

Doubling construction of Calabi–Yau threefolds

Mamoru Doi and Naoto Yotsutani

ABSTRACT. We give a differential-geometric construction and examples of Calabi–Yau threefolds, at least one of which is *new*. Ingredients in our construction are *admissible pairs*, which were dealt with by Kovalev, 2003, and further studied by Kovalev and Lee, 2011. An admissible pair (\bar{X}, D) consists of a three-dimensional compact Kähler manifold \bar{X} and a smooth anticanonical $K3$ divisor D on \bar{X} . If two admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) satisfy the *gluing condition*, we can glue $\bar{X}_1 \setminus D_1$ and $\bar{X}_2 \setminus D_2$ together to obtain a Calabi–Yau threefold M . In particular, if (\bar{X}_1, D_1) and (\bar{X}_2, D_2) are identical to an admissible pair (\bar{X}, D) , then the gluing condition holds automatically, so that we can *always* construct a Calabi–Yau threefold from a *single* admissible pair (\bar{X}, D) by *doubling* it. Furthermore, we can compute all Betti and Hodge numbers of the resulting Calabi–Yau threefolds in the doubling construction.

CONTENTS

1. Introduction	1204
2. Geometry of G_2 -structures	1206
3. The gluing procedure	1208
3.1. Compact complex manifolds with an anticanonical divisor	1208
3.2. Admissible pairs and asymptotically cylindrical Ricci-flat Kähler manifolds	1210
3.3. Gluing admissible pairs	1212
3.4. Gluing construction of Calabi–Yau threefolds	1215
4. Betti numbers of the resulting Calabi–Yau threefolds	1217
5. Two types of admissible pairs	1220
5.1. Fano type	1220
5.2. Nonsymplectic type	1221
6. Appendix: The list of the resulting Calabi–Yau threefolds	1225

Received July 21, 2013.

2010 *Mathematics Subject Classification*. Primary: 53C25, Secondary: 14J32.

Key words and phrases. Ricci-flat metrics, Calabi–Yau manifolds, G_2 -structures, gluing, doubling.

The second author is partially supported by the China Postdoctoral Science Foundation Grant, No. 2011M501045 and the Chinese Academy of Sciences Fellowships for Young International Scientists 2011Y1JB05.

6.1. All possible Calabi–Yau threefolds from Fano type	1226
6.2. All possible Calabi–Yau threefolds from nonsymplectic type	1226
6.3. Graphical chart of our examples	1232
References	1233

1. Introduction

The purpose of this paper is to give a gluing construction and examples of Calabi–Yau threefolds. Before going into details, we recall some historical background behind our gluing construction.

The gluing technique is used in constructing many compact manifolds with a special geometric structure. In particular, it is effectively used in constructing compact manifolds with exceptional holonomy groups G_2 and $\text{Spin}(7)$, which are also called compact G_2 - and $\text{Spin}(7)$ -manifolds respectively. The first examples of compact G_2 - and $\text{Spin}(7)$ -manifolds were obtained by Joyce using Kummer-type constructions in a series of his papers [10, 11, 12]. Also, Joyce gave a second construction of compact $\text{Spin}(7)$ -manifolds using compact four-dimensional Kähler orbifolds with an anti-holomorphic involution. These constructions are based on the resolution of certain singularities by replacing neighborhoods of singularities with ALE-type manifolds. Later, Clancy studied in [5] such compact Kähler orbifolds systematically and constructed more new examples of compact $\text{Spin}(7)$ -manifolds using Joyce’s second construction.

On the other hand, Kovalev gave another construction of compact G_2 -manifolds in [15]. Beginning with a Fano threefold V with a smooth anticanonical $K3$ divisor D , he showed that if we blow up V along a curve representing $D \cdot D$ to obtain \bar{X} , then \bar{X} has an anticanonical divisor isomorphic to D (denoted by D again) with the holomorphic normal bundle $N_{D/\bar{X}}$ trivial. Then $\bar{X} \setminus D$ admits an asymptotically cylindrical Ricci-flat Kähler metric. (We call such (\bar{X}, D) an *admissible pair of Fano type*.) Also, Kovalev proved that if two admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) satisfy a certain condition called the *matching condition*, we can glue together $(\bar{X}_1 \setminus D_1) \times S^1$ and $(\bar{X}_2 \setminus D_2) \times S^1$ along their cylindrical ends in a *twisted* manner to obtain a compact G_2 -manifold. In this construction, Kovalev found many new examples of G_2 -manifolds using the classification of Fano threefolds by Mori and Mukai [19, 20]. Later, Kovalev and Lee [16] found admissible pairs of another type (which are said to be *admissible pairs of nonsymplectic type*) and constructed new examples of compact G_2 -manifolds. They used the classification of $K3$ surfaces with a nonsymplectic involution by Nikulin [22].

In our construction, we begin with two admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) as above. Then each $(\bar{X}_i \setminus D_i) \times S^1$ has a natural asymptotically

cylindrical torsion-free G_2 -structure $\varphi_{i,\text{cyl}}$ using the existence result of an asymptotically cylindrical Ricci-flat Kähler form on $\bar{X}_i \setminus D_i$. Now suppose $\bar{X}_1 \setminus D_1$ and $\bar{X}_2 \setminus D_2$ have the same *asymptotic model*, which is ensured by the *gluing condition* defined later. Then as in Kovalev's construction, we can glue together $(\bar{X}_1 \setminus D_1) \times S^1$ and $(\bar{X}_2 \setminus D_2) \times S^1$, but in a *non-twisted* manner to obtain $M_T \times S^1$. In short, we glue together $\bar{X}_1 \setminus D_1$ and $\bar{X}_2 \setminus D_2$ along their cylindrical ends $D_1 \times S^1 \times (T-1, T+1)$ and $D_2 \times S^1 \times (T-1, T+1)$, and then take the product with S^1 . Moreover, we can glue together torsion-free G_2 -structures to construct a d-closed G_2 -structure φ_T on $M_T \times S^1$. Using the analysis on torsion-free G_2 -structures, we shall prove that φ_T can be deformed into a torsion-free G_2 -structure for sufficiently large T , so that the resulting compact manifold $M_T \times S^1$ admits a Riemannian metric with holonomy contained in G_2 . But if $M = M_T$ is simply-connected, then M must have holonomy $\text{SU}(3)$ according to the Berger-Simons classification of holonomy groups of Ricci-flat Riemannian manifolds. Hence this M is a Calabi-Yau threefold.

For two given admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) , it is difficult to check in general whether the gluing condition holds or not. However, if (\bar{X}_1, D_1) and (\bar{X}_2, D_2) are identical to an admissible pair (\bar{X}, D) , then the gluing condition holds automatically. Therefore we can *always* construct a Calabi-Yau threefold from a *single* admissible pair (\bar{X}, D) by *doubling* it.

Our doubling construction has another advantage in computing Betti and Hodge numbers of the resulting Calabi-Yau threefolds M . To compute Betti numbers of M , it is necessary to find out the intersection of the images of the homomorphisms $H^2(X_i, \mathbb{R}) \rightarrow H^2(D_i, \mathbb{R})$ for $i = 1, 2$ induced by the inclusion $D_i \times S^1 \rightarrow X_i$, where we denote $X_i = \bar{X}_i \setminus D_i$. In the doubling construction, the above two homomorphisms are identical, and the intersection of their images is the same as each one.

With this construction, we shall give 123 topologically distinct Calabi-Yau threefolds (59 examples from admissible pairs of Fano type and 64 from those of nonsymplectic type). Moreover, 54 of the Calabi-Yau threefolds from admissible pairs of nonsymplectic type form mirror pairs (24 mirror pairs and 6 self mirrors). In a word, we construct Calabi-Yau threefolds and their mirrors from $K3$ surfaces. This construction was previously investigated by Borcea and Voisin [3] using algebro-geometric methods. Thus, our doubling construction from nonsymplectic type can be interpreted as a differential-geometric analogue of the Borcea-Voisin construction. Furthermore, the remaining 10 examples from nonsymplectic type contain at least one new example. See 'Discussion' in Section 6.2 for more details. Meanwhile, 59 examples from admissible pairs of Fano type are essentially the same Calabi-Yau threefolds constructed by Kawamata and Namikawa [14] and later developed by Lee [18] using normal crossing varieties. Hence our construction from Fano type provides a differential-geometric interpretation of Lee's construction [18].

This paper is organized as follows. Section 2 is a brief review of G_2 -structures. In Section 3 we establish our gluing construction of Calabi–Yau threefolds from admissible pairs. The rest of the paper is devoted to constructing examples and computing Betti and Hodge numbers of Calabi–Yau threefolds obtained in our doubling construction. The reader who is not familiar with analysis can check Definition 3.6 of admissible pairs, go to Section 3.4 where the gluing theorems are stated, and then proceed to Section 4, skipping Section 2 and the rest of Section 3. In Section 4 we will find a formula for computing Betti numbers of the resulting Calabi–Yau threefolds M in our doubling construction. In Section 5, we recall two types of admissible pairs and rewrite the formula given in Section 4 to obtain formulas of Betti and Hodge numbers of M in terms of certain invariants which characterize admissible pairs. Then the last section lists all data of the Calabi–Yau threefolds obtained in our construction.

The first author is mainly responsible for Sections 1–3, and the second author mainly for Sections 4–6.

Acknowledgements. This joint work was partially developed when the second author attended “2012 Complex Geometry and Symplectic Geometry Conference” which was held at Ningbo University, Zhejiang Province, China in July, 2012. He thanks Professor Xiuxiong Chen for an invitation to give a talk in the conference. Also, he thanks Professor Yuguang Zhang for a helpful comment during the conference. The authors thank Dr. Nam-Hoon Lee for pointing out that our construction partially covers the Borcea–Voisin method [3]. Also, they would like to thank the referee and Dr. Shintaro Kuroki for valuable comments. Especially, the referee pointed out a gap of the proof of Lemma 3.14 and advised them how to correct it, giving a good reference [29].

2. Geometry of G_2 -structures

Here we shall recall some basic facts about G_2 -structures on oriented 7-manifolds. For more details, see Joyce’s book [13].

We begin with the definition of G_2 -structures on oriented vector spaces of dimension 7.

Definition 2.1. Let V be an oriented real vector space of dimension 7. Let $\{\theta^1, \dots, \theta^7\}$ be an oriented basis of V . Set

$$(2.1) \quad \begin{aligned} \varphi_0 &= \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}, \\ g_0 &= \sum_{i=1}^7 \theta^i \otimes \theta^i, \end{aligned}$$

where $\theta^{ij\dots k} = \theta^i \wedge \theta^j \wedge \dots \wedge \theta^k$. Define the $GL_+(V)$ -orbit spaces

$$\begin{aligned} \mathcal{P}^3(V) &= \{ a^* \varphi_0 \mid a \in GL_+(V) \}, \\ \mathcal{M}et(V) &= \{ a^* \mathbf{g}_0 \mid a \in GL_+(V) \}. \end{aligned}$$

We call $\mathcal{P}^3(V)$ the set of *positive 3-forms* (also called the set of G_2 -structures or *associative 3-forms*) on V . On the other hand, $\mathcal{M}et(V)$ is the set of positive-definite inner products on V , which is also a homogeneous space isomorphic to $GL_+(V)/SO(V)$, where $SO(V)$ is defined by

$$SO(V) = \{ a \in GL_+(V) \mid a^* \mathbf{g}_0 = \mathbf{g}_0 \}.$$

Now the group G_2 is defined as the isotropy of the action of $GL(V)$ (in place of $GL_+(V)$) on $\mathcal{P}^3(V)$ at φ_0 :

$$G_2 = \{ a \in GL(V) \mid a^* \varphi_0 = \varphi_0 \}.$$

Then one can show that G_2 is a compact Lie group of dimension 14 which is a Lie subgroup of $SO(V)$ [7]. Thus we have a natural projection

$$(2.2) \quad \mathcal{P}^3(V) \cong GL_+(V)/G_2 \longrightarrow GL_+(V)/SO(V) \cong \mathcal{M}et(V),$$

so that each positive 3-form (or G_2 -structure) $\varphi \in \mathcal{P}^3(V)$ defines a positive-definite inner product $\mathbf{g}_\varphi \in \mathcal{M}et(V)$ on V . In particular, (2.2) maps φ_0 to \mathbf{g}_0 in (2.1). Note that both $\mathcal{P}^3(V)$ and $\mathcal{M}et(V)$ depend only on the orientation of V and are independent of the choice of an oriented basis $\{\theta^1, \dots, \theta^7\}$, and so is the map (2.2). Note also that

$$\dim_{\mathbb{R}} \mathcal{P}^3(V) = \dim_{\mathbb{R}} GL_+(V) - \dim_{\mathbb{R}} G_2 = 7^2 - 14 = 35,$$

which is the same as $\dim_{\mathbb{R}} \wedge^3 V$. This implies that $\mathcal{P}^3(V)$ is an *open* subset of $\wedge^3 V$. The following lemma is immediate.

Lemma 2.2. *There exists a constant $\rho_* > 0$ such that for any $\varphi \in \mathcal{P}^3(V)$, if $\tilde{\varphi} \in \wedge^3 V$ satisfies $|\tilde{\varphi} - \varphi|_{\mathbf{g}_\varphi} < \rho_*$, then $\tilde{\varphi} \in \mathcal{P}^3(V)$.*

Remark 2.3. Here is an alternative definition of G_2 -structures. But the reader can skip the following. Let V be an oriented real vector space of dimension 7 with orientation μ_0 . Let $\Omega \in \wedge^7 V^*$ be a volume form which is positive with respect to the orientation μ_0 . Then $\varphi \in \wedge^3 V^*$ is a positive 3-form on V if an inner product $\mathbf{g}_{\Omega, \varphi}$ given by

$$\iota_u \varphi \wedge \iota_v \varphi \wedge \varphi = 6 \mathbf{g}_{\Omega, \varphi}(u, v) \Omega \quad \text{for } u, v \in V$$

is positive-definite, where ι_u denotes interior product by a vector $u \in V$, from which comes the name ‘positive form’. Whether φ is a positive 3-form depends only on the orientation μ_0 of V , and is independent of the choice of a positive volume form Ω . One can show that if φ is a positive 3-form on (V, μ_0) , then there exists a unique positive-definite inner product \mathbf{g}_φ such that

$$\iota_u \varphi \wedge \iota_v \varphi \wedge \varphi = 6 \mathbf{g}_\varphi(u, v) \text{vol}_{\mathbf{g}_\varphi} \quad \text{for } u, v \in V,$$

where vol_φ is a volume form determined by g_φ and μ_0 . The map $\varphi \mapsto g_\varphi$ gives (2.2) explicitly. One can also prove that there exists an orthogonal basis $\{\theta^1, \dots, \theta^7\}$ with respect to g_φ such that φ and g_φ are written in the same form as φ_0 and g_0 in (2.1).

Now we define G_2 -structures on oriented 7-manifolds.

Definition 2.4. Let M be an oriented 7-manifold. We define $\mathcal{P}^3(M) \rightarrow M$ to be the fiber bundle whose fiber over x is $\mathcal{P}^3(T_x^*M) \subset \wedge^3 T_x^*M$. Then $\varphi \in C^\infty(\wedge^3 T^*M)$ is a *positive 3-form* (also an *associative 3-form* or a G_2 -structure) on M if $\varphi \in C^\infty(\mathcal{P}^3(M))$, i.e., φ is a smooth section of $\mathcal{P}^3(M)$. If φ is a G_2 -structure on M , then φ induces a Riemannian metric g_φ since each $\varphi|_x$ for $x \in M$ induces a positive-definite inner product $g_{\varphi|_x}$ on T_xM . A G_2 -structure φ on M is said to be *torsion-free* if it is parallel with respect to the induced Riemannian metric g_φ , i.e., $\nabla_{g_\varphi}\varphi = 0$, where ∇_{g_φ} is the Levi-Civita connection of g_φ .

Lemma 2.5. Let ρ_* be the constant given in Lemma 2.2. For any $\varphi \in \mathcal{P}^3(M)$, if $\tilde{\varphi} \in C^\infty(\wedge^3 T^*M)$ satisfies $\|\tilde{\varphi} - \varphi\|_{C^0} < \rho_*$, then $\tilde{\varphi} \in \mathcal{P}^3(M)$, where $\|\cdot\|_{C^0}$ is measured using the metric g_φ on M .

The following result is one of the most important results in the geometry of the exceptional holonomy group G_2 , relating the holonomy contained in G_2 with the d- and d*-closedness of the G_2 -structure.

Theorem 2.6 (Salamon [26], Lemma 11.5). *Let M be an oriented 7-manifold. Let φ be a G_2 -structure on M and g_φ the induced Riemannian metric on M . Then the following conditions are equivalent.*

- (1) φ is a torsion-free G_2 -structure, i.e., $\nabla_{g_\varphi}\varphi = 0$.
- (2) $d\varphi = d *_{g_\varphi} \varphi = 0$, where $*_{g_\varphi}$ is the Hodge star operator induced by g_φ .
- (3) $d\varphi = d^*_{g_\varphi} \varphi = 0$, where $d^*_{g_\varphi} = -*_{g_\varphi} d *_{g_\varphi}$ is the formal adjoint operator of d .
- (4) The holonomy group $\text{Hol}(g_\varphi)$ of g_φ is contained in G_2 .

3. The gluing procedure

3.1. Compact complex manifolds with an anticanonical divisor. We suppose that \bar{X} is a compact complex manifold of dimension m , and D is a smooth irreducible anticanonical divisor on \bar{X} . We recall some results in [6], Sections 3.1 and 3.2.

Lemma 3.1. *Let \bar{X} be a compact complex manifold of dimension m and D a smooth irreducible anticanonical divisor on \bar{X} . Then there exists a local coordinate system $\{U_\alpha, (z_\alpha^1, \dots, z_\alpha^{m-1}, w_\alpha)\}$ on \bar{X} such that*

- (i) w_α is a local defining function of D on U_α , i.e., $D \cap U_\alpha = \{w_\alpha = 0\}$.
- (ii) The m -forms $\Omega_\alpha = \frac{dw_\alpha}{w_\alpha} \wedge dz_\alpha^1 \wedge \dots \wedge dz_\alpha^{m-1}$ on U_α together yield a holomorphic volume form Ω on $X = \bar{X} \setminus D$.

Next we shall see that $X = \bar{X} \setminus D$ is a cylindrical manifold whose structure is induced from the holomorphic normal bundle $N = N_{D/\bar{X}}$ to D in \bar{X} , where the definition of cylindrical manifolds is given as follows.

Definition 3.2. Let X be a noncompact differentiable manifold of dimension n . Then X is called a *cylindrical manifold* or a *manifold with a cylindrical end* if there exists a diffeomorphism

$$\pi : X \setminus X_0 \longrightarrow \Sigma \times \mathbb{R}_+ = \{ (p, t) \mid p \in \Sigma, 0 < t < \infty \}$$

for some compact submanifold X_0 of dimension n with boundary $\Sigma = \partial X_0$. Also, extending t smoothly to X so that $t \leq 0$ on $X \setminus X_0$, we call t a *cylindrical parameter* on X .

Let (x_α, y_α) be local coordinates on $V_\alpha = U_\alpha \cap D$, such that x_α is the restriction of z_α to V_α and y_α is a coordinate in the fiber direction. Then one can see easily that $dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^{m-1}$ on V_α together yield a holomorphic volume form Ω_D , which is also called the *Poincaré residue* of Ω along D . Let $\|\cdot\|$ be the norm of a Hermitian bundle metric on N . We can define a cylindrical parameter t on N by $t = -\frac{1}{2} \log \|s\|^2$ for $s \in N \setminus D$. Then the local coordinates (z_α, w_α) on X are asymptotic to the local coordinates (x_α, y_α) on $N \setminus D$ in the following sense.

Lemma 3.3. *There exists a diffeomorphism Φ from a neighborhood V of the zero section of N containing $t^{-1}(\mathbb{R}_+)$ to a tubular neighborhood of U of D in X such that Φ can be locally written as*

$$\begin{aligned} z_\alpha &= x_\alpha + O(|y_\alpha|^2) = x_\alpha + O(e^{-t}), \\ w_\alpha &= y_\alpha + O(|y_\alpha|^2) = y_\alpha + O(e^{-t}), \end{aligned}$$

where we multiply all z_α and w_α by a single constant to ensure $t^{-1}(\mathbb{R}_+) \subset V$ if necessary.

Hence X is a cylindrical manifold with the cylindrical parameter t via the diffeomorphism Φ given in the above lemma. In particular, when

$$H^0(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$$

and $N_{D/\bar{X}}$ is trivial, we have a useful coordinate system near D .

Lemma 3.4. *Let (\bar{X}, D) be as in Lemma 3.1. If $H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ and the normal bundle $N_{D/\bar{X}}$ is holomorphically trivial, then there exists an open neighborhood U_D of D and a holomorphic function w on U_D such that w is a local defining function of D on U_D . Also, we may define the cylindrical parameter t with $t^{-1}(\mathbb{R}_+) \subset U_D$ by writing the fiber coordinate y of $N_{D/\bar{X}}$ as $y = \exp(-t - \sqrt{-1}\theta)$.*

Proof. We deduce from the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow [D] \longrightarrow [D]|_D \longrightarrow 0$$

$$\parallel$$

$$N_{D/\bar{X}} \cong \mathcal{O}_D$$

the long exact sequence

$$\dots \longrightarrow H^0(\bar{X}, [D]) \longrightarrow H^0(D, N_{D/\bar{X}}) \longrightarrow H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \longrightarrow \dots$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$H^0(D, \mathcal{O}_D) \cong \mathbb{C} \qquad \qquad \qquad 0$$

Thus there exists a holomorphic section $s \in H^0(\bar{X}, [D])$ such that $s|_D \equiv 1 \in H^0(D, N_{D/\bar{X}})$. Setting $U_D = \{x \in \bar{X} \mid s(x) \neq 0\}$, we have $[D]|_{U_D} \cong \mathcal{O}_{U_D}$, so that there exists a local defining function w of D on U_D . \square

3.2. Admissible pairs and asymptotically cylindrical Ricci-flat Kähler manifolds.

Definition 3.5. Let X be a cylindrical manifold such that

$$\pi : X \setminus X_0 \longrightarrow \Sigma \times \mathbb{R}_+ = \{(p, t)\}$$

is a corresponding diffeomorphism. If g_Σ is a Riemannian metric on Σ , then it defines a cylindrical metric $g_{\text{cyl}} = g_\Sigma + dt^2$ on $\Sigma \times \mathbb{R}_+$. Then a complete Riemannian metric g on X is said to be *asymptotically cylindrical* (to $(\Sigma \times \mathbb{R}_+, g_{\text{cyl}})$) if g satisfies

$$\left| \nabla_{g_{\text{cyl}}}^j (g - g_{\text{cyl}}) \right|_{g_{\text{cyl}}} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \quad \text{for all } j \geq 0$$

for some cylindrical metric $g_{\text{cyl}} = g_\Sigma + dt^2$, where we regarded g_{cyl} as a Riemannian metric on $X \setminus X_0$ via the diffeomorphism π . Also, we call (X, g) an *asymptotically cylindrical manifold* and $(\Sigma \times \mathbb{R}_+, g_{\text{cyl}})$ the *asymptotic model* of (X, g) .

Definition 3.6. Let \bar{X} be a complex manifold and D a divisor on \bar{X} . Then (\bar{X}, D) is said to be an *admissible pair* if the following conditions hold:

- (a) \bar{X} is a compact Kähler manifold.
- (b) D is a smooth anticanonical divisor on \bar{X} .
- (c) The normal bundle $N_{\bar{X}/D}$ is trivial.
- (d) \bar{X} and $X = \bar{X} \setminus D$ are simply-connected.

From the above conditions, we see that Lemmas 3.1 and 3.4 apply to admissible pairs. Also, from conditions (a) and (b), we see that D is a compact Kähler manifold with trivial canonical bundle. In particular, if $\dim_{\mathbb{C}} \bar{X} = 3$, which case is our main concern, then D must be a $K3$ surface (and so cannot be a complex torus). Let us shortly see this. The short exact sequence $0 \longrightarrow K_{\bar{X}} \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0$ induces the long exact sequence

$$\dots \longrightarrow H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \longrightarrow H^1(D, \mathcal{O}_D) \longrightarrow H^2(\bar{X}, K_{\bar{X}}) \longrightarrow \dots$$

Here $H^2(\bar{X}, K_{\bar{X}})$ is dual to $H^1(\bar{X}, \mathcal{O}_{\bar{X}})$ by the Serre duality and

$$H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \cong H_{\bar{\partial}}^{0,1}(\bar{X})$$

vanishes from $b^1(\bar{X}) = 0$. Thus $H^1(D, \mathcal{O}_D) \cong H_{\bar{\partial}}^{0,1}(D)$ also vanishes, so that we have $b^1(D) = 0$.

Theorem 3.7 (Tian–Yau [27], Kovalev [15], Hein [8]). *Let (\bar{X}, ω') be a compact Kähler manifold and $m = \dim_{\mathbb{C}} \bar{X}$. If (\bar{X}, D) is an admissible pair, then the following is true.*

It follows from Lemmas 3.1 and 3.4, there exist a local coordinate system $(U_{D,\alpha}, (z_{\alpha}^1, \dots, z_{\alpha}^{m-1}, w))$ on a neighborhood $U_D = \cup_{\alpha} U_{D,\alpha}$ of D and a holomorphic volume form Ω on \bar{X} such that

$$(3.1) \quad \Omega = \frac{dw}{w} \wedge dz_{\alpha}^1 \wedge \dots \wedge dz_{\alpha}^{m-1} \quad \text{on } U_{D,\alpha}.$$

Let κ_D be the unique Ricci-flat Kähler form on D in the Kähler class $[\omega'|_D]$. Also let (x_{α}, y) be local coordinates of $N_{D/\bar{X}} \setminus D$ as in Section 3.1 and write y as $y = \exp(-t - \sqrt{-1}\theta)$. Now define a holomorphic volume form Ω_{cyl} and a cylindrical Ricci-flat Kähler form ω_{cyl} on $N_{D/\bar{X}} \setminus D$ by

$$(3.2) \quad \begin{aligned} \Omega_{\text{cyl}} &= \frac{dy}{y} \wedge dx_{\alpha}^1 \wedge \dots \wedge dx_{\alpha}^{m-1} = (dt + \sqrt{-1}d\theta) \wedge \Omega_D, \\ \omega_{\text{cyl}} &= \kappa_D + \frac{dy \wedge d\bar{y}}{|y|^2} = \kappa_D + dt \wedge d\theta. \end{aligned}$$

Then there exist a holomorphic volume form Ω and an asymptotically cylindrical Ricci-flat Kähler form ω on $X = \bar{X} \setminus D$ such that

$$\Omega - \Omega_{\text{cyl}} = d\zeta, \quad \omega - \omega_{\text{cyl}} = d\xi$$

for some ζ and ξ with

$$\left| \nabla_{g_{\text{cyl}}}^j \zeta \right|_{g_{\text{cyl}}} = O(e^{-\beta t}), \quad \left| \nabla_{g_{\text{cyl}}}^j \xi \right|_{g_{\text{cyl}}} = O(e^{-\beta t})$$

for all $j \geq 0$ and $0 < \beta < \min\{1/2, \sqrt{\lambda_1}\}$, where λ_1 is the first eigenvalue of the Laplacian $\Delta_{g_D + d\theta^2}$ acting on $D \times S^1$ with g_D the metric associated with κ_D .

A pair (Ω, ω) consisting of a holomorphic volume form Ω and a Ricci-flat Kähler form ω on an m -dimensional Kähler manifold normalized so that

$$\frac{\omega^m}{m!} = \frac{(\sqrt{-1})^{m^2}}{2^m} \Omega \wedge \bar{\Omega} \quad (= \text{the volume form})$$

is called a *Calabi–Yau structure*. The above theorem states that there exists a Calabi–Yau structure (Ω, ω) on X asymptotic to a cylindrical Calabi–Yau structure $(\Omega_{\text{cyl}}, \omega_{\text{cyl}})$ on $N_{D/\bar{X}} \setminus D$ if we multiply Ω by some constant.

3.3. Gluing admissible pairs. Hereafter we will only consider admissible pairs (\bar{X}, D) with $\dim_{\mathbb{C}} \bar{X} = 3$. Also, we will denote $N = N_{D/\bar{X}}$ and $X = \bar{X} \setminus D$.

3.3.1. The gluing condition. Let (\bar{X}, ω') be a three-dimensional compact Kähler manifold and (\bar{X}, D) be an admissible pair. We first define a natural torsion-free G_2 -structure on $X \times S^1$.

It follows from Theorem 3.7 that there exists a Calabi–Yau structure (Ω, ω) on X asymptotic to a cylindrical Calabi–Yau structure $(\Omega_{\text{cyl}}, \omega_{\text{cyl}})$ on $N \setminus D$, which are written as (3.1) and (3.2). We define a G_2 -structure φ on $X \times S^1$ by

$$(3.3) \quad \varphi = \omega \wedge d\theta' + \text{Im } \Omega,$$

where $\theta' \in \mathbb{R}/2\pi\mathbb{Z}$ is a coordinate on S^1 . Similarly, we define a G_2 -structure φ_{cyl} on $(N \setminus D) \times S^1$ by

$$(3.4) \quad \varphi_{\text{cyl}} = \omega_{\text{cyl}} \wedge d\theta' + \text{Im } \Omega_{\text{cyl}}.$$

The Hodge duals of φ and φ_{cyl} are computed as

$$(3.5) \quad \begin{aligned} *_{g_\varphi} \varphi &= \frac{1}{2} \omega \wedge \omega - \text{Re } \Omega \wedge d\theta', \\ *_{g_{\varphi_{\text{cyl}}}} \varphi_{\text{cyl}} &= \frac{1}{2} \omega_{\text{cyl}} \wedge \omega_{\text{cyl}} - \text{Re } \Omega_{\text{cyl}} \wedge d\theta'. \end{aligned}$$

Then we see easily from Theorem 3.7 and equations (3.3)–(3.5) that

$$(3.6) \quad \begin{aligned} \varphi - \varphi_{\text{cyl}} &= d\xi \wedge d\theta' + \text{Im } d\zeta = d\eta_1, \\ *_{g_\varphi} \varphi - *_{g_{\varphi_{\text{cyl}}}} \varphi_{\text{cyl}} &= \frac{1}{2} (\omega + \omega_{\text{cyl}}) \wedge d\xi - \text{Re } d\zeta \wedge d\theta' = d\eta_2, \\ \text{where } \eta_1 &= \xi \wedge d\theta' + \text{Im } \zeta, \\ \eta_2 &= \frac{1}{2} (\omega + \omega_{\text{cyl}}) \wedge \xi - \text{Re } \zeta \wedge d\theta'. \end{aligned}$$

Thus φ and φ_{cyl} are both torsion-free G_2 -structures, and $(X \times S^1, \varphi)$ is asymptotic to $((N \setminus D) \times S^1, \varphi_{\text{cyl}})$. Note that the cylindrical end of $X \times S^1$ is diffeomorphic to $(N \setminus D) \times S^1 \simeq D \times S^1 \times S^1 \times \mathbb{R}_+ = \{(x_\alpha, \theta, \theta', t)\}$.

Next we consider the condition under which we can glue together X_1 and X_2 obtained from admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) . For gluing X_1 and X_2 to obtain a manifold with an approximating G_2 -structure, we would like (X_1, φ_1) and (X_2, φ_2) to have the same asymptotic model. Thus we put the following

Gluing condition: There exists a diffeomorphism

$$F : D_1 \times S^1 \times S^1 \longrightarrow D_2 \times S^1 \times S^1$$

between the cross-sections of the cylindrical ends such that

$$(3.7) \quad F_T^* \varphi_{2,\text{cyl}} = \varphi_{1,\text{cyl}} \quad \text{for all } T > 0,$$

where $F_T : D_1 \times S^1 \times S^1 \times (0, 2T) \rightarrow D_2 \times S^1 \times S^1 \times (0, 2T)$ is defined by

$$F_T(x_1, \theta_1, \theta'_1, t) = (F(x_1, \theta_1, \theta'_1), 2T - t)$$

for $(x_1, \theta_1, \theta'_1, t) \in D_1 \times S^1 \times S^1 \times (0, 2T)$.

Lemma 3.8. *Suppose that there exists an isomorphism $f : D_1 \rightarrow D_2$ such that $f^*\kappa_2 = \kappa_1$, where $\kappa_i = \kappa_{D_i}$. If we define a diffeomorphism F between the cross-sections of the cylindrical ends by*

$$F_T : D_1 \times S^1 \times S^1 \longrightarrow D_2 \times S^1 \times S^1.$$

$$(x_1, \theta_1, \theta'_1) \longmapsto (x_2, \theta_2, \theta'_2) = (f(x_1), -\theta_1, \theta'_1)$$

Then the gluing condition (3.7) holds, where we change the sign of $\Omega_{2,\text{cyl}}$ (and also the sign of Ω_2 correspondingly).

Proof. It follows by a straightforward calculation using (3.2) and (3.4). \square

Remark 3.9. In the constructions of compact G_2 -manifolds by Kovalev [15] and Kovalev–Lee [16], the map $F : D_1 \times S^1 \times S^1 \rightarrow D_2 \times S^1 \times S^1$ is defined by

$$F(x_1, \theta_1, \theta'_1) = (x_2, \theta_2, \theta'_2) = (f(x_1), \theta'_1, \theta_1) \quad \text{for } (x_1, \theta_1, \theta'_1) \in D_1 \times S^1 \times S^1,$$

so that F twists the two S^1 factors. Then in order for the gluing condition (3.7) to hold, the isomorphism $f : D_1 \rightarrow D_2$ between $K3$ surfaces must satisfy

$$f^*\kappa_2^I = -\kappa_1^J, \quad f^*\kappa_2^J = \kappa_1^I, \quad f^*\kappa_2^K = \kappa_1^K,$$

where $\kappa_i^I, \kappa_i^J, \kappa_i^K$ are defined by

$$\kappa_{D_i} = \kappa_i^I, \quad \Omega_{D_i} = \kappa_i^J + \sqrt{-1}\kappa_i^K.$$

Instead, Kovalev and Lee put a weaker condition (which they call the *matching condition*)

$$f^*[\kappa_2^I] = -[\kappa_1^J], \quad f^*[\kappa_2^J] = [\kappa_1^I], \quad f^*[\kappa_2^K] = [\kappa_1^K],$$

which is sufficient for the existence of f by the global Torelli theorem of $K3$ surfaces. Following Kovalev’s argument in [15], we can weaken the condition $f^*\kappa_2 = \kappa_1$ in Lemma 3.8 to $f^*[\kappa_2] = [\kappa_1]$.

3.3.2. Approximating G_2 -structures. Now we shall glue $X_1 \times S^1$ and $X_2 \times S^1$ under the gluing condition (3.7). Let $\rho : \mathbb{R} \rightarrow [0, 1]$ denote a cut-off function

$$\rho(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 1, \end{cases}$$

and define $\rho_T : \mathbb{R} \rightarrow [0, 1]$ by

$$(3.8) \quad \rho_T(x) = \rho(x - T + 1) = \begin{cases} 1 & \text{if } x \leq T - 1, \\ 0 & \text{if } x \geq T. \end{cases}$$

Setting an approximating Calabi–Yau structure $(\Omega_{i,T}, \omega_{i,T})$ by

$$\Omega_{i,T} = \begin{cases} \Omega_i - d(1 - \rho_{T-1})\zeta_i & \text{on } \{t \leq T - 1\}, \\ \Omega_{i,\text{cyl}} + d\rho_{T-1}\zeta_i & \text{on } \{t \geq T - 2\} \end{cases}$$

and similarly

$$\omega_{i,T} = \begin{cases} \omega_i - d(1 - \rho_{T-1})\xi_i & \text{on } \{t \leq T - 1\}, \\ \omega_{i,\text{cyl}} + d\rho_{T-1}\xi_i & \text{on } \{t \geq T - 2\}, \end{cases}$$

we can define a d-closed (but not necessarily d*-closed) G_2 -structure $\varphi_{i,T}$ on each $X_i \times S^1$ by

$$\varphi_{i,T} = \omega_{i,T} \wedge d\theta'_i + \text{Im } \Omega_{i,T}.$$

Note that $\varphi_{i,T}$ satisfies

$$\varphi_{i,T} = \begin{cases} \varphi_i & \text{on } \{t < T - 2\}, \\ \varphi_{i,\text{cyl}} & \text{on } \{t > T - 1\} \end{cases}$$

and that

$$(3.9) \quad |\varphi_{i,T} - \varphi_{i,\text{cyl}}|_{g_{\varphi_{i,\text{cyl}}}} = O(e^{-\beta T}) \quad \text{for all } 0 < \beta < \min\{1/2, \sqrt{\lambda_1}\}.$$

Let $X_{1,T} = \{t_1 < T + 1\} \subset X_1$ and $X_{2,T} = \{t_2 < T + 1\} \subset X_2$. We glue $X_{1,T} \times S^1$ and $X_{2,T} \times S^1$ along $D_1 \times S^1 \times \{T - 1 < t_1 < T + 1\} \times S^1 \subset X_{1,T} \times S^1$ and $D_2 \times S^1 \times \{T - 1 < t_2 < T + 1\} \times S^1 \subset X_{2,T} \times S^1$ to construct a compact 7-manifold $M_T \times S^1$ using the gluing map F_T (more precisely, $\tilde{F}_T = (\Phi_2, \text{id}_{S^1}) \circ F_T \circ (\Phi_1^{-1}, \text{id}_{S^1})$, where Φ_1 and Φ_2 are the diffeomorphisms given in Lemma 3.3). Also, we can glue together $\varphi_{1,T}$ and $\varphi_{2,T}$ to obtain a 3-form φ_T on M_T . It follows from Lemma 2.5 and (3.9) that there exists $T_* > 0$ such that $\varphi_T \in \mathcal{P}^3(M_T \times S^1)$ for all T with $T > T_*$, so that the Hodge star operator $* = *_{g_{\varphi_T}}$ is well-defined. Thus we can define a 3-form ψ_T on $M_T \times S^1$ with $d^*\varphi_T = d^*\psi_T$ by

$$(3.10) \quad *\psi_T = *\varphi_T - \left(\frac{1}{2} \omega_T \wedge \omega_T - \text{Re } \Omega_T \wedge d\theta' \right).$$

Proposition 3.10. *There exist constants $A_{p,k,\beta}$ independent of T such that for $\beta \in (0, \{1/2, \sqrt{\lambda_1}\})$ we have*

$$\|\psi_T\|_{L_k^p} \leq A_{p,k,\beta} e^{-\beta T},$$

where all norms are measured using g_{φ_T} .

Proof. These estimates follow in a straightforward way from Theorem 3.7 and equation (3.6) by arguments similar to those in [6], Section 3.5. \square

3.4. Gluing construction of Calabi–Yau threefolds. Here we give the main theorems for constructing Calabi–Yau threefolds.

Theorem 3.11. *Let (\bar{X}_1, ω'_1) and (\bar{X}_2, ω'_2) be compact Kähler manifold with $\dim_{\mathbb{C}} \bar{X}_i = 3$ such that (\bar{X}_1, D_1) and (\bar{X}_2, D_2) are admissible pairs. Suppose there exists an isomorphism $f : D_1 \rightarrow D_2$ such that $f^* \kappa_2 = \kappa_1$, where κ_i is the unique Ricci-flat Kähler form on D_i in the Kähler class $[\omega'_i|_{D_i}]$. Then we can glue together X_1 and X_2 along their cylindrical ends to obtain a compact manifold M . The manifold M is a Calabi–Yau threefold, i.e., $b^1(M) = 0$ and M admits a Ricci-flat Kähler metric.*

Corollary 3.12. *Let (\bar{X}, D) be an admissible pair with $\dim_{\mathbb{C}} \bar{X} = 3$. Then we can glue two copies of X along their cylindrical ends to obtain a compact manifold M . The manifold M is a Calabi–Yau threefold.*

Remark 3.13. As stated in Remark 3.9, the condition $f^* \kappa_2 = \kappa_1$ in Theorem 3.11 can be weakened to $f^*[\kappa_2] = [\kappa_1]$ using Kovalev’s argument in [15]. But we don’t go into details here because we don’t need the weaker condition for getting Corollary 3.12 from Theorem 3.11.

Proof of Theorem 3.11. We shall prove the existence of a torsion-free G_2 -structure on $M_T \times S^1$ constructed in Section 3.3 for sufficiently large T . Then $M = M_T$ will be the desired Calabi–Yau threefold according to the following

Lemma 3.14. *If $M \times S^1$ admits a torsion-free G_2 -structure, then M admits a Ricci-flat Kähler metric.*

Proof. Since both X_1 and X_2 are simply-connected by Definition 3.6(d), the resulting manifold $M = M_T$ is also simply-connected. Let us consider a Riemannian metric on $M \times S^1$ with holonomy contained in G_2 , which is induced by a torsion-free G_2 -structure. Then by the Cheeger–Gromoll splitting theorem (see e.g. Besse [2], Corollary 6.67), the universal Riemannian covering of $M \times S^1$ is isometric to a product Riemannian manifold $N \times \mathbb{R}^q$ with holonomy contained in G_2 for some q , where N is a simply-connected $(7 - q)$ -manifold and \mathbb{R}^q has a flat metric. Meanwhile, the natural map $M \times \mathbb{R} \rightarrow M \times S^1$ is also the universal covering. By the uniqueness of the universal covering, we have a diffeomorphism $\phi : M \times \mathbb{R} \rightarrow N \times \mathbb{R}^q$, so that $q = 1$ and N is 6-dimensional. Since the flat metric on \mathbb{R} does not contribute to the holonomy of $N \times \mathbb{R}$, N itself has holonomy contained in G_2 . But the holonomy group of a simply-connected Riemannian 6-manifold is at most $\text{SO}(6)$, and so it must be contained in $\text{SO}(6) \cap G_2 = \text{SU}(3)$. Thus N admits a Ricci-flat Kähler metric.

Now we shall prove that N is indeed diffeomorphic to M . For this purpose, we use the classification of closed, oriented simply-connected 6-manifolds by Wall, Jupp and Zhubr (see the website of the Manifold Atlas Project, 6-manifolds: 1-connected [29] for a good overview which includes further references). Then we see that M and N are diffeomorphic if there is an

isomorphism between the cohomology rings $H^*(M)$ and $H^*(N)$ preserving the second Stiefel–Whitney classes w_2 and the first Pontrjagin classes p_1 (the rest of the invariants are completely determined by the cohomology rings). Such a ring isomorphism is induced by the diffeomorphism $\phi : M \times \mathbb{R} \rightarrow N \times \mathbb{R}$ via the composition

$$H^*(N) \cong H^*(N \times \mathbb{R}) \xrightarrow{\phi^*} H^*(M \times \mathbb{R}) \cong H^*(M).$$

This proves that N is diffeomorphic to M , and hence M admits a Ricci-flat Kähler metric. □

Now it remains to prove the existence of a torsion-free G_2 -structure on $M_T \times S^1$ for sufficiently large T . We recall the following result which reduces the existence of a torsion-free G_2 -structure to the solvability of a nonlinear partial differential equation.

Theorem 3.15 (Joyce [13], Theorem 10.3.7). *Let φ be a G_2 -structure on a compact 7-manifold M' with $d\varphi = 0$. Suppose η is a 2-form on M' with $\|d\eta\|_{C^0} \leq \epsilon_1$, and ψ is a 3-form on M' with $d^*\psi = d^*\varphi$ and $\|\psi\|_{C^0} \leq \epsilon_1$, where ϵ_1 is a constant independent of the 7-manifold M' with $\epsilon_1 \leq \rho_*$. Let η satisfy the nonlinear elliptic partial differential equation*

$$(3.11) \quad (dd^* + d^*d)\eta = d^* \left(1 + \frac{1}{3} \langle d\eta, \varphi \rangle_{g_\varphi} \right) \psi + *dF(d\eta).$$

Here F is a smooth function from the closed ball of radius ϵ_1 in $\wedge^3 T^*M'$ to $\wedge^4 T^*M'$ with $F(0) = 0$, and if $\chi, \xi \in C^\infty(\wedge^3 T^*M')$ and $|\chi|, |\xi| \leq \epsilon_1$, then we have the quadratic estimates

$$\begin{aligned} |F(\chi) - F(\xi)| &\leq \epsilon_2 |\chi - \xi| (|\chi| + |\xi|), \\ |d(F(\chi) - F(\xi))| &\leq \epsilon_3 \{ |\chi - \xi| (|\chi| + |\xi|) |d^*\varphi| \\ &\quad + |\nabla(\chi - \xi)| (|\chi| + |\xi|) + |\chi - \xi| (|\nabla\chi| + |\nabla\xi|) \} \end{aligned}$$

for some constants ϵ_2, ϵ_3 independent of M' . Then $\tilde{\varphi} = \varphi + d\eta$ is a torsion-free G_2 -structure on M' .

To solve (3.11) in our construction, we use the following gluing theorem based on the analysis of Kovalev and Singer [17].

Theorem 3.16 (Kovalev [15], Theorem 5.34). *Let $\varphi = \varphi_T, \psi = \psi_T$ and $M' = M_T \times S^1$ be as constructed in Section 3.3.2, with $d^*\psi_T = d^*\varphi_T$ and the estimates in Proposition 3.10. Then there exists $T_0 > 0$ such that the following is true.*

For each $T > T_0$, there exists a unique smooth 2-form η_T on $M_T \times S^1$ with $\|\eta_T\|_{L^2_p} \leq B_{p,\beta} e^{-\beta T}$ and $\|\eta_T\|_{C^1} \leq C_\beta e^{-\beta T}$ for any $\beta \in (0, \max\{1/2, \sqrt{\lambda_1}\})$ such that $\eta = \eta_T$ satisfies equation (3.11), where $B_{p,\beta}$ and C_β are independent of T .

Proof. The assertion is proved in [15] when $d(\bar{X}_1) = 0$ or $d(\bar{X}_2) = 0$, where $d(\bar{X}_j)$ is the dimension of the kernel of $\iota_j : H^2(X_j, \mathbb{R}) \rightarrow H^2(D_j, \mathbb{R})$ defined in Section 4. This condition applies to admissible pairs of Fano type, but not to ones of nonsymplectic type (see also the proof of Proposition 5.38 in [15] and the remarks after Lemma 2.6 in [16], p. 199). However, the above theorem is still valid in the nonsymplectic case, by a direct application of Kovalev–Singer [17], Proposition 4.2. \square

Applying Theorem 3.16 to Theorem 3.15, we see that $\tilde{\varphi}_T = \varphi_T + d\eta_T$ yields a torsion-free G_2 -structure on $M_T \times S^1$ for sufficiently large T . Combined with Lemma 3.14, this completes the proof of Theorem 3.11. \square

Remark 3.17. In the proof of Theorem 3.11, to solve equation (3.11) given in Theorem 3.15 we may also use Joyce’s book [13], Theorem 11.6.1, where we need uniform bounds of the injectivity radius and Riemann curvature of $M_T \times S^1$ from below and above respectively. Obviously, we have such bounds because X_1 and X_2 are cylindrical manifolds with an asymptotically cylindrical metric.

4. Betti numbers of the resulting Calabi–Yau threefolds

We shall compute Betti numbers of the Calabi–Yau threefolds M obtained in the doubling construction given in Corollary 3.12. Also, we shall see that the Betti numbers of M are completely determined by those of the compact Kähler threefolds \bar{X} .

In our doubling construction, we take two copies (\bar{X}_j, D_j) of an admissible pair (\bar{X}, D) for $j = 1, 2$. Let $X_j = \bar{X}_j \setminus D_j$. We consider a homomorphism

$$(4.1) \quad \iota_j : H^2(X_j, \mathbb{R}) \rightarrow H^2(D_j \times S^1, \mathbb{R}) \xrightarrow{\cong} H^2(D_j, \mathbb{R}),$$

where the first map is induced by the embedding $D_j \times S^1 \rightarrow X_j$ and the second comes from the Künneth theorem. Set $d = d_j = d(\bar{X}_j) = \dim_{\mathbb{R}} \text{Ker } \iota_j$. It is readily seen that

$$(4.2) \quad \dim_{\mathbb{R}} \text{Im } \iota_j = b^2(X) - d.$$

The following formula seems to be well-known for compact Kähler threefolds (see [18], Corollary 8.2).

Proposition 4.1. *Let (\bar{X}_j, D_j) be two copies of an admissible pair (\bar{X}, D) for $j = 1, 2$ and let d be as above. Then the Calabi–Yau threefold M obtained by the doubling construction in Corollary 3.12 has Betti numbers*

$$(4.3) \quad \begin{cases} b^1(M) = 0, \\ b^2(M) = b^2(\bar{X}) + d, \\ b^3(M) = 2(b^3(\bar{X}) + 23 + d - b^2(\bar{X})). \end{cases}$$

Also, the Euler characteristic $\chi(M)$ is given by

$$\chi(M) = 2(\chi(\bar{X}) - \chi(D)).$$

Proof. Obviously, the second statement holds for our construction. Now we restrict ourselves to find the second and third Betti numbers of M because M is simply-connected. Since the normal bundle N_{D_j/\bar{X}_j} is trivial in our assumption, there is a tubular neighborhood U_j of D_j in \bar{X}_j such that

$$(4.4) \quad \bar{X}_j = X_j \cup U_j \quad \text{and} \quad X_j \cap U_j \simeq D_j \times S^1 \times \mathbb{R}_{>0}.$$

Up to a homotopy equivalence, $X_j \cap U_j \sim D_j \times S^1$ as U_j contracts to D_j . Applying the Mayer–Vietoris theorem to (4.4), we see that

$$(4.5) \quad b^2(\bar{X}) = b^2(X) + 1 \quad \text{and} \quad b^3(X) = b^3(\bar{X}) + 22 + d - b^2(X)$$

(see [16], (2.10)). We next consider homotopy equivalences

$$(4.6) \quad M \sim X_1 \cup X_2, \quad X_1 \cap X_2 \sim D \times S^1.$$

Again, let us apply the Mayer–Vietoris theorem to (4.6). Then we obtain the long exact sequence

$$(4.7) \quad 0 \rightarrow H^0(D) \xrightarrow{\delta^1} H^2(M) \xrightarrow{\alpha^2} H^2(X_1) \oplus H^2(X_2) \xrightarrow{\beta^2} H^2(D) \rightarrow \cdots .$$

Note that the map β^2 in (4.7) is given by

$$\iota_1 + f^* \iota_2 : H^2(X_1, \mathbb{R}) \oplus H^2(X_2, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R}),$$

where

$$\iota_j : H^2(X_j, \mathbb{R}) \longrightarrow H^2(D_j, \mathbb{R})$$

are homomorphisms defined in (4.1) and

$$f^* : H^2(D_2, \mathbb{R}) \longrightarrow H^2(D_1, \mathbb{R})$$

is the pullback of the identity $f : D_1 \xrightarrow{\cong} D_2$. Hence we see from (4.2) that

$$\dim_{\mathbb{R}} \text{Im}(\iota_1 + f^* \iota_2) = b^2(X) - d.$$

This yields

$$\begin{aligned} b^2(M) &= \dim_{\mathbb{R}} \text{Ker } \alpha^2 + \dim_{\mathbb{R}} \text{Im } \alpha^2 \\ &= \dim_{\mathbb{R}} \text{Im } \delta^1 + \dim_{\mathbb{R}} \text{Ker}(\iota_1 + f^* \iota_2) \\ &= 1 + 2b^2(X) - (b^2(X) - d) = b^2(\bar{X}) + d, \end{aligned}$$

where we used (4.5) for the last equality. Remark that $b^2(X_1) = b^2(X_2)$ holds for our computation. To find $b^3(M)$, we shall consider a homomorphism

$$(4.8) \quad \tau_j : H^3(X_j, \mathbb{R}) \longrightarrow H^2(D_j, \mathbb{R})$$

which is induced by the embedding $U_j \cap X_j \longrightarrow X_j$ combined with

$$X_j \cap U_j \simeq D_j \times S^1 \times \mathbb{R}_{>0} \quad \text{and} \quad H^3(D_j \times S^1, \mathbb{R}) \cong H^2(D_j, \mathbb{R}).$$

The reader should be aware of the following lemma.

Lemma 4.2 (Kovalev–Lee [16], Lemma 2.6). *Let ι_j and τ_j be homomorphisms defined in (4.1) and (4.8) respectively. Then we have the orthogonal decomposition*

$$H^2(D_j, \mathbb{R}) = \text{Im } \tau_j \oplus \text{Im } \iota_j$$

with respect to the intersection form on $H^2(D_j, \mathbb{R})$ for each $j = 1, 2$.

In an analogous way to the computation of $b^2(M)$, we apply the Mayer–Vietoris theorem to (4.6):

$$(4.9) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H^2(X_1) \oplus H^2(X_2) & \xrightarrow{\iota_1 + f^* \iota_2} & H^2(D) & \xrightarrow{\delta^2} & H^3(M) \longrightarrow \\ & & \xrightarrow{\alpha^3} & H^3(X_1) \oplus H^3(X_2) & \xrightarrow{\beta^3} & H^2(D) & \longrightarrow \cdots \end{array}$$

Similarly, the map β^3 is given by

$$\tau_1 + f^* \tau_2 : H^3(X_1) \oplus H^3(X_2) \longrightarrow H^2(D).$$

On one hand, Lemma 4.2 and (4.2) show that

$$\dim_{\mathbb{R}} \text{Im } \tau_j = 22 + d - b^2(X).$$

Hence we find that

$$(4.10) \quad \begin{aligned} \dim_{\mathbb{R}} \text{Ker}(\tau_1 + f^* \tau_2) &= b^3(X_1) + b^3(X_2) - \dim_{\mathbb{R}} \text{Im}(\tau_1 + f^* \tau_2) \\ &= 2b^3(X) - (22 + d - b^2(X)). \end{aligned}$$

On the other hand, we have the equality

$$22 = \dim_{\mathbb{R}} \text{Im } \delta^2 + \dim_{\mathbb{R}} \text{Im}(\iota_1 + f^* \iota_2)$$

by combining the well-known result on the cohomology of a $K3$ surface D with the Mayer–Vietoris long exact sequence (4.9). Then we have

$$(4.11) \quad \dim_{\mathbb{R}} \text{Ker } \alpha^3 = \dim_{\mathbb{R}} \text{Im } \delta^2 = 22 - b^2(X) + d.$$

Thus we find from (4.10) and (4.11) that

$$b^3(M) = \dim_{\mathbb{R}} \text{Ker } \alpha^3 + \dim_{\mathbb{R}} \text{Ker}(\tau_1 + f^* \tau_2) = 2b^3(X).$$

Substituting the above equation into (4.5), we obtain the assertion. □

Remark 4.3. This formula shows that the topology of the resulting Calabi–Yau threefolds M only depends on the topology of the given compact Kähler threefolds \bar{X} . Also one can determine the Hodge diamond of M from Proposition 4.1 because we already know that $h^{0,0} = h^{3,0} = 1$ and $h^{1,0} = h^{2,0} = 0$ by the well-known result on Calabi–Yau manifolds (see [13], Proposition 6.2.6).

5. Two types of admissible pairs

In this section, we will see the construction of admissible pairs (\bar{X}, D) which will be needed for obtaining Calabi–Yau threefolds in the doubling construction. There are two types of admissible pairs. One is said to be of *Fano type*, and the other of *nonsymplectic type*. We will give explicit formulas for topological invariants of the resulting Calabi–Yau threefolds from these two types of admissible pairs. For the definition of admissible pairs, see Definition 3.6.

5.1. Fano type. Admissible pairs (\bar{X}, D) are ingredients in our construction of Calabi–Yau threefolds and then it is important how to explore appropriate compact Kähler threefolds \bar{X} with an anticanonical $K3$ divisor $D \in |-K_{\bar{X}}|$. In [15], Kovalev constructed such pairs from nonsingular Fano varieties.

Theorem 5.1 (Kovalev [15]). *Let V be a Fano threefold, $D \in |-K_V|$ a $K3$ surface, and let C be a smooth curve in D representing the self-intersection class of $D \cdot D$. Let $\varpi : \bar{X} \dashrightarrow V$ be the blow-up of V along the curve C . Taking the proper transform of D under the blow-up ϖ , we still denote it by D . Then (\bar{X}, D) is an admissible pair.*

Proof. See [15], Corollary 6.43, and also Proposition 6.42. \square

An admissible pair (\bar{X}, D) given in Theorem 5.1 is said to be of *Fano type* because this pair arises from a Fano threefold V . Note that \bar{X} itself is *not* a Fano threefold in this construction.

Proposition 5.2. *Let V be a Fano threefold and (\bar{X}, D) an admissible pair of Fano type given in Theorem 5.1. Let M be the Calabi–Yau threefold constructed from two copies of (\bar{X}, D) by Corollary 3.12. Then we have*

$$\begin{cases} b^2(M) = b^2(V) + 1, \\ b^3(M) = 2(b^3(V) - K_V^3 + 24 - b^2(V)). \end{cases}$$

In particular, the cohomology of M is completely determined by the cohomology of V .

Proof. Let d be the dimension of the kernel of the homomorphism

$$\iota : H^2(X, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R})$$

as in Section 4. Then note that $d = 0$ by the Lefschetz hyperplane theorem whenever (\bar{X}, D) is of Fano type. Applying the well-known result on the cohomology of blow-ups, one can find that

$$H^2(\bar{X}) \cong H^2(V) \oplus \mathbb{R} \quad \text{and} \quad H^3(\bar{X}) \cong H^3(V) \oplus \mathbb{R}^{2g(V)},$$

where $g(V) = \frac{-K_V^3}{2} + 1$ is the genus of a Fano threefold (see [15], (8.52)). This yields

$$b^2(\bar{X}) = b^2(V) + 1 \quad \text{and} \quad b^3(\bar{X}) = b^3(V) + 2g(V).$$

Substituting this into Proposition 4.1, we can show our result. □

Remark 5.3. We have another method to compute the Euler characteristic $\chi(M)$. In fact, we can see easily that if \bar{X} is the blow-up of D along C then the Euler characteristic of \bar{X} is given by

$$\chi(\bar{X}) = \chi(V) - \chi(C) + \chi(E)$$

where E is the exceptional divisor of the blow-up ϖ . Hence we can independently compute $\chi(M)$ by

$$\begin{aligned} \chi(M) &= 2(\chi(\bar{X}) - \chi(D)) \\ &= 2(\chi(V) + \chi(C) - \chi(D)) \end{aligned}$$

because E is a $\mathbb{C}P^1$ -bundle over the smooth curve C . Since the Euler characteristic is also given by $\chi(M) = \sum_{i=0}^{\dim_{\mathbb{R}} M} (-1)^i b^i(M)$, we can check the consistency of our computations.

5.2. Nonsymplectic type. In [16], Kovalev and Lee gave a large class of admissible pairs (\bar{X}, D) from $K3$ surfaces S with a nonsymplectic involution ρ . They also used the classification result of $K3$ surfaces (S, ρ) due to Nikulin [21, 22, 23] for obtaining new examples of compact irreducible G_2 -manifolds. Next we will give a quick review on this construction. For more details, see [16], Section 4.

5.2.1. $K3$ surfaces with a nonsymplectic involution. Let S be a $K3$ surface. Then the vector space $H^{2,0}(S)$ is spanned by a holomorphic volume form Ω , which is unique up to multiplication of a constant. An automorphism ρ of S is said to be *nonsymplectic* if its action on $H^{2,0}(S)$ is nontrivial. We shall consider a nonsymplectic involution:

$$\rho^2 = \text{id} \quad \text{and} \quad \rho^* \Omega = -\Omega.$$

The intersection form of S associates a lattice structure, i.e., a free abelian group of finite rank endowed with a nondegenerate integral bilinear form which is symmetric. We refer to this lattice as the *$K3$ lattice*. It is crucial that the $K3$ lattice has a nice property for a geometrical description of S . Hence we shall review some fundamental concepts of lattice theory which will be needed later.

Recall that the lattice L is said to be *hyperbolic* if the signature of L is $(1, t)$ with $t > 0$. In particular, we are interested in the case where L is *even*, i.e., the quadratic form x^2 is $2\mathbb{Z}$ -valued for any $x \in L$. We can regard L as a sublattice of $L^* = \text{Hom}(L, \mathbb{Z})$ by considering the canonical embedding $i : L \rightarrow L^*$ given by $i(x)y = \langle x, y \rangle$ for $y \in L^*$. Then L is said to be *unimodular* if the quotient group L^*/L is trivial. In general, L^*/L is a finite abelian group and is called the *discriminant group* of L . One can see that the cohomology group $H^2(S, \mathbb{Z})$ of each $K3$ surface S is a unimodular, nondegenerate, even lattice with signature $(3, 19)$. Let H and E_8 denote the

hyperbolic plane lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the root lattice associated to the root system E_8 respectively. Then $H^2(S, \mathbb{Z})$ is isomorphic to $3H \oplus 2(-E_8)$. Let us choose a marking $\phi : H^2(S, \mathbb{Z}) \rightarrow L$ of S , that is, a lattice isomorphism. It is clear that the pullback ρ^* induces an isometry of L with order 2 defined by $\phi \circ \rho^* \circ \phi^{-1}$. Hence we can consider the invariant sublattice L^ρ . Then L is said to be 2-elementary if the discriminant group of L^ρ is isomorphic to $(\mathbb{Z}_2)^a$ for some $a \in \mathbb{Z}_{\geq 0}$.

Theorem 5.4 (Nikulin [21, 22, 23]). *Let (S, ρ) be a K3 surface S with a nonsymplectic involution ρ . Then the deformation class of (S, ρ) depends only on the following triplet $(r, a, \delta) \in \mathbb{Z}^3$ given by:*

- (i) $r = \text{rank } L^\rho$.
- (ii) $(L^\rho)^*/L^\rho \cong (\mathbb{Z}_2)^a$.
- (iii) $\delta(L^\rho) = \begin{cases} 0 & \text{if } y^2 \in \mathbb{Z} \text{ for all } y \in (L^\rho)^*, \\ 1 & \text{otherwise.} \end{cases}$

5.2.2. The cohomology for nonsymplectic type. Let σ be a holomorphic involution of $\mathbb{C}P^1$ given by

$$\sigma : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1, \quad z \mapsto -z$$

in the standard local coordinates. Let G be the cyclic group of order 2 generated by $\rho \times \sigma$. Let X' be the trivial $\mathbb{C}P^1$ -bundle over S . Then the group G naturally acts on X' . Taking a point x in the fixed locus $W = (X')^G$ under the action of G , we denote the stabilizer of x as G_x . Then G_x is an endomorphism of the tangent space $T_x X'$ which satisfies $G_x \subset \text{SL}(T_x X')$. Define the quotient variety

$$Z = X'/G_x$$

and then the above condition $G_x \subset \text{SL}(T_x X')$ yields that the algebraic variety Z admits only Gorenstein quotient singularities [28]. Therefore, there is a crepant resolution $\bar{\pi} : \bar{X} \dashrightarrow Z$ due to Roan’s result (see [25], Main theorem).

Let W be the fixed locus of X' under the action of G as above. We assume that W is nonempty. In fact, this condition always holds unless $(r, a, \delta) = (10, 10, 0)$, i.e., S/ρ is an Enriques surface. Then it is known that W is the disjoint union of some rational curves. Let $\tilde{\pi} : \tilde{X} \dashrightarrow X'$ be the blow-up of $X' = S \times \mathbb{C}P^1$ along the fixed locus W . Then \tilde{X} is simply-connected as X' is simply-connected. Also, the action of G on X' lifts to the action of \tilde{G} on \tilde{X} as follows. Since we have the isomorphism

$$\tilde{X} \setminus \tilde{\pi}^{-1}(W) \cong X' \setminus W,$$

it suffices to consider the action of \tilde{G} on a point $x \in \tilde{\pi}^{-1}(W)$. Setting $g \cdot x = x$ for all $g \in \tilde{G}$ and $x \in \tilde{\pi}^{-1}(W)$, we have the lift \tilde{G} on \tilde{X} . Observe

that $\tilde{X}/\tilde{G} \cong \bar{X}$ as the quotient of the variety \tilde{X} by \tilde{G} . Summing up these arguments, we have the following commutative diagram:

$$\begin{array}{ccc}
 \tilde{G} & \overset{\text{lift}}{\curvearrowright} \tilde{X} & \xrightarrow{\tilde{f}} \bar{X} \\
 & \downarrow \tilde{\pi} & \downarrow \bar{\pi}: \text{crepant} \\
 G & \curvearrowright X' & \xrightarrow{f} Z
 \end{array}$$

where \tilde{f} (resp. f) is the quotient map with respect to \tilde{G} (resp. G). Taking a nonfixed point $z \in \mathbb{C}P^1 \setminus \{0, \infty\}$, let us define $D' = S \times \{z\}$, which is a $K3$ divisor on X' . Setting D as the image of D' in Z , we still denote by D the proper transform of D under $\bar{\pi}$. Then we can see that D is isomorphic to S . Furthermore, the normal bundle $N_{D/\bar{X}}$ is holomorphically trivial. In order to show (\bar{X}, D) is an admissible pair, we need the following three lemmas due to Kovalev and Lee [16].

Lemma 5.5 (Kovalev–Lee [16], Proposition 4.1). *\bar{X} is a compact Kähler threefold. Moreover, there exists a Kähler class $[\omega] \in H^2(\bar{X}, \mathbb{R})$ such that*

$$[\kappa] = [\omega|_D] \in H^2(D, \mathbb{R})$$

where $[\kappa]$ is a ρ -invariant Kähler class on D .

Lemma 5.6 (Kovalev–Lee [16], Lemma 4.2). *\bar{X} and $X = \bar{X} \setminus D$ are simply-connected whenever $(r, a, \delta) \neq (10, 10, 0)$.*

Although the following lemma is also stated in [16], p. 202 without a proof, we will prove it here for the reader’s convenience.

Lemma 5.7. *D is an anticanonical divisor on \bar{X} .*

Proof. To begin with, we consider the divisor $D' = S \times \{z\}$ on $X' = S \times \mathbb{C}P^1$, where $z \in \mathbb{C}P^1 \setminus \{0, \infty\}$. Let $p_1 : X' \rightarrow S$ and $p_2 : X' \rightarrow \mathbb{C}P^1$ be the canonical projections. Then we have the isomorphisms

$$K_{X'} \cong p_1^*K_S \otimes p_2^*K_{\mathbb{C}P^1} \cong p_2^*\mathcal{O}_{\mathbb{C}P^1}(-2),$$

where we used $K_S \cong \mathcal{O}_S$ for the second isomorphism. Similarly, we conclude that

$$[D'] \cong p_2^*[z] \cong p_2^*\mathcal{O}_{\mathbb{C}P^1}(1).$$

This yields

$$K_{X'} \otimes [2D'] \cong \mathcal{O}_{X'}$$

and hence $c_1(K_{X'} \otimes [2D']) = 0$. Since $H^2(Z, \mathbb{Z})$ is the G -invariant part of $H^2(X', \mathbb{Z})$, the pullback map $f^* : H^2(Z, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ is injective. Thus,

$$f^*c_1(K_Z \otimes [D]) = c_1(K_{X'} \otimes [2D']) = 0$$

implies $c_1(K_Z \otimes [D]) = 0$. We remark that

$$(5.1) \quad D \cap \text{Sing}(Z) = \emptyset$$

because $z \in \mathbb{C}P^1$ is a nonfixed point of σ . Since $\bar{\pi}$ is a crepant resolution, we have

$$\bar{\pi}^*K_Z \cong K_{\bar{X}} \quad \text{and} \quad \bar{\pi}^*[D] \cong [D]$$

by (5.1). Hence $c_1(K_Z \otimes [D]) = 0$ implies

$$c_1(K_{\bar{X}} \otimes [D]) = c_1(\bar{\pi}^*K_Z \otimes \bar{\pi}^*[D]) = \bar{\pi}^*c_1(K_Z \otimes [D]) = 0.$$

Now consider the long exact sequence

$$(5.2) \quad \cdots \longrightarrow H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \longrightarrow H^1(\bar{X}, \mathcal{O}_{\bar{X}}^*) \xrightarrow{c_1} H^2(\bar{X}, \mathbb{Z}) \longrightarrow \cdots.$$

It follows from Lemmas 5.5 and 5.6 that $H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \cong H^{0,1}(\bar{X}) = 0$. Thus the map c_1 in (5.2) is injective and so $c_1(K_{\bar{X}} \otimes [D]) = 0$ implies $K_{\bar{X}} \otimes [D] \cong \mathcal{O}_{\bar{X}}$. Hence D is an anticanonical divisor on \bar{X} . \square

Therefore the above constructed pair (\bar{X}, D) is an admissible pair, which is said to be of *nonsymplectic type* except the case of $(r, a, \delta) = (10, 10, 0)$. In order to show the main result Proposition 5.9 in this subsection, we require the following.

Proposition 5.8 (Kovalev–Lee [16], Proposition 4.3).

- (i) $h^{1,1}(\bar{X}) = b^2(\bar{X}) = 3 + 2r - a$ and $h^{1,2}(\bar{X}) = \frac{1}{2}b^3(\bar{X}) = 22 - r - a$.
- (ii) For the restriction map $\iota' : H^2(\bar{X}, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R})$ given by

$$(5.3) \quad \iota' : H^2(\bar{X}, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R}), \quad [\omega] \longmapsto [\omega|_D],$$

we have $\dim_{\mathbb{R}} \text{Im } \iota' = r$.

Proposition 5.9. *Let (S, ρ) be a K3 surface with a nonsymplectic involution ρ which is determined by a K3 invariant (r, a, δ) up to a deformation. Let (\bar{X}, D) be the admissible pair of nonsymplectic type obtained in the above construction from (S, ρ) . Let M denote the Calabi–Yau threefold constructed from two copies of (\bar{X}, D) by Corollary 3.12. Then the number of possibilities of the K3 invariants is 75. The number of topological types of (\bar{X}, D) which are distinguished by Betti or Hodge numbers is 64. Moreover, we have*

$$\begin{cases} h^{1,1}(M) = b^2(M) = 5 + 3r - 2a, \\ h^{2,1}(M) = \frac{1}{2}b^3(M) - 1 = 65 - 3r - 2a. \end{cases}$$

Proof. Recall that we set $d = \dim_{\mathbb{R}} \text{Ker } \iota$, where

$$\iota : H^2(X, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R})$$

is a homomorphism in (4.1). As in (4.3) in [16], we have

$$d = \dim_{\mathbb{R}} \text{Ker } \iota = \dim_{\mathbb{R}} \text{Ker } \iota' - 1,$$

where $\iota' : H^2(\bar{X}, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R})$ is the restriction map defined in (5.3). Since $\dim_{\mathbb{R}} \text{Im } \iota' = r$ by Proposition 5.8(ii), we conclude that

$$d = b^2(\bar{X}) - \dim_{\mathbb{R}} \text{Im } \iota' - 1 = h^{1,1}(\bar{X}) - r - 1.$$

Here we used the equality $h^{2,0}(\bar{X}) = 0$ given by Proposition 2.2 in [16]. Substituting this into (4.3) in Proposition 4.1, we have

$$(5.4) \quad \begin{cases} b^2(M) &= 2h^{1,1}(\bar{X}) - r - 1, \\ b^3(M) &= 2(2h^{2,1}(\bar{X}) + 22 - r). \end{cases}$$

In the above equation, we again used $h^{3,0}(\bar{X}) = 0$ by Proposition 2.2 in [16]. Now the result follows immediately from Proposition 5.8(i). Remark that our result is independent of the integer δ . \square

Remark 5.10. We can also compute the Hodge numbers of the resulting Calabi–Yau threefolds using the Chen-Ruan orbifold cohomology. See [24] for more details. However, Prof. Reidegeld pointed out in a private communication that there is another technical problem in the case of nonsymplectic automorphisms of order $3 \leq p \leq 19$. More precisely, the $K3$ divisors of the compact Kähler threefolds which they have constructed in [24] are in the $p/2$ -multiple of the anticanonical class. This implies that a Ricci-flat Kähler form on $X = \bar{X} \setminus D$ is not asymptotically cylindrical but asymptotically conical. Therefore, their examples of admissible pairs are not applicable to our doubling construction. However, this problem does not affect the method of calculating the Hodge numbers of the resulting Calabi–Yau threefolds, and so an analogous argument of Proposition 5.9 will work.

6. Appendix: The list of the resulting Calabi–Yau threefolds

In this section, we list all Calabi–Yau threefolds obtained in Corollary 3.12. We have the following two choices for constructing Calabi–Yau threefolds M :

- (a) We shall use admissible pairs of *Fano type*. From a Fano threefold V , we obtain an admissible pair (\bar{X}, D) by Theorem 5.1. According to the complete classification of nonsingular Fano threefolds [9, 19, 20], there are 105 algebraic families with Picard number $1 \leq \rho(V) \leq 10$. Then the number of distinct topological types of the resulting Calabi–Yau threefolds is 59 (see Tables 6.1–6.5, and also Figure 6.7 where the resulting Calabi–Yau threefolds are plotted with symbol \times).
- (b) We shall use admissible pairs of *nonsymplectic type*. Starting from a $K3$ surface S with a nonsymplectic involution ρ , we obtain an admissible pair (\bar{X}, D) as in Section 5.2. According to the classification result of (S, ρ) due to Nikulin [21, 22, 23], there are 74 algebraic families. Then the number of distinct topological types of the resulting Calabi–Yau threefolds is 64. Of these Calabi–Yau threefolds, there is at least one new example which is not diffeomorphic to the known ones (see Table 6.6, and also Figure 6.7 where the resulting Calabi–Yau threefolds are plotted with symbols \bullet and \blacksquare).

6.1. All possible Calabi–Yau threefolds from Fano type. In Tables 6.1–6.5, we hereby list the details of the resulting Calabi–Yau threefolds M from admissible pairs of Fano type. These topological invariants are computable by Proposition 5.2, and further details are left to the reader. In the tables below, $\rho = \rho(V)$ denotes the Picard number of the Fano threefold V , and $h^{1,1} = h^{1,1}(M)$, $h^{2,1} = h^{2,1}(M)$ denote the Hodge numbers.

We include the following notes:

1. No. 97 (Table 6.4) was erroneously omitted in [19]. See [20] for the correct table.
2. No. 100 (Table 6.5) is $\mathbb{C}P^1 \times S_6$ where S_6 is a del Pezzo surface of degree 6.

TABLE 6.1. Fano threefolds with $\rho = 1$.

No.	Label in [19]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
1	–	2	52	(2, 128)
2	–	4	30	(2, 86)
3	–	6	20	(2, 68)
4	–	8	14	(2, 58)
5	–	10	10	(2, 52)
6	–	12	7	(2, 48)
7	–	14	5	(2, 46)
8	–	16	3	(2, 44)
9	–	18	2	(2, 44)
10	–	22	0	(2, 44)
11	–	8	21	(2, 72)
12	–	16	10	(2, 58)
13	–	24	5	(2, 56)
14	–	32	2	(2, 58)
15	–	40	0	(2, 62)
16	–	54	0	(2, 76)
17	–	64	0	(2, 86)

6.2. All possible Calabi–Yau threefolds from nonsymplectic type.

In Table 6.6, we hereby list the details of the resulting Calabi–Yau threefolds from admissible pairs of nonsymplectic type. These Hodge numbers are also computable by Proposition 5.9 and further details are left to the reader. In the table below, there is at least *one* new example of Calabi–Yau threefolds, which is listed as the boxed number 64. We also list the number of the mirror partner for each resulting Calabi–Yau threefold in our construction. See Discussion and Section 6.3 below for more details. The symbol – on the list means that the corresponding Calabi–Yau threefold has no mirror partner in this construction.

TABLE 6.2. Fano threefolds with $\rho = 2$.

No.	Label in [19]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
18	1	4	22	(3, 69)
19	2	6	20	(3, 67)
20	3	8	11	(3, 51)
21	4	10	10	(3, 51)
22	5	12	6	(3, 45)
23	6	12	9	(3, 51)
24	7	14	5	(3, 45)
25	8	14	9	(3, 53)
26	9	16	5	(3, 47)
27	10	16	3	(3, 43)
28	11	18	5	(3, 49)
29	12	20	3	(3, 47)
30	13	20	2	(3, 45)
31	14	20	1	(3, 43)
32	15	22	4	(3, 51)
33	16	22	2	(3, 47)
34	17	24	1	(3, 47)
35	18	24	2	(3, 49)
36	19	26	2	(3, 51)
37	20	26	0	(3, 47)
38	21	28	0	(3, 49)
39	22	30	0	(3, 51)
40	23	30	1	(3, 53)
41	24	30	0	(3, 51)
42	25	32	1	(3, 55)
43	26	34	0	(3, 55)
44	27	38	0	(3, 59)
45	28	40	1	(3, 63)
46	29	40	0	(3, 61)
47	30	46	0	(3, 67)
48	31	46	0	(3, 67)
49	32	48	0	(3, 69)
50	33	54	0	(3, 75)
51	34	54	0	(3, 75)
52	35	56	0	(3, 77)
53	36	62	0	(3, 83)

TABLE 6.3. Fano threefolds with $\rho = 3$.

No.	Label in [19]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
54	1	12	8	(4, 48)
55	2	14	3	(4, 40)
56	3	18	3	(4, 44)
57	4	18	2	(4, 42)
58	5	20	0	(4, 40)
59	6	22	1	(4, 44)
60	7	24	1	(4, 46)
61	8	24	0	(4, 44)
62	9	26	3	(4, 52)
63	10	26	0	(4, 46)
64	11	28	1	(4, 50)
65	12	28	0	(4, 48)
66	13	30	0	(4, 50)
67	14	32	1	(4, 54)
68	15	32	0	(4, 52)
69	16	34	0	(4, 54)
70	17	36	0	(4, 56)
71	18	36	0	(4, 56)
72	19	38	0	(4, 58)
73	20	38	0	(4, 58)
74	21	38	0	(4, 58)
75	22	40	0	(4, 60)
76	23	42	0	(4, 62)
77	24	42	0	(4, 62)
78	25	44	0	(4, 64)
79	26	46	0	(4, 66)
80	27	48	0	(4, 68)
81	28	48	0	(4, 68)
82	29	50	0	(4, 70)
83	30	50	0	(4, 70)
84	31	52	0	(4, 72)

TABLE 6.4. Fano threefolds with $\rho = 4$.

No.	Label in [19]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
85	1	24	1	(5, 45)
86	2	28	1	(5, 49)
87	3	30	0	(5, 49)
88	4	32	0	(5, 51)
89	5	32	0	(5, 51)
90	6	34	0	(5, 53)
91	7	36	0	(5, 55)
92	8	38	0	(5, 57)
93	9	40	0	(5, 59)
94	10	42	0	(5, 61)
95	11	44	0	(5, 63)
96	12	46	0	(5, 65)
97	—	26	0	(5, 45)

TABLE 6.5. Fano threefolds with $\rho \geq 5$.

No.	ρ	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
98	5	28	0	(6, 46)
99	5	36	0	(6, 54)
100	5	36	0	(6, 54)
101	6	30	0	(7, 47)
102	7	24	0	(8, 40)
103	8	18	0	(9, 33)
104	9	12	0	(10, 26)
105	10	6	0	(11, 19)

TABLE 6.6. The list of Calabi–Yau threefolds from nonsymplectic type.

No.	$K3$ invariants (r, a, δ)	$(h^{1,1}, h^{2,1})$	Mirror partner
1	(2, 0, 0)	(11, 59)	3
2	(10, 0, 0)	(35, 35)	2
3	(18, 0, 0)	(59, 11)	1
4	(1, 1, 1)	(6, 60)	9
5	(3, 1, 1)	(12, 54)	8
6	(9, 1, 1)	(30, 36)	7
7	(11, 1, 1)	(36, 30)	6
8	(17, 1, 1)	(54, 12)	5
9	(19, 1, 1)	(60, 6)	4
10	(2, 2, 0 or 1)	(7, 55)	18
11	(4, 2, 1)	(13, 49)	17
12	(6, 2, 0)	(19, 43)	16
13	(8, 2, 0)	(25, 37)	15
14	(10, 2, 0 or 1)	(31, 31)	14
15	(12, 2, 1)	(37, 25)	13
16	(14, 2, 0)	(43, 19)	12
17	(16, 2, 1)	(49, 13)	11
18	(18, 2, 0 or 1)	(55, 7)	10
19	(20, 2, 1)	(61, 1)	–
20	(3, 3, 1)	(8, 50)	27
21	(5, 3, 1)	(14, 44)	26
22	(7, 3, 1)	(20, 38)	25
23	(9, 3, 1)	(26, 32)	24
24	(11, 3, 1)	(32, 26)	23
25	(13, 3, 1)	(38, 20)	22
26	(15, 3, 1)	(44, 14)	21
27	(17, 3, 1)	(50, 8)	20
28	(19, 3, 1)	(56, 2)	–
29	(4, 4, 1)	(9, 45)	35
30	(6, 4, 0 or 1)	(15, 39)	34
31	(8, 4, 1)	(21, 33)	33
32	(10, 4, 0 or 1)	(27, 27)	32
33	(12, 4, 1)	(33, 21)	31
34	(14, 4, 0 or 1)	(39, 15)	30
35	(16, 4, 1)	(45, 9)	29

No.	$K3$ invariants (r, a, δ)	$(h^{1,1}, h^{2,1})$	Mirror partner
36	(18, 4, 0 or 1)	(51, 3)	–
37	(5, 5, 1)	(10, 40)	42
38	(7, 5, 1)	(16, 34)	41
39	(9, 5, 1)	(22, 28)	40
40	(11, 5, 1)	(28, 22)	39
41	(13, 5, 1)	(34, 16)	38
42	(15, 5, 1)	(40, 10)	37
43	(17, 5, 1)	(46, 4)	–
44	(6, 6, 1)	(11, 35)	48
45	(8, 6, 1)	(17, 29)	47
46	(10, 6, 0 or 1)	(23, 23)	46
47	(12, 6, 1)	(29, 17)	45
48	(14, 6, 0 or 1)	(35, 11)	44
49	(16, 6, 1)	(41, 5)	–
50	(7, 7, 1)	(12, 30)	53
51	(9, 7, 1)	(18, 24)	52
52	(11, 7, 1)	(24, 18)	51
53	(13, 7, 1)	(30, 12)	50
54	(15, 7, 1)	(36, 6)	–
55	(8, 8, 1)	(13, 25)	57
56	(10, 8, 0 or 1)	(19, 19)	56
57	(12, 8, 1)	(25, 13)	55
58	(14, 8, 1)	(31, 7)	–
59	(9, 9, 1)	(14, 20)	60
60	(11, 9, 1)	(20, 14)	59
61	(13, 9, 1)	(26, 8)	–
62	(10, 10, 1)	(15, 15)	62
63	(12, 10, 1)	(21, 9)	–
64	(11, 11, 1)	(16, 10)	–

Regarding No. 62, we note that $(r, a, \delta) \neq (10, 10, 0)$ by assumption.

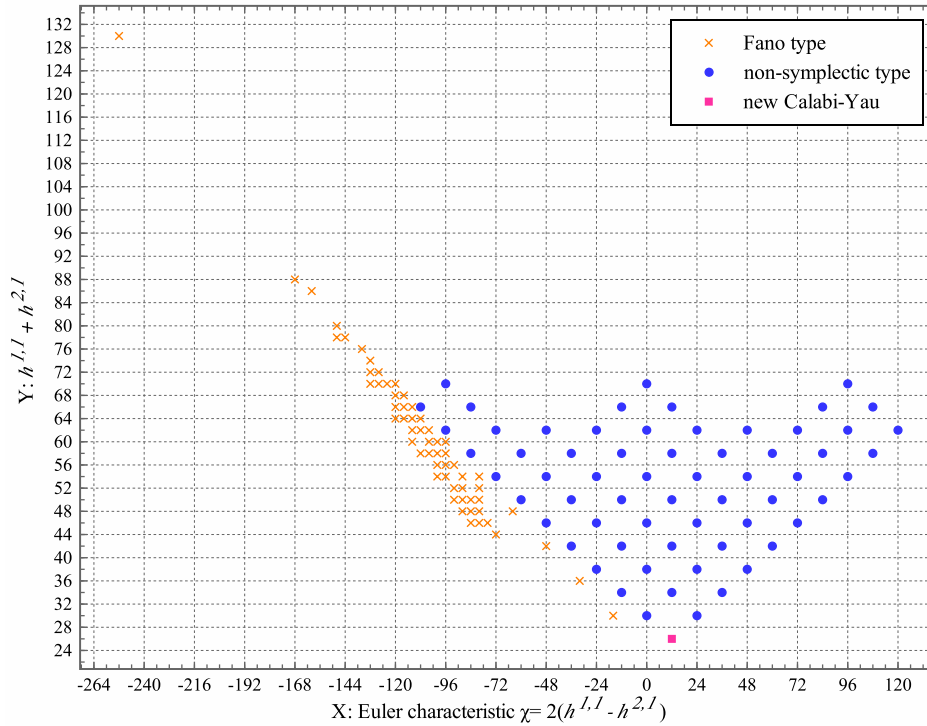


FIGURE 6.7. All resulting Calabi–Yau threefolds.

Discussion. The method of constructing Calabi–Yau threefolds and their mirrors from $K3$ surfaces were originally investigated by Borcea and Voisin [3], Section 4, using algebraic geometry. Our doubling construction is a differential-geometric interpretation of the Borcea–Voisin construction. Observe that Proposition 5.9 gives the condition that two Calabi–Yau threefolds M and M' should be a mirror pair, i.e., $h^{p,q}(M) = h^{3-p,q}(M')$ for all $p, q \in \{0, 1, 2, 3\}$. Let M (resp. M') be a Calabi–Yau threefold from admissible pairs of nonsymplectic type with respect to $K3$ invariants (r, a, δ) (resp. (r', a', δ')). Then $h^{p,q}(M) = h^{3-p,q}(M')$ implies $r + r' = 20$, $a = a'$ by Proposition 5.9. These relations coincide with (11) in [3], p. 723. From these equalities, we can find mirror pairs in our examples of Calabi–Yau threefolds. In particular M is automatically self-mirror when $r = 10$. Thus we find 24 mirror pairs and 6 self-mirror Calabi–Yau threefolds in our examples.

6.3. Graphical chart of our examples. Finally we plot the Hodge numbers of the resulting Calabi–Yau threefolds in Figure 6.7. In the figure, the Calabi–Yau threefolds obtained from Fano type (case (a)) are registered as symbol \times and those from nonsymplectic type (case (b)) are registered as symbol \bullet . Separately, our new example is denoted by solid square \blacksquare in Figure 6.7. We take the Euler characteristic $\chi = 2(h^{1,1} - h^{2,1})$ along the

X -axis and $h^{1,1} + h^{2,1}$ along the Y -axis. We see that all our examples from nonsymplectic type are located on the integral lattice of the form

$$(6.1) \quad (X, Y) = (12, 26) + m(12, 4) + n(-12, 4), \quad m, n \in \mathbb{Z}_{\geq 0}.$$

In this plot the mirror symmetry is considered as the inversion $\mu : (X, Y) \mapsto (-X, Y)$ with respect to the Y -axis. The set of 54 points with $n > 0$ in (6.1) is μ -invariant, and thus the corresponding Calabi–Yau threefolds have a mirror partner in this set.

References

- [1] BARTH, WOLF P.; HULEK, KLAUS; PETERS, CHRIS A. M.; VAN DE VEN, ANTONIUS. Compact complex surfaces. Second edition. A Series of Modern Surveys in Mathematics, 4. *Springer-Verlag, Berlin*, 2004. xii+436 pp. ISBN: 3-540-00832-2. [MR2030225](#) (2004m:14070), [Zbl 1036.14016](#).
- [2] BESSE, ARTHUR L. Einstein manifolds *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3), 10. *Springer-Verlag, Berlin*, 1987. xii+510 pp. ISBN: 3-540-15279-2. [MR0867684](#) (88f:53087), [Zbl 0613.53001](#), doi: [10.1007/978-3-540-74311-8](#).
- [3] BORCEA, CIPRIAN. $K3$ surfaces with involution and mirror pairs of Calabi–Yau manifolds. *Mirror Symmetry II*, 717–743, AMS/IP Stud. Adv. Math., 1. *Amer. Math. Soc., Providence, RI*, 1997. [MR1416355](#) (97i:14023), [Zbl 0939.14021](#).
- [4] CHEN, WEIMIN; RUAN, YONGBIN. A new cohomology theory of orbifold. *Comm. Math. Phys.* **248** (2004), no. 1, 1–31. [MR2104605](#) (2005j:57036), [Zbl 1063.53091](#), [arXiv:math/0004129](#), doi: [10.1007/s00220-004-1089-4](#).
- [5] CLANCY, ROBERT. New examples of compact manifolds with holonomy $\text{Spin}(7)$, *Ann. Global Anal. Geom.* **40** (2011), no. 2, 203–222. [MR2811626](#) (2012h:53111), [Zbl 1245.53047](#), [arXiv:1012.3571](#), doi: [10.1007/s10455-011-9254-4](#).
- [6] DOI, MAMORU. Gluing construction of compact complex surface with trivial canonical bundle. *J. Math. Soc. Japan* **61** (2009), no. 3, 853–884. [MR2552917](#) (2010j:32024), [Zbl 1178.32013](#).
- [7] HARVEY, F. REESE. Spinors and calibrations. *Perspectives in Mathematics*, 9. *Academic Press, Inc., Boston, MA*, 1990. xiv+323 pp. ISBN: 0-12-329650-1. [MR1045637](#) (91e:53056), [Zbl 0694.53002](#).
- [8] HEIN, HANS-JOACHIM. Complete Calabi–Yau metrics from $\mathbb{P}^2 \# 9\bar{\mathbb{P}}^2$. 2010. [arXiv:1003.2646](#).
- [9] ISKOVSKIĬ, V. A. Fano threefolds. I. II, *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 516–562, 717. [MR0463151](#) (80c:14023a). **42** (1978), 506–549. [MR0463151](#) (80c:14023a). English translation: *Math. USSR, Izv.*, **11** (1977), 485–527 and **12** (1978) 469–506.
- [10] JOYCE, D. D. Compact 8-manifolds with holonomy $\text{Spin}(7)$. *Invent. Math.* **123** (1996), no. 3, 507–552. [MR1383960](#) (97d:53052), [Zbl 0858.53037](#), doi: [10.1007/s002220050039](#).
- [11] JOYCE, DOMINIC D. Compact Riemannian 7-manifolds with holonomy G_2 . I. II. *J. Differential Geom.* **43** (1996), no. 2, 291–328, 329–375. [MR1424428](#) (97m:53084), [Zbl 0861.53022](#), [Zbl 0861.53023](#).
- [12] JOYCE, DOMINIC. A new construction of compact 8-manifolds with holonomy $\text{Spin}(7)$. *J. Differential Geom.* **53** (1999), no. 1, 89–130. [MR1776092](#) (2002a:53063), [Zbl 1040.53062](#), [arXiv:math/9910002](#).
- [13] JOYCE, DOMINIC D. Compact manifolds with special holonomy. *Oxford Mathematical Monographs. Oxford University Press, Oxford*, 2000. xii+436 pp. ISBN: 0-19-850601-5. [MR1787733](#) (2001k:53093), [Zbl 1027.53052](#).

- [14] KAWAMATA, YUJIRO; NAMIKAWA, YOSHINORI. Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi–Yau varieties. *Invent. Math.* **118** (1994), no. 3, 395–409. [MR1296351](#) (95j:32030), [Zbl 0848.14004](#), doi:[10.1007/BF01231538](#).
- [15] KOVALEV, ALEXEI. Twisted connected sums and special Riemannian holonomy. *J. Reine Angew. Math.* **565** (2003), 125–160. [MR2024648](#) (2004m:53088), [Zbl 1043.53041](#), [arXiv:math/0012189](#), doi:[10.1515/crll.2003.097](#).
- [16] KOVALEV, ALEXEI; LEE, NAM-HOON. $K3$ surfaces with non-symplectic involution and compact irreducible G_2 -manifolds. *Math. Proc. Camb. Phil. Soc.* **151** (2011), no. 2, 193–218. [MR2823130](#) (2012f:53090), [Zbl 1228.53064](#), [arXiv:0810.0957](#), doi:[10.1017/S030500411100003X](#).
- [17] KOVALEV, A.; SINGER, M. Gluing theorems for complete anti-self-dual spaces. *Geom. Funct. Anal.* **11** (2001), no. 6, 1229–1281. [MR1878320](#) (2002m:53071), [Zbl 1012.53026](#), [arXiv:math/0009158](#), doi:[10.1007/s00039-001-8230-8](#).
- [18] LEE, NAM-HOON. Calabi–Yau construction by smoothing normal crossing varieties. *Internat. J. Math.* **21** (2010), no. 6, 701–725. [MR2658406](#) (2011e:14074), [Zbl 1194.14063](#), [arXiv:math/0604596](#), doi:[10.1142/S0129167X10006173](#).
- [19] MORI, SHIGEFUMI; MUKAI, SHIGERU. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.* **36** (1981/1982), no. 2, 147–162. [MR0641971](#) (83f:14032), [Zbl 0478.14033](#), doi:[10.1007/BF01170131](#).
- [20] MORI, SHIGEFUMI; MUKAI, SHIGERU. Erratum: "Classification of Fano 3-folds with $B_2 \geq 2$ ". *Manuscripta Math.* **110** (2003), no. 3, 407. [MR1969009](#), doi:[10.1007/s00229-002-0336-2](#).
- [21] NIKULIN, V. V. Integral symmetric bilinear forms and some of their applications. *Izv. Akad. Nauk SSSR, Ser. Mat.* **43** (1979), 111–177. English translation: *Math. USSR, Izv.* **14** (1980), 103–167. [Zbl 0427.10014](#), doi:[10.1070/IM1980v014n01ABEH001060](#).
- [22] NIKULIN, V. V. Finite automorphism groups of Kähler $K3$ surfaces. *Trans. Moscow Math. Soc.* **2** (1980), 71–135. [Zbl 0454.14017](#).
- [23] NIKULIN, V. V. Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. *J. Soviet Math.* **22** (1983), 1401–1476. [Zbl 0508.10020](#).
- [24] PUMPERLA, MAX; REIDEGELD, FRANK. G_2 -manifolds from $K3$ surfaces with non-symplectic automorphisms. *J. Geom. Phys.* **62** (2012), no. 11, 2214–2226. [MR2964655](#), [Zbl 1267.53049](#), [arXiv:1110.2623](#), doi:[10.1016/j.geomphys.2012.06.002](#).
- [25] ROAN, SHI-SHYR. On the generalization of Kummer surfaces. *J. Differential Geom.* **30** (1989), no. 2, 523–537. [MR1010170](#) (90j:32032), [Zbl 0661.14031](#).
- [26] SALAMON, SIMON. Riemannian geometry and holonomy groups. Pitman Research Notes in Mathematics Series, 201. *Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York*, 1989. viii+201 pp. ISBN: 0-582-01767-X. [MR1004008](#) (90g:53058), [Zbl 0685.53001](#).
- [27] TIAN, G.; YAU, SHING-TUNG. Complete Kähler manifolds with zero Ricci curvature. I. *J. Amer. Math. Soc.* **3** (1990), no. 3, 579–609. [MR1040196](#) (91a:53096), [Zbl 0719.53041](#), doi:[10.1090/S0894-0347-1990-1040196-6](#).
- [28] WATANABE, KEIICHI. Certain invariant subrings are Gorenstein. I, II. *Osaka J. Math.* **11** (1974), 1–8; *ibid* **11** (1974), 379–388. [MR0354646](#) (50 #7124), [Zbl 0281.13007](#), [Zbl 0292.13008](#).
- [29] 6-manifolds: 1-connected. The Manifold Atlas Project. http://www.map.mpim-bonn.mpg.de/6-manifolds:_1-connected.

(Mamoru Doi) 11-9-302 YUMOTO-CHO, TAKARAZUKA, HYOGO 665-0003, JAPAN
doi.mamoru@gmail.com

(Naoto Yotsutani) SCHOOL OF MATHEMATICAL SCIENCES AT FUDAN UNIVERSITY, SHANGHAI, 200433, P. R. CHINA
naoto-yotsutani@fudan.edu.cn

This paper is available via <http://nyjm.albany.edu/j/2014/20-56.html>.