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# Tangle sums and factorization of A-polynomials

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Abstract. We show that there exist infinitely many examples of pairs of knots, $K_1$ and $K_2$, that have no epimorphism

$$\pi_1(S^3 \setminus K_1) \to \pi_1(S^3 \setminus K_2)$$

preserving peripheral structure although their A-polynomials have the factorization $A_{K_2}(L, M) \mid A_{K_1}(L, M)$. Our construction accounts for most of the known factorizations of this form for knots with 10 or fewer crossings. In particular, we conclude that while an epimorphism will lead to a factorization of A-polynomials, the converse generally fails.

Contents



## 1. Introduction

Cooper et al. [5] introduced the A-polynomial as a knot invariant derived from the $\mathrm{SL}(2, \mathbb{C})$-representations of the fundamental group of the knot's complement. It is a polynomial in the variables $M$ and $L$, which correspond to the eigenvalues of the $\mathrm{SL}(2, \mathbb{C})$-representations of the meridian and longitude respectively. We can obtain a lot of geometric information from A-polynomials including boundary slopes of incompressible surfaces in the knot complement and the nonexistence of Dehn surgeries yielding 3-manifolds with cyclic or finite fundamental groups, see for instance [9, 5, 3].

It is natural to ask if there is a correspondence between epimorphisms among the fundamental groups of knot complements and their A-polynomials.

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Actually, Silver and Whitten [21] showed that if there exists an epimorphism, $\pi_1(S^3 \setminus K_1) \to \pi_1(S^3 \setminus K_2)$, between the fundamental groups of two knot complements, that preserves peripheral structure, then the A-polynomial of $K_1$ has a factor corresponding to the A-polynomial of $K_2$ under a suitable change of coordinates. Here we say an epimorphism preserves peripheral structure if the image of the subgroup generated by the meridian and longitude of $K_1$ is included in the subgroup generated by the meridian and longitude of $K_2$. Hoste and Shanahan [13] refined this by demonstrating that the A-polynomial of $K_1$ has a factor which corresponds to the A-polynomial of $K_2$ under the change of coordinates $(L, M) \mapsto (L^d, M)$ for some $d \in \mathbb{Z}$. Ohtsuki, Riley, and Sakuma [19] made a systematic study of epimorphisms between 2-bridge link groups.

In this paper, we study factorizations of A-polynomials of knots obtained by specific tangle sums and the existence of epimorphisms. To state our main result, let $A_K(L, M)$ be the A-polynomial of a knot $K$ in $S^3$ and $A^\circ_K(L, M)$ the product of the factors of $A_K(L, M)$ containing the variable $L$. We denote by $S + T$ the sum of tangles $S$ and $T$ and by $N(T)$ the numerator closure of $T$.

**Theorem 1.** *Suppose that $N(S + T)$ and $N(T)$ are knots and $N(S)$ is a split link in $S^3$. Then $A^\circ_{N(T)}(L, M) \mid A_{N(S+T)}(L, M)$.*

Note that $A_K(L, M)/A^\circ_K(L, M)$ is a polynomial in only one variable, $M$. While this polynomial is often trivial, the $9_{38}$ knot shows that it need not be. According to a calculation by Culler [4], that knot has $(1 - M^2)^2$ as a factor. We also know that the roots of $A_K(L, M)/A^\circ_K(L, M)$ lie on the unit circle, for instance see [5].

Certain properties arising from the $\mathrm{SL}(2, \mathbb{C})$-representations of the fundamental group of the complement of $N(T)$ are inherited by $N(S + T)$. Specifically, we have the following corollary.

**Corollary 2.** *Suppose that $N(S+T)$, $N(T)$, and $N(S)$ satisfy the conditions of Theorem 1. With the possible exception of $\frac{1}{0}$, the set of boundary slopes of $N(T)$ detected by its character variety is a subset of the boundary slopes of $N(S + T)$.*

**Remark 3.** This has obvious implications for finite/cyclic surgeries and $r$-curves (factors of the form $1 \pm L^b M^a$, for which $r = a/b$, or $L^b \pm M^a$, with $r = -a/b$; see [2]). An $r$–curve of $N(T)$ with $r \neq \frac{1}{0}$ is inherited by $N(S+T)$. On the other hand, the $A$–polynomial of $N(T)$ can often be used to rule out finite/cyclic surgeries of $N(S + T)$ (cf. [14]).

Our second corollary gives an infinite family of pairs of knots where the $A$-polynomial of one factors that of the other even though there is no epimorphism between them. The proof depends on an analogous result for Alexander polynomials which requires the notion of marked tangles.

A marked tangle is one whose four ends have specific orientations as shown on the left in Figure 1. The sum of two marked tangles $S$ and $T$ is a marked tangle obtained as shown on the right, denoted by $S\dot{+}T$. We'll continue to use $N(T)$ and $D(T)$ to denote the numerator and denominator closure of a marked tangle $T$.

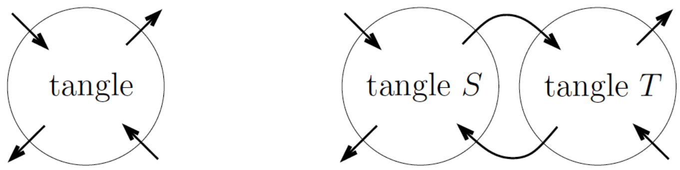


FIGURE 1. A marked tangle and the sum of marked tangles.

Let $\Delta_K(t)$ denote the Alexander polynomial of a knot $K$ in $S^3$. Using his formulation of the Alexander polynomial, Conway observed (cf. [7, Theorem 7.9.1])

$$\Delta_{N(S\dot{+}T)}(t) = \Delta_{N(T)}(t)\Delta_{D(S)}(t) + \Delta_{D(T)}(t)\Delta_{N(S)}(t).$$

In particular, if $N(S)$ is a split link then the Alexander polynomial has a factorization as

$$\Delta_{N(S\dot{+}T)}(t) = \Delta_{N(T)}(t)\Delta_{D(S)}(t) \tag{1}$$

since $\Delta_{N(S)}(t) = 0$.

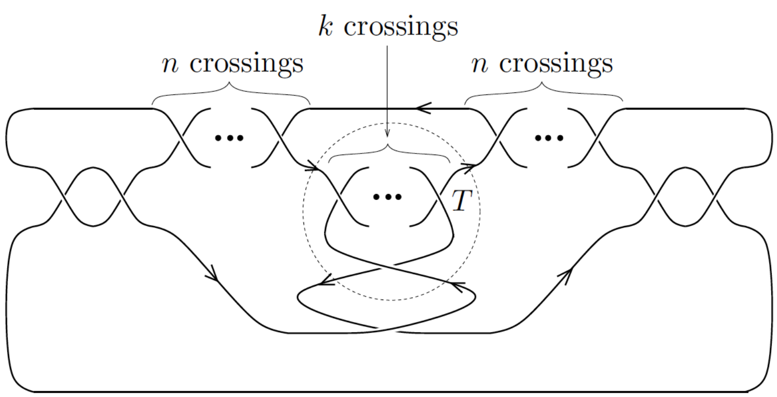


FIGURE 2. The 2-bridge knot $K(\beta/\alpha)$ where $\alpha/\beta = [2, -n, k, n, -2]$.

If, for knots $K_1$ and $K_2$, $\pi_1(S^3 \setminus K_1)$ has an epimorphism onto $\pi_1(S^3 \setminus K_2)$, then $\Delta_{K_2}(t) \mid \Delta_{K_1}(t)$ (e.g., see [8]). However, the converse does not hold in general. Indeed, it is well known that, given knot $K$, there are infinitely many knots $K_i$ with $\Delta_{K_i}(t) \mid \Delta_K(t)$. On the other hand, Agol and Liu [1] show that $\pi_1(S^3 \setminus K)$ surjects only finitely many knot groups.

**Remark 4.** Concretely, the knots described in our second corollary below give an infinite family of pairs for which $\Delta_{K_{2,k}}(t) \mid \Delta_K(t)$ although there is no surjection of the knot groups. Indeed, $K = K(\beta/\alpha)$ has a diagram of the form $N(S\dot{+}T)$ with $N(T) = K_{2,k}$, see Figure 2, while the lack of an epimorphism is easily deduced from work of González-Acuña and Ramírez [11, 12].

**Corollary 5.** *Let $K$ be the 2-bridge knot $K(\beta/\alpha)$ with*

$$\alpha/\beta = [2, -n, k, n, -2]$$

*and $K_{2,k}$ the $(2,k)$-torus knot, where $k > 2$ is odd and $n > 1$. Then $\pi_1(S^3 \setminus K)$ admits no epimorphism onto $\pi_1(S^3 \setminus K_{2,k})$ preserving peripheral structure, although $A_{K_{2,k}}(L, M) \mid A_K(L, M)$.*

The corollary follows from Theorem 1, Remark 4 and that $\infty$ is not a boundary slope of $K_{2,k}$. In [20] Riley discusses three ways in which character varieties of 2-bridge knots and links may become reducible. The examples in Corollary 5 do not fall into any of those three categories.

In the next section, we prove Theorem 1. In Section 3 we list 16 examples of factorizations of A-polynomials as in the theorem among pairs of knots of 10 or fewer crossings.

**Acknowledgements.** We would like to express our gratitude to Makoto Sakuma for his precious comments and for informing us of Riley's result on the reducibility of the character variety of 2-bridge knots. We would like to thank Fumikazu Nagasato for telling us some important details about A-polynomials as relates to Theorem 1. We appreciate the referee's thoughtful suggestions about how to improve our paper.

In this study, we often referred to the list of A-polynomials computed by Hoste and Culler and other knot invariants in the database `KnotInfo` [4]. We also used the program `Knotscape` of Hoste and Thistlethwaite for checking the knot types of given knot diagrams. We thank them for these useful computer programs and their database.

## 2. Proof of Theorem 1

We prove Theorem 1 in this section. Let $F_2$ denote the free group of rank 2. We first introduce a lemma that allows us a specific choice for the generators of $F_2$. We omit the straightforward proof.

**Lemma 6.** *Let $\langle a, b\rangle$ be generators of $F_2$ and $\hat{a}$ be an element in $F_2$ conjugate to $a$. Then there exists $\hat{b} \in F_2$ conjugate to $b$ such that $\hat{a}$ and $\hat{b}$ generate $\langle a, b\rangle = F_2$.*

Let $N(S+T)$, $N(S)$, and $N(T)$ be as in Theorem 1. Since $N(S+T)$ is a knot, the split link $N(S)$ consists of two link components, say $S_1$ and $S_2$. Since

$$\pi_1(S^3 \setminus N(S)) \cong \pi_1(S^3 \setminus S_1) * \pi_1(S^3 \setminus S_2),$$

the abelianizations $\pi_1(S^3 \setminus S_i) \to H_1(S^3 \setminus S_i) \cong \mathbb{Z}$, $i = 1, 2$, define a quotient map $q : \pi_1(S^3 \setminus N(S)) \to F_2$ that sends meridians of the two different components to the two generators $a$ and $b$ of $F_2$. Set $\hat{a}, b'$ to be the elements in $F_2 = \langle a, b\rangle$ corresponding to the meridional loops around the two strands of the numerator closure of the tangle $T$. By replacing $a$ (resp. $b$) by its inverse element if necessary, we may assume that $a$ and $\hat{a}$ (resp. $b$ and $b'$) are conjugate. By Lemma 6, there exists an element $\hat{b}$ conjugate to $b$ such that $\hat{a}$ and $\hat{b}$ generate $F_2 = \langle a, b\rangle$. Since $b'$ is conjugate to $b$, there exists $c \in \langle \hat{a}, \hat{b}\rangle$ such that $b' = c\hat{b}c^{-1}$. We further assume that the elements in $\pi_1(S^3 \setminus N(T))$ corresponding to $\hat{a}$ and $b'$ are conjugate by replacing one of them by its inverse element if necessary.

Let $\rho_0$ be a representation in $\mathrm{Hom}(\pi_1(S^3 \setminus N(T)), \mathrm{SL}(2, \mathbb{C}))$.

**Lemma 7.** *Suppose that* $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ *and that* $\rho_0(b') = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix}$ *satisfies* $b'_{11} \neq M^{\pm 1}$. *Then there is a representation*

$$\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b}\rangle, \mathrm{SL}(2, \mathbb{C}))$$

*such that* $\rho(\hat{a}) = \rho_0(\hat{a})$ *and* $\rho(b') = \rho_0(b')$.

**Proof.** Set $\rho(\hat{a}) = \rho_0(\hat{a})$. We will find a $\rho$ such that $\rho(b') = \rho_0(b')$. Set $\rho(\hat{b}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ and let $f_{11}, f_{12}, f_{21}, f_{22}$ be the polynomial functions, in the variables $M$ and the $b_{ij}$'s, given by

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \rho(c)\rho(\hat{b})\rho(c)^{-1},$$

where $f_{11}f_{22} - f_{12}f_{21} = 1$. We eliminate the variables $b_{22}$ and $b_{21}$ by substituting $b_{21} = \frac{1}{b_{12}}(b_{11}b_{22} - 1)$ and $b_{22} = M + \frac{1}{M} - b_{11}$, where the second equation holds since $\hat{a}$ and $b'$ are conjugate. The remaining variables are $M$, $b_{11}$, and $b_{12}$.

We first prove that $f_{11}$ depends on the variables $b_{11}$ and $b_{12}$. Assume it does not, i.e., $f_{11}$ is constant for each, fixed, choice of $M$. Setting $\rho(\hat{b}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$, resp. $\rho(\hat{b}) = \begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix}$, we have

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix}.$$

Therefore we have $M = M^{-1}$, i.e., $M = \pm 1$ since $f_{11}$ is constant. However, in the case $M = \pm 1$, since $\rho_0(\hat{a}) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, the equality

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \rho(\hat{b}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

is satisfied for any choice of the $b_{ij}$'s, which contradicts the assumption that $f_{11}$ does not depend on $b_{11}$.

Now $f_{11}$ does depend on at least one of the variables $b_{11}$ and $b_{12}$, so we solve the equation $f_{11} = b'_{11}$ in terms of one of these variables. The inequality $f_{11} = b'_{11} \neq M^{\pm 1}$ implies $f_{12} \neq 0$ and $f_{21} \neq 0$, otherwise we cannot have $f_{11}f_{22} - f_{12}f_{21} = 1$. For the same reason, we have $b'_{12} \neq 0$ and $b'_{21} \neq 0$. The conjugation of $\rho$ by the matrix

$$P = \begin{pmatrix} \sqrt{b'_{12}/f_{12}} & 0 \\ 0 & \sqrt{f_{12}/b'_{12}} \end{pmatrix}$$

satisfies

$$P\rho(\hat{a})P^{-1} = \rho(\hat{a}) \quad \text{and} \quad P \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} P^{-1} = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix},$$

where the bottom two equalities in the second matrix equation are automatically satisfied by the equation $f_{11} + f_{22} = b'_{11} + b'_{22}$ and the fact that these matrices are in $\mathrm{SL}(2, \mathbb{C})$. Hence we obtain the representation required. □

Let $f^+(M)$ be the rational function of one variable $M$ that appears as the top-right entry of $\rho(c)\rho(\hat{b})\rho(c)^{-1}$ when

$$\rho(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\hat{b}) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}.$$

Similarly, we define $f^-(M)$ to be the rational function of one variable $M$ that is the top-right entry of $\rho(c)\rho(\hat{b})\rho(c)^{-1}$ when

$$\rho(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\hat{b}) = \begin{pmatrix} M^{-1} & 1 \\ 0 & M \end{pmatrix}.$$

**Lemma 8.** *Either $f^+(M) \equiv 1$ (respectively $f^-(M) \equiv 1$) or $f^+(M)$ (respectively $f^-(M)$) is not constant.*

**Proof.** We can set $\rho(c) = \begin{pmatrix} M^k & c_{12} \\ 0 & M^{-k} \end{pmatrix}$, where $c_{12}$ is a rational function in one variable, $M$, whose denominator, if any, is a power of $M$, and $k \in \mathbb{Z}$. Then

$$\begin{aligned} \rho(c)\rho(\hat{b})\rho(c)^{-1} &= \begin{pmatrix} M^k & c_{12} \\ 0 & M^{-k} \end{pmatrix} \begin{pmatrix} M^{\pm 1} & 1 \\ 0 & M^{\mp 1} \end{pmatrix} \begin{pmatrix} M^{-k} & -c_{12} \\ 0 & M^k \end{pmatrix} \\ &= \begin{pmatrix} M^{\pm 1} & (M^{k\mp 1} - M^{k\pm 1})c_{12} + M^{2k} \\ 0 & M^{\mp 1} \end{pmatrix}, \end{aligned}$$

i.e.,

$$f^{\pm}(M) = (M^{k\mp 1} - M^{k\pm 1})c_{12} + M^{2k}.$$

If $c_{12} = k = 0$ then $f^{\pm}(M) \equiv 1$. Otherwise this cannot be constant since, even if $c_{12}$ has a denominator, it is only a power of $M$. □

**Lemma 9.** *Suppose that $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ and $\rho_0(b') = \begin{pmatrix} M & b'_{12} \\ b'_{21} & M^{-1} \end{pmatrix}$ with $b'_{12}b'_{21} = 0$. Suppose further that $f^+(M) \neq 0$. Then there exists a* *reducible representation* $\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b} \rangle, \mathrm{SL}(2,\mathbb{C}))$ *such that* $\rho(\hat{a}) = \rho_0(\hat{a})$ *and* $\rho(b') = \rho_0(b')$.

**Proof.** Set $\rho(\hat{a}) = \rho_0(\hat{a})$. We will find a reducible representation $\rho$ such that $\rho(b') = \rho_0(b')$. Consider the case where $b'_{21} = 0$. As above, we have $b' = c\hat{b}c^{-1}$. Set $\rho(\hat{b}) = \begin{pmatrix} M & b_{12} \\ 0 & M^{-1} \end{pmatrix}$; then the top-right entry of $\rho(c)\rho(\hat{b})\rho(c)^{-1}$ becomes $f^+(M)b_{12}$. Since $f^+(M) \neq 0$, $b_{12} = b'_{12}/f^+(M)$ gives the required reducible representation. The proof for the case $b'_{12} = 0$ is similar. □

**Lemma 10.** *Suppose that* $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ *and* $\rho_0(b') = \begin{pmatrix} M^{-1} & b'_{12} \\ b'_{21} & M \end{pmatrix}$ *with* $b'_{12}b'_{21} = 0$. *Suppose further that* $f^-(M) \neq 0$. *Then there exists a reducible representation* $\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b} \rangle, \mathrm{SL}(2,\mathbb{C}))$ *such that* $\rho(\hat{a}) = \rho_0(\hat{a})$ *and* $\rho(b') = \rho_0(b')$.

**Proof.** Similar to the proof of Lemma 9. □

**Proof of Theorem 1.** Let $\mathcal{R}(K)$ denote the representation variety $\mathrm{Hom}(\pi_1(S^3 \setminus K), \mathrm{SL}(2,\mathbb{C}))$ of a knot $K$ in $S^3$.

Let $M$ and $M^{-1}$ be the eigenvalues of $\rho_0(\hat{a})$. Assume that $f^{\pm}(M) \neq 0$ and $M \neq \pm 1$. Lemma 8 ensures that, except for a finite number of values, every $M \in \mathbb{R}$ satisfies these conditions. Since $M \neq \pm 1$, $\rho_0(\hat{a})$ is diagonalizable and hence we can set $\rho_0(\hat{a}) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ by conjugation. Then by Lemma 7, Lemma 9, and Lemma 10, for each representation $\rho_0 \in \mathcal{R}(N(T))$, there exists $\rho \in \mathrm{Hom}(\langle \hat{a}, \hat{b} \rangle, \mathrm{SL}(2,\mathbb{C}))$ such that $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. The quotient map $q : \pi_1(S^3 \setminus N(S)) \to \langle \hat{a}, \hat{b} \rangle$ induces a representation $\rho \in \mathcal{R}(N(S))$ which satisfies $\rho(\hat{a}) = \rho_0(\hat{a})$ and $\rho(b') = \rho_0(b')$. Let $D_{N(S+T)}$ be a knot diagram of $N(S+T)$ such that we can see the tangle decomposition into $N(S)$ and $N(T)$ on that diagram. Fix a Wirtinger presentation of $\pi_1(S^3 \setminus N(S+T))$ on $D_{N(S+T)}$. Clearly, $\rho_0$ satisfies the relations of the Wirtinger presentation in the tangle $T$ and $\rho$ also satisfies the relations in the tangle $S$. Therefore these representations satisfy all the relations of the Wirtinger presentation, in other words, we obtain an $\mathrm{SL}(2,\mathbb{C})$-representation of $\pi_1(S^3 \setminus N(S+T))$.

Each irreducible component of $A^{\circ}_{N(T)}(L,M) = 0$ corresponds to an irreducible component $Y$ of $\mathcal{R}(N(T))$ on which $M$ varies. Since each representation $\rho_0 \in Y$ corresponds to a representation $\rho_1 \in \mathcal{R}(N(S+T))$, except for a finite number of $M$ values, there always exists a subvariety $Z$ in $\mathcal{R}(N(S+T))$ which corresponds to $Y$.

Let $Z_\Delta$ be the algebraic subset of $Z$ consisting of all $\rho_1 \in Z$ such that $\rho_1(\ell_1)$ and $\rho_1(m_1)$ are upper triangular, where $(m_1, \ell_1)$ is the meridian-longitude pair of $N(S+T)$. Let $\xi : Z_\Delta \to \mathbb{C}^2$ be the eigenvalue map $\rho_1 \mapsto (L_1, M_1)$, where $L_1$ and $M_1$ are the top-left entries of $\rho_1(\ell_1)$ and $\rho_1(m_1)$ respectively. It is known by [6, Corollary 10.1] that $\dim \xi(Z_\Delta) \leq$

1. Since $M$ varies on $\xi(Z_\Delta)$, we have $\dim \xi(Z_\Delta) = 1$. This means that there exists a factor of the A-polynomial $A_{N(S+T)}(L,M)$ which vanishes at $(L,M) = (L_1, M_1)$.

In summary, for each generic point $(L_0, M_0) \in \{A^\circ_{N(T)}(L,M) = 0\}$, there is a representation $\rho_0 \in \mathcal{R}(N(T))$ such that the top-left entries of $\rho_0(\ell_0)$ and $\rho_0(m_0)$ are $L_0$ and $M_0$ respectively, where $(m_0, \ell_0)$ is the meridian-longitude pair of $N(T)$, and there exists a representation $\rho_1 \in \mathcal{R}(N(S+T))$ corresponding to $\rho_0$ such that the image $(L_1, M_1)$ satisfies $A_{N(S+T)}(L_1, M_1) = 0$. Thus if we have $\rho_0(m_0) = \rho_1(m_1)$ and $\rho_0(\ell_0) = \rho_1(\ell_1)$ then $M_0 = M_1$ and $L_0 = L_1$, and hence we have $A_{N(S+T)}(L_0, M_0) = 0$. This means that the factor $A^\circ_{N(T)}(L,M)$ appears in $A_{N(S+T)}(L,M)$. Since $m_0 = m_1$ from the construction, we have $\rho_0(m_0) = \rho_1(m_1)$. Hence, it is enough to show that $\rho_0(\ell_0) = \rho_1(\ell_1)$.

Let $\Sigma$ be the Seifert surface of $N(S+T)$ described on the diagram $D_{N(S+T)}$ by using Seifert's algorithm. The boundary of $\Sigma$ determines $\ell_1$. Using the Wirtinger presentation of $\pi_1(S^3 \setminus N(S+T))$ on $D_{N(S+T)}$, the longitude $\ell_1$ in $\pi_1(S^3 \setminus N(S+T))$ is represented as a product of words of the generators in the Wirtinger presentation by reading the words along the boundary of $\Sigma$. This word presentation of $\ell_1$ has the form

$$\ell_1 = \ell_{T,1}\ell_{S,1}\ell_{T,2}\ell_{S,2},$$

where, for $i = 1, 2$, $\ell_{T,i}$ is a product of generators in the tangle $T$ and $\ell_{S,i}$ is a product of generators in the tangle $S$. Since each $\ell_{S,i}$ represents one of the boundary components of a Seifert surface of the split link $N(S)$ and the representation $\rho_1$ is defined via the quotient map $q : \pi_1(S^3 \setminus N(S)) \to F_2$, $\rho_1(\ell_{S,i})$ is the identity matrix. Therefore we have

$$\rho_1(\ell_1) = \rho_1(\ell_{T,1})\rho_1(\ell_{T,2}) = \rho_0(\ell_0). \qquad \square$$

## 3. RTR examples of 10 or fewer crossings

**Definition 11.** A knot $K$ in $S^3$ is said to be an RTЯ *knot* if it satisfies the following:

(1) $K$ is of the form $N(\mathrm{R} + T + Я)$, where R is rational, Я is the mirror reflection of R, and $T$ is some tangle.
(2) $K$ is not isotopic to $N(T)$.

The second condition is added to exclude trivialities, for example the case where R consists of two horizontal arcs. Since $N(\mathrm{R} + Я)$ is always a trivial link of two components, $N(\mathrm{R} + T + Я)$ satisfies the conditions of Theorem 1 with $S = \mathrm{R} + Я$.

Here are two simple families of RTЯ knots:

- The 2-bridge knots of the form $[a_1, a_2, a_3, \cdots, a_k, \cdots, a_{2n-1}]$ with $a_i = -a_{2n-i}$ for $i = 1, \cdots, n-1$ and $a_n$ odd.
- Three-tangle Montesinos knots of the form $(p/q, r/s, -p/q)$.

TABLE 1. Factorizations of RTЯ knots

| | RTЯ | type | A-poly. fac. | epi. | Alex. poly. |
|---|---|---|---|---|---|
| $8_{10}$ | $1/3, 3/2, -1/3$ | A | $3_1$ | $8_{10} \to 3_1$ | $(3_1)^3$ |
| $8_{11}$ | $[2, -2, 3, 2, -2]$ | B | $3_1$ | No | $(3_1)(6_1)$ |
| $9_{24}$ | $1/3, 5/2, -1/3$ | A | $4_1$ | $9_{24} \to 3_1$ | $(3_1)^2(4_1)$ |
| $9_{37}$ | $1/3, 5/3, -1/3$ | B | $4_1$ | $9_{37} \to 4_1$ | $(4_1)(6_1)$ |
| $10_{21}$ | $[2, -2, 5, 2, -2]$ | B | $5_1$ | No | $(5_1)(6_1)$ |
| $10_{40}$ | $[2, 2, 3, -2, -2]$ | B | $3_1$ | $10_{40} \to 3_1$ | $(3_1)(8_8)$ |
| $10_{59}$ | $2/5, 3/2, -2/5$ | A | $3_1$ | $10_{59} \to 4_1$ | $(3_1)(4_1)^2$ |
| $10_{62}$ | $1/3, 5/4, -1/3$ | A | $5_1$ | $10_{62} \to 3_1$ | $(3_1)^2(5_1)$ |
| $10_{65}$ | $1/3, 7/4, -1/3$ | A | $5_2$ | $10_{65} \to 3_1$ | $(3_1)^2(5_2)$ |
| $10_{67}$ | $1/3, 7/5, -1/3$ | B | $5_2$ | No | $(5_2)(6_1)$ |
| $10_{74}$ | $1/3, 7/3, -1/3$ | B | $5_2$ | $10_{74} \to 5_2$ | $(5_2)(6_1)$ |
| $10_{77}$ | $1/3, 7/2, -1/3$ | A | $5_2$ | $10_{77} \to 3_1$ | $(3_1)^2(5_2)$ |
| $10_{98}$ | $1/3,\ T_0,\ -1/3$ | B | $3_1\#3_1$ | $10_{98} \to 3_1$ | $(3_1)^2(6_1)$ |
| $10_{99}$ | $1/3,\ T_1,\ -1/3$ | A | $3_1\#3_1^{\text{mir}}$ | $10_{99} \to 3_1$ | $(3_1)^4$ |
| $10_{143}$ | $1/3, 3/4, -1/3$ | A | $3_1$ | $10_{143} \to 3_1$ | $(3_1)^3$ |
| $10_{147}$ | $1/3, 3/5, -1/3$ | B | $3_1$ | No | $(3_1)(6_1)$ |

Note that the infinite collection of pairs of 2-bridge knots of Corollary 5 are included in the first of these families.

In the following, we represent the rational tangle corresponding to the rational number $p/q$ by $R(p/q)$. For example, the Montesinos knot of the form $(p/q, r/s, -p/q)$ is represented as $N(R(p/q) + R(r/s) + R(-p/q))$.

Table 1 lists the RTЯ knots of 10 or fewer crossings of which we know. In the table, $T_0$ is the tangle obtained as the $+\pi/2$-rotation of the tangle sum $R(-1/1) + R(1/3) + R(1/3)$ and $T_1$ is obtained as the $+\pi/2$-rotation of the tangle sum $R(1/3) + R(-1/3)$. We use $3_1^{\text{mir}}$ to denote the mirror image of $3_1$ and use # for the connected sum of two knots. In the table, we include information of epimorphisms among the knot groups and Alexander polynomials for convenience. The epimorphism column shows the existence of an epimorphism to a knot group up to ten crossings, as proven in [15]. In the column "Alex. poly.," we represent a knot's Alexander polynomial by enclosing the knot's symbol in parenthesis.

There are two types of RTЯ knots depending on how the strands enter and leave the tangle $T$. We say that the RTЯ knot $N(\mathrm{R} + T + Я)$ is *of type A* if the tangle $T$ is a marked tangle. Otherwise we say it is *of type B*.

**Lemma 12.** *Let* $K = N(\mathrm{R} + T + Я)$ *be an* RTЯ *knot with* $\mathrm{R} = R(p/q)$ *and* $q > 0$. *Then:*

(i) $q > 1$.
(ii) *If* $K$ *is of type A then* $\Delta_K(t) = \Delta_{N(T)}(t)\Delta_{D(\mathrm{R})}(t)^2$.
(iii) *If* $K$ *is of type B then* $\Delta_K(t) = \Delta_{N(T)}(t)\Delta_{N(\mathrm{R}+R(1/1)+Я)}(t)$.

(iv) *The knot determinant of $K$ is divisible by $q^2$.*

**Proof.** If $q = 1$ then we have $N(\mathrm{R} + T + Я) = N(T)$. Such a knot is not RTЯ by definition. Thus we have assertion (i). Assertion (ii) follows from Equation (1) and the equations

$$\Delta_{D(S)}(t) = \Delta_{D(\mathrm{R})\#D(Я)}(t) = \Delta_{D(\mathrm{R})}(t)^2.$$

Next we prove assertion (iii). Since $K$ is of type B, we need to modify the diagram of $N(\mathrm{R}+T+Я)$ as shown in Figure 3 such that it becomes the sum of marked tangles. We denote the marked tangle obtained from $T$ by $T'$ and the complementary tangle of $T'$ by $S'$. From the figure, we can see that $D(S') = N(\mathrm{R}+R(1/1)+Я)$. Thus assertion (iii) follows from Equation (1).

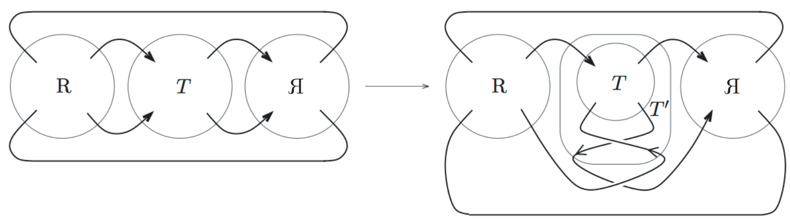


FIGURE 3. Changing an RTЯ knot of type B into the sum of two marked tangles.

Finally, we check the last assertion. It is known that the knot determinant of a knot is equal to the absolute value of its Alexander polynomial evaluated at $t = -1$ (see for instance [18, Proposition 6.1.5]). We also know that the knot determinant of $D(R(p/q))$ is $q$. Thus, if $K$ is of type A then assertion (iv) follows immediately from the factorization in (ii). Suppose $K$ is of type B. Then, from

$$N(R(p/q) + R(1/1) + R(-p/q)) = N(R(p/q) + R((q-p)/q)),$$

the knot determinant of $D(S') = N(\mathrm{R} + R(1/1) + Я)$ is calculated as

$$|pq + (q-p)q| = q^2$$

(see [10] and also [18, Theorem 9.3.5]). Thus, from the factorization in (iii), we again have assertion (iv). □

Using Lemma 12, we can check that most of the knots up to 10 crossings are not RTЯ knots. We first consider the case of type A. Set $R = R(p/q)$ with $q > 0$. By Lemma 12(i), we have $q > 1$. We check if the Alexander polynomial of a knot, up to 10 crossings, has a factorization of the form in Lemma 12(ii). Since the knot determinant of $D(R)$ is equal to $\Delta_{D(R)}(-1)$, $q > 1$ implies that the polynomial $\Delta_{D(R)}(t)^2$ in Lemma 12(ii) is nontrivial.

Now we check if the Alexander polynomial of a knot has such a nontrivial, multiple factor corresponding to a knot up to 10 crossings. The only candidate knots $K$ are

$$8_{10}, 8_{18}, 8_{20}, 9_{24}, 9_{40}, 10_{59}, 10_{62}, 10_{65}, 10_{77},$$
$$10_{82}, 10_{87}, 10_{98}, 10_{99}, 10_{123}, 10_{137}, 10_{140}, 10_{143}.$$

Next we consider RTЯ knots of type B. It is shown in the proof of Lemma 12(iv) that the knot determinant of $D(S)$ is $q^2 > 1$. Hence $D(R)$ is a 2-bridge knot with denominator $q > 1$. Moreover, since $N(T)$ is assumed to be a nontrivial knot of 10 or fewer crossings, $\Delta_{N(T)}(t)$ is nontrivial. Hence we know that the factorization of $\Delta_K(t)$ in Lemma 12(iii) is nontrivial.

Now we make a list of knots $K$, up to 10 crossings that satisfy

- the Alexander polynomial of $K$ factors into two nontrivial Alexander polynomials, and
- the knot determinant of $K$ is divisible by $q^2$ for some integer $q > 1$.

The following knots satisfy these conditions:

$$8_{10}, 8_{11}, 8_{18}, 8_{20}, 9_1, 9_6, 9_{23}, 9_{24}, 9_{37}, 9_{40}, 10_{21}, 10_{40}, 10_{59}, 10_{62}, 10_{65}, 10_{66}, 10_{67},$$
$$10_{74}, 10_{77}, 10_{82}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, 10_{123}, 10_{137}, 10_{140}, 10_{143}, 10_{147}.$$

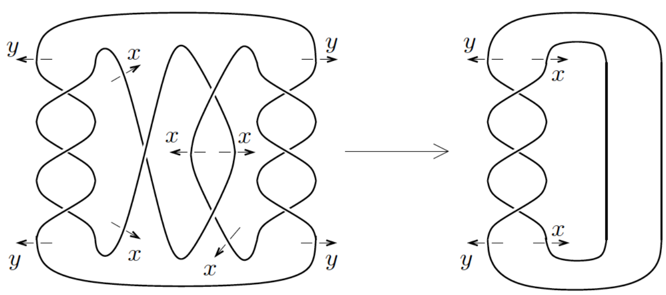


FIGURE 4. An epimorphism $\pi_1(S^3 \setminus 8_{10}) \to \pi_1(S^3 \setminus 3_1)$. Construct a Seifert surface and observe that the longitude of $8_{10}$ vanishes in $\pi_1(S^3 \setminus 3_1)$.

**Remark 13.** There is no direct relationship between the RTЯ construction and the list of epimorphisms in [15]. First of all, we can see from Table 1 that the following 9 knots

$$8_{11}, 9_{24}, 10_{21}, 10_{59}, 10_{62}, 10_{65}, 10_{67}, 10_{77}, 10_{147}$$

have the factorization of the A-polynomials but have no epimorphisms to the corresponding knot groups.

Even for the other knots in Table 1, we believe that there is no relationship for the following reason: In [16] it is written that there is an epimorphism $\pi_1(S^3 \setminus 8_{10}) \to \pi_1(S^3 \setminus 3_1)$ which maps the longitude of $8_{10}$ to $1 \in \pi_1(S^3 \setminus 3_1)$, see Figure 4, while Theorem 1 shows that the longitude of $8_{10}$ corresponds to that of $3_1$ in our construction. In this example, the epimorphism is given by the tangle R and the factorization of the A-polynomial is given by the tangle $T$. In general, for any RTЯ knot of type A, there is an epimorphism from $\pi_1(S^3 \setminus N(\mathrm{R}+T+\text{Я}))$ to $\pi_1(S^3 \setminus D(\mathrm{R}))$ such that the image of the longitude of this RTЯ knot is $1 \in \pi_1(S^3 \setminus D(\mathrm{R}))$; however, the longitude of $N(\mathrm{R}+T+\text{Я})$ corresponds to that of $D(\mathrm{R})$ when we compare their A-polynomials. This shows that the type A examples do not correspond to the epimorphisms. We remark that there may exist other epimorphisms from $\pi_1(S^3 \setminus N(\mathrm{R}+T+\text{Я}))$ to $\pi_1(S^3 \setminus D(\mathrm{R}))$ preserving peripheral structure. This is why we cannot exclude the possibility that there is a relationship between the factorization of A-polynomials and epimorphisms for these examples.

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