On pointwise estimates involving sparse operators

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Abstract. We obtain an alternative approach to recent results by M. Lacey and Hytönen–Roncal–Tapiola about a pointwise domination of ω-Calderón–Zygmund operators by sparse operators. This approach is rather elementary and it also works for a class of nonintegral singular operators.

Contents

1. Introduction

This paper is motivated by several recent works about a domination of Calderón–Zygmund operators by sparse operators. Such a domination was first established by the author [11] in terms of X-norms, where X is an arbitrary Banach function space. This result was used in order to give an alternative proof of the A_2 theorem obtained earlier by T. Hytönen [6].

The X-norm estimate in [11] was proved for a class of ω-Calderón–Zygmund operators with the modulus of continuity ω satisfying the logarithmic Dini condition $\int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty$. After that, under the same assumption on ω, the X-norm bound was improved by a pointwise bound independently and simultaneously by J. Conde-Alonso and G. Rey [2], and by the author and F. Nazarov [12].

Later, M. Lacey [10] found a new method allowing him to relax the log-Dini condition in the pointwise bound to the classical Dini condition $\int_0^1 \omega(t)\frac{dt}{t} < \infty$. Very recently, T. Hytönen, L. Roncal and O. Tapiola [9] elaborated the proof in [10] to get a precise linear dependence on the Dini constant with a subsequent application to rough singular integrals.

In the present note we modify a main idea from Lacey's work [10] with the aim to give a rather short and elementary proof of the result in [9]. This yields a further simplification of the A_2 theorem and related bounds. Our modification consists in a different cubic truncation of a Calderón–Zygmund operator T with the help of an auxiliary "grand maximal truncated" operator $\mathcal{M}_T$. This way of truncation allows to get a very simple recursive relation for T, and, as a corollary, the pointwise bound by a sparse operator.

Notice also that our proof has an abstract nature, and, in particular, it is easily generalized to a class of singular non-kernel operators.

2. Main definitions

2.1. Sparse families and operators. By a cube in $\mathbb{R}^n$ we mean a half-open cube $Q = \prod_{i=1}^n [a_i, a_i + h), h > 0$. Given a cube $Q_0 \subset \mathbb{R}^n$, let $\mathcal{D}(Q_0)$ denote the set of all dyadic cubes with respect to Q_0, that is, the cubes obtained by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes.

We say that a family $\mathcal{S}$ of cubes from $\mathbb{R}^n$ is η-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \eta|Q|$, and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint. Usually η will depend only on the dimension, and when this parameter is unessential we will skip it.

Denote $f_Q = \frac{1}{|Q|} \int_Q f$. Given a sparse family $\mathcal{S}$, define a sparse operator $\mathcal{A}_\mathcal{S}$ by

$$\mathcal{A}_\mathcal{S} f(x) = \sum_{Q\in\mathcal{S}} f_Q \chi_Q(x).$$

2.2. ω-Calderón–Zygmund operators. Let $\omega : [0,1] \to [0,\infty)$ be a modulus of continuity, that is, ω is increasing, subadditive and $\omega(0) = 0$.

We say that T is an ω-Calderón-Zygmung operator if T is L^2 bounded, represented as

$$T f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad \text{for all } x \notin \operatorname{supp} f$$

with kernel K satisfying the size condition $|K(x,y)| \leq \frac{C_K}{|x-y|^n}, x \neq y$, and the smoothness condition

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \omega\left(\frac{|x-x'|}{|x-y|}\right)\frac{1}{|x-y|^n}$$

for $|x - y| > 2|x - x'|$.

We say that ω satisfies the Dini condition if $\|\omega\|_{\text{Dini}} = \int_0^1 \omega(t)\frac{dt}{t} < \infty$.

3. The Lacey–Hytönen–Roncal–Tapiola Theorem

As was mentioned in the introduction, we give here an alternative proof of a recent result by T. Hytönen *et al.* [9], which in turn is a revised version of Lacey's domination theorem [10].

Theorem 3.1. *Let T be an ω-Calderón–Zygmund operator with ω satisfying the Dini condition. Then, for every compactly supported $f \in L^1(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$(3.1) \qquad |Tf(x)| \leq c_n(\|T\|_{L^2 \to L^2} + C_K + \|\omega\|_{\mathrm{Dini}})\mathcal{A}_{\mathcal{S}}|f|(x).$$

We will need a number of auxiliary maximal operators. The key role in the proof is played by the maximal operator $\mathcal{M}_T$ defined by

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi \in Q} |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. This object can be called the grand maximal truncated operator. Recall that the standard maximal truncated operator is defined by

$$T^\star f(x) = \sup_{\varepsilon > 0}\left|\int_{|y-x|>\varepsilon} K(x,y)f(y)dy\right|.$$

Given a cube Q_0, for $x \in Q_0$ define a local version of $\mathcal{M}_T$ by

$$\mathcal{M}_{T,Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \operatorname*{ess\,sup}_{\xi \in Q} |T(f\chi_{3Q_0 \setminus 3Q})(\xi)|.$$

Finally, let M be the standard Hardy–Littlewood maximal operator.

Lemma 3.2. *The following pointwise estimates hold:*

(i) *For a.e. $x \in Q_0$,*

$$|T(f\chi_{3Q_0})(x)| \leq c_n\|T\|_{L^1 \to L^{1,\infty}}|f(x)| + \mathcal{M}_{T,Q_0}f(x).$$

(ii) *For all $x \in \mathbb{R}^n$,*

$$\mathcal{M}_T f(x) \leq c_n(\|\omega\|_{\mathrm{Dini}} + C_K)Mf(x) + T^\star f(x).$$

Proof. (i) Suppose that $x \in \operatorname{int} Q_0$, and let x be a point of approximate continuity of $T(f\chi_{3Q_0})$ (see, e.g., [4, p. 46]). Then for every $\varepsilon > 0$, the sets

$$E_s(x) = \{y \in B(x,s) : |T(f\chi_{3Q_0})(y) - T(f\chi_{3Q_0})(x)| < \varepsilon\}$$

satisfy $\lim_{s \to 0} \frac{|E_s(x)|}{|B(x,s)|} = 1$, where $B(x,s)$ is the open ball centered at x of radius s.

Denote by $Q(x,s)$ the smallest cube centered at x and containing $B(x,s)$. Let $s > 0$ be so small that $Q(x,s) \subset Q_0$. Then for a.e. $y \in E_s(x)$,

$$|T(f\chi_{3Q_0}(x)| < |T(f\chi_{3Q_0})(y)| + \varepsilon \leq |T(f\chi_{3Q(x,s)})(y)| + \mathcal{M}_{T,Q_0}f(x) + \varepsilon.$$

Therefore, applying the weak type $(1, 1)$ of T yields

$$|T(f\chi_{3Q_0}(x)| \leq \operatorname*{ess\,inf}_{y \in E_s(x)} |T(f\chi_{3Q(x,s)})(y)| + \mathcal{M}_{T,Q_0} f(x) + \varepsilon$$

$$\leq \|T\|_{L^1 \to L^{1,\infty}} \frac{1}{|E_s(x)|} \int_{3Q(x,s)} |f| + \mathcal{M}_{T,Q_0} f(x) + \varepsilon.$$

Assuming additionally that x is a Lebesgue point of f and letting subsequently $s \to 0$ and $\varepsilon \to 0$, we obtain (i).

(ii) Let $x, \xi \in Q$. Denote by B_x the closed ball centered at x of radius $2 \operatorname{diam} Q$. Then $3Q \subset B_x$, and we obtain

$$|T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \leq |T(f\chi_{\mathbb{R}^n \setminus B_x})(\xi) - T(f\chi_{\mathbb{R}^n \setminus B_x})(x)|$$
$$+ |T(f\chi_{B_x \setminus 3Q})(\xi)| + |T(f\chi_{\mathbb{R}^n \setminus B_x})(x)|.$$

By the smoothness condition,

$$|T(f\chi_{\mathbb{R}^n \setminus B_x})(\xi) - T(f\chi_{\mathbb{R}^n \setminus B_x})(x)|$$
$$\leq \int_{|y-x|>2\operatorname{diam} Q} |f(y)| \omega \left(\frac{\operatorname{diam} Q}{|x-y|} \right) \frac{1}{|x-y|^n} dy$$
$$\leq \sum_{k=1}^{\infty} \left(\frac{1}{(2^k \operatorname{diam} Q)^n} \int_{2^k B_x} |f| \right) \omega(2^{-k}) \leq c_n \|\omega\|_{\operatorname{Dini}} Mf(x).$$

Next, by the size condition,

$$|T(f\chi_{B_x \setminus 3Q})(\xi)| \leq c_n C_K \frac{1}{|B_x|} \int_{B_x} |f| \leq c_n C_K Mf(x).$$

Finally, $|T(f\chi_{\mathbb{R}^n \setminus B_x})(x)| \leq T^\star f(x)$. Combining the obtained estimates proves (ii). □

Denote $C_T = \|T\|_{L^2 \to L^2} + C_K + \|\omega\|_{\operatorname{Dini}}$. An examination of standard proofs (see, e.g., [5, Ch. 8.2]) shows that

$$(3.2) \qquad \max(\|T\|_{L^1 \to L^{1,\infty}}, \|T^\star\|_{L^1 \to L^{1,\infty}}) \leq c_n C_T.$$

Proof of Theorem 3.1. Fix a cube $Q_0 \subset \mathbb{R}^n$. Let us show that there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,

$$(3.3) \qquad |T(f\chi_{3Q_0})(x)| \leq c_n C_T \sum_{Q \in \mathcal{F}} |f|_{3Q} \chi_Q(x).$$

It suffices to prove the following recursive claim: there exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and

$$(3.4) \qquad |T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leq c_n C_T |f|_{3Q_0} + \sum_j |T(f\chi_{3P_j})|\chi_{P_j}$$

a.e. on Q_0. Indeed, iterating this estimate, we immediately get (3.3) with $\mathcal{F} = \{P_j^k\}, k \in \mathbb{Z}_+$, where $\{P_j^0\} = \{Q_0\}$, $\{P_j^1\} = \{P_j\}$ and $\{P_j^k\}$ are the cubes obtained at the k-th stage of the iterative process.

Next, observe that for arbitrary pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,

$$|T(f\chi_{3Q_0})|\chi_{Q_0} = |T(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f\chi_{3Q_0})|\chi_{P_j}$$

$$\leq |T(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j}$$

$$+ \sum_j |T(f\chi_{3P_j})|\chi_{P_j}.$$

Hence, in order to prove the recursive claim, it suffices to show that one can select pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ with $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and such that for a.e. $x \in Q_0$,

$$(3.5) \qquad |T(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j} \leq c_n C_T |f|_{3Q_0}.$$

By Lemma 3.2(ii) and by (3.2), $\|\mathcal{M}_T\|_{L^1 \to L^{1,\infty}} \leq \alpha_n C_T$. Therefore, one can choose c_n such that the set

$$E = \{x \in Q_0 : |f| > c_n|f|_{3Q_0}\} \cup \{x \in Q_0 : \mathcal{M}_{T,Q_0}f > c_n C_T|f|_{3Q_0}\}$$

will satisfy $|E| \leq \frac{1}{2^{n+2}}|Q_0|$.

The Calderón–Zygmund decomposition applied to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$ produces pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and $|E \setminus \cup_j P_j| = 0$. It follows that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and $P_j \cap E^c \neq \emptyset$. Therefore,

$$\operatorname*{ess\,sup}_{\xi \in P_j} |T(f\chi_{3Q_0\setminus 3P_j})(\xi)| \leq c_n C_T |f|_{3Q_0}.$$

Also, by Lemma 3.2(i) and by (3.2), for a.e. $x \in Q_0 \setminus \cup_j P_j$,

$$|T(f\chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0},$$

which, along with the previous estimate, proves (3.5) and so (3.3).

Take now a partition of $\mathbb{R}^n$ by cubes R_j such that $\operatorname{supp}(f) \subset 3R_j$ for each j. For example, take a cube Q_0 such that $\operatorname{supp}(f) \subset Q_0$ and cover $3Q_0 \setminus Q_0$ by $3^n - 1$ congruent cubes R_j. Each of them satisfies $Q_0 \subset 3R_j$. Next, in the same way cover $9Q_0 \setminus 3Q_0$, and so on. The union of resulting cubes, including Q_0, will satisfy the desired property.

Having such a partition, apply (3.3) to each R_j. We obtain a $\frac{1}{2}$-sparse family $\mathcal{F}_j \subset \mathcal{D}(R_j)$ such that (3.3) holds for a.e. $x \in R_j$ with $|Tf|$ on the left-hand side. Therefore, setting $\mathcal{F} = \cup_j \mathcal{F}_j$, we obtain that $\mathcal{F}$ is $\frac{1}{2}$-sparse and for a.e. $x \in \mathbb{R}^n$,

$$|Tf(x)| \leq c_n C_T \sum_{Q \in \mathcal{F}} |f|_{3Q}\chi_Q(x).$$

Thus, (3.1) holds with a $\frac{1}{2\cdot 3^n}$-sparse family $\mathcal{S} = \{3Q : Q \in \mathcal{F}\}$. $\qquad\square$

4. Remarks and complements

Remark 4.1. Given a sparse family $\mathcal{S}$ and $1 \le r < \infty$, define for $f \ge 0$,

$$\mathcal{A}_{r,\mathcal{S}}f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f^r \right)^{1/r} \chi_Q(x).$$

Next, notice that the definition of $\mathcal{M}_T$ can be given for arbitrary (non-kernel) sublinear operator T. Then, the proof of Theorem 3.1 with minor modifications allows to get the following result.

Theorem 4.2. *Assume that T is of weak type (q, q) and $\mathcal{M}_T$ is of weak type (r, r), where $1 \le q \le r < \infty$. Then, for every compactly supported $f \in L^r(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$(4.1) \qquad |Tf(x)| \le K\mathcal{A}_{r,\mathcal{S}}|f|(x),$$

where $K = c_{n,q,r}(\|T\|_{L^q \to L^{q,\infty}} + \|\mathcal{M}_T\|_{L^r \to L^{r,\infty}})$.

Indeed, Lemma 3.2(i) works with $\|T\|_{L^1 \to L^{1,\infty}}$ replaced by $\|T\|_{L^q \to L^{q,\infty}}$. Next, Lemma 3.2(ii) (the only part in the proof of Theorem 3.1 where the kernel assumptions were used) is replaced by the postulate that $\mathcal{M}_T$ is of weak type (r, r). Finally, under trivial changes in the definition of the set E, we obtain that the key estimate (3.4) holds with $c_n C_T |f|_{3Q_0}$ replaced by $K\left(\frac{1}{|3Q_0|} \int_{3Q_0} |f|^r\right)^{1/r}$. The rest of the proof is identically the same.

Recently, F. Bernicot, D. Frey and S. Petermichl [1] obtained sharp weighted estimates for a large class of singular nonintegral operators in a rather general setting using similar ideas based on a domination by sparse operators. However, the main result in [1] and Theorem 4.2 include some nonintersecting cases.

Remark 4.3. It is easy to see that the cubes of the resulting sparse family $\mathcal{S}$ in Theorem 3.1 (and so in Theorem 4.2) are not dyadic. But approximating an arbitrary cube by cubes from a finite number of dyadic grids (as was shown in [7, Lemma 2.5] or [12, Theorem 3.1]) yields

$$(4.2) \qquad \mathcal{A}_{r,\mathcal{S}}f(x) \le c_{n,r} \sum_{j=1}^{3^n} \mathcal{A}_{r,\mathcal{S}_j}f(x),$$

where $\mathcal{S}_j$ is a sparse family from a dyadic grid $\mathscr{D}_j$.

Remark 4.4. Recall that a weight (that is, a nonnegative locally integrable function) w satisfies the $A_p, 1 < p < \infty$, condition if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty.$$

The following lemma is well known (its variations and extensions can be found in [3, 8, 12, 13]).

Lemma 4.5. *For every η-sparse family $\mathcal{S}$ and for all $1 \le r < p < \infty$,*

$$(4.3) \qquad \|\mathcal{A}_{r,\mathcal{S}}f\|_{L^p(w)} \le c_{n,p,r,\eta}[w]_{A_{p/r}}^{\max\left(1,\frac{1}{p-r}\right)}\|f\|_{L^p(w)}.$$

Notice that $\mathcal{S}$ in the above mentioned works is a sparse family of dyadic cubes. Thus, the case of an arbitrary sparse family can be treated by means of (4.2). On the other hand, (4.2) is not necessary for deriving Lemma 4.5. In order to keep this paper essentially self-contained, we give a proof of Lemma 4.5 (avoiding (4.2)) in the Appendix.

Theorem 4.2 along with Lemma 4.5 implies the following.

Corollary 4.6. *Assume that T is of weak type (q,q) and $\mathcal{M}_T$ is of weak type (r,r), where $1 \le q \le r < \infty$. Then, for all $r < p < \infty$,*

$$\|T\|_{L^p(w)} \le C[w]_{A_{p/r}}^{\max\left(1,\frac{1}{p-r}\right)},$$

where $C = c_{n,p,q,r}(\|T\|_{L^q \to L^{q,\infty}} + \|\mathcal{M}_T\|_{L^r \to L^{r,\infty}}).$

Remark 4.7. Consider a class of rough singular integrals $Tf = \text{p.v.} f * K$, where $K(x) = \frac{\Omega(x)}{|x|^n}$ with Ω homogeneous of degree zero, $\Omega \in L^\infty(S^{n-1})$ and $\int_{S^{n-1}} \Omega(x)d\sigma(x) = 0$.

It was shown in [9] that $\|T\|_{L^2(w)} \le c_n\|\Omega\|_{L^\infty}[w]_{A_2}^2$, and it was conjectured there that the squared dependence on $[w]_{A_2}$ can be replaced by the linear one. Since T is of weak type $(1,1)$ (as was proved by A. Seeger [14]), by Corollary 4.6, it would suffice to prove that $\mathcal{M}_T$ is of weak type $(1,1)$, too. However, it is even not clear to us whether $\mathcal{M}_T$ is an L^2 bounded operator in this setting.

5. Appendix

Let us prove (4.3). Denote $\sigma = w^{-\frac{1}{p-1}}$ and $\nu = w^{-\frac{r}{p-r}}$. Let E_Q be pairwise disjoint subsets of $Q \in \mathcal{S}$.

Since $\frac{1}{w(3Q)}\int_Q g \le \inf_Q M_w^c(gw^{-1})$, where M_w^c is the centered weighted maximal operator with respect to w, we obtain

$$\sum_{Q \in \mathcal{S}} \left(\frac{1}{w(3Q)}\int_Q g\right)^{p'} w(E_Q) \le \sum_{Q \in \mathcal{S}} \int_{E_Q} M_w^c(gw^{-1})^{p'}w$$

$$\le \|M_w^c(gw^{-1})\|_{L^{p'}(w)}^{p'} \le c_{n,p}\|g\|_{L^{p'}(\sigma)}^{p'}.$$

Similarly,

$$\sum_{Q \in \mathcal{S}} \left(\frac{1}{\nu(3Q)}\int_Q f^r\right)^{p/r} \nu(E_Q) \le c_{n,p,r}\|f\|_{L^p(w)}^p.$$

Therefore, multiplying and dividing by

$$T_{p,r}(w;Q) = \frac{w(3Q)}{w(E_Q)^{1/p'}}\frac{\nu(3Q)^{1/r}}{\nu(E_Q)^{1/p}}\frac{1}{|Q|^{1/r}}$$

along with Hölder's inequality yields

$$\sum_{Q\in\mathcal{S}} \Big(\frac{1}{|Q|}\int_Q f^r\Big)^{1/r} \int_Q g \le c_{n,p,r}\sup_Q T_{p,r}(w;Q)\|f\|_{L^p(w)}\|g\|_{L^{p'}(\sigma)},$$

which, by duality, is equivalent to

$$\|\mathcal{A}_{r,\mathcal{S}}f\|_{L^p(w)} \le c_{n,p,r}\sup_Q T_{p,r}(w;Q)\|f\|_{L^p(w)}.$$

It remains to show that

$$\sup_Q T_{p,r}(w;Q) \le c_{n,p,r,\eta}[w]_{A_{p/r}}^{\max\left(1,\frac{1}{p-r}\right)}.$$

By Hölder's inequality,

$$|Q|^{p/r} \le \eta^{-p/r}|E_Q| \le \eta^{-p/r}w(E_Q)\nu(E_Q)^{\frac{p}{r}-1}.$$

From this,

$$\frac{w(3Q)}{w(E_Q)}\Big(\frac{\nu(3Q)}{\nu(E_Q)}\Big)^{\frac{p}{r}-1} \le \eta^{-p/r}\frac{w(3Q)}{|Q|}\Big(\frac{\nu(3Q)}{|Q|}\Big)^{\frac{p}{r}-1} \le (3^n/\eta)^{p/r}[w]_{A_{p/r}},$$

and therefore,

$$\begin{aligned}
T_{p,r}(w;Q) &= \left[\frac{w(3Q)}{|Q|}\Big(\frac{\nu(3Q)}{|Q|}\Big)^{\frac{p}{r}-1}\right]^{1/p}\Big(\frac{w(3Q)}{w(E_Q)}\Big)^{1/p'}\Big(\frac{\nu(3Q)}{\nu(E_Q)}\Big)^{1/p} \\
&\le 3^{n/r}[w]_{A_{p/r}}^{1/p}\left[\frac{w(3Q)}{w(E_Q)}\Big(\frac{\nu(3Q)}{\nu(E_Q)}\Big)^{\frac{p}{r}-1}\right]^{\max\left(\frac{1}{p'},\frac{r}{p(p-r)}\right)} \\
&\le c_{n,p,r,\eta}[w]_{A_{p/r}}^{\frac{1}{p}+\max\left(\frac{1}{p'},\frac{r}{p(p-r)}\right)} = c_{n,p,r,\eta}[w]_{A_{p/r}}^{\max\left(1,\frac{1}{p-r}\right)}.
\end{aligned}$$

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