite places of \mathbb{A}/\mathbb{Q} .

Proposition 2.2. *Let $\varepsilon > 0$ and suppose that $X \subseteq [0, 1)$ is an ε -dense subset of $[0, 1)$. For each prime p let*

$$(2.1) \quad k_p = k_p(\varepsilon) := \max(0, \lceil -\log_p \varepsilon \rceil - 1),$$

and set

$$(2.2) \quad N = N(\varepsilon) := \prod_p p^{k_p}.$$

Then, with respect to the metric d , the set

$$\{(x; n, n, n, \dots) : x \in X, 1 \leq n \leq N\} \subseteq \mathcal{F}$$

is an ε -dense subset of \mathbb{A}/\mathbb{Q} .

Proof. The diameter of the space \mathbb{A}/\mathbb{Q} is $1/2$, so we may assume that $\varepsilon < 1/2$. For any $\alpha \in \mathcal{F}$ we can clearly choose $x \in X$ so that $|\alpha_\infty - x|_\infty \leq \varepsilon$. Furthermore by the Chinese Remainder Theorem we can choose an integer $1 \leq n \leq N$ so that

$$n = \alpha_p \pmod{p^{k_p}}$$

for all primes p (note that $k_p = 0$ for all primes $p \geq 1/\varepsilon$). Then by the definition of d we have that

$$d(\alpha, (x; n, n, n, \dots)) = \max \left(|\alpha_\infty - x|_\infty, \max_p \frac{|\alpha_p - n|_p}{p} \right) \leq \max \left(\varepsilon, \max_p p^{-k_p-1} \right).$$

By the choice of k_p this is less than or equal to ε . □

3. Proof of Theorem 1.1

The outline of our proof of Theorem 1.1 is based on Furstenberg’s original proof of his orbit closure theorem [4] and also on Boshernitzan’s exposition of this result [2]. However there are many features of our proof which are new, which do not appear when one is considering the group \mathbb{R}/\mathbb{Z} .

To begin, by replacing γ with γ^2 and δ with δ^2 if necessary, we assume without loss of generality that $\gamma > 0$ and $\delta > 1$. We first have the following proposition.

Proposition 3.1. *For any $\varepsilon > 0$ there exist $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ with*

$$(3.1) \quad 1 < \frac{\gamma^a}{\delta^b} < 1 + \varepsilon.$$

Proof. Taking logarithms in (3.1) and dividing by $b \log \gamma$, this is equivalent to the statement that, for any $\varepsilon' > 0$, there are $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ with

$$0 < \frac{a}{b} - \frac{\log \delta}{\log \gamma} < \frac{\varepsilon'}{b}.$$

Now we can take $a/b = p_{2n+1}/q_{2n+1}$ to be the $(2n + 1)^{st}$ convergent in the continued fraction expansion of $\log \delta / \log \gamma$, with n chosen large enough that $q_{2n+1}^{-1} < \varepsilon'$. If $\gamma < 1$ then we choose $a > 0, b < 0$, while if $\gamma > 1$ we choose $a, b > 0$. □

Now let \mathcal{P} be the collection of primes p for which $|\gamma|_p > 1$, define

$$(3.2) \quad P = \prod_{p \in \mathcal{P}} |\gamma|_p,$$

and write $P = p_1^{a_1} \cdots p_m^{a_m}$ with p_1, \dots, p_m distinct primes (we can assume that \mathcal{P} is nonempty, otherwise we are in the case of Furstenberg’s theorem). Let

$$X_{\mathcal{P}} = \left(\mathbb{R} \times \prod_{i=1}^m \mathbb{Q}_{p_i} \right) / \iota(\mathbb{Z}[1/(p_1 \cdots p_m)]),$$

where

$$\iota : \mathbb{Z}[1/(p_1 \cdots p_m)] \rightarrow \mathbb{R} \times \prod_{i=1}^m \mathbb{Q}_{p_i}$$

denotes the diagonal embedding, and let

$$\pi : \mathbb{A}/\mathbb{Q} \rightarrow X_{\mathcal{P}}$$

be the map obtained by projection of \mathcal{F} onto $X_{\mathcal{P}}$. We take $X_{\mathcal{P}}$ with its usual quotient topology and note that the map π is continuous. Furthermore the elements of Σ act both on \mathbb{A}/\mathbb{Q} and on $X_{\mathcal{P}}$ as endomorphisms, and these actions commute with π . We have the following proposition.

Proposition 3.2. *If $\alpha \in \mathbb{A}/\mathbb{Q}$ satisfies $\alpha_{\infty} \notin \mathbb{Q}$, and if there exists an $s/t \in \mathbb{Q}$ such that $\iota(s/t)$ is an accumulation point of the set $\pi(\Sigma\alpha)$, then $\Sigma\alpha$ is dense in \mathbb{A}/\mathbb{Q} .*

Proof. We first claim that we can replace $\iota(s/t)$ with an accumulation point satisfying the same hypotheses, which is also invariant under the action of suitable powers of γ and δ . If it happens that $s/t = 0$ then this claim is trivially satisfied, so let us assume temporarily that $s/t \neq 0$.

Write $\gamma = r/P$ with $(r, P) = 1$. We can assume that s/t lies in the fundamental domain

$$(3.3) \quad (-1, 0] \times \prod_{p \in \mathcal{P}} \mathbb{Z}_p,$$

and we can also assume (by multiplying by an element of Σ if necessary), that $(r, t) = (\delta, t) = 1$. Here and throughout the proof we are using the fact, mentioned before the proposition, that

$$\pi(\sigma\alpha) = \sigma\pi(\alpha), \text{ for any } \alpha \in \mathbb{A}/\mathbb{Q} \text{ and } \sigma \in \Sigma.$$

We can view s/t as an element of the ring

$$\lim_{\leftarrow} \mathbb{Z}/P^i\mathbb{Z},$$

with an eventually periodic expansion. For each $i \in \mathbb{N}$ choose $0 \leq A_i < P^i$ with $s/t = A_i \pmod{P^i}$. Then there is an $I \in \mathbb{N}$ such that for all $i > I$, the rational number

$$(3.4) \quad \xi_i = P^{-i} \left(\frac{s}{t} - A_i \right) = \frac{s}{P^i t} - \frac{A_i}{P^i} \in \mathbb{Z}_p$$

has a purely periodic p -adic expansion, for all primes $p \in \mathcal{P}$. We may assume that I is large enough so that $\xi_i \in (-1, 0)$ for all $i > I$. Now choose $\mu > I$ such that $(r\delta)^{\mu} = 1 \pmod{t}$, and let $\beta' = (r\delta)^{\mu}(s/t)$ and $\beta'' = (\gamma\delta)^{\mu}(s/t)$. Then, choosing a representative in (3.3) for β' , we have that

$$\beta'_{\infty} = \frac{(r\delta)^{\mu}s}{t} - \left(1 + \left\lfloor \frac{(r\delta)^{\mu}s}{t} \right\rfloor \right) = \frac{s}{t},$$

and therefore also that $\beta'_p = s/t$ for all $p \in \mathcal{P}$. Since $\beta'' = P^{-\mu}\beta'$, we then have that $\beta''_{\infty} = \xi_{\mu}$ and that $\beta''_p = \xi_{\mu}$ for all $p \in \mathcal{P}$. Here our representative for β'' is determined uniquely by the requirements that $\beta''_{\infty} \in (-1, 0]$ and $\beta''_p \in \mathbb{Z}_p$ for all p . Therefore, for the remainder of this proof we will assume (by replacing s/t with ξ_{μ}) that the hypotheses of this proposition are satisfied for a fraction $s/t \in (-1, 0)$ with $(r\delta, t) = 1$, with $(p, t) = 1$ for all $p \in \mathcal{P}$, and with the property that the p -adic expansion of s/t is purely periodic, for all $p \in \mathcal{P}$.

Let $\nu \in \mathbb{N}$ be the least common multiple of $\varphi(t)$ and all of the period lengths of s/t modulo primes $p \in \mathcal{P}$. Set $\beta^{(0)} = r^\nu(s/t)$, and for each $1 \leq i \leq m$ set $\beta^{(i)} = p_i^{-\nu a_i} \beta^{(i-1)}$. We will work with representatives for these elements which lie in the fundamental domain (3.3).

First of, as in the argument above, notice that

$$\beta_\infty^{(0)} = \frac{r^\nu s}{t} - \left(1 + \left\lfloor \frac{r^\nu s}{t} \right\rfloor\right) = \frac{s}{t},$$

and therefore that $\beta_p^{(0)} = s/t$ for all $p \in \mathcal{P}$. Similarly for each $1 \leq i \leq m$ we have that

$$\beta_{p_i}^{(i)} = \frac{s}{t},$$

because the p_i -adic expansion of s/t is periodic with period dividing ν . Therefore we have that $\beta_\infty^{(i)} = s/t$ and $\beta_p^{(i)} = s/t$ for all $p \in \mathcal{P}$ and for $1 \leq i \leq m$. In particular, this is true for $i = m$, when $\beta^{(m)} = \gamma^\nu(s/t)$. The same argument applies with γ replaced by δ , and this verifies the claim which we made at the beginning of the proof.

Let us take s/t as above, satisfying our claim, and for notational convenience let us replace γ and δ by γ^ν and δ^ν , and Σ by the semigroup generated by the new choices of γ and δ . The next part of the proof is somewhat technical, but the central idea is to find a point $y \in \mathbb{A}/\mathbb{Q}$ for which $\pi(y)$ is close to $\iota(s/t)$, and then multiply y by a sequence of elements of Σ , selected with the aid of Proposition 3.1, to move away from s/t in the Archimedean direction in very small steps. We continue on in small steps, controlling the p -adic directions so that they stay in \mathbb{Z}_p , until the Archimedean component of our point is close to an integer N which is as large as we need. Then we translate everything back into a fundamental domain. The real components will remain in a sufficiently dense set in the real direction, and the p -adic components will simultaneously wind around enough in the p -adic directions to allow us to apply Proposition 2.2.

Now we fill in the details of this argument. Let $M \geq 2$ be an even integer, for each prime p let $k_p = k_p(M^{-1})$ be defined as in (2.1), and let $N = N(M^{-1})$ be defined as in (2.2). Then let

$$L = \text{lcm}_{p \leq M} \left(p^{k_p - 1} (p - 1) \right),$$

and apply Proposition 3.1 to find integers $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ with

$$1 < \frac{\gamma^a}{\delta^b} < 1 + \frac{1}{LMN}.$$

Next set $\sigma = \gamma^a/\delta^b$ and let

$$K = \lceil 2\delta MN(\sigma - 1)^{-1} \sigma^L \rceil.$$

If $b < 0$ then choose $\gamma_0 \in \Sigma$ to be any element satisfying $\gamma_0 > 1$, and if $b > 0$ choose $\gamma_0 = \delta^{Kb}$. Then set

$$\gamma_i = \sigma^i \gamma_0 \text{ for } 1 \leq i \leq K,$$

and note that each of these fractions is in Σ . Finally choose a point $y \in \Sigma\alpha$ in the fundamental domain

$$(3.5) \quad (-1, 0] \times \prod_p \mathbb{Z}_p$$

satisfying

$$(3.6) \quad \left| y_\infty - \frac{s}{t} \right|_\infty < \frac{1}{2\gamma_0\sigma^L M}, \quad \text{and}$$

$$(3.7) \quad \left| y_p - \frac{s}{t} \right|_p < \frac{\min(1, |\sigma|_p^{-K})}{p^{k_p}} \quad \text{for all } p \in \mathcal{P}.$$

By multiplying y by a suitable power of δ , which does not increase the p -adic absolute values of any of the y_p -coordinates, we may assume that we have chosen y so that y_∞ also satisfies the lower bound

$$(3.8) \quad \left| y_\infty - \frac{s}{t} \right|_\infty > \frac{1}{2\delta\gamma_0\sigma^L M}.$$

We point out that this is where we are using the fact that $\alpha_\infty \notin \mathbb{Q}$. Now for each $0 \leq i \leq K$ define $y^{(i)} = \gamma_i y$. If we write $y_\infty^{(0)} = s/t + x_\infty$ and $y_p^{(0)} = s/t + x_p$ for each $p \in \mathcal{P}$, then we have for each $0 \leq i \leq K$ that

$$\begin{aligned} y_\infty^{(i)} &= \sigma^i \cdot \frac{s}{t} + \sigma^i x_\infty, \\ y_p^{(i)} &= \sigma^i \cdot \frac{s}{t} + \sigma^i x_p \quad \text{for } p \in \mathcal{P}, \quad \text{and} \\ y_p^{(i)} &= \sigma^i y_p^{(0)} \quad \text{for } p \notin \mathcal{P}. \end{aligned}$$

By our choice of s/t there is a fraction s'/t' which is in \mathbb{Z}_p for all $p \notin \mathcal{P}$ and which satisfies

$$\sigma \cdot \frac{s}{t} = \frac{s}{t} + \frac{s'}{t'}.$$

Noting that inequality (3.7) guarantees that $|\sigma^i x_p|_p < p^{-k_p}$ for all $p \in \mathcal{P}$ and $0 \leq i \leq K$, we have after translating back into the fundamental domain (3.5) that

$$\begin{aligned} y_\infty^{(i)} &= \frac{s}{t} + \sigma^i x_\infty - n_i, \\ y_p^{(i)} &= \frac{s}{t} + \sigma^i x_p - n_i \quad \text{for } p \in \mathcal{P}, \quad \text{and} \\ y_p^{(i)} &= \sigma^i y_p^{(0)} - \frac{s'}{t'} \cdot \sum_{j=0}^{i-1} \sigma^j - n_i \quad \text{for } p \notin \mathcal{P}, \end{aligned}$$

where

$$n_i = 1 + \left\lfloor \frac{s}{t} + \sigma^i x_\infty \right\rfloor.$$

Now by our lower bound (3.8) we have for any $0 \leq i < K$ that

$$|\sigma_{i+1} x_\infty - \sigma_i x_\infty|_\infty = |(\sigma - 1)\sigma^i x_\infty|_\infty > \frac{(\sigma - 1)}{2\delta\sigma^L M},$$

and by our choice for K this implies that $\sigma^K x_\infty > N$. On the other hand for any i with $\sigma^i < N$ we have that

$$|\sigma_{i+1}x_\infty - \sigma_i x_\infty|_\infty < \frac{1}{LM}.$$

This together with our upper bound (3.6) ensures that for any interval $\mathcal{I} \subseteq [0, N]$ of length $2/M$, there is an integer $0 \leq j \leq K - L$ with the property that $\sigma^i x_\infty \in \mathcal{I}$ for all $j \leq i \leq j + L$. One of these integers must equal 0 modulo L and for this integer we have that

$$\sigma^i = 1 \pmod{p^{k_p}} \text{ for all } p \notin \mathcal{P},$$

and therefore also that

$$\sum_{j=0}^{i-1} \sigma^j = \frac{\sigma^i - 1}{\sigma - 1} = 0 \pmod{p^{k_p}} \text{ for all } p \notin \mathcal{P}.$$

To finish the argument, partition the interval $(-1, 0]$ into disjoint intervals of length $2/M$. By what we have said above, for each such interval \mathcal{I} and for each integer $1 \leq n \leq N$ there is an $0 \leq i \leq K$ with $y_\infty^{(i)} \in \mathcal{I}$ and $|y_p^{(i)} - n|_p \leq p^{-k_p}$ for all primes p . By Proposition 2.2 the set $\Sigma\alpha$ is $2/M$ -dense in \mathbb{A}/\mathbb{Q} , and M can be taken arbitrary large. \square

Now we come to the proof of our main result.

Proof of Theorem 1.1. We will show that if $\alpha \in \mathbb{A}/\mathbb{Q}$ satisfies $\alpha_\infty \notin \mathbb{Q}$, then there is a rational s/t satisfying the hypothesis of Proposition 3.2. Suppose, by way of contradiction, that this is not the case. As before write $\gamma = r/P$ with $(r, P) = 1$. Then for any integer $v \geq 3$ which is coprime to $Pr\delta$, there exists an integer u with $-v < u \leq -1$ and $(u, v) = 1$, for which u/v has a purely p -adic expansion for all $p \in \mathcal{P}$. This follows from the same argument used to construct the fractions ξ_i in the previous proof (cf. (3.4)). By replacing γ and δ by suitable powers, as in the previous proof, we may assume that $\iota(u/v)$ is invariant under the action of Σ .

Define $Y_0 \subseteq X_{\mathcal{P}}$ by

$$Y_0 = \overline{\pi(\Sigma\alpha)},$$

and for each $i \in \mathbb{N}$ define $Y_i \subseteq Y_{i-1}$ by

$$Y_i = \{y \in Y_{i-1} : y + \iota(u/v) \in Y_{i-1}\}.$$

For any $i \in \mathbb{N}$ any for any $y \in Y_i$ we have that

$$y + j \cdot \iota(u/v) \in Y_0 \text{ for all } 0 \leq j \leq i.$$

Therefore if we can show that all of the sets Y_i are nonempty then, since i can be taken arbitrarily large, it will follow from Proposition 2.2 that the set Y_0 must be $1/v$ -dense in $X_{\mathcal{P}}$. Since Y_0 is closed and v can be taken arbitrarily large, this would mean that $Y_0 = X_{\mathcal{P}}$, which would contradict our initial assumption and complete the proof.

All that remains is to show that the sets Y_i are always nonempty, and we will do this by an inductive argument. It is clear that Y_0 is nonempty, closed, and Σ -invariant. Since $\alpha_\infty \notin \mathbb{Q}$, it follows that Y_0 contains an accumulation point y and a sequence of points $(y^{(j)})_{j=1}^\infty$ approaching y , with the property that $y_\infty^{(j)} \neq y_\infty$ for any j . This implies that there is a sequence of points $(x^{(j)})_{j=1}^\infty$ in the difference set

$$D_0 := Y_0 - Y_0$$

with $x^{(j)} \rightarrow 0$ and $x_\infty^{(j)} \neq 0$ for any j . Since D_0 is closed, this allows us to apply the same argument as in the previous proposition to conclude that $D_0 = X\mathcal{P}$. We note that the property that $x_\infty^{(j)} \neq 0$ for any j is crucial here (cf. (3.8)). Since $u/v \in D_0$ we conclude that Y_1 is nonempty.

Next we show that Y_2 is nonempty. The set Y_1 is nonempty, closed, and Σ -invariant (since $\iota(u/v)$ is Σ -invariant). If there is an element $y \in Y_1$ with $y_\infty \notin \mathbb{Q}$ then the argument is exactly the same as before. If not, then choose any element $y \in Y_1$ and, considering it as an element of the fundamental domain (3.3), choose a prime $p \in \mathcal{P}$ with the property that $y_p \neq y_\infty$. This is possible since we know that $\iota(y_\infty) \notin Y_0$. Then $y_p \in \mathbb{Z}_p$ and $y_\infty \in \mathbb{Q} \cap \mathbb{Q}_p$, and we can write their p -adic expansions as

$$y_\infty = \sum_{\ell=-\infty}^{\infty} b_\ell p^\ell \quad \text{and} \quad y_p = \sum_{\ell=0}^{\infty} c_\ell p^\ell,$$

with $b_\ell = 0$ for all ℓ less than some bound. Now write $\gamma = p^{-a}\gamma'$ with $|\gamma'|_p = 1$, and set $z^{(i)} = p^{-ia}y$ and $y^{(i)} = \gamma^i y$. Working in the fundamental domain (3.3), we find that for all sufficiently large i ,

$$|z_\infty^{(i)}|_p = p^{ia} \cdot \left| \sum_{\ell=-\infty}^{-1} b_\ell p^\ell + \sum_{\ell=0}^{ia-1} (b_\ell - c_\ell) p^\ell \right|_p.$$

Here we are using the fact that $b_\ell \neq c_\ell$ for some ℓ , and this also implies that $|z_\infty^{(i)}|_p$ tends to infinity with i . Since $|\gamma'|_p = 1$, and since translation by a p -adic integer (in order to bring all other coordinates back into the fundamental domain) does not change the p -adic absolute value of an element of $\mathbb{Q}_p \setminus \mathbb{Z}_p$, we have that $|y_\infty^{(i)}|_p$ tends to infinity with i . This means that there are infinitely many points in Y_1 whose representatives in the fundamental domain (3.3) have different Archimedean coordinates, and therefore there is a sequence of points $(x^{(j)})_{j=1}^\infty$ in the difference set $D_1 := Y_1 - Y_1$ with $x^{(j)} \rightarrow 0$ and $x_\infty^{(j)} \neq 0$ for any j . The rest of the argument for showing that Y_2 is nonempty is exactly the same as before, and the same argument then shows that Y_i is nonempty for all $i \in \mathbb{N}$. This concludes the proof of Theorem 1.1. \square

4. Examples of nondense orbits

Finally we give examples which illustrate various ways in which one can fail to have dense orbits. Firstly, if all the elements of Σ are contractions on \mathbb{R} , then there is one obvious degeneracy.

Proposition 4.1. *Suppose that $\gamma, \delta \in \mathbb{Q}$ satisfy*

$$\max(|\gamma|, |\delta|) < 1,$$

let Σ denote the multiplicative semigroup which they generate, and let P be defined as in (3.2). If $\alpha \in \mathbb{A}/\mathbb{Q}$ is any point with coordinates in the fundamental domain

$$(4.1) \quad [-1/2, 1/2) \times \prod_p \mathbb{Z}_p$$

which satisfies $\alpha_p = 0$ for all $p \mid P$, then all accumulation points x of the set $\Sigma\alpha$ have $x_\infty = 0$.

Proof. If α is any point which satisfies the hypotheses of this proposition then for any $\sigma \in \Sigma$ and for any prime p , we will have that $\sigma\alpha_p \in \mathbb{Z}_p$. However as a and b tend to infinity along any sequence in \mathbb{N} , we will have that

$$\gamma^a \delta^b \alpha_\infty \rightarrow 0,$$

so that 0 is the only possible real coordinate of any accumulation point of $\Sigma\alpha$ in the fundamental domain (4.1). □

Secondly, there are some semigroups of contractions which give rise to orbits whose real coordinates become dense in fractal sets.

Proposition 4.2. *Let Σ be the multiplicative semigroup generated by $1/4$ and $5/64$. There are points $\alpha \in \mathbb{A}/\mathbb{Q}$ with the property that*

$$\overline{\{(\sigma\alpha)_\infty \in [0, 1) : \sigma \in \Sigma\}}$$

is a set of Hausdorff dimension equal to $1/2$.

Proof. Define affine contractions $T_1, T_2, T_3, T_4 : [0, 1) \rightarrow [0, 1)$ by setting

$$\begin{aligned} T_1(x) &:= x/4 + 1/4, & T_2(x) &:= x/4 + 1/2 \\ T_3(x) &:= 5x/64 + 41/64, & \text{and } T_4(x) &:= 5x/64 + 9/32. \end{aligned}$$

Then there exists a unique nonempty compact set F such that

$$F = \bigcup_{i=1}^4 T_i(F)$$

(see Theorem 9.1 in [3]). Noting that

$$\begin{aligned} T_3([0, 1)) &\subset T_2([0, 1)), \\ T_4([0, 1)) &\subset T_1([0, 1)), \end{aligned}$$

we may use Hutchinson's Formula (see Theorem 9.3 in [3]) to calculate the Hausdorff dimension of the attractor F . We have that $\dim_{\text{H}}(F) = s$, where $4^{-s} + 4^{-s} = 1$, that is, $s = 1/2$.

Let $x \in [0, 1)$ be a point whose orbit under the collection of maps T_i is dense in F , and choose $\alpha \in \mathbb{A}/\mathbb{Q}$ to be any point in \mathcal{F} with $\alpha_{\infty} = x$ and $\alpha_2 = 1/3$. The action of multiplying α (or any point in \mathcal{F} with 2-adic coordinate $1/3$) by $1/4$, and then translating back to \mathcal{F} , corresponds to the map T_1 , extended coordinate-wise to \mathbb{A}/\mathbb{Q} . Also, the 2-adic coordinate of the image is $1/3$.

Similarly, multiplication of any point in \mathcal{F} with 2-adic coordinate $1/3$ by $5/64$ corresponds to the map T_3 , and the 2-adic coordinate of the image is $2/3$. The same argument applies to points with 2-adic coordinates $2/3$, and the resulting maps correspond to T_2 and T_4 . Therefore the real part of the closure of the orbit of α under the semigroup Σ equals F . \square

Finally, if Σ is generated by finitely many integers, then there will be other types of infinite nondense orbits (of points α with $\alpha_{\infty} \in \mathbb{Q}$). For example the orbit of any point $\alpha \in \mathbb{A}/\mathbb{Q}$ with $\alpha_{\infty} = 2/7$ and $\alpha_p \neq 2/7$ for any p under the action of the semigroup generated by 8 and 729 will be infinite, but all points in the closure will have real coordinate $2/7$. We point this out only in order to emphasize that the correct interpretation must be applied when using Berend's hypotheses, for example in [1, Theorems II.1, IV.1].

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