Fields of definition and Belyi type theorems for curves and surfaces

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ABSTRACT. We study the relationship between the (effective) fields of definition of a complex projective variety and the orbit $\{X^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ where X^σ is the "twisted" variety obtained by applying σ to the equations defining X. Furthermore we present some applications of this theory to smooth curves and smooth minimal surfaces.

CONTENTS

Introduction

A complex projective variety X is defined over a subfield F of $\mathbb{C}$ if it is abstractly isomorphic to a projective variety which is cut out by polynomials with coefficients in F. As González–Diez showed in [16], the property of being defined over a number field is closely related to the structure of the set $\{X^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$, where X^σ is obtained by applying the field automorphism σ to the equations of X. It is worth mentioning that X^σ and X are not isomorphic as complex varieties but only as schemes, therefore they are essentially different objects, but on the other hand they share many important geometric properties.

A first aim of this paper is to give a revisitation of [16] in the language of scheme theory. The proof of the main result of [16] (which is Theorem 1.29 in this paper) is rather technical and long, but here is given a shorter proof based on a theorem in [15] about the existence of the minimal algebraically closed field of definition. Furtermore, Theorem 1.28 gives a criterion for

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deciding when X is actually cut out by polynomials with coefficients in a number field.

Section 2 is basically a survey about Belyi's well-known theorem for curves and shows a concrete application of the theory developed in the first section.

[17] presents an extension of Belyi's theorem for surfaces, and here, in Section 3, there is a slightly different version of it. In particular the proof of Proposition 3.16 is new and Proposition 3.17 does not seem to be published in the literature.

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0. Notations and preliminary results

This section is a mere collection of definitions and results useful in the paper, so it is logically independent from the other sections. The author advises against reading it from scratch; the reader should use the following list as reference material.

Every field in the paper is of characteristic 0, and the algebraic closure of a field K is $\overline{K}$.

Given a polynomial $g = \sum a_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_n^{i_n} \in K[T_1,\ldots,T_n]$ and $\sigma \in \mathrm{Aut}\,(K)$, the symbol g^σ denotes the polynomial

$$g^\sigma := \sum \sigma(a_{i_1,\ldots,i_n}) X_1^{i_1} \ldots X_n^{i_n}$$

A *variety* over a field K is a K-scheme of finite type, separated and geometrically integral. An *affine variety* over K is a variety (over K) such that the K-scheme is an affine scheme and a *projective variety* over K is a variety (over K) such that the K-scheme is a projective scheme. A *complex variety* is a variety over $K = \mathbb{C}$. A *curve* over K is a variety over K of dimension 1 and a *surface* over K is a variety over K of dimension 2.

$\mathbb{A}_K^n := \mathrm{Spec}(K[T_1,\ldots,T_n])$ and $\mathbb{P}_K^n := \mathrm{Proj}(K[T_0,\ldots,T_n])$. They are different from $\mathbb{A}^n(K)$ and $\mathbb{P}^n(K)$ which are respectively the affine n-dimensional space and the projective n-dimensional space over K. However if k is an algebraically closed field, then there is a bijective correspondence between the closed points of $\mathbb{A}_k^n$ (resp. $\mathbb{P}_k^n$) and $\mathbb{A}^n(k)$ (resp. $\mathbb{P}^n(k)$).

When the ground field is $\mathbb{C}$, one can associate in a canonical way to any complex projective variety a *complex projective manifold* $X(\mathbb{C})$. Basically $X(\mathbb{C})$ is obtained by equipping an algebraic set $\mathscr{X}$ with the sheaf of holomorphic functions (cf. [2, Corollary 2.5.16]).

Let $\varphi : X \to Y$ a morphism of varieties, we say that φ is *étale at* $x \in X$ if it is flat and unramified at x. Moreover φ is *étale* if it is étale at every point of X.

The following properties hold for étale morphisms between varieties:

- The set of points where a morphism is étale is open (possibly empty).

- The composition of two morphisms which are étale is étale.
- The base change of a morphism which is étale is étale.
- An étale morphism is open.

Let Y be a variety over k. A *finite covering of Y* is a couple (X, φ) where X is a variety over k and $\varphi : X \to Y$ is a surjective finite étale morphism.

0.1. Theorem (Riemann existence theorem)**.** *Let Y be a nonsingular projective complex variety and consider the associated complex manifold $Y(\mathbb{C})$. Then there is an equivalence of categories between the category of finite covering of Y (up to equivalence) and the category of finite holomorphic coverings of $Y(\mathbb{C})$ (up to equivalence).*

Proof. See [19, éxpose XII] or [32, Proposition 4.5.13]. $\qquad\qquad\square$

A nonconstant morphism between two nonsingular projective curves $\varphi : X \to Y$ is a *branched covering* (i.e., X is a branched covering of Y) if the open set $U \subseteq X$ where φ is étale, is nonempty. The finite set $\mathrm{Ram}(\varphi) := X \setminus U$ is called the *ramification locus* and moreover $\mathrm{Br}(\varphi) := \varphi(X \setminus U)$ is the *branch locus*. A point of $\mathrm{Ram}(\varphi)$ is a ramification point and a point of $\mathrm{Br}(\varphi)$ is a *branch point.*

Let B be a fixed k-scheme (k-algebraically closed). A *family of curves* over B (or a *fibration* over B) is a surjective proper flat morphism of k-schemes $\pi : X \to B$ such that the fibres are connected (maybe nonintegral) curves.

A family of curves $\pi : X \to B$ is said to be:

- *smooth* if all fibres are nonsingular;
- *isotrivial* if there exists a dense open set $U \subseteq B$ such that $f^{-1}(x) \cong f^{-1}(y)$ for every $x, y \in U$;
- *locally trivial* if it is smooth and all the fibers are isomorphic;
- *relatively minimal* if no fibre contains a (-1)-curve.

Let B be a nonsingular complex projective curve, S a nonsingular complex projective surface and $\pi : S \to B$ a relatively minimal family of curves with genus g; then the following numbers are well defined:

$$K_\pi^2 := K_S^2 - 8(g-1)(g(B)-1),$$
$$\chi_\pi := \chi(\mathcal{O}_S) - (g-1)(g(B)-1),$$
$$e_\pi := \chi_{\mathrm{top}}(S) - 4(g-1)(g(B)-1).$$

Let B be a nonsingular complex projective curve, S a nonsingular complex projective surface and $\pi : S \to B$ a nonisotrivial family of curves. Let moreover $\Delta \subset B$ be a finite set of points such that $\pi : S \setminus \pi^{-1}(\Delta) \to B \setminus \Delta$ is smooth. In this case π is called an *admissible family* with respect to the couple (B, Δ). What follows is a very deep theorem that was also known as the Shafarevich conjecture (see [24] for generalizations):

0.2. Theorem (Parshin–Arakelov theorem)**.** *Let B be a nonsingular complex projective curve of genus $g(B)$ and let $\Delta \subset B$ be a finite set, then:*

- *Up to B-isomorphism there is only a finite number of admissible families with respect to (B, Δ), of genus $g \geq 2$.*
- *If $2g(B) - 2 + \#(\Delta) \leq 0$, then there are no such admissible families.*

Proof. See [1]. □

Let X be a k-scheme over an algebraically closed field. A closed point $x \in X$ is called an *ordinary double point* if the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ has the following property: there exists some $n \in \mathbb{N}$ such that

$$\widehat{\mathcal{O}}_{X,x} \cong \frac{k[[T_1, \ldots, T_n]]}{(f)}$$

where $f \in \mathfrak{m}^2$ (here $\mathfrak{m}$ is the maximal ideal of $k[[T_1, \ldots, T_n]]$) and $f = Q + g$ for a nonsingular quadratic form Q and an element $g \in \mathfrak{m}^3$.

An ordinary double point is a singular point. In particular, if $\dim(X) = 1$, an ordinary double point is usually called a *node*.

A projective curve C over an algebraically closed field k is said *stable* if the following conditions hold:

- C is reduced and connected.
- $p_a(C) \geq 2$.
- The singularities of C, if present, are nodes.
- If $E \subsetneq C$ (note that must be $E \neq C$) is a rational component of C, then $\# \left(E \cap \overline{C \setminus E} \right) \geq 3$.

A family of curves $\pi : X \to B$ is said *stable* if the fibres of π are stable curves.

0.3. Proposition. *Let $\pi : X \to B$ be a stable family of curves (with nonsingular generic fibre) over a nonsingular integral projective curve B. If π is isotrivial then it is locally trivial.*

Proof. This follows easily from the theory of moduli of (stable) curves. □

0.4. Theorem. *Let B be a nonsingular complex projective curve, S a nonsingular complex projective surface and $\pi : S \to B$ a relatively minimal family of curves with genus $g \geq 2$. Then $K_\pi^2 \geq 0$, $\chi_\pi \geq 0$ and $e_\pi \geq 0$. Moreover, under the additional hypothesis that π is a stable family of curves, then $K_\pi^2 = 0$ if and only if π is locally trivial.*

Proof. See [5] and [1]. □

1. General theory

1.1. Definition. Let $F \subseteq K$ be a field extension. A projective variety X over K with a fixed closed immersion

$$j : X \hookrightarrow \mathbb{P}_K^n = \mathbb{P}_F^n \times_{\operatorname{Spec} F} \operatorname{Spec} K$$

is effectively defined over F if there exists a projective variety $X_{(F)}$ with a closed immersion $j_{(F)} : X_{(F)} \hookrightarrow \mathbb{P}^n_F$ such that $X_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K \cong X$ and moreover the following diagram is commutative:

$$
\begin{array}{ccc}
 & \mathbb{P}^n_K & \\
{}^{j}\nearrow & & \overset{j_{(F)} \times_{\operatorname{Spec} F} \operatorname{id}_{\operatorname{Spec} K}}{\longleftarrow} \\
X & \xrightarrow[\cong]{} & X_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K \ .
\end{array}
$$

In this case F is called *an effective field of definition of* $X \hookrightarrow \mathbb{P}^n_K$.

The meaning of the adjective "effective" associated to a field of definition is explained by the following proposition.

1.2. Proposition. *A projective variety* $X \hookrightarrow \mathbb{P}^n_K$ *is effectively defined over* $F \subseteq K$ *if and only if there exist some homogeneous polynomials* $f_1, \ldots, f_m \in F[T_0, \ldots T_n]$ *such that* $X \cong \operatorname{Proj} \frac{K[T_0, \ldots T_n]}{(f_1, \ldots, f_m)}$ *as subvarieties of* $\mathbb{P}^n_K$.

Proof. ($\Rightarrow$) Suppose that X is effectively defined over F, namely that there is a variety $X_{(F)} \hookrightarrow \mathbb{P}^n_F$ over F such that $X \cong X_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K$ as subvarieties of $\mathbb{P}^n_K$. Therefore

$$
X \cong \operatorname{Proj} \frac{F[T_0, \ldots T_n]}{(f_1, \ldots, f_m)} \times_{\operatorname{Spec} F} \operatorname{Spec} K
$$

where $f_1, \ldots, f_m$ are homogeneous polynomials in $F[T_0, \ldots T_n]$. Now by [27, Proposition 3.1.9]:

$$
\operatorname{Proj} \frac{F[T_0, \ldots T_n]}{(f_1, \ldots, f_m)} \times_{\operatorname{Spec} F} \operatorname{Spec} K \cong \operatorname{Proj} \left(\frac{F[T_0, \ldots T_n]}{(f_1, \ldots, f_m)} \otimes_F K \right) ,
$$

but $\frac{F[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)} \otimes_F K \cong \frac{K[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)}$, therefore $X \cong \operatorname{Proj} \frac{K[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)}$.

($\Leftarrow$) In order to show that $X \cong X_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K$, it is enough to retrace backward the above proof starting from the fact that $X \cong \operatorname{Proj} \frac{K[T_0, \ldots T_n]}{(f_1, \ldots, f_m)}$, where $f_1, \ldots, f_m, \in F[T_0, \ldots T_m]$. Furthermore, the commutativity of the diagram

$$
\begin{array}{ccc}
 & \mathbb{P}^n_K & \\
\nearrow & & \nwarrow \\
X & \xrightarrow[\cong]{} & X_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K
\end{array}
$$

is evident from the commutativity of the following diagram of K-algebras

$$
\begin{array}{ccc}
 & F[T_0, \ldots, T_n] \otimes_F K & \\
\swarrow & & \searrow \\
\dfrac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)} & \xleftarrow{\quad \cong \quad} & \dfrac{F[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)} \otimes_F K \ . \ \square
\end{array}
$$

1.3. Remark. For any projective variety $X \hookrightarrow \mathbb{P}_K^n$ *with a fixed closed immersion* there is an ideal $(f_1, \ldots, f_m) \subset K[T_0, \ldots, T_n]$ such that $X \cong$ Proj $\frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)}$ *as subvarieties of* $\mathbb{P}_K^n$. So, from now on we can work inside $\mathbb{P}_K^n$ and write by abuse of notations:

$$X = \text{Proj} \, \frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)} \subseteq \mathbb{P}_K^n \,.$$

For example Theorem 1.2 can be stated as follows: $X \subseteq \mathbb{P}_K^n$ is effectively defined over F if and only if there exist some homogeneous polynomials $f_1, \ldots, f_m \in F[T_0, \ldots T_n]$ such that $X = \text{Proj} \, \frac{K[T_0, \ldots T_n]}{(f_1, \ldots, f_m)}$.

The concept of effective field of definition of a projective variety is very important in Diophantine geometry, but it has a serious drawback: it is not preserved by isomorphisms. Indeed a projective variety Y isomorphic to Proj $\frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)}$ with $f_1, \ldots, f_m \in F[T_0, \ldots, T_n]$ may not be cut out by polynomials with coefficients in F (see Example 1.8). Therefore a weaker concept of field of definition is needed.

1.4. Definition. Let $F \subseteq K$ be a field extension and let X be a variety over K. *X is defined over F* if there exists a variety $X_{(F)}$ over F such that $X \cong X_{(F)} \times_{\text{Spec} \, F} \text{Spec} \, K$ (isomorphism over K), where the fibre product is taken along the morphism $\text{Spec} \, K \to \text{Spec} \, F$.

$$
\begin{array}{ccc}
X \cong X_{(F)} \times_{\text{Spec} \, F} \text{Spec} \, K & \longrightarrow & \text{Spec} \, K \\
\downarrow & & \downarrow \\
X_{(F)} & \longrightarrow & \text{Spec} \, F \,.
\end{array}
$$

Note that Definition 1.4 works for any variety, not necessarily projective. Moreover a projective variety $X \subseteq \mathbb{P}_K^n$ is defined over F exactly when it is isomorphic to a projective variety effectively defined over F.

1.5. Remark. The literature is somewhat confusing as far as Definitions 1.1 and 1.4 are concerned. Some sources mix the two definitions, but there seems to be a tacit agreement on calling "a field of definition" what is depicted in Definition 1.4. On the other hand the terminology used in Definition 1.1 is not standard.

In [16] and [17] the author uses the terms "*X is* defined over F" for Definition 1.1 and "*X can be* defined over F" for Definition 1.4.

1.6. Definition. Let $F \subseteq K$ be a field extension and let X, Y be two varieties over K. A morphism $\varphi : X \to Y$ *is defined over F* if there exist two varieties $X_{(F)}, Y_{(F)}$ over F with a morphism $\varphi_{(F)} : X_{(F)} \to Y_{(F)}$ such that:

- $X \cong X_{(F)} \times_{\text{Spec} \, F} \text{Spec} \, K$.
- $Y \cong Y_{(F)} \times_{\text{Spec} \, F} \text{Spec} \, K$.

- The following diagram is commutative:

$$
\begin{array}{ccc}
Y_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K & \xleftarrow{\ \cong\ } & Y \\
{\scriptstyle \varphi_{(F)} \times_{\operatorname{Spec} F} \operatorname{id}_{\operatorname{Spec} K}} \Big\uparrow & & \Big\uparrow {\scriptstyle \varphi} \\
X_{(F)} \times_{\operatorname{Spec} F} \operatorname{Spec} K & \xleftarrow{\ \cong\ } & X \ .
\end{array}
$$

1.7. Remark. It is evident that if φ is defined over F, then both X and Y are defined over F. Moreover if Γ_φ is the graph of the morphism φ, then φ is defined over F if and only if the immersion $j : \Gamma_\varphi \hookrightarrow X \times_{\operatorname{Spec} K} Y$ is a morphism defined over F.

The following very simple example shows that the concept of effective field of definition is truly stronger than the concept of field of definition.

1.8. Example. Consider the one-point variety $p = (eX_0 + X_1) \subseteq \mathbb{P}^1_{\mathbb{R}}$ where $e = \exp(1)$; x is defined over $\mathbb{Q}$, but on the other hand x is not effectively defined over $\mathbb{Q}$. Note that the same example works if we substitute e with any irrational number.

An important question consists in asking if there exists a minimal field of definition or a minimal effective field of definition for a given projective variety.

1.9. Definition. Let $X \subseteq \mathbb{P}^n_K$ be a projective variety over K, then a subfield $K_0 \subseteq K$ is *the (effective) minimal field of definition* of X if the following conditions hold:

- K_0 is an (effective) field of definition of X.
- If F is any (effective) field of definition of X contained in K, then $K_0 \subseteq F$.

In the "effective case" we have an affirmative answer thanks to the following theorem due to Weil.

1.10. Theorem (Weil, 1962). *Consider a field extension $F \subseteq K$ and a nonzero ideal $\mathfrak{a} \subseteq K[T_1, \ldots, T_n]$, then there exists a field K_0 between F and K with the following properties:*

(1) *$\mathfrak{a}$ has a system of generators in $K_0[T_1, \ldots, T_n]$.*
(2) *If K' is any field between F and K such that $\mathfrak{a}$ has a system of generators in $K'[T_1, \ldots, T_n]$, then $K_0 \subseteq K'$.*

Proof. Assume that a monomial order is fixed (in general one considers the graded lexicographic ordering), then the key point is the uniqueness of the reduced Gröbner basis for a nonzero polynomial ideal ([11, 2 Proposition 6]).

If $G = \{g_1, \ldots, g_t\}$ is the reduced Gröbner basis for $\mathfrak{a}$, and S is the set of all coefficients of the polynomials in G, one simply defines $K_0 := F(S)$. Since G is a set of generators for $\mathfrak{a}$, K_0 clearly satisfies the condition (1). Let K' be a field between F and K such that $\mathfrak{a}$ has a system of generators

$$
h_1, \ldots, h_s \in K'[T_1, \ldots, T_n]
$$

and consider the ideal $\mathfrak{a}' = (h_1, \ldots, h_s) \subseteq K'[T_1, \ldots, T_n]$. If G' is the reduced Gröbner basis for $\mathfrak{a}'$, then it is also the reduced Gröbner basis for $\mathfrak{a}$: this is true since the reduced Gröbner basis of the ideal $(h_1, \ldots, h_s)$ can be obtained by an algorithm which manipulates only the polynomials $h_1, \ldots, h_s$ without going out from the field generated by their coefficients (Buchberger's algorithm plus a reduction process, cf. [11]). By the uniqueness of the reduced Gröbner basis it follows that $G' = G$, but the polynomials in G' have coefficients in K', so $S \subseteq K'$. This means that $K_0 \subseteq K'$ and therefore also the condition (2) is satisfied. $\qquad\square$

The above proof of Theorem 1.10 is shorter than the classical ones: see for example [33, I.7 Lemma 2] (this is the original proof of Weil) or [26, III.2 Theorem 7]. By the way the author is not aware of any argument at all based on Gröbner bases in the literature.

1.11. Corollary. *If $X \subseteq \mathbb{P}^n_K$ is a projective variety, then there exists an effective minimal field of definition of X.*

Proof. Let
$$X = \mathrm{Proj}\, \frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)}$$
be a projective variety where $\mathfrak{a} = (f_1, \ldots, f_m)$ is a homogeneous prime ideal of $K[T_0, \ldots, T_n]$. If F is the prime field of K, thanks to Theorem 1.10 one can find a minimal field $K_0 \subseteq K$ such that $\mathfrak{a}$ has a system of generators $g_1, \ldots, g_r \in K_0[T_0, \ldots, T_n]$. If $g_j^{(d)}$ is the homogeneous part of degree d of g_j, then $g_j^{(d)} \in \mathfrak{a}$ because $\mathfrak{a}$ is a homogeneous ideal, therefore $\mathfrak{a} = \left(\left\{ g_j^{(d)} \right\}_{j,d} \right)$ where j and d run in their range. Thanks to Proposition 1.2 K_0 is the effective minimal field of definition for X. $\qquad\square$

On the other hand, regarding the existence of the minimal field of definition of X, we can state a partial affirmative result if we restrict to algebraically closed fields of definition.

1.12. Theorem. *Let $X \subseteq \mathbb{P}^n_k$ a projective variety over an algebraically closed field k, then there exists a minimal algebraically closed field of definition of X.*

Proof. See [15, Theorem 2] for a proof of a more general result. $\qquad\square$

1.13. Definition. Let $s : X \to \mathrm{Spec}\, K$ be a variety over K and let $\sigma \in \mathrm{Aut}(K)$. The variety X^σ is defined (up to isomorphism) as the base change of X with respect to the morphism $\mathrm{Spec}\, \sigma : \mathrm{Spec}\, K \to \mathrm{Spec}\, K$.

$$
\begin{array}{ccc}
X^\sigma = X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, K & \xrightarrow{\ p_2\ } & \mathrm{Spec}\, K \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \mathrm{Spec}\, \sigma} \\
X & \xrightarrow{\quad s \quad} & \mathrm{Spec}\, K\ .
\end{array}
$$

1.14. Remark. If Var_K is the set of all varieties over K, then the mapping

$$\begin{aligned} \mathrm{Var}_K \times \mathrm{Aut}(K) &\to \mathrm{Var}_K \\ (X, \sigma) &\longmapsto X^\sigma \end{aligned}$$

is not a function since X^σ is defined up to isomorphism. The problem can be avoided by assuming a fixed canonical choice of X^σ among all isomorphic fibre products. With this clarification in mind, it is evident that the rule $(X, \sigma) \mapsto X^\sigma$ defines a group action of $\mathrm{Aut}(K)$ on the set Var_K.

The structural morphism of the variety X^σ is *always* understood to be $p_2 : X^\sigma \to \mathrm{Spec}\, K$. Note that $p_1 : X^\sigma \to X$ *is not* a morphism of varieties, and one can only say that p_1 is an isomorphism of schemes. The inverse map can be obtained by taking the base change through $\mathrm{Spec}\, \sigma^{-1}$. This is a crucial point: in general X and X^σ are two nonisomorphic varieties but two isomorphic schemes.

Let $\varphi : X \to Y$ be a morphism of varieties over K and consider an element $\sigma \in \mathrm{Aut}(K)$. Then by equipping $\mathrm{Spec}\, K$ with the K-scheme structure $\mathrm{Spec}\, \sigma : \mathrm{Spec}\, K \to \mathrm{Spec}\, K$, the following canonical morphism of varieties:

$$\varphi^\sigma := \varphi \times_{\mathrm{Spec}\, K} \mathrm{id}_{\mathrm{Spec}\, K} : X^\sigma \to Y^\sigma \,.$$

1.15. Proposition. *Let* $X = \mathrm{Proj}\, \frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)}$ *be a projective variety over a field* K, *then we can choose* $X^\sigma = \mathrm{Proj}\, \frac{K[T_0, \ldots, T_n]}{(f_1^\sigma, \ldots, f_m^\sigma)}$.

Proof. Suppose for the moment that $\sigma : K \to K'$ is an isomorphism of fields where K and K' are not necessarily equal, then consider the K'-scheme $X \times_\sigma \mathrm{Spec}\, K'$ given by the following diagram

$$\begin{array}{ccc} X \times_\sigma \mathrm{Spec}\, K' & \xrightarrow{p_2} & \mathrm{Spec}\, K' \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \mathrm{Spec}\, \sigma} \\ X & \xrightarrow{\ \ s\ \ } & \mathrm{Spec}\, K \,. \end{array}$$

Since $X = \mathrm{Proj}\, \frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)}$, by [27, 3 Proposition 1.9]:

$$X \times_\sigma \mathrm{Spec}\, K' \cong \mathrm{Proj}\left(\frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)} \otimes_\sigma K' \right).$$

But $\frac{K[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)} \otimes_\sigma K'$ is a K'-algebra isomorphic to $\frac{K'[T_0, \ldots, T_n]}{(f_1^\sigma, \ldots, f_m^\sigma)}$, so it follows that

$$X \times_\sigma \mathrm{Spec}\, K' \cong \mathrm{Proj}\, \frac{K'[T_0, \ldots, T_n]}{(f_1^\sigma, \ldots, f_m^\sigma)}$$

and we can put $X^\sigma = \mathrm{Proj}\, \frac{K'[T_0, \ldots, T_n]}{(f_1^\sigma, \ldots, f_m^\sigma)}$. Finally, by using the fact that $K = K'$ the proof is complete. $\qquad\square$

1.16. Remark. If $X = \operatorname{Proj} \frac{K[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)}$, from now on we always put $X^\sigma = \operatorname{Proj} \frac{K[T_0,\ldots,T_n]}{(f_1^\sigma,\ldots,f_m^\sigma)}$. In particular $(\mathbb{P}^n_K)^\sigma = \mathbb{P}^n_K$.

So, at least for projective varieties over an algebraically closed field k, the abstract switch from X to X^σ, is equivalent to transforming the projective algebraic set $Z(f_1,\ldots,f_m) \subseteq \mathbb{P}^n(k)$ in $Z(f_1^\sigma,\ldots,f_m^\sigma) \subseteq \mathbb{P}^n(k)$. Since X and X^σ are in general not isomorphic as varieties, then $Z(f_1,\ldots,f_m)$ and $Z(f_1^\sigma,\ldots,f_m^\sigma)$ are not isomorphic as algebraic sets. If $X = \operatorname{Proj} \frac{k[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)}$ (k algebraically closed), it is not difficult to see that the scheme isomorphism $p_1 : X^\sigma \to X$ induced by the fibre product construction, in classical terms is described by the map:

$$Z(f_1^\sigma,\ldots,f_m^\sigma) \quad \to \quad Z(f_1,\ldots,f_m)$$
$$p \quad \longmapsto \quad \sigma^{-1}(p).$$

Moreover if $Y \cong \operatorname{Proj} \frac{k[T_0,\ldots,T_n]}{(g_1,\ldots,g_h)}$, $\varphi : X \to Y$ is a morphism of varieties, and $\widetilde{\varphi} : Z(f_1,\ldots,f_m) \to Z(g_1,\ldots,g_h)$ is its corresponding morphism of projective algebraic sets, then it follows that $\varphi^\sigma : X^\sigma \to Y^\sigma$ corresponds to the morphism of projective algebraic sets defined by:

$$\sigma \circ \widetilde{\varphi} \circ \sigma^{-1} : Z(f_1^\sigma,\ldots,f_m^\sigma) \to Z(g_1^\sigma,\ldots,g_h^\sigma)$$

If around a point $p \in Z(f_1,\ldots,f_m)$ the morphism $\widetilde{\varphi}$ is defined by the polynomials $h_1,\ldots,h_r$, then around $q = \sigma(p) \in Z(f_1^\sigma,\ldots,f_m^\sigma)$ the morphism $\sigma \circ \widetilde{\varphi} \circ \sigma^{-1}$ is defined by $h_1^\sigma,\ldots,h_r^\sigma$. In the Example 1.18 it will be clear how it is not difficult to encounter two nonisomorphic varieties which are two isomorphic schemes.

1.17. Lemma. *Let k be an algebraically closed field. If F is a subfield of k, then every element of $\operatorname{Aut}(F)$ extends to an element of $\operatorname{Aut}(k)$.*

Proof. If S is a transcendence basis for k/F, then every element $\sigma \in \operatorname{Aut}(F)$ extends naturally to an element $\widetilde{\sigma} \in \operatorname{Aut}(F(S))$. Moreover, since k is an algebraic closure of $F(S)$, the isomorphism extension theorem (see [29, Theorem I.3.20]) ensures that $\widetilde{\sigma}$ extends to an element of $\operatorname{Aut}(k)$. $\qquad\square$

1.18. Example. Consider the following two varieties over $\mathbb{C}$:

$$X = \mathbb{P}^1_{\mathbb{C}} \setminus \{(T_0),\, (T_0 - T_1),\, (T_0 - \pi T_1),\, (T_1)\}\,,$$

$$Y = \mathbb{P}^1_{\mathbb{C}} \setminus \{(T_0),\, (T_0 - T_1),\, (T_0 - e^\pi T_1),\, (T_1)\}\,.$$

The elements π and e^π are known to be algebraically independent over $\mathbb{Q}$, so the function $\{\pi, e^\pi\} \to \{\pi, e^\pi\}$ that exchanges them can be extended to an automorphism $\sigma \in \operatorname{Aut}(\mathbb{Q}(\pi, e^\pi))$. By Lemma 1.17, σ extends to an element $\widetilde{\sigma} \in \operatorname{Aut}(\mathbb{C})$, but $\widetilde{\sigma}$ in turn extends in the obvious way to an automorphism of graded rings $\overline{\sigma} : \mathbb{C}[T_0, T_1] \to \mathbb{C}[T_0, T_1]$. Thanks to the properties of the $\operatorname{Proj}(\cdot)$ construction, the graded automorphism $\overline{\sigma}$ induces an automorphism $\operatorname{Proj}\overline{\sigma}$ of the scheme $\mathbb{P}^1_{\mathbb{C}}$ such that $(\operatorname{Proj}\overline{\sigma})(X) = Y$. Practically X and Y

are two isomorphic schemes. We can identify them with the open sets of the projective line given by

$$\mathscr{X} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \pi, \infty\},$$

$$\mathscr{Y} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, e^\pi, \infty\}.$$

$\mathscr{X}$ and $\mathscr{Y}$ are not isomorphic, because if they were, then would exist a birational map $f : \mathbb{P}^1(\mathbb{C}) \dashrightarrow \mathbb{P}^1(\mathbb{C})$. Such a map induces an automorphism $\overline{f}$ of $\mathbb{P}^1(\mathbb{C})$ sending by construction $\{0, 1, \pi, \infty\}$ in $\{0, 1, e^\pi, \infty\}$. But the automorphisms of $\mathbb{P}^1(\mathbb{C})$ are Möbius transformations, so they preserve the cross-ratio. This leads to the contradiction because all cross-ratios obtainable by $\{0, 1, \pi, \infty\}$ and $\{0, 1, e^\pi, \infty\}$ are different. Now since $\mathscr{X}$ and $\mathscr{Y}$ aren't isomorphic, then X and Y can't be two isomorphic varieties over $\mathbb{C}$.

The key point of the example is that $\overline{\sigma} : \mathbb{C}[T_0, T_1] \to \mathbb{C}[T_0, T_1]$ is an automorphism of graded rings, but it is not an automorphism of $\mathbb{C}$-algebras.

1.19. Definition. Let X be a variety over K, and consider the group

$$U(X) := \{\sigma \in \mathrm{Aut}(K) \,:\, X^\sigma \cong X \text{ as varieties over } K\} \subseteq \mathrm{Aut}(K),$$

then the field $M(X) := \mathrm{Fix}_K(U(X)) \subseteq K$ is the *field of moduli of X*.

1.20. Lemma. *Let k be an algebraically closed field, then*

$$\mathrm{Fix}_k\left(\mathrm{Gal}(k/F)\right) = F.$$

Proof. Obviously it is enough to prove that for every $x \in k \setminus F$ there is an element $\sigma \in \mathrm{Gal}(k/F)$ such that $\sigma(x) \neq x$. There are two cases:

Case 1. x is transcendental over F. Consider the field $F(x)$, then the assignment $x \mapsto -x$ induces a unique element $\sigma \in \mathrm{Gal}(F(x)/F)$ that moves x. By Lemma 1.17 this σ extends to an element of $\mathrm{Gal}(k/F)$.

Case 2. x is algebraic over F. Since in characteristic 0 every polynomial is separable, if $f = \min(x, F)$ then there exists in k (remember that f splits over k) a root y of f such that $x \neq y$. Let M be the splitting field of f over F and look at the inclusions $F \subseteq F(x) \subseteq M \subseteq k$; M is normal over F and the canonical F-isomorphism $\sigma : F(x) \to F(y)$ can be viewed as an immersion $\sigma : F(x) \to k$. By [29, Proposition I.3.28 (3.)] there is an element $\tau \in \mathrm{Gal}(M/F)$ such that $\tau_{|F(x)} = \sigma$ (in particular $\tau(x) = y$), therefore by Lemma 1.17 τ extends to an element of $\mathrm{Gal}(k/F)$ which moves x.   □

1.21. Proposition. *Let $X = \mathrm{Proj}\, \frac{k[T_0, \ldots, T_n]}{(f_1, \ldots, f_m)}$ be a projective variety over an algebraically closed field k, then every effective field of definition of X contains $M(X)$.*

Proof. Let $F \subseteq k$ be an effective field of definition of X and consider $\sigma \in \mathrm{Gal}(k/F)$. Thanks to Proposition 1.15 $X^\sigma = \mathrm{Proj}\, \frac{k[T_0, \ldots, T_n]}{(f_1^\sigma, \ldots, f_m^\sigma)}$. But X is effectively defined over F, so, since σ fixes F, then $X^\sigma = X$. By Lemma 1.20 it follows that $M(X) \subseteq \mathrm{Fix}_k\left(\mathrm{Gal}(k/F)\right) = F$.   □

1.22. Corollary. *If a projective variety X over an algebraically closed field k is effectively defined over $M(X)$, then $M(X)$ is the minimal effective field of definition of X.*

In general is a difficult problem to show that a given complex variety is effectively defined over $\overline{\mathbb{Q}}$ without finding explicitly the adequate equations.

1.23. Remark. Note that if $X = \text{Proj} \frac{\mathbb{C}[T_0,...,T_n]}{(f_1,...,f_m)}$, and $\{\alpha_{ij}\}_{j \in J}$ are the coefficients of the polynomial f_i, then X is effectively defined over the field $\mathbb{Q}\left(\bigcup_{i=1}^m \{\alpha_{ij}\}\right)$. Therefore X is defined over $\overline{\mathbb{Q}}$ if and only if X is defined over a number field.

Remember that if $F \subseteq k$ is a field extension with k algebraically closed, then the algebraic closure of F in k is $\overline{F}$. This simple fact will be tacitly used throughout the paper.

1.24. Definition. Two fields K_1 and K_2 containing a field F are said *algebraically disjoint* over F if for every pair of sets $S_1 \subseteq K_1$ and $S_2 \subseteq K_2$, both algebraically independent over F, it holds that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2$ is algebraically independent over F.

1.25. Remark. By the above definition it follows that the intersection of two fields algebraically disjoint over F is algebraic over F. Indeed, if there exists $x \in K_1 \cap K_2$ such that x is not algebraic over F, then $\{x\}$ is an algebraically independent subset of both K_1 and K_2 over F.

1.26. Lemma. *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a complex projective variety and consider a subfield F of $\mathbb{C}$. If X is effectively defined [resp. defined] over two subfields K_1 and K_2 of $\mathbb{C}$ which are algebraically disjoint over F, then X is effectively defined [resp. defined] over $\overline{F}$.*

Proof. If X is effectively defined over K_1 and K_2 thanks to Corollary 1.11 there exists the effective minimal field of definition K_0 of X, therefore $K_0 \subseteq K_1$ and $K_0 \subseteq K_2$. This means that $K_0 \subseteq K_1 \cap K_2$, so X is effectively defined over $K_1 \cap K_2$, but $K_1 \cap K_2$ is algebraic over F (see Remark 1.25), then X is effectively defined over $\overline{F}$.

If X is defined over K_1 and K_2 then it is defined over the algebraic closures $\overline{K_1}, \overline{K_2} \subseteq \mathbb{C}$. Therefore X is defined over $\overline{K_1} \cap \overline{K_2} \subseteq \mathbb{C}$ because of the existence of the minimal algebraically closed field of definition $k_0 \subseteq \overline{K_1} \cap \overline{K_2}$ (Theorem 1.12). By [10, Corollary of prop. 12, page A.V. 113] $\overline{K_1}$ and $\overline{K_2}$ are algebraically disjoint over F, so the extension $F \subseteq \overline{K_1} \cap \overline{K_2}$ is algebraic. This means that X is defined over $\overline{F}$. $\square$

1.27. Lemma. *If K is a countable subfield of $\mathbb{C}$ then every transcendence basis of $\mathbb{C}$ over K is uncountable.*

Proof. Suppose that the proposition is false, namely there exists is a countable transcendence basis B of $\mathbb{C}$ over K. Clearly $K(B) = \bigcup_{b \in B} K(b)$, but since K is countable, also $K(b)$ is countable and it follows that $K(B)$ is

countable. Now $\mathbb{C}$ is algebraic over $K(B)$, so by [29, Lemma I.3.13] it follows that $\mathbb{C}$ is countable which is absurd. $\square$

Below there are the main results of this first section; they are stated for a generic countable subfield F of $\mathbb{C}$, but we are mainly interested in the case $F = \mathbb{Q}$. The proofs are heavily based on Lemma 1.26 [1].

1.28. Theorem. *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a complex projective variety and let F be a countable subfield of $\mathbb{C}$, then the following conditions are equivalent:*

(1) *X is effectively defined over $\overline{F}$.*
(2) *The set $\{X^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C}/F)}$ is finite.*
(3) *The set $\{X^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C}/F)}$ is countable.*

Proof. (1)$\Rightarrow$(2). Let $X = \frac{\mathbb{C}[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)}$ where $f_1,\ldots,f_m \in \overline{F}[T_0,\ldots T_n]$. Denote by K the field generated over F by the coefficients of the polynomials $f_1,\ldots,f_m$. $F \subseteq K \subseteq \overline{F}$, so the extension $F \subseteq K$ is finitely generated and algebraic, therefore it is finite. Suppose that the degree of the extension $F \subseteq K$ is $[K : F] = r$ and that $\{b_1,\ldots,b_r\}$ is a basis for K over F; if $\sigma \in \mathrm{Gal}(\mathbb{C}/F)$ and $i \in \{1,\ldots,r\}$, then $\sigma(b_i)$ is a root of $\min(b_i,F)$. But $\min(b_i,F)$ can have at most r roots, therefore $\{\sigma|_K\}_{\sigma\in\mathrm{Gal}(\mathbb{C}/F)}$ is a finite set as well as $\{X^\sigma\}_{\sigma\in\mathrm{Gal}(\mathbb{C}/F)}$.

(2)$\Rightarrow$(3). Obvious.

(3)$\Rightarrow$(1). If $X = \mathrm{Proj}\,\frac{\mathbb{C}[T_0,\ldots,T_n]}{(f_1,\ldots,f_m)}$ where $f_1,\ldots,f_m \in \mathbb{C}[T_0,\ldots T_n]$, denote with K the field generated over F by the coefficients of the polynomials $f_1,\ldots,f_m$. If K is algebraic over F there is nothing to prove, so suppose that $\{\pi_1,\ldots,\pi_d\}$ is a transcendence basis of K over F with $d \geq 1$. Since F is countable, by Lemma 1.27 there is an uncountable number of sets $A_\alpha = \{\alpha_1,\ldots,\alpha_d\} \subseteq \mathbb{C}$ algebraically independent over F and such that $A_\alpha \cap \{\pi_1,\ldots,\pi_d\} = \emptyset$ and $A_\alpha \cap A_\beta = \emptyset$ for every pair of indexes α and β. Consider the field

$$L = F\left(\pi_1,\ldots,\pi_d,\bigcup_\alpha A_\alpha\right) \subset \mathbb{C}\,;$$

for any α, there is certainly $\sigma_\alpha \in \mathrm{Gal}(L/F)$ such that $\sigma_\alpha(\pi_i) = \alpha_i$ and $\sigma_\alpha(\alpha_i) = \pi_i$ for every $i = 1,\ldots,d$. Now by Lemma 1.17 every σ_α extends to an element $\tau_\alpha \in \mathrm{Gal}(\mathbb{C}/F)$. All the τ_α are distinct by construction, therefore $\{X^{\tau_\alpha}\}_\alpha$ is an uncountable set. But by hypothesis there is only a countable number of elements in the orbit, so there exist certainly τ_α and τ_β such that $X^{\tau_\alpha} = X^{\tau_\beta}$. If $\sigma := \tau_\beta\tau_\alpha^{-1}$, it follows that $X = X^\sigma$. Now X is effectively defined over K thanks to Proposition 1.2; but on the other hand $X = X^\sigma = \frac{\mathbb{C}[T_0,\ldots,T_n]}{(f_1^\sigma,\ldots,f_m^\sigma)}$ by Proposition 1.15, therefore again Proposition 1.2

---

[1]See [16, Theorem 2.12] for a less general version of Lemma 1.26; it is proved using specializations of k-algebras and some very technical "ε-δ argumentations". In this paper, on the contrary, Lemma 1.26 follows immediately from the existence of the (effective) minimal fields of definition.

implies that X is also effectively defined over $\sigma(K)$. $F \subset K$ and $F \subset \sigma(K)$ since σ fixes F, and $\{\sigma(\pi_1), \ldots, \sigma(\pi_d)\}$ is a transcendence basis of $\sigma(K)$ over F such that

$$\{\sigma(\pi_1), \ldots, \sigma(\pi_d)\} \cap \{\pi_1, \ldots, \pi_d\} = \emptyset \,.$$

Thus, by construction K and $\sigma(K)$ are two algebraically disjoint fields over F and by Lemma 1.26 it follows that X is effectively defined over $\overline{F}$. $\square$

1.29. Theorem (González–Diez, 2006). *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a complex projective variety and let F be a countable subfield of $\mathbb{C}$, then the following conditions are equivalent:*

(1) *X is defined over $\overline{F}$.*
(2) *The set $\{X^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C}/F)}$ contains at most finitely many isomorphism classes.*
(3) *The set $\{X^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C}/F)}$ contains at most countably many isomorphism classes.*

Proof. (1)$\Rightarrow$(2). Follows easily from the implication (1)$\Rightarrow$(2) of Theorem 1.28.

(2)$\Rightarrow$(3). Obvious.

(3)$\Rightarrow$(1). We repeat word by word the construction made in the last implication of Theorem 1.28 except for the following obvious changes. Here by hypothesis we have only a countable number of isomorphism classed in the orbit $\{X^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C}/F)}$, hence there exist τ_α and τ_β such that $X^{\tau_\alpha} \cong X^{\tau_\beta}$. If $\sigma := \tau_\beta \tau_\alpha^{-1}$, then $X \cong X^\sigma = \frac{\mathbb{C}[T_0, \ldots, T_n]}{(f_1^\sigma, \ldots, f_m^\sigma)}$. It follows that X is defined over K and $\sigma(K)$ which are algebraically disjoint over F, so again by Lemma 1.26 we can conclude that X is defined over $\overline{F}$. $\square$

2. Curves defined over $\overline{\mathbb{Q}}$

For the rest of the paper we deal with fields of definition (not necessarily effective). In practice, it is not a feasible problem to decide when a variety X is defined over $\overline{\mathbb{Q}}$ directly from Theorem 1.29. Indeed one should count the elements in the orbit $\{X^\sigma : \sigma \in \mathrm{Aut}(\mathbb{C})\}$, but $\mathrm{Aut}(\mathbb{C})$ is an uncountable group, so this "task" in general can't be easily performed. Here is presented a beautiful characterization for nonsingular complex projective curves defined over $\overline{\mathbb{Q}}$ in terms of morphisms to $\mathbb{P}^1_{\mathbb{C}}$ and their branch points:

2.1. Theorem (Definability over $\overline{\mathbb{Q}}$ for curves). *A nonsingular complex projective curve $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$, with at most three branch points.*

The "if direction" of the theorem is also called "the obvious implication" because it is a known result for specialists in the field, and "only if direction" is referred as the Belyi's theorem. Despite of the name, the obvious implication can be approached in different ways and all the proofs are far from straightforward. On the other hand the proof of Belyi's theorem dates

back only to 1979 and is very simple and short, in spite of the problem was considered hard to solve. What follows is the translation of an original quote by Alexander Grothendieck, taken from [18], about Belyi's proof; note that even Grothendieck was struck by the simplicity of the Belyi's argument:

> Every finite oriented map gives rise to a projective nonsingular algebraic curve defined over $\overline{\mathbb{Q}}$, and one immediately asks the question: which are the algebraic curves over $\overline{\mathbb{Q}}$ obtained in this way — do we obtain them all, who knows? (...) could it be true that every projective nonsingular algebraic curve defined over a number field occurs as a possible "modular curve" parametrising elliptic curves equipped with a suitable rigidification? Such a supposition seemed so crazy that I was almost embarrassed to submit it to the competent people in the domain. (...) Bielyi announced exactly that result, with a proof of disconcerting simplicity which fit into two little pages of a letter of Deligne — never, without a doubt, was such a deep and disconcerting result proved in so few lines!
>
> In the form in which Bielyi states it, his result essentially says that every algebraic curve defined over a number field can be obtained as a covering of the projective line ramified only over the points 0, 1 and ∞. This result seems to have remained more or less unobserved. Yet it appears to me to have considerable importance. To me, its essential message is that there is a profound identity between the combinatorics of finite maps on the one hand, and the geometry of algebraic curves defined over number fields on the other. This deep result, together with the algebraic geometric interpretation of maps, opens the door onto a new, unexplored world — within reach of all, who pass by without seeing it.

The proof of the "obvious implication" presented here, like in [16], is based on standard results about covering spaces and on Theorem 1.29. For other approaches see [23], [32], [34], [9] or [25].

2.2. Lemma. *Let $d \in \mathbb{N}$; if G is a finitely generated group then there is a finite number of subgroups $H \leq G$ such that $|G : H| = d$.*

Proof. Suppose that $\{H_\alpha\}$ is the set of all subgroups of G of index d, where α ranges in some index set. Now define the sets of right cosets G/H_α for every α, and for each of them fix a bijection with $\{1, \ldots, d\}$ such that the identity coset $H_\alpha \in G/H_\alpha$ corresponds to the number 1.

To any subgroup H_α, one can associate a group action of G on G/H_α by right multiplication, therefore, in the above setting, for each H_α is well defined a group homomorphism $\varphi_\alpha : G \to S_d$ (here S_d is the group of permutations of d elements). Vice versa, given $\varphi_\alpha : G \to S_d$, one can recover

H_α as:
$$H_\alpha = \mathrm{Stab}_G(1) = \{g \in G \ : \ \varphi_\alpha(g)(1) = 1\}.$$
In this way e have defined a injective map from $\{H_\alpha\}$ to $\mathrm{Hom}(G, S_d)$. Since G is finitely generated, then $\mathrm{Hom}(G, S_d)$ is a finite set and $\{H_\alpha\}$ is a finite set too. $\qquad\square$

2.3. Lemma. *Fix a finite set $B = \{y_1, \dots, y_t\} \subset \mathbb{P}^1(\mathbb{C})$ and a number $d \in \mathbb{N}$. Then there is a finite number of equivalence classes of degree d connected topological coverings of $V = \mathbb{P}^1(\mathbb{C}) \setminus B$ (in the complex topology).*

Proof. The classification theorem of covering spaces (cf. [21, Theorem 1.38]) says that the following two sets are in bijective correspondence:

$$R_1 := \{\text{classes of deg. } d \text{ path-connected topological coverings of } V\}$$

$$R_2 := \{\text{conjugacy classes of subgroups } H \subseteq \pi_1(V) \text{ of index } d\}$$

Moreover as a consequence of the van Kampen theorem

$$\pi_1(V) \cong \left\langle g_1, \dots, g_t \ : \ \prod_{i=1}^{t} g_i = 1 \right\rangle.$$

By applying Lemma 2.2 on $G = \pi_1(V)$, it follows that R_2 is a a finite set. $\qquad\square$

2.4. Lemma. *Fix a finite set $B = \{y_1, \dots, y_t\} \subset \mathbb{P}^1_{\mathbb{C}}$ and a number $d \in \mathbb{N}$. Then there is a finite number of equivalence classes of degree d branched coverings of $\mathbb{P}^1_{\mathbb{C}}$ whose branch locus is contained in B.*

Proof. Let $V = \mathbb{P}^1_{\mathbb{C}} \setminus B$. Every degree d branched covering $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ with branch locus contained in B induces a (degree d) finite covering $\pi|_U : U \to V$ where $U = X \setminus \pi^{-1}(B)$. The claim follows immediately from the Riemann existence theorem and Lemma 2.3. $\qquad\square$

2.5. Theorem ("If direction" of Theorem 2.1)**.** *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a nonsingular complex projective curve and suppose that there exists a branched covering $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ with at most three branch points, then X is defined over $\overline{\mathbb{Q}}$.*

Proof. Fix φ with degree d. By possibly composing φ with an appropriate automorphism of $\mathbb{P}^1_{\mathbb{C}}$, one can always suppose that the branch locus of φ is contained in $\{(T_0), (T_0 - T_1), (T_1)\}$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ consider the morphism of varieties $\varphi^\sigma : X^\sigma \to \left(\mathbb{P}^1_{\mathbb{C}}\right)^\sigma$ and the following commutative diagram:

$$
\begin{array}{ccc}
X^\sigma & \xrightarrow{\ \varphi^\sigma\ } & \left(\mathbb{P}^1_{\mathbb{C}}\right)^\sigma \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\
X & \xrightarrow{\ \varphi\ } & \mathbb{P}^1_{\mathbb{C}} \ .
\end{array}
$$

Clearly $(X^\sigma, \varphi^\sigma)$ is a finite covering of degree d of $\left(\mathbb{P}^1_{\mathbb{C}}\right)^\sigma = \mathbb{P}^1_{\mathbb{C}}$. Moreover, since the set $\{(T_0), (T_0 - T_1), (T_1)\}$ is fixed pointwise by h_2, the two morphisms φ and φ^σ have the same branch locus by construction. So for any $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$(X^\sigma, \varphi^\sigma)$ is a finite covering of degree d of $\mathbb{P}^1_{\mathbb{C}}$ whose branch locus is contained in $\{(T_0), (T_0 - T_1), (T_1)\}$. By Lemma 2.4, $\{X^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ contains only a finite number of isomorphism classes of curves, so Theorem 1.29 implies that X is defined over $\overline{\mathbb{Q}}$. $\square$

The proof of the only if direction of Theorem 2.1 follows Belyi's argument (he presented two different proofs: [7] and [8]).

The idea is simple: given a branched covering of Riemann surfaces whose branch-locus is contained in $\mathbb{P}^1(\overline{\mathbb{Q}})$, there is a systematic way to reduce the branch-locus to a set of three points. The reduction "algorithm" can be divided in two steps, presented here as two lemmas: in the first step the branch-locus is sent onto a finite set of $\mathbb{P}^1(\mathbb{Q})$ and in the second step it is shrunk to $\{0, 1, \infty\}$.

2.6. Remark. If K is any field and $f \in K[T]$ is a polynomial, then it clearly induces a map $\widetilde{f} : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ as follows: $\widetilde{f}((x : 1)) := (f(x) : 1)$ and $\widetilde{f}(\infty) = \infty$. By an abuse of notation f is identified with $\widetilde{f}$, so from now a polynomial of $K[T]$ can be considered as a morphism from $\mathbb{P}^1(K)$ to itself.

2.7. Lemma. *Let B be a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$. If B is invariant under the natural action of $\mathrm{Aut}(\overline{\mathbb{Q}})$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$, then there exists a polynomial $f \in \mathbb{Q}[T]$ with the following properties:*

- *$f(B) \subseteq \mathbb{P}^1(\mathbb{Q})$.*
- *The branch points of f lie in $\mathbb{P}^1(\mathbb{Q})$.*

Proof. One can proceed by induction on $\#(B) = n$. If $n = 1$, then $B = \{\alpha\}$ and α is a fixed point of $\mathrm{Aut}(\overline{\mathbb{Q}})$, therefore by Lemma 1.20 $\alpha \in \mathbb{Q}$. By choosing $f = z$, the base step of the induction is done.

Let $n > 1$, then consider the set of polynomials

$$\mathcal{S} = \{\min(\alpha, \mathbb{Q}) \, : \, \alpha \in B\}$$

and define

$$g(T) = \prod_{p(T) \in \mathcal{S}} p(T) \, .$$

Firstly note that the polynomial g doesn't have repeated roots, indeed every $p(T)$ in the product is separable and moreover if β was a root of $p(T)$, and $q(T)$ both in M, then $q(T) = p(T) = \min(\beta, \mathbb{Q})$. If $Z(g)$ is the zero-locus of g, clearly $B \subseteq Z(g)$. Vice versa if $\beta \in Z(g)$, then there exists some $p(T) = \min(\alpha, \mathbb{Q}) \in \mathcal{S}$ such that $p(\beta) = 0$; but $\mathrm{Aut}(\overline{\mathbb{Q}})$ acts transitively on the roots of $p(T)$ [13, 8.1.4], therefore there is some $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}})$ such that $\sigma(\alpha) = \beta$, and $\beta \in B$ since B is an invariant set under $\mathrm{Aut}(\overline{\mathbb{Q}})$. It has been proved that B is exactly the set of all distinct roots of g, so $\deg(g) = \#(B) = n$. Define

$$B' = g\left(\left\{z \in \overline{\mathbb{Q}} \, : \, g'(z) = 0\right\}\right) \, ,$$

then $\#(B') \leq n-1$ and moreover $B' \cup \{\infty\}$ is exactly the set of branch points of g. Clearly also B' is closed under the action of $\mathrm{Aut}(\overline{\mathbb{Q}})$, so by inductive hypothesis: there exists a polynomial $h \in \mathbb{Q}[T]$ such that $h(B') \subseteq \mathbb{P}^1(\mathbb{Q})$ and its branch points lie in $\mathbb{P}^1(\mathbb{Q})$. Now look at the composition $f = h \circ g \in \mathbb{Q}[T]$:

$$\mathrm{Br}(f) = \mathrm{Br}(h) \cup h(\mathrm{Br}(g)) = \mathrm{Br}(h) \cup h(B' \cup \{\infty\}) \subseteq \mathbb{P}^1(\mathbb{Q}).$$

Moreover $f(B) = h(g(B)) = h(0) \in \mathbb{P}^1(\mathbb{Q})$, hence f is the required polynomial. $\qquad\square$

2.8. Remark. Actually it has been shown that f maps B onto 0.

2.9. Lemma. *Let D be a finite subset of $\mathbb{P}^1(\mathbb{Q})$. Then there exists a polynomial $f \in \mathbb{Q}[T]$ such that:*

- $f(D) \subseteq \{0, 1, \infty\}$.
- *The branch points of f lie in $\{0, 1, \infty\}$.*

Proof. Here one works by induction on $\#(D) = n$. If $n \leq 3$ there is an appropriate Möbius transformation M such that $M(D) \subseteq \{0, 1, \infty\}$. So suppose that $n > 3$; in this case by applying an appropriate Möbius transformation, one can suppose that $\{0, 1, \infty\} \subseteq D$ an that there is a fourth point $\frac{m}{m+n} \in D$ with $m, n \in \mathbb{N}$. Indeed take $P_1, P_2, P_3, P_4 \in D$, then there exists a Möbius transformation M such that $M(\{P_1, P_2, P_3, P_4\}) = \{0, 1, \infty, P'\}$ where $P' = (1 : x)$ with $x \in \mathbb{Q} \cap]0, 1[$. Now consider the polynomial

$$g(T) := \frac{(m+n)^{m+n}}{m^m n^n} T^m (1-T)^m \in \mathbb{Q}[T];$$

It holds that $g\left(\left\{0, \frac{m}{m+n}, 1, \infty\right\}\right) = \{0, 1, \infty\}$ and moreover since

$$g'(T) = -\frac{(m+n)^{m+n}}{m^m n^n}(1-T)^{(n-1)} T^{(m-1)}[(m+n)T - m],$$

the branch points of g lie in $\{0, 1, \infty\}$. By the induction hypothesis (the case when $n = 3$) applied to $g(D)$, there exists a polynomial $h \in \mathbb{Q}[T]$ such that $h(g(D)) \subseteq \{0, 1, \infty\}$ and moreover the branch points of h lie in $\{0, 1, \infty\}$. Finally, the function $f := h \circ g$ satisfies the required conditions and the proof is complete:

$$\mathrm{Br}(f) = \mathrm{Br}(h) \cup h(\mathrm{Br}(g)) \subseteq \mathrm{Br}(h) \cup h(\{0, 1, \infty\}) \subseteq \{0, 1, \infty\}. \qquad\square$$

2.10. Theorem (Belyi, 1979. "Only if direction" of Theorem 2.1)**.** *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be nonsingular complex projective curve defined over $\overline{\mathbb{Q}}$, then there exists a branched covering $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ with at most three branch points.*

Proof. Since by hypothesis X is defined over $\overline{\mathbb{Q}}$, then one can choose a finite nonconstant morphism $\psi : X \to \mathbb{P}^1_{\mathbb{C}}$ defined over $\overline{\mathbb{Q}}$: indeed if $\psi_1 : X_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is any finite morphism of varieties, it is enough to take $\psi := \psi_1 \times_{\mathrm{Spec}\,\overline{\mathbb{Q}}} \mathrm{id}_{\mathrm{Spec}\,\mathbb{C}}$.

Now consider the Riemann surface $X(\mathbb{C}) \subseteq \mathbb{P}^n(\mathbb{C})$ associated to X and the holomorphic map $\psi(\mathbb{C}) : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. For any $\sigma \in \mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$, consider

$\psi^\sigma(\mathbb{C}) = \sigma \circ \psi(\mathbb{C}) \circ \sigma^{-1}$, then $\mathrm{Br}(\psi^\sigma(\mathbb{C})) = \sigma(\mathrm{Br}(\psi(\mathbb{C})))$. But since $X(\mathbb{C})$ and $\psi(\mathbb{C})$ are both defined over $\overline{\mathbb{Q}}$, then $\psi^\sigma(\mathbb{C}) = \psi(\mathbb{C})$ and it follows that

$$\mathrm{Br}(\psi^\sigma(\mathbb{C})) = \sigma(\mathrm{Br}(\psi(\mathbb{C}))) = \mathrm{Br}(\psi(\mathbb{C})) \,.$$

It means that $\mathrm{Br}(\psi(\mathbb{C}))$ is a finite set fixed by all $\sigma \in \mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$, so by Lemma 1.20 $\mathrm{Br}(\psi(\mathbb{C})) \subseteq \mathbb{P}^1(\overline{\mathbb{Q}})$. If $B \subseteq \mathbb{P}^1(\overline{\mathbb{Q}})$ is the smallest set invariant under the action of $\mathrm{Aut}(\overline{\mathbb{Q}})$ containing $\mathrm{Br}(\psi(\mathbb{C}))$, it has clearly finite cardinality because B can be obtained by adding to each point $\alpha \in \mathrm{Br}(\psi(\mathbb{C}))$ its finite orbit under $\mathrm{Aut}(\overline{\mathbb{Q}})$. Now by Lemma 2.7 there exists a polynomial $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $D := \mathrm{Br}(h) \cup h(B) \subseteq \mathbb{P}^1(\mathbb{Q})$ with $\#(D)$ finite. Finally by applying Lemma 2.9 on D one obtains a polynomial $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $g(D) \cup \mathrm{Br}(g) \subseteq \{0, 1, \infty\}$. The branched covering of Riemann surfaces

$$\varphi(\mathbb{C}) := g \circ h \circ \psi(\mathbb{C}) : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$

has at most three branch points, so by the Riemann existence theorem it induces a rational map $\varphi : X \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ which is a finite covering of the projective line minus at most three points, where it is well defined. But every rational map between complex nonsingular projective curves is everywhere defined, therefore $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ is a branched covering with at most three branch points. $\qquad\square$

3. Minimal Surfaces defined over $\overline{\mathbb{Q}}$

Theorem 2.1 establishes a sufficient and necessary condition for a nonsingular complex projective curve to be defined over $\overline{\mathbb{Q}}$, so one would like to have a similar theorem for minimal complex surfaces. Things here are more complicated since most of the tools used in section 2 are not available for surfaces, therefore a completely different approach is needed. The results of this section are inspired by [17] which employs the theory of Lefschetz pencils. The idea is the following: if for a curve X the definability over $\overline{\mathbb{Q}}$ depends on the critical values of a morphism $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$, here for a minimal surface S the definability over $\overline{\mathbb{Q}}$ depends on the critical values of a Lefschetz pencil.

By the way, the author believes that in the case of minimal ruled surfaces (i.e., geometrically ruled) over a base curve B, the conditions imposed in [17] on both S and B, are too strong. For this reason, we formulate a new sufficient condition based on *the number* of base points of a Lefschetz pencil. The drawback is that the statement does not guarantee the definability over $\overline{\mathbb{Q}}$ in the case of elliptic Lefschetz fibrations or when there is a number of base points which is a multiple of 8. These cases should be treated separately. Alternative generalizations of Theorem 2.1 for surfaces are given in [30] and [31].

For the purposes of this paper it is enough to present the theory of Lefschetz fibrations on complex surfaces, but the definitions can be extended

to any variety over an algebraically closed field. For a more general treatment of this argument the reader may consult [12] and [14].

3.1. Definition. Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a nonsingular complex projective surface. A rational map $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ which is not defined on a nonempty set Ξ_λ, is called a *Lefschetz pencil* on S if the following conditions are satisfied:

(a) All but finitely many fibres of λ are nonsingular (i.e., the generic fibre is nonsingular).
(b) The singular fibres have only one singular point, and this point is a node.
(c) The closures in S of the fibres of λ meet pairwise transversally at the points of Ξ_λ.

The finite (nonempty) subset $\Xi_\lambda \subset S$ is called the *base locus* and each point of Ξ_λ is called a *base point* of the Lefschetz pencil.

If a Lefschetz pencil is given on S, then S can be "approximated" with a family of curves over $\mathbb{P}^1_{\mathbb{C}}$:

3.2. Definition. Let $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ be a Lefschetz pencil. Thanks to Castelnuovo's elimination of indeterminacy (see [6]) there exists a nonsingular complex projective surface $\widetilde{S}$ with a birational morphism $\rho : \widetilde{S} \to S$ and a morphism $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\widetilde{S} & \dashrightarrow^{\rho} & S \\
{\scriptstyle\Lambda}\downarrow & \swarrow{\scriptstyle\lambda} & \\
\mathbb{P}^1_{\mathbb{C}} & .
\end{array}
$$

The triple $(\widetilde{S}, \rho \,; \Lambda)$ is called a *Lefschetz fibration* associated to λ.

When λ and ρ are clear from the context, a Lefschetz fibration is indicated as a morphism $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$.

3.3. Proposition. *Let $(\widetilde{S}, \rho; \Lambda)$ be a Lefschetz fibration associated to the Lefschetz pencil $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ and let $b \in \Xi_\lambda$ any base point. If $E_b := \rho^{-1}(b)$, then every fibre of Λ meets E_b.*

Proof. Certainly $\Lambda(E_b)$ can't be a point, indeed if $\Lambda(E_b) = p$, then λ could be defined at b as $\lambda(b) := p$. It follows that $\Lambda(E_b) = \mathbb{P}^1_{\mathbb{C}}$, so for every $x \in \mathbb{P}^1_{\mathbb{C}}$ the intersection $\Lambda^{-1}(x) \cap E_b$ is not empty. $\square$

The geometric picture of the property $c)$ of Definition 3.1 needs to be clarified:

3.4. Proposition. *Let $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ be a Lefschetz pencil, then the closure of $\lambda^{-1}(x)$ in S is $\lambda^{-1}(x) \cup \Xi_\lambda$ for any $x \in \mathbb{P}^1_{\mathbb{C}}$.*

Proof. Let x be any point of $\mathbb{P}^1_{\mathbb{C}}$. The inclusion $\overline{\lambda^{-1}(x)} \subseteq \lambda^{-1}(x) \cup \Xi_\lambda$ is quite trivial: $\lambda^{-1}(x) = C \cap (S \setminus \Xi_\lambda)$ where C is a closed subset of S, so

$\lambda^{-1}(x) \cup \Xi_\lambda = C \cap (S \setminus \Xi_\lambda) \cup \Xi_\lambda = C$. Practically $\lambda^{-1}(x) \cup \Xi_\lambda$ is a closed subset containing $\lambda^{-1}(x)$.

In order to prove that $\overline{\lambda^{-1}(x)} \supseteq \lambda^{-1}(x) \cup \Xi_\lambda$, it is enough to show that if $b \in \Xi_\lambda$, then $b \in \overline{\lambda^{-1}(x)}$. Let $(\widetilde{S}, \rho; \Lambda)$ be a Lefschetz fibration associated to λ and let $E_b := \rho^{-1}(b)$. The intersection $\Lambda^{-1}(x) \cap E_b$ is not empty thanks to Proposition 3.3, therefore if $y \in \Lambda^{-1}(x) \cap E_b$, then $y \in \overline{\rho^{-1}(\lambda^{-1}(x))}$. Since ρ is continuous, this implies that

$$b = \rho(y) \in \overline{\rho(\rho^{-1}(\lambda^{-1}(x)))} = \overline{\lambda^{-1}(x)}\,. \qquad \square$$

3.5. Theorem. *Let $S \subseteq \mathbb{P}^n_\mathbb{C}$ be a nonsingular complex projective surface. Then there exists a Lefschetz pencil $\lambda : S \dashrightarrow \mathbb{P}^1_\mathbb{C}$ on S such that an associated Lefschetz fibration $\Lambda : \widetilde{S} \to \mathbb{P}^1_\mathbb{C}$ is a family of curves which admits a section. Moreover the Lefschetz pencil λ can be chosen in such a way that the fibres of Λ are irreducible.*

Sketch of proof. The complete proof can be found in [28, V.3] and [22]. Thanks to the Bertini's theorem (cf. [20, Theorem II.8.18]) there exists a hyperplane $H \subseteq \mathbb{P}^n_\mathbb{C}$ such that $H \cap S$ is a nonsingular projective curve and furthermore for the generic element H' of the complete linear system $|H|$, the intersection $X \cap H'$ is a nonsingular projective curve. There is a bijection $\phi : |H| \to \mathbb{P}^1_\mathbb{C}$ and moreover that $S \subseteq \bigcup_{H' \in |H|} H'$. So one can define

$$\lambda : S \setminus \bigcap_{H' \in |H|} \left(S \cap H'\right) \to \mathbb{P}^1_\mathbb{C}$$

in the following way: $\lambda(x) := \phi(H')$ if $x \in H'$ for the unique element $H' \in |H|$ containing x. Note that the base locus of λ is the finite set $\Xi_\lambda := \bigcap_{H' \in |H|} \left(S \cap H'\right)$. $\qquad \square$
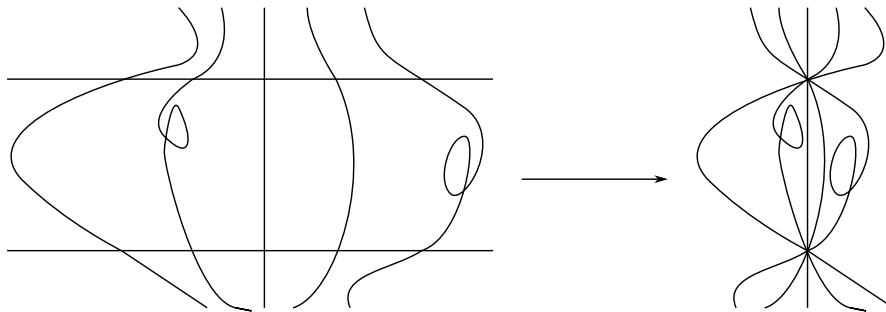

FIGURE 1. A schematization of a Lefschetz fibration associated to a Lefschetz pencil. In this case there are are two base points b_1 and b_2 and two knotted curves. The birational morphism ρ is such that $\rho(E_1) = b_1$ and $\rho(E_2) = b_2$.

3.6. Remark. From now on, by simplicity any Lefschetz pencil $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ satisfies the conditions of Theorem 3.5. In particular one assume that any Lefschetz fibration $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$ is a family of curves with irreducible fibres and with a section.

3.7. Remark. Actually, from the properties of $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$, it follows that its fibres are also reduced.

3.8. Proposition. *A Lefschetz fibration of genus $g \geq 2$ is a stable family of curves.*

Proof. Let F be a fibre of a Lefschetz fibration $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$. By Remarks 3.6 and 3.7 F is connected and reduced. By definition F has at most nodes as singularities and moreover by hypothesis $p_a(F) \geq 2$. Finally, since F is irreducible, the condition on the rational components is vacuously true. $\square$

Let S be a nonsingular complex projective surface and consider a Lefschetz fibration $(\widetilde{S}, \rho\,; \Lambda)$, then $\widetilde{S}$ can't be a minimal surface since it is obtained by some blow-ups of S. On the other hand one can at least ensure the relative minimality of Λ:

3.9. Proposition. *Any Lefschetz fibration $\Lambda : S \to \mathbb{P}^1_{\mathbb{C}}$ is relatively minimal.*

Proof. Suppose by contradiction that a fibre F of Λ contains a (-1)-curve E. Since F is irreducible $E = F$, namely E is an entire fibre of Λ. But this means that $E^2 = 0$ which is in contradiction with $E^2 = -1$. $\square$

3.10. Theorem. *Let $\pi : S \to B$ be a relatively minimal fibration (with nonsingular generic fibre) of genus g where S is a nonsingular complex projective surface and B is a nonsingular complex projective curve. Let moreover E be a (-1)-curve on S such that $E.F > 2g - 2$ for a nonsingular fibre F of π, then S is ruled.*

Proof. If $g = 0$, then S is rational thanks to Noether-Enriques theorem (cf. [6, Theorem III.4]); so one can suppose that $g \geq 1$. Let $\beta : S \to X$ the contraction of the curve E and let $F \subset S$ be a nonsingular fibre of π such that $E.F > 2g - 2$ and $\beta^* F' = F + mE$ for $F' = \beta(F)$. Therefore:

$$F'.K_X = (\beta^* F').(\beta^* K_X) = (F + mE).(K_S - E)$$
$$= F.K_S - F.E + mE.K_S - mE^2 = F.K_S - F.E + mE.K_S + m\,.$$

Now by adjunction $E.K_S = -2 - E^2 = -1$ and $F.K_S = 2g - 2 - F^2 = 2g - 2$ (here $F^2 = 0$ since F is a nonsingular fibre of a surjective morphism), hence by continuing the previous chain of inequalities:

$$F'.K_X \leq F.K_S - F.E + mE.K_S + m$$
$$= 2g - 2 - F.E < 2g - 2 - (2g - 2) = 0\,.$$

Practically $F'.K_X < 0$ and this implies that X is ruled (cf. [6, Corollary VI.18]). But X and S are birational, so S is ruled too. $\square$

3.11. Corollary. *Let $\pi : S \to B$ be a relatively minimal fibration (with nonsingular generic fibre) of genus $g = 1$ where S is a nonruled surface and B is a nonsingular complex projective curve. Then S is minimal.*

Proof. Suppose by contradiction that E is a (-1)-curve in S and let F be a nonsingular fibre of π such that $\#(E \cap F) > 0$. Since E is irreducible and E is not contained in F then $E.F > 0 = 2g - 2$. Now Theorem 3.10 ensures that S is ruled and this is a contradiction.   □

3.12. Proposition. *If S is a nonruled minimal surface, then a Lefschetz fibration over S can't be elliptic.*

Proof. Suppose by contradiction that $(\widetilde{S}, \rho\,;\Lambda)$ is an elliptic Lefschetz fibration over a nonruled surface S. Clearly $\widetilde{S}$ is also nonruled, and moreover by Proposition 3.9 the fibration is relatively minimal. From Corollary 3.11 it follows that $\widetilde{S}$ is a minimal surface which is an absurd.   □

Suppose that a nonsingular complex projective surface $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$, what can be said about the fields of definition of the divisors on S and about the fields of definition of the minimal models of S? The following propositions try to answer to these questions.

3.13. Proposition. *Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a nonsingular complex projective surface defined over $\overline{\mathbb{Q}}$, then every irreducible effective divisor of S with negative self intersection number is defined over $\overline{\mathbb{Q}}$.*

Proof. Suppose by contradiction that there exists an effective divisor $D \subset S$ with $D^2 < 0$ which is not defined over $\overline{\mathbb{Q}}$. By Theorem 1.29 the orbit set $S^{\mathrm{Aut}(\mathbb{C})}$ contains only a finite number of surfaces up to isomorphism and on the other hand $D^{\mathrm{Aut}(\mathbb{C})}$ contains an uncountable numbers of divisors up to isomorphism. So it follows that for a certain $\sigma \in \mathrm{Aut}(\mathbb{C})$, S^σ contains an uncountable numbers of effective divisors with negative self intersection (remember that the intersection number is invariant under base extensions of the ground field). But the Neron–Severi group $\mathrm{NS}(S^\sigma)$ is finitely generated, so in S^σ one can always find two nonisomorphic curves D_1 and D_2 with negative self intersection which are numerically equivalent; in particular $D_1.D_2 = D_2^2 < 0$. So D_1 and D_2 must have a common irreducible component. But since they are both irreducible, then $D_1 = D_2$ contradicting the fact that D_1 and D_2 are not isomorphic.   □

3.14. Proposition. *Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a nonsingular complex projective surface defined over $\overline{\mathbb{Q}}$, then every minimal model of S is defined over $\overline{\mathbb{Q}}$.*

Proof. It is enough to prove that if S is defined over $\overline{\mathbb{Q}}$ and $\beta : S \to X$ is a contraction of a (-1)-curve $E \subseteq S$ such that $\beta(E) = x \in X$, then X is a surface defined over $\overline{\mathbb{Q}}$.

Suppose by contradiction that X is not defined over $\overline{\mathbb{Q}}$ and moreover consider the set $\{\beta^\sigma : S^\sigma \to X^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$. Thanks to Theorem 1.29, there

exists an element $\tau \in \mathrm{Aut}(\mathbb{C})$ such that $S \cong S^\tau$ and $X \not\cong X^\tau$. Denote with $\theta : S \to S^\tau$ the isomorphism between S and S^τ, then one obtains the following diagram:

$$
\begin{array}{ccc}
S & \xrightarrow{\ \theta\ } & S^\tau \\
\beta \downarrow & & \downarrow \beta^\tau \\
X & & X^\tau \ .
\end{array}
$$

The morphism $\beta' := \beta^\tau \circ \theta$ is a contraction of E such that $\beta'(E) = x' \in X^\tau$, therefore is well defined a birational map $\psi = \beta' \circ \beta^{-1} : X \dashrightarrow X^\tau$ which actually is an isomorphism between the open sets $X \setminus \{x\}$ and $X^\tau \setminus \{x'\}$.

This implies that ψ is everywhere defined, in particular it is an isomorphism. This contradicts the fact that X and X^τ are not isomorphic.   □

Finally, there are all the ingredients to study in detail the definability over $\overline{\mathbb{Q}}$. One can distinguish 3 cases:

- *Case* 1: *rational surfaces.* This will be quite trivial.
- *Case* 2: *nonruled surfaces.* The statement in this paper is the same of [17], but here is given an original proof.
- *Case* 3: *ruled nonrational surfaces.* This will be a new sufficient condition on the number of base points of a Lefschetz pencil.

On the other hand there is a single necessary condition.

3.15. Proposition (Suff. Cond. Case 1)**.** *Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a rational minimal surface, then it is defined over $\overline{\mathbb{Q}}$.*

Proof. S is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ or to some Hirzebruch surface Σ_m with $m \neq 1$, but they are both defined over $\overline{\mathbb{Q}}$.   □

3.16. Proposition (Suff. cond. Case 2)**.** *Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a nonruled minimal surface. If S admits a Lefschetz pencil $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ with critical values in $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, then S is defined over $\overline{\mathbb{Q}}$.*

Proof. Suppose that $(\widetilde{S}, \rho; \Lambda)$ is the Lefschetz fibration associated to λ. If the genus of the Lefschetz fibration over S is $g = 0$, by [6, Theorem III.4] it follows that $\widetilde{S}$ is a rational surface, and then S is a rational surface too. The case $g = 1$ can be excluded thanks to Proposition 3.12, so it is enough to prove the claim when $g \geq 2$.

The first step consists in showing that $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$ is not an isotrivial fibration. Suppose by contradiction that Λ is isotrivial, then by propositions 3.8 and 0.3 Λ is locally trivial, and Theorem 0.4 implies that $K^2_\Lambda = 0$. So

$$(1) \qquad\qquad 0 = K^2_\Lambda = K^2_{\widetilde{S}} + 8(g-1)\,.$$

On the other hand, if the Lefschetz pencil has $r = \#(\Xi_\lambda)$ base points, then $\widetilde{S}$ is obtained from S by r repeated blow-ups. The fact that $\widetilde{S}$ is obtained through a finite sequence of blow-ups follows by Castelnuovo's elimination of indeterminacy, but the number of blow-ups is exactly r thanks to the property $c)$ of Definition 3.1. Indeed if two effective divisors C_1 and C_2 of S meet transversally, then their strict transforms $\widehat{C}_1$ and $\widehat{C}_2$ don't meet in $\widetilde{S}$. So $K^2_S - K^2_{\widetilde{S}} = r$, but since S is minimal and nonruled, then K_S is nef (cf. [4, III Corollary 2.4], [6, Appendix C]) and therefore [3, Theorem 1.25] implies that $K^2_S \geq 0$. It follows that $0 \leq r + K^2_{\widetilde{S}}$, and by equation (1) $8(g-1) \leq r$. There exists a nonsingular curve $C \subset S$ of genus g such that $C^2 \geq r$: let $F_0 \subset \widetilde{S}$ be a nonsingular fibre of Λ and consider $C := \rho(F_0)$. F_0 meets all the r (-1)-curves of $\widetilde{S}$ which are mapped by ρ onto a base point of λ (cf. Proposition 3.3); furthermore $\rho = \beta_r \circ \beta_{r-1} \circ \ldots \circ \beta_1$ where each β_i is a contraction of a (-1)-curve. In particular, if $F_1 = \beta_1(F_0)$, then $F_0 = \beta_1^* F_1 - mE$ where E is the (-1)-curve contracted by β_1, and $m \geq 1$. By using the properties of the blow-up:

$$F_0^2 = (\beta_1^* F_1 - mE)^2 = (\beta_1^* F_1)^2 - 2mE.(\beta_1^* F_1) + m^2 E^2 = F_1^2 - m^2 < F_1^2\,.$$

One can repeat the same argument on F_1 to get $F_2 = \beta_2(F_1)$ such that $F_1^2 < F_2^2$; by continuing to generate the curves $F_i = \beta_i(F_{i-1})$ in this fashion, one eventually obtains $C = \rho(F_0) = \beta_r(F_{r-1}) = F_r$. The finite sequence of curves $F_0, F_1, \ldots, F_{r-1}, C$ has the following property (remember that F_0 is a fibre, so $F_0^2 = 0$):

$$0 = F_0^2 < F_1^2 < \ldots < F_{r-1}^2 < C^2\,.$$

Therefore it is evident that $C^2 \geq r$. Now, since K_S is nef, by adjunction one gets the inequality

$$2g - 2 = C^2 + C.K_S \geq r \geq 8g - 8$$

which gives $g \leq 1$, contradicting the choice of g.

Finally, suppose by contradiction that S is not defined over $\overline{\mathbb{Q}}$, then $\widetilde{S}$ is not defined over $\overline{\mathbb{Q}}$ (Proposition 3.14) and by the Theorem 1.29, the set $\{\widetilde{S}^\sigma\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ contains infinitely many isomorphism classes. Suppose that Δ is the set of critical points of λ (so the set of critical points of of Λ). By hypothesis Δ is contained in $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, so it has a finite orbit under the action of $\mathrm{Aut}(\mathbb{C})$; therefore there exists $\tau \in \mathrm{Aut}(\mathbb{C})$ such that the set

$$F := \left\{ \Lambda^\sigma : \widetilde{S}^\sigma \to \left(\mathbb{P}^1_{\mathbb{C}}\right)^\sigma = \mathbb{P}^1_{\mathbb{C}} \text{ s.t. } \sigma \in \mathrm{Aut}(\mathbb{C}) \text{ and } \mathrm{Crit}\,(\Lambda^\sigma) \subseteq \Delta^\tau \right\}$$

contains an *infinite number* of nonisomorphic fibrations. In other words
F contains an infinite number of nonisomorphic admissible families with
respect to $\left(\mathbb{P}^1_{\mathbb{C}}, \Delta^\tau\right)$, but this contradicts the Parshin–Arakelov theorem
(Theorem 0.2). Note that the nonisotriviality of each member of F follows
from the nonisotriviality of Λ shown above. $\qquad \square$

3.17. Proposition (Suff. Cond. Case 3). *Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a ruled nonrational
minimal surface over a curve B. Suppose that there exists a Lefschetz
fibration $\Lambda : \widetilde{S} \to \mathbb{P}^1_{\mathbb{C}}$ associated to a Lefschetz pencil $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ such that
the following properties are satisfied:*

- *A nonsingular fibre of Λ has genus $g \geq 2$.*
- *λ has critical values in $\mathbb{P}^1_{\overline{\mathbb{Q}}}$.*
- *$\#(\Xi_\lambda) \neq 8t$ for $t \in \mathbb{N} \setminus \{0\}$.*

Then S is defined over $\overline{\mathbb{Q}}$.

Proof. It is enough to show that Λ is not isotrivial. Indeed the rest of
the proof follows by applying the Parshin–Arakelov theorem as in Proposi-
tion 3.16.

Assume by contradiction that Λ is isotrivial, then it is locally trivial and
$0 = K^2_\Lambda = K^2_{\widetilde{S}} + 8(g - 1)$ which implies that $K^2_{\widetilde{S}} = 8(1 - g)$. On the other
hand S is a geometrically ruled surface over B, hence by [6, proposition
III.21] $K^2_S = 8(1 - g(B))$. By substituting these values of $K^2_{\widetilde{S}}$ and K^2_S in the
equation $K^2_S - K^2_{\widetilde{S}} = \#(\Xi_\lambda)$ one gets $g - g(B) = \frac{\Xi_\lambda}{8}$. But $g - g(B)$ is an
integer, hence $\Xi_\lambda = 8t$ for some $t \in \mathbb{N} \setminus \{0\}$ contradicting the hypotheses of
the theorem. $\qquad \square$

Finally the necessary condition for the definability over $\overline{\mathbb{Q}}$:

3.18. Proposition (Nec. Cond.). *Let $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a minimal surface defined
over $\overline{\mathbb{Q}}$, then S admits a Lefschetz pencil $\lambda : S \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ with critical values in
$\mathbb{P}^1_{\overline{\mathbb{Q}}}$.*

Proof. $S \cong S_{\overline{\mathbb{Q}}} \times_{\mathrm{Spec}\,\overline{\mathbb{Q}}} \mathrm{Spec}\,\mathbb{C}$ for a surface $S_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$. Now let $H_{\overline{\mathbb{Q}}} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}}$
be a hyperplane which gives a Lefschetz pencil $\lambda_{\overline{\mathbb{Q}}}$ over $S_{\overline{\mathbb{Q}}}$; clearly the
critical values of $\lambda_{\overline{\mathbb{Q}}}$ are contained in $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ and moreover the hyperplane
$H = H_{\overline{\mathbb{Q}}} \times_{\mathrm{Spec}\,\overline{\mathbb{Q}}} \mathrm{Spec}\,\mathbb{C}$ gives a Lefschetz pencil λ over S with the same
critical values of $\lambda_{\overline{\mathbb{Q}}}$. $\qquad \square$

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