Asymptotically optimal configurations for Chebyshev constants with an integrable kernel

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ABSTRACT. We show that if a lower-semicontinuous kernel K satisfies some mild additional hypotheses, then configurations that are asympotitically optimal for the extremal problems defining the Chebyshev constants are precisely those whose counting measures converge to the equilibrium measure for the corresponding minimum energy problem.

CONTENTS

1. Background and Results

Suppose $\mathcal{A}$ is a compact set in some Euclidean space $\mathbb{R}^t$. Let

$$K(x,y) : \mathcal{A} \times \mathcal{A} \to [0, \infty]$$

be a symmetric and lower semi-continuous kernel. We will let $\mathcal{M}(\mathcal{A})$ denote the set of positive probability measures with support in $\mathcal{A}$. For any $\mu \in \mathcal{M}(\mathcal{A})$, the kernel generates a potential U^μ by

$$U^\mu(x) = \int_{\mathcal{A}} K(x,y) d\mu(y), \quad x \in \mathcal{A},$$

which is also nonnegative and lower semi-continuous (see [9, Lemma 2.2.1]). For any configuration $\omega_N = (a_1, \ldots, a_N)$ of N (possibly not distinct) points in $\mathcal{A}$, we define the quantity $\underline{Q}(\omega_N)$ by

$$\underline{Q}(\omega_N) := \min_{x \in \mathcal{A}} \frac{1}{N} \sum_{y \in \omega_N} K(x, y)$$

(for motivation, see [7, Definition 2.9]). Equivalently, $\underline{Q}(\omega_N)$ is the minimum of the potential generated by the probability measure ν_N that assigns weight N^{-1} to each point in ω_N (counting multiplicities). If we associate such N-point configurations with the space $\mathcal{A}^N$, then we are interested in finding the Nth Chebyshev constant:

(1) $$\mathcal{Q}(\mathcal{A}, N) := \sup_{\omega_N \in \mathcal{A}^N} \underline{Q}(\omega_N),$$

which is a version of the quantities considered in [7, Equation 2.5], but restricted to a particular $N \in \mathbb{N}$ (see also [14]). Indeed, if $\mathcal{M}_N(\mathcal{A})$ denotes the set of all probability measures ν of the form

$$\nu = \frac{1}{N} \sum_{j=1}^{N} \delta_{a_j}, \quad a_j \in \mathcal{A}, \quad j = 1, \ldots, N,$$

then $\mathcal{Q}(\mathcal{A}, N)$ can be defined as

$$\mathcal{Q}(\mathcal{A}, N) = \sup_{\nu \in \mathcal{M}_N(\mathcal{A})} \min_{x \in \mathcal{A}} U^\nu(x).$$

Any configuration $\omega_N \in \mathcal{A}^N$ for which the supremum on the right-hand side of (1) is attained will be called a *Chebyshev N-point system*. We will be interested in configurations ω_N that attain or nearly attain the supremum on the right-hand side of (1). To do so, we will need the notion of asymptotic optimality, which we define as in [2]. A sequence of configurations $\{\omega_N\}_{N=1}^\infty$ (where each $\omega_N \in \mathcal{A}^N$) is said to be *asymptotically optimal* if

$$\lim_{N \to \infty} \frac{\underline{Q}(\omega_N)}{\mathcal{Q}(\mathcal{A}, N)} = 1.$$

Chebyshev constants and their generalizations have a lengthy history, with many substantial results appearing in [1, 4, 5, 6, 7, 8, 9, 10, 14]. Much of the previous work on the subject is devoted to understanding the asymptotics of the Chebyshev constants as N becomes large. One of the most fundamental results is [14, Theorem 2], which asserts that

(2) $$\lim_{N \to \infty} \mathcal{Q}(\mathcal{A}, N) = \sup_{\mu \in \mathcal{M}(\mathcal{A})} \min_{x \in \mathcal{A}} U^\mu(x).$$

By comparison, relatively few results discuss Chebyshev N-point systems. In [2], Borodachov and Bosuwan showed that if $K(x, y) = |x - y|^{-d}$ and $\mathcal{A}$ is a d-dimensional manifold, then any sequence of Chebyshev N-point systems is asymptotically equidistributed on $\mathcal{A}$ as $N \to \infty$ (see also [3]). Another notable result is the recent work appearing in [10], which shows that if $\mathcal{A} =$

$\mathbb{S}^1$ and $K(x, y) = f(|x - y|)$ and f satisfies certain convexity, continuity, and monotonicity properties, then all Chebyshev N-point systems are equally spaced points on the unit circle. Our results here are motivated by a desire to prove an analogous fact for higher dimensional spheres. This leads us to our first theorem.

Theorem 1.1. *Let $\mathcal{A} = \mathbb{S}^d \subset \mathbb{R}^{d+1}$ and suppose $K(x, y) = |x - y|^{-s}$ for some $s \in (0, d)$. For each $N \geq 1$, choose some $\omega_N \in \mathcal{A}^N$ and let ν_N be the probability measure that assigns mass N^{-1} to each point in ω_N (counting multiplicities). The following are equivalent:*

    (a) *The measures $\{\nu_N\}_{N \in \mathbb{N}}$ converge in the weak-$*$ topology to normalized surface-area measure σ on $\mathbb{S}^d$ as $N \to \infty$.*

    (b) *It holds that*

$$\lim_{N \to \infty} \underline{Q}(\omega_N) = \int_{\mathcal{A}} \int_{\mathcal{A}} |x - y|^{-s} \, d\sigma(x) \, d\sigma(y) = 2^{d-s-1} \frac{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{d-s}{2}\right)}{\sqrt{\pi} \, \Gamma\left(d - \frac{s}{2}\right)}$$

    (c) *It holds that*

$$\lim_{N \to \infty} \left(\frac{1}{N} \sum_{y \in \omega_N} |x - y|^{-s} - \underline{Q}(\omega_N) \right) = 0,$$

    in $L^1(\sigma)$.

Remark. The formula for the integral in part (b) is from [5, Equation 3.2].

We will prove Theorem 1.1 by proving a more general result, which we will formulate as Theorem 1.2. Any measure μ that achieves the supremum on the right-hand side of (2) will be referred to as an *extremal measure*. One consequence of Theorem 1.2 is a demonstration of the uniqueness of the extremal measure for a large class of kernels K and compact sets $\mathcal{A}$ and a proof that these extremal measures are also extremal for the minimum energy problem, which we now describe.

For any measure $\mu \in \mathcal{M}(\mathcal{A})$ we define its K-energy by

$$I[\mu] := \int_{\mathcal{A}} \int_{\mathcal{A}} K(x, y) d\mu(x) d\mu(y).$$

Following the notation of [16], the set of K-equilibrium measures (also called capacitary distributions of unit mass in [9]) is given by

$$\left\{ \mu \in \mathcal{M}(\mathcal{A}) : I[\mu] = \inf_{\nu \in \mathcal{M}(\mathcal{A})} I[\nu] \right\}$$

and is of most interest when there exists a $\nu \in \mathcal{M}(\mathcal{A})$ satisfying $I[\nu] < \infty$. In Corollary 1.3 below, we will connect extremal measures to K-equilibrium measures under the appropriate assumptions.

To prove Theorem 1.2, we will need the following conditions on K and $\mathcal{A}$:

(A1) There is a $\mu \in \mathcal{M}(\mathcal{A})$ so that $I[\mu] < \infty$.

(A2) There is a unique K-equilibrium measure, which we denote by μ_{eq}.

(A3) The support of μ_{eq} is all of $\mathcal{A}$.

(A4) The potential function

$$U^{eq}(x) := \int_{\mathcal{A}} K(x,y) d\mu_{\text{eq}}(y)$$

is equal to a positive constant, which we denote by R, on all of $\mathcal{A}$.

The condition (A1) is often referred to as the *nonpolarity* of $\mathcal{A}$. Condition (A2) is satisfied when the kernel K is strictly definite in the sense of [9, Section 2.4] (see also the discussion of strictly positive definite kernels in [11]). The condition (A4) is a statement about continuity of the equilibrium potential and is often called K-invariance of the equilibrium measure (see [8]). By [9, Theorem 2.4c], the condition (A3) implies the condition (A4) for continuous kernels K. It is clear that $R = I[\mu_{\text{eq}}]$. In general, the conditions (A1)–(A4) are not trivial to verify, though we will highlight some situations in which these conditions can be verified and also discuss a case when they are not satisfied (see Example 2.1).

Now we are ready to state our main result.

Theorem 1.2. *Let the compact set $\mathcal{A}$ and symmetric, nonnegative, and lower semi-continuous kernel K satisfy conditions (A1)–(A4) . For each $N \geq 1$, choose some $\omega_N \in \mathcal{A}^N$ and let ν_N be the probability measure that assigns mass N^{-1} to each point in ω_N (counting multiplicities). The following are equivalent:*

(a) *The measures $\{\nu_N\}_{N \in \mathbb{N}}$ converge in the weak-$*$ topology to μ_{eq} as $N \to \infty$.*

(b) *It holds that*

$$\lim_{N \to \infty} \underline{\text{Q}}(\omega_N) = R.$$

(c) *It holds that*

$$\lim_{N \to \infty} \left(\frac{1}{N} \sum_{y \in \omega_N} K(x,y) - \underline{\text{Q}}(\omega_N) \right) = 0,$$

in $L^1(\mu_{eq})$.

To prove this result, we will need to utilize the well-known Principle of Descent [16, Theorem I.6.8], which we will shortly state. Although stated in [16] for logarithmic potentials, a similar proof works in compact metric spaces for more general kernels by appealing to [17, Theorem 2.3.15] (see also [16, Theorem 0.1.4]).

Principle of Descent. *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures all having support in $\mathcal{A}$ and converging as $n \to \infty$ to some measure μ in the weak-* topology. Suppose also that $\{z_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{A}$ so that z_n converges to z_∞ as $n \to \infty$. Then*

$$U^\mu(z_\infty) \leq \liminf_{n \to \infty} U^{\mu_n}(z_n).$$

Proof of Theorem 1.2. For every $N \geq 1$, define

$$U_N(x) := \int K(x,y) d\nu_N(y) = \frac{1}{N} \sum_{y \in \omega_N} K(x,y).$$

It is clear (by Fubini's Theorem) that

(3) $$\underline{Q}(\omega_N) = \min_{x \in \mathcal{A}} U_N(x) \leq \int_{\mathcal{A}} U_N(x) d\mu_{eq}(x) = R.$$

Assume that (a) is true. Let x_N be a point in $\mathcal{A}$ where U_N attains its minimum. By passing to a subsequence if necessary, we may assume that x_N converges to some x_∞ (also in $\mathcal{A}$) and $U_N(x_N)$ converges to $\liminf U_N(x_N)$ as $N \to \infty$. The Principle of Descent shows

$$\liminf_{N \to \infty} U_N(x_N) \geq R,$$

where we used assumption (A4). This proves part (b).

Now let us assume (b) is true. We know from (3) that

(4) $$\int_{\mathcal{A}} U_N(x) d\mu_{eq}(x) = R.$$

However, our assumption (b) implies $\min_{\mathcal{A}} U_N(x) \to R$ as $N \to \infty$. We then calculate

$$\int_{\mathcal{A}} \left| U_N(x) - \min_{z \in \mathcal{A}} U_N(z) \right| d\mu_{eq}(x) = \int_{\mathcal{A}} \left(U_N(x) - \min_{z \in \mathcal{A}} U_N(z) \right) d\mu_{eq}(x),$$

which tends to zero as $N \to \infty$ by (4), which proves (c).

Now, let us assume that (c) is true. By appealing to (4), we can write

$$R - \underline{Q}(\omega_N) = \int_{\mathcal{A}} \left(U_N(x) - \min_{z \in \mathcal{A}} U_N(z) \right) d\mu_{eq}(x) \to 0,$$

as $N \to \infty$, which proves (b).

Finally, assume (b) is true and let $\mathcal{N} \subseteq \mathbb{N}$ be a subsequence through which ν_N converges in the weak-$*$ topology to a limit ν_∞ as $N \to \infty$ through $\mathcal{N}$. We have already seen that (b) implies (c), so $U_N - R$ converges to 0 in probability (with respect to μ_{eq}) as $N \to \infty$ through $\mathcal{N}$. We may therefore take a further subsequence $\mathcal{N}_1 \subseteq \mathcal{N}$ so that U_N converges to R μ_{eq}-almost everywhere as $N \to \infty$ through $\mathcal{N}_1$ (see [18, page 169]). Again using the Principle of Descent, we calculate for μ_{eq}-almost every x:

(5) $$R = \lim_{\substack{N \to \infty \\ N \in \mathcal{N}_1}} U_N(x) \geq U^{\nu_\infty}(x)$$

μ_{eq}-almost everywhere, in particular at all isolated points of $\mathcal{A}$ (by (A3)). Finally, we note that the potential on the far right-hand side of (5) is lower-semicontinuous as a function of x. Therefore (5) holds for all $x \in \mathcal{A}$. From this, it follows that ν_∞ has the same K-energy as μ_{eq}, and the uniqueness of the K-equilibrium measure implies that ν_∞ must be μ_{eq}. We have thus shown that μ_{eq} is the unique weak-* limit point of the sequence $\{\nu_N\}_{N \in \mathbb{N}}$

and hence the whole sequence must converge to μ_{eq} in the weak-* topology, which proves (a). $\qquad\square$

Remark. Notice that the equivalence (b)$\Leftrightarrow$(c) in Theorem 1.2 does not make use of assumption (A3).

Remark. Theorem 1.1 is an immediate consequence of Theorem 1.2 because all four conditions (A1)–(A4) are satisfied when $\mathcal{A} = \mathbb{S}^d \subset \mathbb{R}^{d+1}$ and $K(x, y) = |x - y|^{-s}$ for any $s \in (0, d)$. In this case, the K-equilibrium measure is normalized surface-area measure on $\mathbb{S}^d$.

Theorem 1.2 has the following important consequence.

Corollary 1.3. *Assume the hypotheses of Theorem* 1.2 *on* $\mathcal{A}$ *and* K.
  (i) *For any asymptotically optimal sequence* $\{\omega_N\}_{N\in\mathbb{N}}$ *of configurations having corresponding counting measures* $\{\nu_N\}_{N\in\mathbb{N}}$*, it holds that* ν_N *converges in the weak-$*$ topology to* μ_{eq} *as* $N \to \infty$.
  (ii) μ_{eq} *is the unique extremal measure.*

Proof. (i) Suppose $\{\gamma_N\}_{N\in\mathbb{N}}$ is a sequence of configurations, where each $\gamma_N \in \mathcal{A}^N$ and the corresponding counting measures $\{\rho_N\}_{N\in\mathbb{N}}$ converge in the weak-* topology to μ_{eq} as $N \to \infty$. Then combining Theorem 1.2 and (3) shows

$$(6) \qquad R \geq \lim_{N\to\infty} \mathcal{Q}(\mathcal{A}, N) \geq \lim_{N\to\infty} \underline{\mathrm{Q}}(\gamma_N) = R.$$

Therefore, the desired conclusion follows from the equivalence of (a) and (b) in Theorem 1.2.

(ii) First note that (6) implies μ_{eq} is an extremal measure. Let μ_p be an extremal measure and $U^{\mu_p}(x)$ the corresponding potential. Then by definition and (2) we have,

$$\min_{x\in\mathcal{A}} U^{\mu_p}(x) = \lim_{N\to\infty} \mathcal{Q}(\mathcal{A}, N) = R.$$

However, $\int U^{\mu_p}(x)d\mu_{\text{eq}}(x) = R$, so $U^{\mu_p}(x) = R$ μ_{eq}-almost everywhere. Since (A3) implies $\text{supp}(\mu_{\text{eq}}) = \mathcal{A}$ and $U^{\mu_p}(x)$ is lower-semicontinuous, this implies $U^{\mu_p}(x) \leq R$ on all of $\mathcal{A}$. Therefore, $I[\mu_p] = R$ and hence $\mu_p = \mu_{\text{eq}}$ by (A2). $\qquad\square$

As a consequence of Corollary 1.3, we see that Chebyshev N-point systems on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ for the kernel $K(x, y) = |x-y|^{-s}$ with $s \in (0, d)$ are asymptotically equidistributed over $\mathbb{S}^d$ as $N \to \infty$ and that normalized surface-area measure on $\mathbb{S}^d$ is the unique extremal measure.

2. Examples

In this section we will explore some examples that highlight the utility and some subtleties of the results of Section 1. Our first example concerns the conditions (A1)–(A4) and shows that without conditions of at least comparable strength, Theorem 1.2 would fail.

2.1. Example: Riesz potentials on the solid ball. Assume $d \geq 3$. Set $\mathcal{A} = \{x \in \mathbb{R}^d : |x| \leq 1\}$ and consider the Riesz kernel $K(x, y) = |x - y|^{-s}$ for some $0 < s \leq d - 2$. It was shown in [5, Section 3] that the N-point configuration consisting of N points at the origin is in fact a Chebyshev N-point system with this choice of kernel. It is obvious that a point mass has infinite K-energy, so the counting measures for these configurations do not, in this case, converge in the weak-$*$ topology to the equilibrium measure. Thus we see that it is not clear how asymptotically optimal sequences of configurations behave when the conditions (A1)–(A4) are not satisfied. This example shows that the equivalences stated in Theorem 1.2 need not hold in general.

2.2. Example: Random and greedy point configurations. Suppose that $\mathcal{A}$ and K are such that conditions (A1)–(A4) are satisfied. Let

$$\{x_1, x_2, x_3, \ldots\}$$

be a sequence of points in $\mathcal{A}$ chosen independently and at random with distribution μ_{eq}. For each $N \geq 1$ set $\omega_N = (x_1, \ldots, x_N)$ and let ν_N be the probability measure assigning weight N^{-1} to each point in ω_N. The Strong Law of Large Numbers implies that as $N \to \infty$, the measures $\{\nu_N\}_{N \in \mathbb{N}}$ almost surely converge in the weak-$*$ topology to μ_{eq}. Theorem 1.2 implies that $\underline{Q}(\omega_N) \to R$ as $N \to \infty$. Therefore, randomly chosen points from the appropriate distribution almost surely create an asymptotically optimal sequence.

In [13], López-García and Saff studied greedy energy points, which are sequences of N-point configurations $\{\omega_N\}_{N \in \mathbb{N}}$ that are optimal for the energy problem subject to the constraint that $\omega_{N-1} \subseteq \omega_N$ (these are sometimes called Leja points after [12]). More precisely, we define a sequence $\{a_n\}_{n=1}^{\infty}$ by choosing $a_1 \in \mathcal{A}$ arbitrarily, and then for each $n > 1$ we choose $a_n \in \mathcal{A}$ so that

$$\frac{1}{n-1} \sum_{i=1}^{n-1} K(a_n, a_i) = \underline{Q}((a_i)_{i=1}^{n-1}).$$

The set ω_N is then taken to be $(a_i)_{i=1}^{N}$. Part (iii) of [13, Theorem 2.1] says that under the assumptions (A1)–(A4) , it holds that

$$\lim_{n \to \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} K(a_n, a_i) = R.$$

In other words, the sequence of configurations $\{\omega_N\}_{N \in \mathbb{N}}$ is asymptotically optimal. By Theorem 1.2, we conclude that the measures

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{a_i}$$

converge in the weak-$*$ topology to μ_{eq}, wihch is the same conclusion as [13, Theorem 2.1(ii)].

2.3. Example: Logarithmic potentials on curves in the plane. Consider the case when $\mathcal{A}$ is a union of $M \geq 1$ disjoint and mutually exterior Jordan curves in $\mathbb{R}^2$ and $K(x, y) = -\log(c|x - y|)$, where $c > 0$ is a constant chosen to ensure that $K(x, y) > 0$ when $x, y \in \mathcal{A}$. In this case, it is easily seen that condition (A1) is satisfied and [16, Theorem I.1.3] assures us that (A2) is satisfied. By [16, Theorem IV.1.3] and an application of Mergelyan's Theorem (see [15, Theorem 20.5]), one can check that $\text{supp}(\mu_{\text{eq}}) = \mathcal{A}$, so condition (A3) is satisfied as well.

The only condition that remains to verify before we can apply our results is (A4). There are several criteria that imply continuity of the logarithmic equilibrium potential. The criterion that we will use is [16, Theorem I.4.8ii], which applies to every point of $\mathcal{A}$ because every point of $\mathcal{A}$ is on the boundary of two components of $\mathbb{R}^2 \setminus \mathcal{A}$, one of which is bounded and one of which is unbounded. Applying this result shows condition (A4) is satisfied, and hence Theorem 1.2 applies in this setting.

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