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$\mathcal{U}$-invariant kernels, defect operators, and graded submodules

Sameer Chavan and **Rani Kumari**

Abstract. Let κ be an $\mathcal{U}$-invariant reproducing kernel and let $\mathscr{H}(\kappa)$ denote the reproducing kernel Hilbert $\mathbb{C}[z_1, \ldots, z_d]$-module associated with the kernel κ. Let M_z denote the d-tuple of multiplication operators $M_{z_1}, \ldots, M_{z_d}$ on $\mathscr{H}(\kappa)$. For a positive integer ν and d-tuple $T = (T_1, \ldots, T_d)$, consider the defect operator

$$D_{T^*,\nu} := \sum_{l=0}^{\nu} (-1)^l \binom{\nu}{l} \sum_{|p|=l} \frac{l!}{p!} T^p T^{*p}.$$

The first main result of this paper describes all $\mathcal{U}$-invariant kernels κ which admit finite rank defect operators $D_{M_z^*,\nu}$. These are $\mathcal{U}$-invariant polynomial perturbations of $\mathbb{R}$-linear combinations of the kernels κ_ν, where $\kappa_\nu(z,w) = \frac{1}{(1-\langle z, w\rangle)^\nu}$ for a positive integer ν. We then formulate a notion of pure row ν-hypercontraction, and use it to show that certain row ν-hypercontractions correspond to an $\mathcal{A}$-morphism. This result enables us to obtain an analog of Arveson's Theorem F for graded submodules of $\mathscr{H}(\kappa_\nu)$. It turns out that for $\mu < \nu$, there are no nonzero graded submodules M of $\mathscr{H}(\kappa_\nu)$ ($\nu \geq 2$) with finite rank defect $D_{(M_z|_M)^*,\mu}$.

Contents

1. $\mathcal{U}$-invariant kernels and spherical tuples

The starting point of the investigations in this paper is the observation that the multiplication tuple $M_{z,\nu}$ acting on the reproducing kernel Hilbert space $\mathscr{H}(\kappa_\nu)$ associated with the kernel $\kappa_\nu(z,w) = \frac{1}{(1-\langle z, w\rangle)^\nu}$ is a rank one

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row ν-hypercontraction, where ν is a positive integer. Needless to say, the multiplication tuple $M_{z,1}$ (commonly known as the *Drury–Arveson shift*) acting on $\mathscr{H}(\kappa_1)$ is the most outstanding example of a rank one row contraction. A beautiful result in the theory of finite rank row contractions (to be referred to as *Arveson's Theorem F*) says that any finite rank graded module of $\mathscr{H}(\kappa_1)$ is necessarily of finite codimension [4]. In the same paper, W. Arveson asked whether this result remains true for general submodules. This question was settled affirmatively by K. Guo in [14]. The paper [14] also contains a version of Arveson's Theorem F for the submodules of $\mathscr{H}(\kappa_\nu)$, where the finite rank condition is replaced by finiteness of the ranks of cross-commutators of $M_{z_i,1}$ and the orthogonal projection onto the submodules. One of the main results in this paper provides an analog of Arveson's Theorem F that takes into consideration the notion of row ν-hypercontraction. Indeed, the present paper is an attempt to develop the theory of finite rank row ν-hypercontractions parallel to that of finite rank row contractions. It is worth noting that the class of row ν-hypercontractions is precisely the class of adjoints of joint ν-hypercontractions. The later one is studied extensively in one and several variables (refer to [1], [5], [18], [3], etc.).

In this preliminary section, we discuss basics of $\mathcal{U}$-invariant reproducing kernel Hilbert spaces (RKHS) and weighted symmetric Fock spaces (the reader is referred to [15] and [17]). For future reference, we see below that the bounded multiplication tuple M_z on an $\mathcal{U}$-invariant RKHS is unitarily equivalent to the creation tuple on certain weighted symmetric Fock space (Proposition 1.9). As far as we know, this fact appears to be unnoticed in the literature.

Throughout this paper, we use the following notations. For the set $\mathbb{N}$ of nonnegative integers, let $\mathbb{N}^d$ denote the Cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (d times). Let $p \equiv (p_1, \ldots, p_d)$ and $n \equiv (n_1, \ldots, n_d)$ be in $\mathbb{N}^d$. We write $|p| := \sum_{i=1}^d p_i$ and $p \leq n$ if $p_i \leq n_i$ for $i = 1, \ldots, d$. For $n \in \mathbb{N}^d$, we let $n! := \prod_{i=1}^d n_i!$.

The open unit ball $\{z \in \mathbb{C}^d : \|z\|_2 < 1\}$ will be denoted by $\mathbb{B}$ while the unit sphere $\{z \in \mathbb{C}^d : \|z\|_2 = 1\}$ will be denoted by $\partial\mathbb{B}$, where $\|z\|_2$ denotes the Euclidean norm of z in $\mathbb{C}^d$.

Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space. Let M be closed subspace of $\mathcal{H}$. We reserve the notation P_M to denote an orthogonal projection from $\mathcal{H}$ onto M, and the symbol $M^\perp$ for the orthogonal complement of M in $\mathcal{H}$. By $\dim M$, we understand the Hilbert space dimension of M.

For a Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. Let $T \in B(\mathcal{H})$. If T is a positive operator then by $T^{1/2}$ (resp. $\operatorname{trace} T$), we mean the positive square-root (resp. trace) of T. We reserve the notation $\operatorname{ran} T$ for the range of T. If T is a finite rank operator then the rank $\operatorname{rank} T$ of T is defined as $\dim \operatorname{ran} T$. If $x, y \in \mathcal{H}$ then the rank one operator $x \otimes y$ is defined through $x \otimes y(h) = \langle h, y\rangle x$ for $h \in \mathcal{H}$. For $S \in B(\mathcal{H})$, set $[S, T] := ST - TS$.

Let $\kappa(z, w)$ be a reproducing kernel defined for z, w in the open unit ball $\mathbb{B}$ in $\mathbb{C}^d$. We say that $\kappa(z, w)$ is *$\mathcal{U}$-invariant* if

$$\kappa(Uz, Uw) = \kappa(z, w) \text{ for any unitary } d \times d \text{ matrix } U \text{ and } z, w \in \mathbb{B}.$$

The following fact is certainly known, which we include for the sake of completeness (refer to [7, Section 1], [15, Section 4]).

Lemma 1.1. *Assume that κ is holomorphic separately in the variables z and $\overline{w}$ on the unit ball $\mathbb{B}$ in $\mathbb{C}^d$. If κ is $\mathcal{U}$-invariant then there exists a sequence $\{a_n\}_{n\geq 0}$ of nonnegative numbers such that*

$$\kappa(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n \quad (z, w \in \mathbb{B}), \tag{1.1}$$

where $\langle z, w\rangle = \sum_{i=1}^{d} z_i \overline{w}_i$ for $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$ in $\mathbb{C}^d$.

Proof. Since κ is holomorphic separately in z and $\overline{w}$, by a result of Hartogs [20, Pg 6], $\kappa(z, w)$ is holomorphic in $(z, \overline{w})$. By general theory of Reinhardt domains [20, Theorem 1.5, Ch II], one can expand κ as a power series in z and $\overline{w}$ on the open unit ball $\mathbb{B}$:

$$\kappa(z, w) = \sum_{p,q \in \mathbb{N}^d} b_{pq} z^p \overline{w}^q \quad (z, w \in \mathbb{B}).$$

Suppose now that κ is $\mathcal{U}$-invariant. In particular, κ is invariant under diagonal unitary $d \times d$ matrices. It is now easy to see by integrating term by term in polar coordinates that $b_{pq} = 0$ if $p \neq q$. Thus κ takes the form $\kappa(z, w) = \sum_{p \in \mathbb{N}^d} b_{pp} z^p \overline{w}^p$ $(z, w \in \mathbb{B})$. Let U be a $d \times d$ unitary matrix that sends $z = (z_1, \ldots, z_d) \in \mathbb{B}$ to $(\|z\|, 0, \ldots, 0) \in \mathbb{B}$. Note that

$$\begin{aligned}\kappa(z, w) = \kappa(Uz, Uw) &= \sum_{\substack{p \,\in\, \mathbb{N}^d,\\ p_2 \,=\, \cdots \,=\, p_d \,=\, 0}} b_{pp} \|z\|^{p_1} \overline{(Uw)}_1^{p_1} \\ &= \sum_{p_1 \in \mathbb{N}} a_{p_1} \langle Uz, Uw\rangle^{p_1} = \sum_{p_1 \in \mathbb{N}} a_{p_1} \langle z, w\rangle^{p_1}\end{aligned}$$

for some scalar sequence $\{a_n\}_{n\geq 0}$. If $\mathscr{H}(\kappa)$ denotes the reproducing kernel Hilbert space associated with κ then $a_{|\alpha|} = \frac{1}{\|z^\alpha\|^2} \frac{\alpha!}{|\alpha|!}$ provided $a_{|\alpha|} \neq 0$ [15, Proposition 4.1]. In particular, each a_n is nonnegative. $\square$

We always assume that any $\mathcal{U}$-invariant kernel κ satisfies the hypothesis of the preceding lemma. We reserve the notation κ_a for the kernel κ associated with $\{a_n\}_{n\geq 0}$ as given in (1.1).

Throughout this paper, we let E_n stand for the orthogonal projection of the reproducing kernel Hilbert space $\mathscr{H}(\kappa_a)$ onto the space H_n generated by homogeneous polynomials of degree n in the variables $z_1, \ldots, z_d$, where it is tacitly assumed that H_n is a subspace of $\mathscr{H}(\kappa_a)$.

In what follows, we need frequently the following lemma, which is essentially included in [15, Propositions 4.1, 4.3 and Corollary 4.4].

Lemma 1.2. *Let κ_a be an $\mathcal{U}$-invariant kernel. Then the orthonormal basis of the reproducing kernel Hilbert space $\mathscr{H}(\kappa_a)$ associated with κ_a is given by*

$$\left\{\sqrt{a_{|\alpha|}\frac{|\alpha|!}{\alpha!}}\ z^\alpha : a_{|\alpha|} \neq 0\right\}.$$

Let M_z denote the d-tuple of the multiplications operators $M_{z_1}, \ldots, M_{z_d}$ defined on $\mathscr{H}(\kappa_a)$. If s is the smallest nonnegative integer such that the sequence $\{a_n\}_{n \geq s}$ consists of positive numbers then we have the following:

(1) *For $i = 1, \ldots, d$, M_{z_i} is bounded if and only if $\sup_{n \geq s} \frac{a_n}{a_{n+1}} < \infty$.*

If (1) holds true then

(2) $\sum_{i=1}^d M_{z_i} M_{z_i}^* = \sum_{n=s+1}^\infty \frac{a_{n-1}}{a_n} E_n$.

(3) $\sum_{i=1}^d M_{z_i} M_{z_i}^* \leq I$ *if and only if the sequence $\{a_{n-1}\}_{n \geq s+1}$ is increasing.*

(4) *Let l be a positive integer such that $l \geq s$.*

$$\sum_{|\beta|=l} \frac{l!}{\beta!} M_z^\beta (M_z^*)^\beta z^\alpha = \begin{cases} \frac{a_{|\alpha|-l}}{a_{|\alpha|}} z^\alpha & \text{if } |\alpha| \geq l, \\ 0 & \text{if } |\alpha| < l. \end{cases}$$

Note. Throughout this paper, we will assume that the multiplication operators $M_{z_1}, \ldots, M_{z_d}$ are bounded linear operators on $\mathscr{H}(\kappa_a)$.

It is easy to describe all z-invariant spaces $\mathscr{H}(\kappa_a)$ (cf. [9, Theorem 2.12]).

Lemma 1.3. *Let κ_a be an $\mathcal{U}$-invariant kernel and let $\mathscr{H}(\kappa_a)$ be the reproducing kernel Hilbert space associated with κ_a. Then the following are equivalent:*

(1) *$\mathscr{H}(\kappa_a)$ is z_i-invariant for $i = 1, \ldots, d$.*

(2) *There exist a nonnegative integer s and a sequence $\{a_n\}_{n \geq s}$ of positive numbers such that*

$$\kappa_a(z, w) = \sum_{n=s}^\infty a_n \langle z, w \rangle^n \quad (z, w \in \mathbb{B}).$$

(3) *There exists a nonnegative integer s such that $\{z^\alpha : \alpha \in \mathbb{Z}_+^d, |\alpha| \geq s\}$ forms an orthogonal basis for $\mathscr{H}(\kappa_a)$.*

Proof. (1) implies (2): Let s be the least nonnegative integer such that $a_s \neq 0$. By Lemma 1.2, $z^\alpha \in \mathscr{H}(\kappa_a)$ for all $|\alpha| = s$. As $\mathscr{H}(\kappa_a)$ is z_i-invariant, for all α such that $|\alpha| \geq s$, $z^\alpha \in \mathscr{H}(\kappa_a)$. By Lemma 1.2, we must have $a_k \neq 0$ for all $k \geq s$.

(2) implies (3): This is immediate from Lemma 1.2.

(3) implies (1): This is obvious. $\square$

The multiplication tuples M_z on $\mathcal{U}$-invariant reproducing kernel Hilbert space provide important examples of so-called spherical tuples. Before we recall the definition of spherical tuples, let us introduce some notations.

By a *commuting d-tuple* T on $\mathcal{H}$, we mean the tuple $(T_1, \dots, T_d)$ of commuting bounded linear operators $T_1, \dots, T_d$ on $\mathcal{H}$. For a commuting d-tuple T on $\mathcal{H}$, we interpret T^* to be $(T_1^*, \dots, T_d^*)$, and T^p to be $T_1^{p_1} \dots T_d^{p_d}$ for $p = (p_1, \dots, p_d) \in \mathbb{N}^d$.

Let T be a d-tuple on $\mathcal{H}$ and let $\mathcal{U}(d)$ denote the group of complex $d \times d$ unitary matrices. For $U = (u_{jk})_{1 \leq j,k \leq d} \in \mathcal{U}(d)$, the commuting operator d-tuple T_U is given by

$$(T_U)_j = \sum_{k=1}^{d} u_{jk} T_k \quad (1 \leq j \leq d).$$

Following [9], we say that T is *spherical* if for every $U \in \mathcal{U}(d)$, there exists a unitary operator $\Gamma(U) \in B(\mathcal{H})$ such that $\Gamma(U) T_j = (T_U)_j \Gamma(U)$ for all $j = 1, \dots, d$. If, further, Γ can be chosen to be a strongly continuous unitary representation of $\mathcal{U}(d)$ on $\mathcal{H}$ then we say that T is *strongly spherical*.

Remark 1.4. If $(T_1, \dots, T_d)$ is a spherical tuple then so is $(\pi(T_1), \dots, \pi(T_d))$ for any unital $*$-homomorphism $\pi : B(\mathcal{H}) \to B(\mathcal{K})$.

The reader is referred to [9] for the basics of spherical tuples. We remark that spherical tuples are nothing but $\mathcal{U}(d)$-homogeneous tuples (cf. [6]).

It turns out that creation tuple on any weighted symmetric Fock space can be modelled as a spherical multiplication tuple on an $\mathcal{U}$-invariant reproducing kernel Hilbert space (see Proposition 1.9 below). Before we see that, let us reproduce some notations and notions from [17].

Let $E_0 := \mathbb{C}$. Let $E^{\otimes n}$ and E^n denote the full and symmetric tensor product of n copies of $E = \mathbb{C}^d$ for $n \geq 1$ respectively. Recall that

$$\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} : 1 \leq i_1, \dots, i_n \leq d\}$$

is an orthogonal basis for $E^{\otimes n}$ and

$$\{e_{i_1} e_{i_2} \dots e_{i_n} : 1 \leq i_1 \leq \cdots \leq i_n \leq d\}$$

is an orthogonal basis for E^n, where $\xi\eta$ denotes the symmetric tensor product of ξ and η. Set

$$\mathcal{F}^{\otimes}(E) := \bigoplus_{n=0}^{\infty} E^{\otimes n}, \ \mathcal{F}(E) := \bigoplus_{n=0}^{\infty} E^n.$$

Definition 1.5. A *weighted full Fock space* $\mathcal{F}_a^{\otimes}(E)$ associated with a sequence of nonnegative real numbers $\{a_n\}_{n \geq 0}$ is defined as the completion of finite sums of elements in $E^{\otimes n}, n \geq 0$. Note that for $\oplus\xi_n, \oplus\eta_n \in \mathcal{F}_a^{\otimes}(E)$,

$$\langle \oplus\xi_n, \oplus\eta_n \rangle := \sum_{n=0}^{\infty} a_n \langle \xi_n, \eta_n \rangle_{E^{\otimes n}}.$$

Similarly, a *weighted symmetric Fock space* $\mathcal{F}_a(E)$ associated with a sequence of nonnegative real numbers $\{a_n\}_{n\geq 0}$ is defined as the completion of finite sums of elements in $E^n, n \geq 0$. Note that for $\oplus\xi_n, \oplus\eta_n \in \mathcal{F}_a(E)$,

$$\langle \oplus\xi_n,\, \oplus\eta_n\rangle := \sum_{n=0}^{\infty} a_n \langle \xi_n,\, \eta_n\rangle_{E^n}.$$

Remark 1.6. The weighted symmetric Fock (resp. full Fock) space $\mathcal{F}_a(E)$ (resp. $\mathcal{F}_a^{\otimes}(E)$) is a Hilbert space.

Since we are not aware of an appropriate reference, we include the following with details:

Lemma 1.7. *For integers $1 \leq j_1, \ldots, j_n \leq d$, we have*

$$\sum_{i_1,\ldots,i_n=1}^{d} a_{n,\nu} \langle e_{i_1} e_{i_2} \ldots e_{i_n}, e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n}\rangle_{\mathcal{F}_\nu^{\otimes}(E)} = 1.$$

where $a_{n,\nu} := \frac{\nu(\nu+1)\ldots(\nu+n-1)}{n!}$ $(n \geq 0)$ and $\mathcal{F}_\nu^{\otimes}(E)$ denotes the weighted full Fock space endowed with the inner-product

$$\langle \oplus\xi_n,\, \oplus\eta_n\rangle := \sum_{n=0}^{\infty} \frac{1}{a_{n,\nu}} \langle \xi_n,\, \eta_n\rangle_{E^{\otimes n}}.$$

Proof. Without loss of generality, suppose that $j_1, \ldots, j_m$ are distinct integers in the finite sequence $\{j_1, \ldots, j_n\}$ such that j_p appears k_p times, where $p = 1, \ldots, m$. Clearly, $k_1 + \cdots + k_m = n$. Note that

$$\begin{aligned}
&\sum_{i_1,\ldots,i_n=1}^{d} a_{n,\nu} \langle e_{i_1} \ldots e_{i_n}, e_{j_1} \otimes \cdots \otimes e_{j_n}\rangle_{\mathcal{F}_\nu^{\otimes}(E)} \\
&= \sum_{i_1,\ldots,i_n=1}^{d} a_{n,\nu} \langle e_{i_1} \ldots e_{i_n}, e_{j_1} \ldots e_{j_n}\rangle_{\mathcal{F}_\nu(E)} \\
&= a_{n,\nu} \frac{n!}{k_1! \ldots k_m!} \|e_{j_1} \ldots e_{j_n}\|^2_{\mathcal{F}_\nu(E)} \\
&= a_{n,\nu} \frac{n!}{k_1! \ldots .. k_m!} \|e_{j_1}^{k_1} \ldots . e_{j_m}^{k_m}\|^2_{\mathcal{F}_\nu(E)}.
\end{aligned}$$

The desired conclusion now follows from the formula

$$\|e_1^{n_1} \ldots e_d^{n_d}\|^2_{\mathcal{F}_\nu(E)} = \frac{1}{a_{n,\nu}} \frac{n_1! \ldots n_d!}{(n_1 + \cdots + n_d)!}$$

(cf. [3, Lemma 3.8]). □

Definition 1.8. The *creation d-tuple* $S = (S_1, \ldots, S_d)$ on the weighted symmetric Fock space $\mathcal{F}_a(E)$ is defined as follows : For $1 \leq i \leq d$, $S_i : \mathcal{F}_a(E) \longrightarrow \mathcal{F}_a(E)$ is given by

$$S_i(\xi_n) = e_i \xi_n \quad \text{for } \xi_n \in E^n.$$

Proposition 1.9. *Let $\mathcal{F}_a(E)$ denote a weighted symmetric Fock space associated with the sequence $\{a_n\}_{n\geq 0}$. Then the following statements are equivalent:*

(1) *The weighted symmetric Fock space is invariant under the creation d-tuple S.*
(2) *There is a smallest nonnegative integer s such that*
$$\{e_1^{n_1}e_2^{n_2}\dots e_d^{n_d} : |n| \geq s\}$$
is an orthogonal basis for $\mathcal{F}_a(E)$.
(3) *There exist a smallest integer $s \geq 0$ such that the sequence $\{a_k\}_{k\geq s}$ consists of positive numbers and a unitary mapping*
$$U : \mathscr{H}(\kappa_b) \longrightarrow \mathcal{F}_a(E)$$
such that $S_iU = UM_{z_i}$ for $i = 1,\dots,d$, where $b = \{1/a_k : k \geq s\}$.

Proof. (1) implies (2): Note that $\mathcal{F}_a(E)$ is the completion of linear span of $\{e_{i_1}\dots e_{i_n} \in E_n : i_1,\dots,i_n \in \{1,\dots,d\}, a_n \neq 0\}$. Since $\mathcal{F}_a(E)$ is invariant under the creation d-tuple S, (2) follows.

(2) implies (3): Since
$$\|e_1^{n_1}e_2^{n_2}\dots e_d^{n_d}\|^2_{\mathcal{F}_a(E)} = a_{|n|}\|e_1^{n_1}e_2^{n_2}\dots e_d^{n_d}\|^2_{E^{|n|}},$$
$a_k > 0$ for $k \geq s$. Let $b = \{1/a_k : k \geq s\}$ and define $U : \mathscr{H}(\kappa_b) \longrightarrow \mathcal{F}_a(E)$ by
$$U(z_1^{\alpha_1}\dots z_d^{\alpha_d}) = e_1^{\alpha_1}\dots e_d^{\alpha_d} \quad \text{for } (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d,$$
and extend U linearly to the subspace spanned by $\{z^n : |n| \geq s\}$. Since
$$\begin{aligned}\|U(z_1^{\alpha_1}\dots z_d^{\alpha_d})\|^2_{\mathcal{F}_a(E)} &= \|e_1^{\alpha_1}\dots e_d^{\alpha_d}\|^2_{\mathcal{F}_a(E)}\\ &= \frac{a_{|\alpha|}\alpha_1!\dots\alpha_d!}{|\alpha|!} = \|z_1^{\alpha_1}\dots z_d^{\alpha_d}\|^2_{\mathcal{H}_{(\kappa_b)}},\end{aligned}$$
we may extend U continuously to $\mathscr{H}(\kappa_b)$. A routine verification shows that $S_iU = UM_{z_i}$ for $i = 1,\dots,d$.

(3) implies (1): This follows from Lemma 1.3. □

Remark 1.10. By Lemma 1.2(1), S_i is bounded for $i = 1,\dots,d$ if and only if
$$\sup_{n\geq s}\frac{a_n}{a_{n+1}} < \infty.$$

With the notations of Proposition 1.9, we agree to say that $\mathcal{F}_a(E)$ is the *Fock space realization of* $\mathcal{H}(\kappa_b)$. We further refer to the creation d-tuple S as the *Fock space realization of the multiplication d-tuple M_z*. For $\nu \geq 1$, we denote by $\mathcal{F}_\nu$ the Fock space realization of the reproducing kernel Hilbert space $\mathscr{H}(\kappa_\nu)$ associated with the $\mathcal{U}$-invariant kernel
$$\kappa_\nu(z,w) = \frac{1}{(1-\langle z,\, w\rangle)^\nu} \quad (z,w \in \mathbb{B}).$$

Sometimes, we use the simpler notation $\mathscr{H}_\nu$ in place of $\mathscr{H}(\kappa_\nu)$. The Fock space realization of the multiplication d-tuple $M_{z,\nu}$ on $\mathscr{H}(\kappa_\nu)$ will be denoted by $S^{(\nu)}$. We use these notations interchangeably.

Here is the plan of the present paper. In Section 2, we describe all $\mathcal{U}$-invariant kernels which admit finite rank defect operators $D_{M_z^*,\nu}$ (Theorem 2.10). Loosely speaking, the $\mathcal{U}$-invariant kernels $\{\kappa_\nu : \nu \geq 1\}$ form basis for these kernels. In Section 3, we introduce a new notion of pure row ν-contraction. This notion combined with the theory of weighted symmetric Fock spaces [17] enables us to show that certain row ν-contractions correspond to an $\mathcal{A}$-morphism in the sense of Arveson (Theorem 3.6). As a consequence, we recover a ball analog of von Neumann inequality for row ν-hypercontractions (refer to [11], [3], [13], [17], [19] for variants of von Neumann-type inequalities). In the last section, we obtain an analog of Arveson's Theorem F for finite rank graded submodules of $\mathscr{H}(\kappa_\nu)$ (Theorem 4.9). We remark that the Arveson's method of proof of Theorem F does not readily generalize to the kernels κ_ν. This is perhaps due to the fact that the kernel κ_ν is not a complete NP kernel for $\nu \geq 2$. Our method, build off of the ideas of K. Guo [14], gives an alternative proof of Arveson's Theorem F. One rather striking consequence of Theorem 4.9 asserts that for $\nu \geq 2$ and $1 \leq \mu \leq \nu - 1$, the defect operator $D_{(M_z|_M)^*,\mu}$ can never be of finite rank for any nonzero graded submodule M of $\mathscr{H}(\kappa_\nu)$ (Corollary 4.15).

2. Finite rank defect operators

Given a commuting d-tuple $T = (T_1, \ldots, T_d)$ on $\mathcal{H}$, we set

$$Q_T(X) := \sum_{i=1}^{d} T_i^* X T_i \quad (X \in B(\mathcal{H})). \tag{2.2}$$

It is easy to see that $Q_T^n(I) = \sum_{|p|=n} \frac{n!}{p!} T^{*p} T^p$. Consider the *defect operator* $D_{T,k}$ of order k given by

$$D_{T,k} := \sum_{l=0}^{k} (-1)^l \binom{k}{l} Q_T^l(I). \tag{2.3}$$

Unless it is specified, the sequence $\{a_n\}_{n\geq 0}$ associated with the $\mathcal{U}$-invariant kernel κ_a consists of positive numbers. The main result (Theorem 2.10) of this section extends naturally to the case in which first finitely many elements of $\{a_n\}_{n\geq 0}$ are 0 (see Remark 2.11).

Let κ_a be an $\mathcal{U}$-invariant reproducing kernel and let $\mathscr{H}(\kappa_a)$ denote the reproducing kernel Hilbert space associated with κ_a. Let M_z denote the d-tuple of bounded linear multiplication operators $M_{z_1}, \ldots, M_{z_d}$ on $\mathscr{H}(\kappa_a)$. Recall that E_n denotes the orthogonal projection of $\mathscr{H}(\kappa_a)$ onto the space H_n generated by homogeneous polynomials of degree n. We see in Lemma 2.3 below that there exists a sequence $\{\alpha_n\}_{n\geq 0}$ of real numbers such that

$$D_{M_z^*,k} = \sum_{n=0}^{\infty} \alpha_n E_n.$$

We are interested in the spaces $\mathscr{H}(\kappa_a)$ which admit finite rank defect operators $D_{M_z^*,k}$. Before we see concrete examples of such spaces, we find it convenient to introduce the following family $\mathcal{D}_{k,l}$ of $\mathcal{U}$-invariant reproducing kernels κ_a for positive integers k, l:

$$\{\kappa_a : D_{M_z^*,k} = \alpha_0 E_0 + \cdots + \alpha_{l-1} E_{l-1} \text{ for some scalars } \alpha_0, \dots, \alpha_{l-1} \in \mathbb{R}\}.$$

Caution. In the definition of $\mathcal{D}_{k,l}$, k is the order of the defect operator $D_{M_z^*,k}$, but l is *not* the rank of $D_{M_z^*,k}$.

Remark 2.1. For any integer $m \geq l$, $\mathcal{D}_{k,l} \subseteq \mathcal{D}_{k,m}$.

The results in this section are motivated mainly by the following basic question.

Question 2.2. *What is the structure of the cone $\mathcal{D}_{k,l}$ of $\mathcal{U}$-invariant reproducing kernels?*

Before we answer this question, we gather some preliminary results. The first of which provides a handy formula for the defect operator $D_{M_z^*,k}$.

Lemma 2.3. *Let M_z be the multiplication d-tuple on $\mathscr{H}(\kappa_a)$. Then*

$$D_{M_z^*,k} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} (-1)^i \binom{k}{i} \frac{a_{n-i}}{a_n} \right) E_n,$$

where we used the standard convention that $\binom{k}{i} = 0$ for any positive integer $i > k$.

Proof. By Lemma 1.2, $Q^l_{M_z^*}(I) = \sum_{n=l}^{\infty} \frac{a_{n-l}}{a_n} E_n$. It follows that

$$\begin{aligned} D_{M_z^*,k} &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sum_{n=l}^{\infty} \frac{a_{n-l}}{a_n} E_n \\ &= \sum_{n=0}^{\infty} E_n - \binom{k}{1} \sum_{n=1}^{\infty} \frac{a_{n-1}}{a_n} E_n + \cdots + (-1)^k \sum_{n=k}^{\infty} \frac{a_{n-k}}{a_n} E_n. \end{aligned}$$

We can see that for $n \leq k$, the coefficient of E_n is

$$1 - \binom{k}{1} \frac{a_{n-1}}{a_n} + \binom{k}{2} \frac{a_{n-2}}{a_n} + \cdots + (-1)^n \binom{k}{n} \frac{a_0}{a_n}.$$

Otherwise, the coefficient of E_n is

$$1 - \binom{k}{1} \frac{a_{n-1}}{a_n} + \binom{k}{2} \frac{a_{n-2}}{a_n} + \cdots + (-1)^k \frac{a_{n-k}}{a_n}.$$

This completes the proof of the lemma. $\square$

Remark 2.4. Note that the coefficient of E_0 equals 1.

Lemma 2.5. *Let κ_a be an $\mathcal{U}$-invariant kernel. Suppose the defect operator $D_{M_z^*,k} = \sum_{n=0}^{\infty} \alpha_n E_n$ for a scalar sequence $\{\alpha_n\}_{n \geq 0}$. Then we have:*

(1) *If $0 < n < k$, then $\alpha_n = 0$ if and only if there exists a nonnegative polynomial p in i of degree at most $n-1$ such that*
$$a_{n-i} = p(i)(n+1-i)\dots(k-i) \quad (0 \leq i \leq n),$$

(2) *If $n \geq k$, then $\alpha_n = 0$ if and only if a_{n-i} is a polynomial in i of degree at most $k-1$.*

Proof. The proof relies on the following well-known fact, which may be derived from Newton's Interpolation Formula: For a sequence $\{b_k\}_{k=0}^n$ of positive real numbers, $\sum_{k=0}^n (-1)^k \binom{n}{k} b_k = 0$ if and only if b_k is a polynomial in k of degree less than or equal to $n-1$.

Suppose that $0 < n < k$. By Lemma 2.3, $\alpha_n = 0$ if and only if
$$\sum_{i=0}^{n} (-1)^i \binom{k}{i} a_{n-i} = 0.$$
The later one is equivalent to
$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\binom{k}{i}}{\binom{n}{i}} a_{n-i} = 0,$$
and hence by the observation stated in the last paragraph, $\alpha_n = 0$ if and only if there exists a polynomial p in i of degree less than or equal to $n-1$ such that $\frac{n!}{k!} \frac{\binom{k}{i}}{\binom{n}{i}} a_{n-i} = p(i)$. The desired conclusion in (1) is now immediate. The same argument yields the conclusion in (2). □

Remark 2.6. The Dirichlet kernel κ_a with $a_n = \frac{1}{n+1}$ does not belong to $\mathcal{D}_{k,l}$ for any $k, l \geq 1$.

We are now ready to give examples of $\mathcal{U}$-invariant kernels with finite rank defect operators, which in some sense play the role of building blocks for the family $\mathcal{D}_{k,l}$.

Example 2.7. For an integer $\nu \geq 1$, consider the $\mathcal{U}$-invariant kernel
$$\kappa_\nu(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\nu} \quad (z, w \in \mathbb{B})$$
and let $M_{z,\nu}$ be the multiplication d-tuple on $\mathcal{H}(\kappa_\nu)$. We contend that $D_{M_{z,\nu}^*,\nu} = E_0$, that is, $\kappa_\nu \in \mathcal{D}_{\nu,1}$.

We may rewrite $\kappa_\nu(z, w)$ as $\sum_{n=0}^{\infty} a_{n,\nu} \langle z, w \rangle^n$ for a sequence $\{a_{n,\nu}\}_{n \geq 0}$ of nonnegative real numbers (see (1.1)). It is easy to see using Multinomial Theorem that
$$a_{n,\nu} = \frac{\nu(\nu+1)\dots(\nu+n-1)}{n!} \quad (n \geq 0).$$

Note that $a_{n-i,\nu} = \frac{(n-i+1)\dots(n-i+\nu-1)}{(\nu-1)!}$ is a polynomial in i of degree $\nu-1$. By Lemma 2.5, the coefficient of E_n is 0 for $n \geq \nu$. Also, for $1 \leq n \leq \nu$, we have

$$\begin{aligned} \frac{a_{n-i,\nu}}{(n+1-i)\dots(\nu-i)} &= \frac{1}{(\nu-1)!}\frac{(n-i+1)\dots(n-i+\nu-1)}{(n-i+1)\dots(\nu-i)} \\ &= \frac{(\nu-i+1)\dots(\nu-i+n-1)}{(\nu-1)!}, \end{aligned}$$

which is a polynomial in i of degree $n-1$. By another application of Lemma 2.5, the coefficient of E_n is 0 for $1 \leq n \leq \nu$.

Remark 2.8. Let ν be a positive integer bigger than 1 and let μ be a positive integer less than ν. Since $a_{n-i,\nu}$ is a polynomial in i of degree $\nu-1$, by Lemma 2.5(2), the defect operator $D_{M^*_{z,\nu,\mu}}$ is of infinite rank.

The conclusion of Example 2.7 is well-known in the case of Drury–Arveson kernel. In the special case of Bergman kernel on the unit disc, it is observed in [16, Pg 618] with the help of Berezin transform.

Here is an example of $\kappa_a \in \mathcal{D}_{k,l}$, which does not belong to $\mathcal{D}_{k,m}$ for any $m = 1, \dots, l-1$.

Example 2.9. For $k \geq 2$, consider the $\mathcal{U}$-invariant kernel κ_a with $a_0 = 1$ and $a_i = i^{k-1}$ for $i \geq 1$. It may be concluded from Lemma 2.5 that $\kappa_a \in \mathcal{D}_{k,k+1}$, but κ_a does not belong to $\mathcal{D}_{k,m}$ for any $1 \leq m \leq k$.

We remark that up to a scalar multiple, κ_ν is the *only* $\mathcal{U}$-invariant kernel in $\mathcal{D}_{\nu,1}$. Although this follows from the main result of this section, we regard this observation as the starting point of our investigations here, and hence we wish to outline a direct proof of it.

Suppose $\kappa_b \in \mathcal{D}_{\nu,1}$. By Lemma 2.3, for $n \geq 1$, the coefficient of E_n is 0, that is,

$$\sum_{i=0}^{n} (-1)^i \binom{\nu}{i} \frac{b_{n-i}}{b_n} = 0.$$

Clearly, $b_1 = \nu b_0 = a_{1,\nu} b_0$. We will prove by induction that $b_n = a_{n,\nu} b_0$ for $n \geq 1$. Suppose $b_j = a_{j,\nu} b_0$ for $1 \leq j \leq n-1$. Now

$$b_n = \left(\sum_{i=1}^{n} (-1)^{i-1} \binom{\nu}{i} a_{n-i,\nu} \right) b_0.$$

In view of the calculations of Example 2.7, we have $b_n = a_{n,\nu} b_0$. This completes the induction. We thus obtain $\kappa_b(z,w) = b_0 \kappa_\nu(z,w)$.

We now state the main result of this section.

Theorem 2.10. *Let κ_a be an $\mathcal{U}$-invariant kernel. Recall that*

$$\kappa_\nu(z,w) = \sum_{n=0}^{\infty} a_{n,\nu} \langle z, w \rangle^n, \quad \textit{where} \quad a_{n,\nu} = \frac{\nu(\nu+1)\dots(\nu+n-1)}{n!}.$$

Then any one of the following cases occurs:

(1) $k < l$: *$\kappa_a \in \mathcal{D}_{k,l}$ if and only if there exist real numbers $\alpha_1, \ldots, \alpha_k$ and a complex polynomial p_m of degree at most m in one variable such that*

$$\kappa_a(z, w) = \sum_{\nu=1}^{k} \alpha_\nu \kappa_\nu(z, w) + p_{l-k-1}(\langle z,\, w\rangle).$$

(2) $k = l$: *$\kappa_a \in \mathcal{D}_{k,l}$ if and only if there exist real numbers $\alpha_1, \ldots, \alpha_k$ such that*

$$\kappa_a(z, w) = \sum_{\nu=1}^{k} \alpha_\nu \kappa_\nu(z, w).$$

(3) $k > l$: *$\kappa_a \in \mathcal{D}_{k,l}$ if and only if there exist real numbers $\alpha_1, \ldots, \alpha_k$ such that*

$$\kappa_a(z, w) = \sum_{\nu=1}^{k} \alpha_\nu \kappa_\nu(z, w),$$

$$\sum_{\nu=1}^{k-1} \alpha_\nu \Big(\sum_{i=0}^{n} (-1)^i \binom{k}{i} a_{n-i,\nu} \Big) = 0 \quad \textit{for } n = l, l+1, \ldots, k-1.$$

Remark 2.11. Suppose κ_b is an $\mathcal{U}$-invariant kernel in $\mathcal{D}_{k,l}$. Let s be the smallest positive integer such that $\{b_n\}_{n \geq s}$ consists of positive numbers. Consider the kernel κ_a with positive coefficients $\{a_n\}$ such that $a_n := a_{n,k}$ for $n = 0, \ldots, s-1$ and $a_n = b_n$ for $n \geq s$. Then κ_a belongs to $\mathcal{D}_{k,l}$. Since $\kappa_b(z, w) = \kappa_a(z, w) - \sum_{n=0}^{s-1} a_{n,k} \langle z,\, w\rangle^n$, κ_b is also a polynomial perturbation of a linear combination of κ_ν ($\nu = 1, \ldots, k$).

Before we present a proof of Theorem 2.10, we would like to discuss some of its consequences.

Example 2.12. Let us see two instructive examples:

(1) The $\mathcal{U}$-invariant kernel

$$\kappa_a(z, w) = \frac{2 - \langle z,\, w\rangle}{(1 - \langle z, w\rangle)^2} + \langle z,\, w\rangle$$

belongs to $\mathcal{D}_{2,4}$ with $\alpha_1 = 1 = \alpha_2$ and $p_1(x) = x$:

$$\kappa_a(z, w) = \kappa_1(z, w) + \kappa_2(z, w) + \langle z,\, w\rangle.$$

(2) The $\mathcal{U}$-invariant kernel $\frac{\langle z, w\rangle}{(1 - \langle z, w\rangle)^2}$ belongs to $\mathcal{D}_{2,2}$ with $\alpha_1 = -1$ and $\alpha_2 = 1$. However, this kernel can not be in $\mathcal{D}_{2,1}$ since

$$\sum_{i=0}^{1} (-1)^i \binom{2}{i} a_{1-i,1} \neq 0.$$

Corollary 2.13. *Let $\kappa_a \in \mathcal{D}_{1,\nu}$ for $\nu \geq 2$. Then there exists a positive number α and a complex polynomial p_m of degree at most m in one variable such that $\kappa_a(z, w) = \alpha \kappa_1(z, w) + p_{\nu-2}(\langle z,\, w\rangle)$.*

Corollary 2.14. *Let $\kappa_a \in \mathcal{D}_{\nu,1}$. Then there exists a positive number α such that $\kappa_a(z,w) = \alpha\kappa_\nu(z,w)$.*

We now turn to proofs of Theorem 2.10 and Corollary 2.14, which involves several lemmas.

Lemma 2.15. *We have the inclusion $\mathcal{D}_{k,l} \subseteq \mathcal{D}_{k+1,l+1}$. In particular, the $\mathcal{U}$-invariant kernel κ_ν belongs to $\mathcal{D}_{\nu+m,m+1}$ for any integer $m \geq 1$.*

Proof. Consider the defect operator $D_{T,k}$ as defined in (2.3). Note that $D_{T,k+1} = D_{T,k} - Q_T(D_{T,k})$, where Q_T is as given in (2.2). Suppose that κ_a is in $\mathcal{D}_{k,l}$, that is, $D_{M_z^*,k} = \sum_{i=0}^{l-1}\alpha_i E_i$ for some scalars $\alpha_0, \dots, \alpha_{l-1}$. It follows that

$$D_{M_z^*,k+1} = \sum_{i=0}^{l-1}\alpha_i E_i - \sum_{i=0}^{l-1}\alpha_i Q_{M_z^*}(E_i).$$

However, $Q_{M_z^*}(E_i) = \frac{a_i}{a_{i+1}}E_{i+1}$. Thus we obtain

$$\begin{aligned} D_{M_z^*,k+1} &= \sum_{i=0}^{l-1}\alpha_i E_i - \sum_{i=0}^{l-1}\alpha_i \frac{a_i}{a_{i+1}}E_{i+1} \\ &= \alpha_0 E_0 + \sum_{i=1}^{l-1}\Big(\alpha_i - \alpha_{i-1}\frac{a_{i-1}}{a_i}\Big)E_i - \alpha_{l-1}\frac{a_{l-1}}{a_l}E_l. \end{aligned}$$

Thus $\kappa_a \in \mathcal{D}_{k+1,l+1}$. The remaining part is immediate from the fact $\kappa_\nu \in \mathcal{D}_{\nu,1}$, as recorded in Example 2.7. □

Remark 2.16. In general, the inclusion $\mathcal{D}_{k,l} \subseteq \mathcal{D}_{k+1,l+1}$ is strict: For instance, take $\kappa_a(z,w) = \frac{1}{(1-\langle z,w\rangle)} + \frac{1}{(1-\langle z,w\rangle)^2} - 1$. Then $a_0 = 1$ and $a_n = n+2$ for $n \geq 1$. By Lemma 2.5, $\kappa_a \in \mathcal{D}_{2,3}$ but $\kappa_a \notin \mathcal{D}_{1,2}$.

Lemma 2.17. *Let k and l be positive integers such that $k \leq l$. Let κ_a be an $\mathcal{U}$-invariant kernel of the form*

$$\kappa_a(z,w) = \sum_{\nu=1}^{k}\alpha_\nu\kappa_\nu(z,w) + p_{l-k-1}(\langle z,\, w\rangle)$$

for some real numbers $\alpha_1, \dots \alpha_k$, and a complex polynomial p_m of degree at most m (with the interpretation that the term p_m is absent if $m < 0$). Then κ_a belongs to $\mathcal{D}_{k,l}$.

Proof. Fix $1 \leq \nu \leq k$. By Lemma 2.15, κ_ν belongs to $\mathcal{D}_{\nu+m,m+1}$ for any integer $m \geq 1$. Letting $m := k - \nu$, we obtain that κ_ν belongs to $\mathcal{D}_{k,k-\nu+1}$. Thus for $n \geq k - \nu + 1$, the coefficient $\beta_{n,\nu}$ of E_n in $D_{M_{z,\nu}^*,k}$ is zero. By Lemma 2.5, for $n \geq k$, $a_{n-i,\nu}$ is a polynomial in i of degree at most $k-1$. By hypothesis, $a_n = \sum_{\nu=1}^{k}\alpha_\nu a_{n,\nu}$ for any integer $n \geq l-k$. Thus for $n \geq l \geq k$, a_{n-i} is a polynomial in i of degree at most $k-1$. By another application of Lemma 2.5, for $n \geq l$, the coefficient β_n of E_n in $D_{M_z^*,k}$ is zero. The desired conclusion is immediate. □

Lemma 2.18. *For $\nu = 1, \dots, k$ and for $l \geq k$, consider $a_{l-i,\nu}$ as an $\mathbb{R}$-valued polynomial in i from $\{0, 1, \dots, k-1\}$. Then the set $\{a_{l-i,\nu} : \nu = 1, \dots, k\}$ is linearly independent in the following sense: If for real numbers α_ν ($1 \leq \nu \leq k$), we have*

$$\sum_{\nu=1}^{k} \alpha_\nu a_{l-i,\nu} = 0 \quad (0 \leq i \leq k-1),$$

then $\alpha_1 = 0, \dots, \alpha_k = 0$.

Proof. Note that $a_{l-i,\nu}$ is a polynomial in i of degree $\nu - 1$. In particular,

$$\Delta^\nu(a_{l-i,\nu}) = 0 \quad \text{and} \quad \Delta^{\nu-1}(a_{l-i,\nu}) \neq 0,$$

where the difference operator Δ is given by $\Delta\gamma_i = \gamma_{i+1} - \gamma_i$ for a scalar sequence $\{\gamma_i\}_{i\geq 0}$. Let

$$\gamma_i := \sum_{\nu=1}^{k} \alpha_\nu a_{l-i,\nu} = 0 \quad (0 \leq i \leq k-1).$$

Note that $\Delta^{k-1}\gamma_i = \sum_{\nu=1}^{k} \alpha_\nu \Delta^{k-1}(a_{l-i,\nu})$. Since $\Delta^{k-1}(a_{l-i,\nu}) = 0$ for $1 \leq \nu \leq k-1$, and $\Delta^{k-1}(a_{l-i,k}) \neq 0$, it follows that $\alpha_k = 0$. A finite inductive argument now gives the required linear independence. □

Proof of Theorem 2.10. We discuss first the case in which $k \leq l$. The easier half is precisely Lemma 2.17. We see the necessary part. To see that, fix $n \geq l$. As $\kappa_a \in \mathcal{D}_{k,l}$, the coefficient β_n of E_n is zero. Hence by Lemma 2.5, a_{n-i} is a polynomial in i of degree less than or equals to $k-1$. We note that $a_{n-i,\nu}$ is a polynomial in i of degree $\nu-1$ as noted in Example 2.7. It follows that $\{a_{n-i,1}, \dots, a_{n-i,k}\}$ forms a basis for the vector space of polynomials in i of degree less than or equal to $k-1$. In particular, a_{n-i} belongs to the $\mathbb{R}$-linear span of $\{a_{n-i,1}, \dots, a_{n-i,k}\}$. Thus there exist scalars $\alpha_{1,n}, \dots, \alpha_{k,n} \in \mathbb{R}$ such that

$$a_{n-i} = \sum_{\nu=1}^{k} \alpha_{\nu,n} a_{n-i,\nu} \quad (0 \leq i \leq k, n \geq l). \tag{2.4}$$

We claim that the sequence $\{\alpha_{\nu,m} : m \geq l\}$ is constant for any $\nu = 1, \dots, k$. We achieve this by verifying that $\alpha_{\nu,l+j} = \alpha_{\nu,l+j+1}$ for any integer $j \geq 0$. Fix an integer $j \geq 0$. If we take $n = l + j$ in Equation (2.4) then we get $a_{l+j-i} = \sum_{\nu=1}^{k} \alpha_{\nu,l+j} a_{l+j-i,\nu}$ for any $0 \leq i \leq k$. Further, if we take $n = l + j + 1$ and replace i by $i + 1$ in Equation (2.4) then we get $a_{l+j-i} = \sum_{\nu=1}^{k} \alpha_{\nu,l+j+1} a_{l+j-i,\nu}$ for any $0 \leq i \leq k-1$. Thus we obtain for any $0 \leq i \leq k-1$,

$$\sum_{\nu=1}^{k} (\alpha_{\nu,l+j} - \alpha_{\nu,l+j+1}) a_{l+j-i,\nu} = 0.$$

But the set $\{a_{l+j-i,\nu} : \nu = 1, \dots, k\}$ is linearly independent (Lemma 2.18). This implies that $\alpha_{\nu,l+j} = \alpha_{\nu,l+j+1}$ for $\nu = 1, \dots, k$. Thus the claim stands verified, and hence

$$a_n = \sum_{\nu=1}^{k} \alpha_{\nu,l} a_{n,\nu} \quad \text{for any integer } n \geq l-k.$$

To complete the proof, note that if $l > k$,

$$\begin{aligned}
\kappa_a(z,w) &= \sum_{n=0}^{l-k-1} a_n \langle z, w\rangle^n + \sum_{n=l-k}^{\infty} a_n \langle z, w\rangle^n \\
&= \sum_{n=0}^{l-k-1} a_n \langle z, w\rangle^n + \sum_{n=l-k}^{\infty} \left(\sum_{\nu=1}^{k} \alpha_{\nu,l} a_{n,\nu} \right) \langle z, w\rangle^n \\
&= \sum_{\nu=1}^{k} \alpha_{\nu,l} \sum_{n=0}^{\infty} a_{n,\nu} \langle z, w\rangle^n + \sum_{n=0}^{l-k-1} \left(a_n - \sum_{\nu=1}^{k} \alpha_{\nu,l} a_{n,\nu} \right) \langle z, w\rangle^n \\
&= \sum_{\nu=1}^{k} \alpha_{\nu,l} \kappa_\nu(z,w) + p_{l-k-1}(\langle z,\, w\rangle).
\end{aligned}$$

The same calculation yields the desired conclusion in case $l = k$ as well.

Finally, we treat the case in which $k > l$. Clearly, $\mathcal{D}_{k,l} \subseteq \mathcal{D}_{k,k}$, and hence by the case $k = l$, there exist real numbers $\alpha_1, \dots, \alpha_k$ such that

$$\kappa_a(z,w) = \sum_{\nu=1}^{k} \alpha_\nu \kappa_\nu(z,w).$$

One may use Lemma 2.3 to deduce that κ_a in $\mathcal{D}_{k,k}$ belongs to $\mathcal{D}_{k,l}$ if and only if

$$\sum_{\nu=1}^{k} \alpha_\nu \left(\sum_{i=0}^{n} (-1)^i \binom{k}{i} a_{n-i,\nu} \right) = 0 \quad \text{ for } l \leq n \leq k-1.$$

Since $\kappa_k \in \mathcal{D}_{k,1}$ (Example 2.7), $\sum_{i=0}^{n} (-1)^i \binom{k}{i} a_{n-i,k} = 0$ for any $n \geq 1$. The required equivalence in case $k > l$ is now immediate. This also completes the proof of the theorem. □

Remark 2.19. The coefficients of the polynomial p_{l-k-1}, as appearing in (1), are real.

Proof of Corollary 2.14. This is the case in which $k > l = 1$. By Theorem 2.10, there exist real numbers $\alpha_1, \dots, \alpha_k$ such that

$$\kappa_a(z,w) = \sum_{\nu=1}^{k} \alpha_\nu \kappa_\nu(z,w),$$

$$\sum_{\nu=1}^{k-1}\alpha_\nu\left(\sum_{i=0}^{n}(-1)^i\binom{k}{i}a_{n-i,\nu}\right)=0 \quad \text{for } 1\le n\le k-1.$$

If $c_{n,\nu} := \sum_{i=0}^{n}(-1)^i\binom{k}{i}a_{n-i,\nu}$ $(1\le n,\nu\le k-1)$ then we have a system $AX=0$ of $k-1$ equations in $k-1$ variables $\alpha_1,\dots,\alpha_{k-1}$, where A is the $(k-1)\times(k-1)$ matrix $(c_{n,\nu})$ and X is the column vector $[\alpha_1\dots\alpha_{k-1}]^T$. Since the system $AX=0$ admits a trivial solution, it suffices to check that it has a unique solution. Note that $c_{n,\nu}$ is precisely the coefficient of E_n in $D_{M^*_{z,\nu},k}$. Since κ_ν belongs to $\mathcal{D}_{k,k-\nu+1}$, $c_{n,\nu}=0$ for $n\ge k-\nu+1$. In particular, the matrix A is a lower triangular matrix. Also, since κ_ν does not belong to $\mathcal{D}_{k,k-\nu}$, the off-diagonal entries of A are nonzero. Thus we get $\alpha_1=0,\dots,\alpha_{k-1}=0$, and hence $\kappa_a=\alpha_k\kappa_k$ as desired. □

We discuss some applications of the classification result.

Corollary 2.20. *Let $\kappa_a\in\mathcal{D}_{k,k}$. Then there exist finite sequences*

$$\{\nu_1,\dots,\nu_n\} \quad \textit{and} \quad \{\mu_1,\dots,\mu_m\}$$

of positive integers such that any $f\in\mathscr{H}(\kappa_a)$ admits the decomposition

$$f=\sum_{i=1}^{n}\alpha_{\nu_i}f_i+\sum_{j=1}^{m}\alpha_{\mu_j}g_j$$

for some finite sequences $\{\alpha_{\nu_i}\}_{i=1}^n\subseteq(0,\infty)$, $\{\alpha_{\mu_j}\}_{i=1}^m\subseteq(-\infty,0)$, $f_i\in\mathscr{H}_{\nu_i}$ $(i=1,\dots,n)$ and $g_j\in\mathscr{H}_{\mu_j}$ $(j=1,\dots,m)$. Moreover,

$$\begin{aligned}\|f\|^2 \ge \min\Big\{\sum_{i=1}^{n}|\alpha_{\nu_i}|^2\|\tilde f_i\|^2_{\mathscr{H}_{\nu_i}} : \sum_{i=1}^{n}\alpha_{\nu_i}\tilde f_i=\sum_{i=1}^{n}\alpha_{\nu_i}f_i\Big\}\\ -\min\Big\{\sum_{j=1}^{m}\|\alpha_{\mu_j}|^2|\tilde g_j\|^2_{\mathscr{H}_{\mu_j}} : \sum_{i=1}^{m}\alpha_{\mu_j}\tilde g_j=\sum_{i=1}^{m}\alpha_{\mu_j}g_j\Big\}.\end{aligned} \tag{2.5}$$

Proof. By Theorem 2.10, there exist real numbers $\alpha_1,\dots,\alpha_k$ such that

$$\kappa_a(z,w)=\sum_{\nu=1}^{k}\alpha_\nu\kappa_\nu(z,w).$$

Consider the finite sequence $\{\nu_1,\dots,\nu_n\}$ for which the corresponding coefficients $\alpha_{\nu_1},\dots,\alpha_{\nu_n}$ are positive, and also the finite sequence $\{\mu_1,\dots,\mu_n\}$ for which the corresponding coefficients $\alpha_{\mu_1},\dots,\alpha_{\mu_n}$ are negative. Then $\kappa_a(z,w)$ satisfies

$$\sum_{i=1}^{n}\alpha_{\nu_i}\kappa_{\nu_i}(z,w)=\kappa_a(z,w)+\sum_{i=1}^{m}(-\alpha_{\mu_i})\kappa_{\mu_i}(z,w).$$

This yields the first part. Further, by Aronszajn's Theorem on sum of reproducing kernels [2, Section 6],

$$\begin{aligned}
&\min\left\{\sum_{i=1}^{n}|\alpha_{\nu_i}|^2\|\tilde{f}_i\|^2_{\mathscr{H}_{\nu_i}} : \sum_{i=1}^{n}\alpha_{\nu_i}\tilde{f}_i = \sum_{i=1}^{n}\alpha_{\nu_i}f_i\right\}\\
&= \min\left\{\|\tilde{f}\|^2_{\mathscr{H}(\kappa_a)} + \sum_{j=1}^{m}|\alpha_{\mu_j}|^2\|\tilde{g}_j\|^2_{\mathscr{H}_{\mu_j}}\right.\\
&\qquad\qquad\qquad \left. : \tilde{f} + \sum_{j=1}^{n}(-\alpha_{\mu_j})\tilde{g}_j = f + \sum_{j=1}^{n}(-\alpha_{\mu_j})g_j\right\}\\
&\leq \|f\|^2_{\mathscr{H}(\kappa_a)} + \min\left\{\sum_{j=1}^{m}|\alpha_{\mu_j}|^2\|\tilde{g}_j\|^2_{\mathscr{H}_{\mu_j}} : \sum_{j=1}^{n}\alpha_{\mu_j}\tilde{g}_j = \sum_{j=1}^{n}\alpha_{\mu_j}g_j\right\},
\end{aligned}$$

which gives the desired norm estimate. □

Remark 2.21. The case in which the finite sequence $\{\mu_1, \ldots, \mu_m\}$ is absent, equality holds in (2.5) (refer to [2]).

We conclude this section with one application of Theorem 2.10 to operator theory.

Recall that an d-tuple T of commuting bounded linear operators $T_1, \ldots, T_d$ is *p-essentially normal* if the cross-commutators $[T_i^*, T_j]$ belong to the Schatten p-class for all $i, j = 1, \ldots, d$.

Corollary 2.22. *If $\kappa_a \in \mathcal{D}_{k,l}$ then the multiplication d-tuple $M_{z,a}$ acting on $\mathscr{H}(\kappa_a)$ is p-essentially normal for any $p > d$.*

Proof. Consider the weight multi-sequence $\{w_\alpha^{(i)} : \alpha \in \mathbb{N}^d, i = 1, \ldots, d\}$ given by

$$w_\alpha^{(i)} = \frac{\bar{\beta}_{|\alpha|+1}}{\bar{\beta}_{|\alpha|}}\sqrt{\frac{\alpha_i + 1}{|\alpha| + d}} \quad (\alpha \in \mathbb{N}^d, 1 \leq i \leq d),$$

where the scalar sequence $\{\bar{\beta}_k\}$ is given by

$$\bar{\beta}_k^2 = \frac{(d-1+k)!}{(d-1)!\,k!}\frac{1}{a_k},\ \ k \geq 0.$$

Note that $\delta_k := \bar{\beta}_{k+1}/\bar{\beta}_k = \sqrt{\frac{d+k}{k+1}}\sqrt{\frac{a_k}{a_{k+1}}}$ $(k \in \mathbb{N})$. It is easy to see that $M_{z,a}$ is a weighted multi-shift: For $i = 1, \ldots, d$ and $\alpha \in \mathbb{N}^d$, $M_{z_i}\frac{z^\alpha}{\|z^\alpha\|} = w_\alpha^{(i)}\frac{z^{\alpha+\epsilon_i}}{\|z^{\alpha+\epsilon_i}\|}$, where ϵ_i is the d-tuple with 1 in the ith place and zeros elsewhere. In view of [9, Theorem 4.2], it suffices to check that

$$\sum_{k=1}^{\infty}\delta_k^{2p}\,k^{d-p-1} + \sum_{k=1}^{\infty}\left|\delta_k^2 - \delta_{k-1}^2\right|^p k^{d-1} < \infty. \tag{2.6}$$

Let $N = \max\{l-k, 0\}$. By Theorem 2.10, for any $n \geq N$, a_n is a polynomial of degree, say m. Note that for any polynomial $q(x)$ of degree n, the degree of $q(x+1)-q(x)$ is at most $n-1$. It follows that $\delta_n^2 - \delta_{n-1}^2 = \frac{r(n)}{s(n)}$, where $r(n)$ is a polynomial in n of degree at most $2m+1$ and $s(n)$ is a polynomial in n of degree $2m+2$. Thus there exists a scalar $C > 0$ such that $|\delta_n^2 - \delta_{n-1}^2| \leq C/n$ for $n \geq N$. It is now easy to see that (2.6) holds for any integer $p > d$. □

Remark 2.23. Let M be a z-invariant subspace of $\mathscr{H}(\kappa_a)$ such that $M^\perp$ is finite dimensional. Then for each i, the self-commutator of $M_{z_i}|_M$ is a finite rank perturbation of the self-commutator of M_{z_i}. In particular, $M_z|_M$ is p-essentially normal for any integer $p > d$.

We will see in the last section that for any graded submodule M of $\mathscr{H}(\kappa_\nu)$, the defect operator $D_{(M_{z,\nu}|_M)^*,\nu}$ is of finite rank if and only if $M^\perp$ is of finite dimension. Consequently, $M_{z,\nu}|_M$ is p-essentially normal for any integer $p > d$ in this case. This supports Arveson–Douglas conjecture [12].

3. Pure row ν-contractions

In this section, we introduce a notion of pure row ν-contraction. We then combine the theory of weighted symmetric spaces as developed in [17] with the powerful techniques from [3] to show that certain row ν-contractions correspond to a unique $\mathcal{A}$-morphism in a natural way. This can be used to obtain a version of von Neumann inequality for tuples which are row k-contractions for $1 \leq k \leq \nu$. The latter one is certainly known as a consequence of a dilation theorem of Müller and Vasilescu [18] (The reader is referred to [11] for a ball analog of von Neumann inequality in case $\nu = 1$, and also to [13], [17] and [19] for some interesting variants of von Neumann inequality). These results also form basis for our analysis of finite rank graded submodules of $\mathscr{H}(\kappa_\nu)$ as carried out in Section 4.

Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. Recall that Q_T is given by

$$Q_T(X) := \sum_{i=1}^{d} T_i^* X T_i \quad (X \in B(\mathcal{H})).$$

We also recall that the *defect operator* $D_{T,k}$ *of order* k is given by

$$D_{T,k} := \sum_{l=0}^{k} (-1)^l \binom{k}{l} Q_T^l(I).$$

Definition 3.1. Let T be a commuting d-tuple of operators $T_1, \ldots, T_d$ in $B(\mathcal{H})$. We say that T is a *row ν-contraction* if $D_{T^*,\nu}$ is a positive operator. We say that T is a *row ν-hypercontraction* if T is a row k-contraction for $k = 1, \ldots, \nu$. In case $\nu = 1$ then we refer to T as a *row contraction.* We say that T is a *row ν-contraction of finite rank* if $D_{T^*,\nu}$ is a positive operator of finite rank.

Remark 3.2. The multiplication tuple $M_{z,\nu}$ on $\mathscr{H}(\kappa_\nu)$ is a row ν-contraction of rank 1.

Recall that $a_{0,\nu} = 1$ and $a_{n,\nu} := \frac{\nu(\nu+1)\dots(\nu+n-1)}{n!}$ for integers $n \geq 1$.

Definition 3.3. Let $T := (T_1, \dots, T_d)$ be a d-tuple of bounded linear operators $T_1, \dots, T_d$ on a Hilbert space $\mathcal{H}$. We say that T is a *pure row ν-contraction* if T is a row ν-contraction such that

$$p(T, \nu, l) := \sum_{j=0}^{\nu-1} c(j+1, \nu, l) Q_{T^*}^{l+j}(I) \longrightarrow 0 \quad \text{as } l \longrightarrow \infty \quad (\text{sot}), \tag{3.7}$$

where the coefficients $c(j+1, \nu, l)$ is given by

$$c(j+1, \nu, l) := \sum_{i=1}^{l} (-1)^{i+j} \binom{\nu}{i+j} a_{l-i,\nu} \quad \text{if } 0 \leq j \leq \nu - 1.$$

If in addition T is a row ν-hypercontraction then we say that T is a *pure row ν-hypercontraction.*

The notion above is partly motivated by the considerations in [18]. Note that for $\nu = 1$, this coincides with the notion of pure row contraction discussed in [3]. We will see soon that this fits well with the notion of row ν-contraction (see, for example, Lemma 3.8).

Lemma 3.4. *Let $T := (T_1, \dots, T_d)$ be a row ν-contraction on a Hilbert space $\mathcal{H}$, and define the operator*

$$\Delta_{T^*,k} := \Big(\sum_{i=0}^{k} (-1)^i \binom{k}{i} Q_{T^*}^i(I) \Big)^{1/2}.$$

Then $p(T, \nu, l) = \sum_{n=0}^{l-1} a_{n,\nu} Q_{T^*}^n(D_{T^*,\nu}) - I$. In particular, $p(T, \nu, l)$ is an increasing function of l.

Proof. Note that $\Delta_{T^*,k}^2 = D_{T^*,k}$. By Example 2.7,

$$c(0, \nu, l) := c(1, \nu, l) + a_{l,\nu} = \sum_{i=0}^{l} (-1)^i \binom{\nu}{i} a_{l-i,\nu} = 0$$

for every integer $l \geq 1$. It is also easy to see that

$$c(j+1, \nu, l) + (-1)^j \binom{\nu}{j} a_{l,\nu} = c(j, \nu, l+1) \quad (j = 1, \dots, \nu - 1).$$

One may now use these observations to establish the following identity by a routine inductive argument on $l \geq 1$:

$$\sum_{n=0}^{l-1} a_{n,\nu} Q_{T^*}^n(D_{T^*,\nu}) = I + \sum_{j=0}^{\nu-1} c(j+1, \nu, l) Q_{T^*}^{l+j}(I) = I + p(T, \nu, l).$$

This completes the proof of the first part. The second part is now immediate from the positivity of the defect operator $D_{T^*,\nu}$. $\square$

Let us see examples of pure row ν-contractions, with which we are primarily concerned.

Example 3.5. Consider the multiplication d-tuple $M_{z,\nu}$ on $\mathcal{H}(\kappa_\nu)$. We check that $M_{z,\nu}$ is a pure row ν-contraction. We have already seen that $M_{z,\nu}$ is a row ν-contraction. Let us see that $M_{z,\nu}$ satisfies (3.7). For convenience, let $M_{z,\nu} = M_z$. We know from Example 2.7 that $D_{M_z^*,\nu} = E_0$. By Lemma 1.2(4), $Q^n_{M_z^*}(D_{M_z^*,\nu}) = \frac{a_{0,\nu}}{a_{n,\nu}} E_n$ $(n \in \mathbb{N})$. This implies that $\sum_{n=0}^{l} a_{n,\nu} Q^n_{M_z^*}(D_{M_z^*,\nu}) = a_{0,\nu} \sum_{n=0}^{l} E_n$. Since $a_{0,\nu} = 1$, it follows that

$$\lim_{l\to\infty} \sum_{n=0}^{l} a_n Q^n_{M_z^*}(D_{M_z^*,\nu}) = \sum_{n=0}^{\infty} E_n = I \quad \text{(sot)}.$$

The purity of M_z follows from the preceding lemma.

Let T be a commuting d-tuple on $\mathcal{H}$. By *the C^*-algebra generated by T*, we mean the norm closure of all noncommutative polynomials in the $(2d)$-variables $T_1, \ldots, T_d, T_1^*, \ldots, T_d^*$. Let $\mathcal{A}$ denote a unital C^*-algebra. By an *operator system*, we mean a self-adjoint subspace of $\mathcal{A}$ containing the unit. Let $M_n(\mathcal{A})$ denote the C^*-algebra of all $n \times n$ matrices with entries from $\mathcal{A}$. A mapping ϕ from $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$ is said to be *positive* if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. Let $\mathcal{S} \subseteq \mathcal{A}$ denote an operator system. If $\phi : \mathcal{S} \to \mathcal{B}$ is a linear map, then we define $\phi_n : M_n(\mathcal{S}) \to M_n(\mathcal{B})$ by $\phi_n([a_{i,j}]) := [\phi(a_{i,j})]$, where $[a_{i,j}] \in M_n(\mathcal{S})$. We say that ϕ is *completely positive* if ϕ_n is positive for all $n \geq 1$. Let $\mathcal{A}$ be a subalgebra of a unital C^*-algebra $\mathcal{B}$ which contains the unit of $\mathcal{B}$. An *$\mathcal{A}$-morphism* is a completely positive linear map $\phi : \mathcal{B} \to B(\mathcal{H})$ such that $\phi(1) = I$ and $\phi(AX) = \phi(A)\phi(X)$ for $A \in \mathcal{A}, X \in \mathcal{B}$.

Recall that $\mathcal{F}_\nu$ denotes the Fock space realization of the reproducing kernel Hilbert space $\mathcal{H}(\kappa_\nu)$. Recall further that the Fock space realization of $M_{z,\nu}$ is denoted by $S^{(\nu)}$. The *Toeplitz C^*-algebra* generated by $S^{(\nu)}$ will be denoted by $\mathcal{T}_{d,\nu}$.

We now state the main result of this section.

Theorem 3.6. *Let $\mathcal{A}$ be the subalgebra of the Toeplitz C^*-algebra $\mathcal{T}_{d,\nu}$ consisting of all polynomials in $S^{(\nu)}$. If T is a row ν-hypercontraction on a Hilbert space $\mathcal{H}$ then there is a unique $\mathcal{A}$-morphism $\phi : \mathcal{T}_{d,\nu} \to B(\mathcal{H})$ such that $\phi(M_{z_i,\nu}) = T_i$ for $i = 1, \ldots, d$. Conversely, every $\mathcal{A}$-morphism $\phi : \mathcal{T}_{d,\nu} \to B(\mathcal{H})$ gives rise to row ν-hypercontraction T by way of $\phi(M_{z_i,\nu}) = T_i$, $i = 1, \ldots, d$.*

Note that the case $\nu = 1$ is exactly [3, Theorem 6.2]. The proof of Theorem 3.6 involves several lemmas. Before we present the proof, let us introduce a few notations. Let $C(X)$ denote the C^*-algebra of continuous functions on the compact Hausdorff space X, endowed with the sup norm $\|\cdot\|_\infty$. Let $M_{z,a}$ denote the multiplication tuple on the RKHS associated with the $\mathcal{U}$-invariant kernel κ_a. We will refer to $\mathcal{T}_d := C^*(M_{z,a})$ as the *Toeplitz C^*-algebra* generated by $M_{z,a}$. Whenever there is no role of the sequence $\{a_n\}$, we suppress the suffix a and use the simpler notation M_z for $M_{z,a}$. We further assume that $a_n > 0$ for every integer $n \geq 0$.

We now present a structure result for Toeplitz C^*-algebra of certain finite rank row ν-contractions. This generalizes [3, Theorem 5.7]. An analog of the first part of Lemma 3.7 is obtained in [15, Theorem 4.6] for a family of spherical multi-shifts, which includes finite rank row ν-contractions. The description of the Toeplitz C^*-algebra, as given below in part (2), is essential in the proof of the main theorem.

Lemma 3.7. *Let M_z be the tuple of multiplication operators $M_{z_1}, \ldots, M_{z_d}$ on $\mathcal{H}(\kappa_a)$. Let $\mathcal{T}_d$ denote the Toeplitz C^*-algebra generated by M_z and let $\mathcal{A}$ denote the commutative algebra generated by $I, M_{z_1}, \ldots, M_{z_d}$. If $\kappa_a \in \mathcal{D}_{k,1}$ then the following statements are true:*

(1) *$\mathcal{T}_d$ contains the algebra $\mathcal{K}$ of all compact operators on $\mathcal{H}(\kappa_a)$, and we have an exact sequence of C^*-algebras*

$$0 \longmapsto \mathcal{K} \overset{i}{\hookrightarrow} \mathcal{T}_d \overset{\pi}{\longmapsto} C(\partial\mathbb{B}) \longmapsto 0,$$

where $i : \mathcal{K} \hookrightarrow \mathcal{T}_d$ is the inclusion map and $\pi : \mathcal{T}_d \to C(\partial\mathbb{B})$ is the unital-homomorphism defined by $\pi(M_{z_j}) = z_j$ $(j = 1, \ldots, d)$.*

(2) *$\mathcal{T}_d = \overline{\mathrm{span}}\mathcal{A}\mathcal{A}^*$, where* $\mathrm{span}\, W$ *denotes the linear span of W in $B(\mathcal{H}(\kappa_a))$.*

Proof. The proof is a simple modification of the proof of [3, Theorem 5.7]. Suppose that $\kappa_a \in \mathcal{D}_{k,1}$. Thus $D_{M_z^*,k} = E_0$, which belongs to $\overline{\mathrm{span}}\mathcal{A}\mathcal{A}^*$. Since $f \otimes \bar{g} = M_f E_0 M_g^*$ for any polynomial f and g, it is easy to see from $D_{M_z^*,k} = E_0$ that $\overline{\mathrm{span}}\mathcal{A}\mathcal{A}^*$ contains all finite rank, and hence all compact operators. Since $M_{z_i}M_{z_j}^* - M_{z_j}^* M_{z_i} \in \mathcal{K}$ by Corollary 2.22, $\overline{\mathrm{span}}\mathcal{A}\mathcal{A}^*$ is closed under multiplication. This implies that $\overline{\mathrm{span}}\mathcal{A}^*\mathcal{A}$ is contained in $\overline{\mathrm{span}}\mathcal{A}\mathcal{A}^*$, and hence $\mathcal{T}_d = \overline{\mathrm{span}}\mathcal{A}\mathcal{A}^*$.

Let Z denote the d-tuple $(\pi(M_{z_1}), \ldots, \pi(M_{z_1}))$, where π is the Calkin map. Since M_z is essentially normal, Z is a commuting normal d-tuple. Also, since M_z is a spherical tuple, by Remark 1.4, Z is also spherical (after embedding the Calkin algebra into $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$). By [9, Proposition 3.7], the joint spectrum of Z has spherical symmetry. On the other hand, since $D_{M_z^*,k}$ is a finite rank operator, $D_{Z^*,k} = 0$. Hence by elementary spectral theory, Z^* is a joint isometry (that is, $\sum_{i=1}^d Z_i Z_i^* = I$). It follows that the joint spectrum of Z^*, and hence that of Z is contained in the unit sphere [10]. In particular, the joint spectrum of Z is the entire unit sphere. It is now easy to finish the proof of (1). $\square$

In case $\nu = 1$, the following result is obtained in [3, Theorem 4.5] (cf. [17, Theorem 7.5]).

Lemma 3.8. *Let $T := (T_1, \dots, T_d)$ be a pure row ν-contraction on a Hilbert space $\mathcal{H}$, and consider the positive operator $\Delta_{T^*,\nu} = (D_{T^*,\nu})^{1/2}$, and the subspace $\mathcal{K} := \overline{\Delta_{T^*,\nu}\mathcal{H}}$. Then there is a unique co-isometry $L : \mathcal{F}_\nu \otimes \mathcal{K} \to \mathcal{H}$ satisfying $L(1 \otimes \xi) = \Delta_{T^*,\nu}\xi$ and*

$$L(e_{i_1}e_{i_2}\dots e_{i_n} \otimes \xi) = T_{i_1}T_{i_2}\dots T_{i_n}\Delta_{T^*,\nu}\xi$$

for every $i_1, \dots, i_n \in \{1, 2, \dots, d\}$ and integer $n \geq 1$.

Proof. By Lemma 3.4, the series $\sum_{n=0}^{\infty} a_{n,\nu}Q^n_{T^*}(D_{T^*,\nu})$ converges to the identity operator in the strong operator topology. Define a linear operator L from $\mathcal{F}_\nu \otimes \mathcal{K}$ into $\mathcal{H}$ by setting $L(1 \otimes \xi) = \Delta_{T^*,\nu}\xi$, and

$$L(e_{i_1}e_{i_2}\dots e_{i_n} \otimes \xi) = T_{i_1}T_{i_2}\dots T_{i_n}\Delta_{T^*,\nu}\xi$$

for every $i_1, \dots, i_n \in \{1, 2, \dots, d\}$, $n = 1, 2, \dots$. Clearly, L is well defined. If we prove that L is a bounded linear operator then the uniqueness of L follows from the fact that

$$\{e^{n_1}_{i_1}e^{n_2}_{i_2}\dots e^{n_d}_{i_d} : n_1 + n_2 + \dots + n_d = n\}$$

is an orthogonal basis for E^n $(n \geq 1)$. Indeed, we will see that the adjoint L^* of L is an isometry.

For nonnegative integer n, consider the element ξ_n in $\mathcal{F}^{\otimes}_\nu \otimes \mathcal{K}$ given by

$$\xi_n = \sum_{i_1,\dots,i_n=1}^{d} e_{i_1} \otimes \dots \otimes e_{i_n} \otimes \Delta_{T^*,\nu}T^*_{i_1}T^*_{i_2}\dots T^*_{i_n}\eta.$$

Define $A : \mathcal{H} \to \mathcal{F}^{\otimes}_\nu \otimes \mathcal{K}$ by

$$A(\eta) = \{a_{0,\nu}\xi_0, a_{1,\nu}\xi_1, a_{2,\nu}\xi_2 \dots\}.$$

Note that A actually maps $\mathcal{H}$ into $\mathcal{F}_\nu \otimes \mathcal{K}$. We check that

$$\langle e_{j_1}e_{j_2}\dots e_{j_n} \otimes \xi,\, A(\eta)\rangle_{\mathcal{F}^{\otimes}_\nu \otimes \mathcal{K}} = \langle L(e_{j_1}e_{j_2}\dots e_{j_n} \otimes \xi),\, \eta\rangle_{\mathcal{H}}$$

for every $\xi \in \mathcal{K}$ and $\eta \in \mathcal{H}$. To see this, note first that

$$a_{n,\nu}\|e_{j_1} \otimes \dots \otimes e_{j_n}\|^2_{\mathcal{F}^{\otimes}_\nu} = 1$$

(Lemma 1.7). Thus

$$\begin{aligned}
&\langle e_{j_1}e_{j_2}\dots e_{j_n} \otimes \xi,\, A(\eta)\rangle_{\mathcal{F}^{\otimes}_\nu \otimes \mathcal{K}} \\
&= \langle e_{j_1}e_{j_2}\dots e_{j_n} \otimes \xi,\, a_{n,\nu}\xi_n\rangle_{\mathcal{F}^{\otimes}_\nu \otimes \mathcal{K}} \\
&= \sum_{i_1,\dots,i_n=1}^{d} a_{n,\nu}\langle e_{j_1}e_{j_2}\dots e_{j_n} \otimes \xi,\, e_{i_1} \otimes \dots \otimes e_{i_n} \otimes \Delta_{T^*,\nu}T^*_{i_1}T^*_{i_2}\dots T^*_{i_n}\eta\rangle_{\mathcal{F}^{\otimes}_\nu \otimes \mathcal{K}} \\
&= \sum_{i_1,\dots,i_n=1}^{d} a_{n,\nu}\langle e_{j_1}e_{j_2}\dots e_{j_n},\, e_{i_1} \otimes \dots \otimes e_{i_n}\rangle_{\mathcal{F}^{\otimes}_\nu}\langle \xi,\, \Delta_{T^*,\nu}T^*_{i_1}T^*_{i_2}\dots T^*_{i_n}\eta\rangle_{\mathcal{H}} \\
&= \langle \xi,\, \Delta_{T^*,\nu}T^*_{j_1}T^*_{j_2}\dots T^*_{j_n}\eta\rangle_{\mathcal{H}} = \langle L(e_{j_1}e_{j_2}\dots e_{j_n} \otimes \xi),\, \eta\rangle_{\mathcal{H}}.
\end{aligned}$$

Now

$$\begin{aligned}\|\xi_n\|^2 &= \sum_{i_1,\dots,i_n=1}^{d} \|e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes \Delta_{T^*,\nu}T_{i_n}^*\dots T_{i_1}^*\eta\|^2\\ &= \frac{1}{a_{n,\nu}}\sum_{i_1,\dots,i_n=1}^{d}\|\Delta_{T^*,\nu}T_{i_n}^*\dots T_{i_1}^*\eta\|^2\\ &= \frac{1}{a_{n,\nu}}\langle Q_{T^*}^n(D_{T^*,\nu})\eta,\eta\rangle.\end{aligned}$$

Since T is a pure row ν-contraction, by Lemma 3.4, we obtain

$$\begin{aligned}\|A\eta\|^2 &= \lim_{l\to\infty}\sum_{n=0}^{l} a_{n,\nu}^2\|\xi_n\|^2 = \lim_{l\to\infty}\sum_{n=0}^{l} a_{n,\nu}\langle Q_{T^*}^n(D_{T^*,\nu})\eta,\eta\rangle\\ &= \lim_{l\to\infty}\langle\sum_{n=0}^{l} a_{n,\nu}Q_{T^*}^n(D_{T^*,\nu})\eta,\eta\rangle = \|\eta\|^2.\end{aligned}$$

Since $A = L^*$, the proof is over. □

We also need the following well-known fact in the proof of Theorem 3.6.

Lemma 3.9. *Suppose that $T = (T_1,\dots,T_d)$ is a row ν-hypercontraction. Then, for any positive number $r < 1$, the d-tuple $rT = (rT_1,\dots,rT_d)$ is a pure row ν-hypercontraction.*

Proof. The first assertion is derived in [5, Proof of Theorem 4.2] (with roles of T and T^* interchanged). To see the purity of rT, note that the general term in $p(rT,\nu,l)$ is of the form

$$a_{l-i,\nu}Q_{rT^*}^{l+j}(I) \approx (l-i)^{\nu-1}r^{l+j}Q_{T^*}^{l+j}(I),$$

which converges to 0 in operator norm topology provided $\{Q_{T^*}^l(I)\}_{l\geq 0}$ is bounded in $B(\mathcal{H})$. □

Proof of Theorem 3.6. Most of the work required for the proof is already done. The rest of the proof is imitation of that of [3, Theorem 6.2].

The uniqueness of ϕ follows from Lemma 3.7(2). In view of Lemma 3.9, it suffices to treat the case in which T is a pure row ν-contraction on $\mathcal{H}$. By Lemma 3.8, there is a unique co-isometry $L : \mathcal{F}_\nu\otimes\mathcal{K}\to\mathcal{H}$ satisfying $L(1\otimes\xi) = \Delta_{T^*,\nu}\xi$ and

$$L(e_{i_1}e_{i_2}\dots e_{i_n}\otimes\xi) = T_{i_1}T_{i_2}\dots T_{i_n}\Delta_{T^*,\nu}\xi$$

for every $i_1,\dots,i_n\in\{1,2,\dots,d\}$ and integer $n\geq 1$. It is easy to see that

$$L(p(S^{(\nu)})\otimes I_{\mathcal{K}}) = p(T)L$$

for every complex polynomial p in d variables. Define the completely positive map $\phi : \mathcal{T}_{d,\nu}\to B(\mathcal{H})$ by

$$\phi(X) = L(X\otimes I_{\mathcal{K}})L^*,\ X\in\mathcal{T}_{d,\nu}.$$

Clearly, $\phi(I_{\mathcal{F}_\nu}) = I$. If X belongs to span$\mathcal{A}\mathcal{A}^*$ then one may use Agler's hereditary functional calculus to see that

$$\phi(p(S^\nu)X) = p(T)\phi(X).$$

By Lemma 3.7(2), ϕ is an $\mathcal{A}$-morphism having required properties. $\square$

Since restriction of an $\mathcal{A}$-morphism to $\mathcal{A}$ is a completely contractive representation of the subalgebra $\mathcal{A}$ on $\mathcal{H}$, we immediately obtain the following:

Corollary 3.10. *Let T be a row ν-hypercontractive d-tuple on a Hilbert space $\mathcal{H}$. Then, for every complex polynomial p in d variables, we have*

$$\|p(T)\| \leq \|p(S^{(\nu)})\|.$$

A particular consequence of the last corollary is worth-notable: Since the row ν-contraction $S^{(\nu)}$ is also a row μ-contraction for any $\mu = 1, \ldots, \nu$, for every complex polynomial p, we have

$$\|p(S^{(\nu)})\| \leq \|p(S^{(\nu-1)})\| \leq \cdots \leq \|p(S^{(1)})\|.$$

The conclusion of the last corollary is applicable to all row 2-contractions on $\mathscr{H}(\kappa_a)$.

Proposition 3.11. *Let M_z be a row 2-contraction d-tuple on an $\mathcal{U}$-invariant RKHS $\mathscr{H}(\kappa_a)$. Then, for every complex polynomial p is d variables, we have*

$$\|p(M_z)\| \leq \|p(S^{(2)})\|.$$

Proof. Suppose that M_z is a row 2-contraction. In view of Corollary 3.10, it now suffices to check that M_z is a row contraction. By Lemma 1.2(3), M_z is a row contraction if and only if the sequence $\{a_n\}_{n\geq 0}$ is increasing. It is easy to see from Lemma 2.3 that

$$1 - 2\frac{a_0}{a_1} \geq 0, \quad \text{and} \quad 1 - 2\frac{a_{n-1}}{a_n} + \frac{a_{n-2}}{a_n} \geq 0 \quad (n \geq 2). \tag{3.8}$$

We prove by induction that $a_n \leq a_{n+1}$ for $n \geq 0$. Clearly, $a_0 \leq 2a_0 \leq a_1$. Suppose $a_{n-2} \leq a_{n-1}$. Note that by (3.8),

$$0 \leq 1 - 2\frac{a_{n-1}}{a_n} + \frac{a_{n-2}}{a_n} = 1 - 2\frac{a_{n-1}}{a_n} + \frac{a_{n-2}}{a_{n-1}}\frac{a_{n-1}}{a_n} \leq 1 - \frac{a_{n-1}}{a_n},$$

and hence the sequence $\{a_n\}_{n\geq 0}$ is increasing. $\square$

4. Finite rank graded submodules

Let $\mathcal{H}$ be a Hilbert space and let T be a commuting d-tuple on $\mathcal{H}$. Then $\mathcal{H}$ can be considered as a *Hilbert module* over the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$ as follows: The module action is given by

$$(p, h) \in \mathbb{C}[z_1, \ldots, z_d] \times \mathcal{H} \longmapsto p(T)h \in \mathcal{H}.$$

(refer to [21] for the basic theory of Hilbert modules over function algebras).

Consider the Hilbert module $\mathscr{H}(\kappa_\nu)$ associated with the kernel $\kappa_\nu(z,w) = \frac{1}{(1-\langle z,w\rangle)^\nu}$ and the multiplication tuple M_z on $\mathscr{H}(\kappa_\nu)$, where ν is a positive integer.

Definition 4.1. By a *submodule* M of $\mathscr{H}(\kappa_\nu)$, we understand a closed subspace M of $\mathscr{H}(\kappa_\nu)$ which is also a Hilbert module over the polynomial ring $\mathbb{C}[z_1,\dots,z_d]$. We say that the submodule M *is of finite rank* if the defect operator $D_{(M_z|_M)^*,\nu}$ in $B(M)$ is of finite rank.

Remark 4.2. If $D_{(M_z|_M)^*,\mu}$ is a finite rank (resp. trace-class) operator then so is $D_{(M_z|_M)^*,\nu}$ for any integer $\nu \geq \mu$. This follows from the identity $D_{T,k+1} = D_{T,k} - Q_T(D_{T,k})$.

Remark 4.3. For a submodule M of $\mathscr{H}(\kappa_1)$, let

$$D(M) := P_M - \sum_{i=1}^{d} M_{z_i} P_M M_{z_i}^*.$$

By the standard definition of finite rank submodules M of Drury–Arveson space, M is of finite rank if the rank of the defect operator $\Delta(M) := D(M)^{1/2}$ is finite (see, for instance, [14, Pg 1]). Our definition differs from this in two aspects:

(1) Note that $D(M)$ is a bounded linear operator from $\mathcal{H}$ into M. On the other hand, our defect operator $D_{(M_z|_M)^*,1}$ is a bounded linear operator from M into M. It is easy to see that $D(M)$ is of finite rank if and only if $D_{(M_z|_M)^*,1}$ is of finite rank.

(2) Unlike the case $\nu = 1$, the defect operator $D_{(M_z|_M)^*,\nu}$ may not be positive. Our definition is consistent with the standard definition for submodules of Drury–Arveson Hilbert module, since $D(M)$ and $\Delta(M)$ have the same rank in this case.

Consider the d-tuple M_z of multiplication operators $M_{z_1},\dots,M_{z_d}$ on $\mathscr{H}(\kappa_\nu)$. Then M_z is *strongly circular* in the following sense: The strongly continuous unitary representation $\Gamma : \mathbb{T} \to B(\mathscr{H}(\kappa_\nu))$ of the circle group $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ given by $(\Gamma(\lambda)f)(z_1,\dots,z_d) = f(\lambda z_1,\dots,\lambda z_d)$ satisfies

$$\Gamma(\lambda)M_{z_i} = \lambda M_{z_i}\Gamma(\lambda) \quad (i = 1,\dots,d, \lambda \in \mathbb{T}).$$

Let M be a submodule of the Hilbert module $\mathscr{H}(\kappa_\nu)$. We say that M is a *graded submodule* if $\Gamma(\lambda)M \subseteq M$ for every $\lambda \in \mathbb{T}$.

Remark 4.4. Any graded submodule M admits the decomposition $M = \oplus_{n=0}^\infty V_n$, where

$$V_n := \{f \in M : \Gamma(\lambda)f = \lambda^n f \text{ for every } \lambda \in \mathbb{T}\}.$$

Note that V_n is a subspace of H_n, V_n is orthogonal to V_m for $m \neq n$, and $M_{z_i}V_n \subseteq V_{n+1}$ for every $i = 1,\dots,d$.

A simple application of Hilbert Basis Theorem shows that M is a graded submodule if and only if it is generated by finitely many homogeneous polynomials. We include a variant of this fact for ready reference (refer to [4], [14]).

Lemma 4.5. *Let M be a graded submodule of $\mathscr{H}(\kappa_\nu)$. Then there exists an orthonormal set consisting of finitely many homogeneous polynomials $p_1, \ldots, p_k$ such that $M = \overline{\langle p_1, \ldots, p_k \rangle}$, where $\langle p_1, \ldots, p_k \rangle$ denotes the ideal generated by $p_1, \ldots, p_k$.*

Proof. Note that $M = \oplus_{n=0}^{\infty} V_n$ for some subspaces V_n of H_n. For $n \geq 1$, let $\{f_{n,1}, \ldots, f_{n,k_n}\}$ be an orthonormal basis of V_n. Let I denote an ideal in $\mathbb{C}[z]$ generated by $\{f_{n,1}, \ldots, f_{n,k_n} : n \geq 1\}$. Then we have $M = \bar{I}$. By Hilbert Basis Theorem [8, Theorem 7.21], there exist finitely many polynomial $g_1, \ldots, g_l$ (which may not be homogeneous) such that $I = \langle g_1, \ldots, g_l \rangle$. For each $i = 1, \ldots, l$ we have $g_i = \sum_{j=1}^{n_i} h_j g_{i,j}$, where $g_{i,j} \in \{f_{n,1}, \ldots, f_{n,k_n} : n \geq 1\}$. Let J be an ideal generated by $\{g_{i,j} : j = 1, \ldots, n_i, i = 1, \ldots, l\}$. Clearly, $I \subseteq J$. It follows that $M = \bar{I} \subseteq \bar{J} \subseteq M$. Thus we obtain $M = \bar{J}$ as desired. $\square$

The main result in this section generalizes the following result of Arveson:

Theorem 4.6. [4, Theorem F] *Any graded submodule of the Drury–Arveson space $\mathscr{H}(\kappa_1)$ of finite rank is of finite codimension. Moreover, if $M = \oplus_{n=0}^{\infty} V_n$ then there exists an integer $N \geq 1$ such that*

$$\dim V_{n+1} = \left(\frac{n+d}{n+1}\right) \dim V_n \text{ for all integers } n \geq N.$$

Remark 4.7. The second half of Theorem 4.6 is essentially obtained in the proof of [14, Theorem 2.1], which forces that $V_n = H_n$ $(n \geq N)$ [14, Proposition 2.3].

Remark 4.8. We make several remarks in order.

(1) Arveson's proof of Theorem 4.6 relies basically on the fact that κ_1 is a complete NP kernel (see also [14, Example 1]), and hence it does not readily generalize to the kernels κ_ν for $\nu \geq 2$.
(2) K. Guo obtained a remarkable generalization of Theorem 4.6 for all submodules M of $\mathcal{H}(\kappa_\nu)$ in dimension $d \geq 2$ [14, Theorem 4.1]: M is of finite codimension if and only if $\sum_{j=1}^{d} \mathrm{rank}[P_M, M_{z_j}] < \infty$. However, characterization of finite codimensionality of (graded) submodules in terms of a single defect operator (as in the case of Arveson's Theorem F) is unnoticed.
(3) It may happen that $\mathrm{rank}\, D_{T^*,\nu}$ is finite for some $\nu \geq 2$, but still rank $D_{T^*,1}$ is infinite. In fact, we will see that for any nonzero graded submodule M of $\mathcal{H}(\kappa_\nu)$ $(\nu \geq 2)$ of finite rank, $\mathrm{rank}\, D_{(M_z|_M)^*,1}$ is always infinite (Corollary 4.15).

(4) If $\nu \geq 2$, then $D_{M_z^*,\nu} \geq 0$ need not imply $D_{(M_z|_M)^*,\nu} \geq 0$ for a graded submodule M of $\mathscr{H}(\kappa_\nu)$. For example, consider the graded submodule $M = \oplus_{n=2}^{\infty} V_n$ of $\mathscr{H}(\kappa_2)$, where V_n is linear subspace of H_n generated by $\{z_1^{n_1} z_2^{n_2} \ : \ n_1 + n_2 = n, \ , n_1 \neq 0, \text{ and } n_2 \neq 0\}$. Then, for any integer $n_2 > 2$, one has $\langle D_{(M_z|_M)^*,2}(z_1^2 z_2^{n_2}), \ z_1^2 z_2^{n_2}\rangle < 0$.

We now state an analog of Arveson's Theorem F for the graded submodules of $\mathscr{H}(\kappa_\nu)$.

Theorem 4.9. *Any graded submodule of $\mathscr{H}(\kappa_\nu)$ of finite rank is of finite codimension. Moreover, if $M = \oplus_{n=0}^{\infty} V_n$ then there exists an integer $N \geq \nu$ such that for every $n \geq N$,*

$$\dim V_{n+1} = \sum_{i=1}^{\nu} (-1)^{i-1} \binom{\nu}{i} \frac{(n+1-i+d)\dots(n+d)}{(n+1-i+\nu)\dots(n+\nu)} \ \dim V_{n+1-i}.$$

Remark 4.10. The dimension formula takes a nice form specially in case dimension $d = \nu$:

$$\sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} \dim V_{n-i} = 0 \quad (n > N).$$

We discuss one immediate consequence of the preceding theorem, which recovers a special case of [14, Theorem 4.1].

Corollary 4.11. *Let M denote a nonzero graded submodule of $\mathscr{H}(\kappa_\nu)$. Then the following statements are equivalent:*

(1) *M is of finite rank.*
(2) *M is of finite codimension.*
(3) $\sum_{j=1}^{d}$ *rank*$[P_M, M_{z_j}] < \infty$.

Proof. (1) implies (2) follows from Theorem 4.9 while (2) implies (3) is immediate from the identity $[P_M, M_{z_j}] = P_M M_{z_j} P_{M^\perp}$. To see that (3) implies (1), note that $D_{M_z^*,\nu}$ is a rank one operator (Example 2.7), and $D_{(M_z|_M)^*,\nu}$ is a finite rank perturbation of $P_M D_{M_z^*,\nu}|_M$. $\square$

The proof of Theorem 4.9 is a combination of ideas of [14] and a topological argument based on Lemma 3.7. In this proof, we need several lemmas.

Lemma 4.12. *Let S be a finite rank self-adjoint operator on a reproducing kernel Hilbert space $\mathscr{H}$ of holomorphic functions defined on unit ball in $\mathbb{C}^d$. Suppose that S sends polynomials to polynomials. Then there exist polynomials $p_1, \dots, p_k, q_1, \dots, q_k$ such that*

$$S = \sum_{i=1}^{k} p_i \otimes q_i.$$

Moreover, the polynomials $p_1, \dots, p_k$ form an orthonormal subset of $\mathscr{H}$. If in addition S is an orthogonal projection then $S = \sum_{i=1}^{k} p_i \otimes p_i$.

Proof. Since S sends polynomials to polynomials, $S\mathbb{C}[z] \subseteq \mathbb{C}[z]$, where $\mathbb{C}[z]$ denotes the complex vector space of polynomials in $z_1, \ldots, z_d$. Also, since $\operatorname{ran} S$ is finite dimensional, so is $S\mathbb{C}[z]$. Thus there exist polynomials $r_1, \ldots, r_k$ such that $S\mathbb{C}[z] = \operatorname{span}\{r_1, \ldots, r_k\}$. If follows that

$$\operatorname{ran} S = \overline{S\mathbb{C}[z]} = \operatorname{span}\{r_1, \ldots, r_k\}.$$

By Gram–Schmidt Process, there exist an orthonormal basis consisting of polynomials $p_1, \ldots, p_k$ such that $\operatorname{ran} S = \operatorname{span}\{p_1, \ldots, p_k\}$. Thus, for any $f \in \mathscr{H}$,

$$S(f) = \sum_{i=1}^{k} \langle S(f), p_i\rangle p_i = \sum_{i=1}^{k} \langle f, S(p_i)\rangle p_i = \sum_{i=1}^{k} p_i \otimes q_i(f),$$

where $q_i = S(p_i)$ is also a polynomial.

If in addition S is an orthogonal projection then $q_i = S(p_i) = p_i$ for $i = 1, \ldots, k$. This completes the proof of the lemma. □

Lemma 4.13. *Let M be a submodule of $\mathscr{H}(\kappa_a)$. Then*

$$D_{(M_z|_M)^*,\nu} = \sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} Q^i_{M_z^*}(P_M)|_M,$$

where $P_M : \mathscr{H}(\kappa_a) \to M$ denotes the orthogonal projection of $\mathscr{H}(\kappa_a)$ onto M.

Proof. Note that $(M_z|_M)^* = S_z$, where $S_z = (S_{z_1}, \ldots, S_{z_d})$ denote the d-tuple $(P_M M_{z_1}^*|_M, \ldots, P_M M_{z_d}^*|_M)$. We claim that for any $\alpha \geq 0$,

$$S_z^{\alpha} = P_M(M_z^{*\alpha})|_M.$$

Let $K_\lambda := \kappa_a(\cdot, \lambda)$ and $K_\lambda^M := P_M K_\lambda$ for $\lambda \in \mathbb{B}$. We first note that

$$\{K_\mu^M : \mu \in \mathbb{B}\}$$

is a spanning set for M. We next check that

$$S_z^{\alpha}(K_\lambda^M) = \overline{\lambda}^{\alpha} K_\lambda^M \quad (\alpha \geq 0).$$

This follows from

$$\langle S_z^{\alpha}(K_\lambda^M), K_\mu^M\rangle = \langle K_\lambda^M, z^\alpha K_\mu^M\rangle = \overline{\lambda}^{\alpha} \langle K_\lambda^M, K_\mu^M\rangle \quad (\alpha \geq 0).$$

Since $P_M M_z^{*\alpha}|_M(K_\lambda^M) = \overline{\lambda}^{\alpha} K_\lambda^M$, the claim now stands verified. We immediately obtain

$$\begin{aligned} Q^i_{(M_z|_M)^*}(I) = \sum_{|\alpha|=i} \frac{i!}{\alpha!} (M_z|_M)^{\alpha} S_z^{\alpha} &= \sum_{|\alpha|=i} \frac{i!}{\alpha!} M_z^{\alpha} P_M M_z^{*\alpha}|_M \\ &= Q^i_{M_z^*}(P_M)|_M. \end{aligned}$$

The desired conclusion now follows from the very definition of the defect operator. □

Lemma 4.14. *Let M be a graded submodule of $\mathscr{H}(\kappa_\nu)$ with decomposition $M = \oplus_{n=0}^{\infty} V_n$. Then for integers $n \geq 1$ and $i \geq 1$,*

$$\text{trace } Q^i_{M_z^*}(P_{V_n}) = \frac{(n+d)\dots(n+d+i-1)}{(n+\nu)\dots(n+\nu+i-1)} \ \dim V_n.$$

Proof. Let $d_n = \dim V_n$. By Lemma 4.12, there exists an orthonormal basis for V_n consisting polynomials $p_{n_1}, \dots, p_{n_{d_n}}$ such that

$$P_{V_n} = \sum_{j=1}^{d_n} p_{n_j} \otimes p_{n_j}.$$

It follows that

$$\text{trace } M_z^\alpha P_{V_n} M_z^{*\alpha} = \sum_{j=1}^{d_n} \text{ trace } M_z^\alpha (p_{n_j} \otimes p_{n_j}) M_z^{*\alpha}.$$

Note that $M_z^\alpha (p_{n_j} \otimes p_{n_j}) M_z^{*\alpha}$ is a rank one operator with action

$$f \longmapsto \langle f, \, M_z^\alpha p_{n_j} \rangle M_z^\alpha p_{n_j},$$

and hence

$$\text{trace } M_z^\alpha (p_{n_j} \otimes p_{n_j}) M_z^{*\alpha} = \|M_z^\alpha p_{n_j}\|^2.$$

A straightforward inductive argument shows that for any integer $i \geq 1$,

$$Q^i_{M_z}(I) = \sum_{n=0}^{\infty} \frac{(n+d)\dots(n+d+i-1)}{(n+\nu)\dots(n+\nu+i-1)} E_n.$$

Combining last two observations, we obtain

$$\begin{aligned} \text{trace } Q^i_{M_z^*}(P_{V_n}) &= \sum_{|\alpha|=i} \frac{i!}{\alpha!} \text{trace } M_z^\alpha P_{V_n} M_z^{*\alpha} \\ &= \sum_{|\alpha|=i} \frac{i!}{\alpha!} \sum_{j=1}^{d_n} \|M_z^\alpha p_{n_j}\|^2 \\ &= \sum_{j=1}^{d_n} \left\langle Q^i_{M_z}(I) p_{n_j}, p_{n_j} \right\rangle \\ &= \frac{(n+d)\dots(n+d+i-1)}{(n+\nu)\dots(n+\nu+i-1)} \ \dim V_n. \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 4.9. Let M be a graded submodule such that the rank of $D_{(M_z|_M)^*,\nu}$ is finite. Since M is a graded submodule, M has an orthonormal basis consisting of homogeneous polynomials, and hence the orthogonal projection $P_M : \mathscr{H}(\kappa_\nu) \to M$ maps polynomials to polynomials. This and

Lemma 4.13 imply that the defect operator $D_{(M_z|_M)^*,\nu}$ maps polynomials to polynomials. By Lemma 4.12, there exist polynomials

$$p_1, \ldots, p_k, q_1, \ldots, q_k \in M$$

such that

$$D_{(M_z|_M)^*,\nu} = \sum_{i=1}^{k} p_i \otimes q_i.$$

Let $K_\lambda := \kappa_\nu(\cdot, \lambda)$ and $K_\lambda^M := P_M K_\lambda$ for $\lambda \in \mathbb{B}$. Note that

$$\begin{aligned}\langle D_{(M_z|_M)^*,\nu} K_\lambda^M,\, K_\mu^M\rangle &= \sum_{i=1}^{k} \langle p_i \otimes q_i K_\lambda^M,\, K_\mu^M\rangle \\ &= \sum_{i=1}^{k} \langle K_\lambda^M,\, q_i\rangle\langle p_i,\, K_\mu^M\rangle = \sum_{i=1}^{k} \overline{q_i(\lambda)} p_i(\mu).\end{aligned}$$

Interchanging the roles of μ and λ, we get

$$\langle K_\lambda^M,\, D_{(M_z|_M)^*,\nu} K_\mu^M\rangle = \sum_{i=1}^{k} q_i(\mu)\overline{p_i(\lambda)}.$$

Thus

$$\sum_{i=1}^{k} \overline{q_i(\lambda)} p_i(\mu) = \sum_{i=1}^{k} q_i(\mu)\overline{p_i(\lambda)} \quad (\lambda, \mu \in \mathbb{B}).$$

This implies that $\langle \sum_{i=1}^{k} M_{p_i} M_{q_i}^* K_\lambda,\, K_\mu\rangle = \langle K_\lambda,\, \sum_{i=1}^{k} M_{p_i} M_{q_i}^* K_\mu\rangle$ for every $\lambda, \mu \in \mathbb{B}$. Since all M_{p_i}, M_{q_j} are bounded operators, $\sum_{i=1}^{k} M_{p_i} M_{q_i}^*$ is a self-adjoint operator in $B(\mathcal{H})$. We note that

$$\begin{aligned}K_\lambda^M \left(D_{(M_z|_M)^*,\nu} K_\lambda^M\right) &= K_\lambda^M \left(\sum_{i=1}^{k} p_i \otimes q_i(K_\lambda^M)\right) \\ &= \sum_{i=1}^{k} M_{p_i} M_{q_i}^* K_\lambda^M.\end{aligned} \tag{4.9}$$

By Lemma 4.13, we have

$$D_{(M_z|_M)^*,\nu} = \sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} Q^i_{M_z^*}(P_M)|_M. \tag{4.10}$$

On the other hand,

$$\begin{aligned}Q^i_{M_z^*}(P_M) K_\lambda^M = \sum_{|\alpha|=i} \frac{i!}{\alpha!} M_z^\alpha P_M M_z^{*\alpha} K_\lambda^M &= \sum_{|\alpha|=i} \frac{i!}{\alpha!} z^\alpha \overline{\lambda}^\alpha K_\lambda^M \\ &= \langle z, \lambda\rangle^i K_\lambda^M.\end{aligned}$$

Combining this with (4.10), we obtain

$$K_\lambda^M D_{M_z^*|_M,\nu} K_\lambda^M = K_\lambda^M \sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} Q^i_{M_z^*}(P_M) K_\lambda^M = K_\lambda^M.$$

Since $\lambda \in \mathbb{B}$ is arbitrary, we conclude from (4.9) that $\sum_{i=1}^k M_{p_i} M_{q_i}^*$ is identity on M. Also, since $q_i \in \mathbb{C}[z]$ and $p_i \in M$, the range of $M_{p_i} M_{q_i}^*$ is contained in M for every $i = 1, \ldots, k$. Since $\sum_{i=1}^k M_{p_i} M_{q_i}^*$ is a self-adjoint operator, it follows that $P_M = \sum_{i=1}^k M_{p_i} M_{q_i}^*$. Also, since P_M is a projection, we have

$$\sum_{i=1}^k M_{p_i} M_{q_i}^* = \Big(\sum_{i=1}^k M_{p_i} M_{q_i}^*\Big)^2.$$

By Lemma 3.7(1), the continuous function $g(z) = \sum_{i=1}^k p_i(z)\overline{q}_i(z)$ satisfies

$$g(z)(1 - g(z)) = 0 \text{ for any } z \in \partial\mathbb{B}.$$

Let

$$A := \{\lambda \in \partial\mathbb{B}_d \ : g(\lambda) = 0\}, \quad B := \{\lambda \in \partial\mathbb{B}_d \ : g(\lambda) = 1\}.$$

Then A, B are closed subsets of $\partial\mathbb{B}_d$ such that

$$\partial\mathbb{B}_d = A \cup B, \ A \cap B = \emptyset.$$

Now the connectedness of $\partial\mathbb{B}_d$ implies that either $A = \emptyset$ or $B = \emptyset$. If $B = \emptyset$ then $g(\lambda) = 0$ on $\partial\mathbb{B}_d$. By another application of Lemma 3.7, we must have $P_M = \sum_{i=1}^k M_{p_i} M_{q_i}^*$ is a compact operator, and hence P_M is a finite rank operator. Since a nonzero submodule of a Hilbert module is infinite dimensional, we must have $A = \emptyset$, that is, $g(\lambda) = 1$ on $\partial\mathbb{B}_d$. Again, by Lemma 3.7, $P_{\mathscr{H}(\kappa_\nu)\ominus M} = I - \sum_{i=1}^k M_{p_i} M_{q_i}^*$ is compact, and hence M is of finite codimension.

We now see the remaining part. Since $D_{(M_z|_M)^*,\nu}$ is a finite rank operator with range spanned by polynomials, there exists a positive integer $N \geq 1$ such that $P_{V_n} D_{(M_z|_M)^*,\nu} E_{V_n} = 0$ for every $n \geq N$, where $E_{V_n} : V_n \to M$ denotes the inclusion map from V_n into M. It follows from Lemma 4.13 that

$$\sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} Q^i_{M_z^*}(P_{V_{n+1-i}}) = 0 \quad (n \geq \max\{N, \nu\}).$$

The desired dimension formula is now immediate from Lemma 4.14. □

Here is a rigidity statement about graded submodules of $\mathscr{H}(\kappa_\nu)$.

Corollary 4.15. *Let M be a nonzero graded submodule of $\mathscr{H}(\kappa_\nu)$. If the defect operator $D_{(M_z|_M)^*,\mu}$ is of finite rank then $\mu \geq \nu$.*

Proof. Suppose for some positive integer $\mu < \nu$, $D_{(M_z|_M)^*,\mu}$ is of finite rank. Then, by Remark 4.2, $D_{(M_z|_M)^*,\nu}$ has finite rank. Also, by Theorem 4.9, M is of finite codimension. An application of Lemma 4.13 and Corollary 4.11 shows that $P_M D_{M_z^*,\mu}|_M - D_{(M_z|_M)^*,\mu}$ is necessarily a finite rank operator, where P_M denotes the orthogonal projection of $\mathscr{H}(\kappa_\nu)$ onto M. Since $M^\perp$ is finite dimensional, $D_{M_z^*,\mu}$ itself is a finite rank operator. This is not possible in view of Remark 2.8. □

It is not known whether there exists a finite rank nongraded submodule of $\mathscr{H}(\kappa_\nu)$ $(\nu \geq 2)$ such that rank of $[P_M, M_{z_j}]$ is infinite for some $j = 1, \ldots, d$.

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(Sameer Chavan) Indian Institute of Technology Kanpur, Kanpur- 208016, India
chavan@iitk.ac.in

(Rani Kumari) Indian Institute of Technology Kanpur, Kanpur- 208016, India
rani@iitk.ac.in

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