

# Almost flat bundles and homological invariance of infinite K-area

Benedikt Hunger

ABSTRACT. We extend the notion of an almost flat bundle over a closed Riemannian manifold to bundles over simplicial complexes, and prove that up to a constant factor, this notion is invariant under pullback via maps which induce isomorphisms on fundamental groups. As an application, we show that the property of having infinite K-area only depends on the image of the fundamental class under the classifying map of the universal cover.

## CONTENTS

1. Introduction and statement of results	687
2. Almost flat bundles	691
3. The trivialization lemma	697
4. Applications of the trivialization lemma	701
5. Almost flat bundles over Riemannian manifolds	704
6. Pullbacks of almost flat bundles	708
7. Homological invariance of infinite K-area	710
8. Almost representations and quasi-representations	712
Appendix A. Parallel transport and curvature	717
Appendix B. Unitary elements of C*-algebras	718
References	721

## 1. Introduction and statement of results

Connes, Gromov and Moscovici [CGM90] introduced the notion of almost flat K-theory classes in order to give a unified approach to different special cases of the Novikov conjecture. We first recall the definition of an

---

Received January 9, 2019.

2010 *Mathematics Subject Classification.* 53C23.

*Key words and phrases.* infinite K-area; homological invariance; almost flat bundles; almost representations.

asymptotically flat K-theory class<sup>1</sup> as introduced by Connes, Gromov and Moscovici.

Let  $M$  be a differentiable manifold (not necessarily closed). The *curvature* of a smooth vector bundle  $E \rightarrow M$  with connection  $\nabla$  is the endomorphism-valued 2-form  $\mathcal{R}^\nabla \in \Omega^2(M; \text{End}(E))$  given by the formula

$$\mathcal{R}^\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]}s.$$

Suppose in addition that  $E$  is a Hermitian vector bundle (i. e. a complex vector bundle where the fibres are equipped with a smoothly varying Hermitian inner product) and the connection  $\nabla$  is compatible with the metric (i. e. parallel transport is an isometry). Then  $\text{End}(E)$  is a bundle of normed spaces where the norm over each fibre is simply the operator norm. If, additionally,  $M$  is a Riemannian manifold with metric  $g$ , the norm of the curvature of  $E$  is defined by

$$\|\mathcal{R}^\nabla\|_g = \sup_{x \in M} \sup_{\substack{X \wedge Y \in \Lambda^2 T_x M \\ \|X \wedge Y\|_g \leq 1}} \|\mathcal{R}^\nabla(X, Y)\|_{\text{op}}.$$

Here the norm on  $\Lambda^2 TM$  is given by  $\|X \wedge Y\|_g^2 = \|X\|_g^2 \|Y\|_g^2 - g(X, Y)^2$ .

Now a class  $\eta \in K^0(M)$  is *asymptotically flat* if there are sequences  $(E_n, \nabla_n)$  and  $(\tilde{E}_n, \tilde{\nabla}_n)$  of Hermitian vector bundles with compatible connection over  $M$ , such that  $\eta = [E_n] - [\tilde{E}_n]$  for every  $n \in \mathbb{N}$ , and such that

$$\lim_{n \rightarrow \infty} \|\mathcal{R}^{\nabla_n}\|_g = \lim_{n \rightarrow \infty} \|\mathcal{R}^{\tilde{\nabla}_n}\|_g = 0.$$

A priori, this notion of asymptotic flatness depends on the choice of metric on  $M$ . However, if  $M$  is compact, then any two metrics are bi-Lipschitz equivalent. Thus, given metrics  $g$  and  $\tilde{g}$  on a compact manifold  $M$ , there is a constant  $c > 0$  such that

$$\|\mathcal{R}^\nabla\|_g \leq c \|\mathcal{R}^\nabla\|_{\tilde{g}}$$

for all Hermitian bundles  $(E, \nabla)$  with compatible connection. In particular, K-theory classes are asymptotically flat with respect to  $g$  if and only if they are almost flat with respect to  $\tilde{g}$ .

Now suppose that  $(E, \nabla)$  is a Hermitian vector bundle with compatible connection over  $M$  which satisfies  $\|\mathcal{R}^\nabla\|_g \leq \epsilon$ . We will call such a bundle an  $\epsilon$ -flat bundle (or an  $\epsilon$ -almost flat bundle) over  $M$ . A fundamental result from classical Riemannian geometry states that if we are given a bundle with small curvature then parallel transport along a nullhomotopic curve  $\gamma$  is  $c\epsilon$ -close to the identity, where  $c$  is a constant depending only on  $M$  and  $\gamma$ . In fact,  $c$  may be chosen as the area of a disk filling  $\gamma$ .

<sup>1</sup>The term in [CGM90] is “fibré presque plat”, while the standard term is “almost flat K-theory class”. However, following Manuilov and Mishchenko [MM01], we will use the word *almost* for notions depending on a parameter  $\epsilon > 0$ , and *asymptotic* for any kind of limit as  $\epsilon \rightarrow 0$ .

We may use this to construct a map  $f: \pi_1(M, x_0) \rightarrow U(l)$  as follows: We fix a trivialization of the fibre over the base-point  $x_0$ . For each class  $c \in \pi_1(M, x_0)$ , we choose a representing loop  $\gamma$ . We define  $f(c)$  to be the parallel transport along  $\gamma$  with respect to the trivialization of the fibre over the base-point. Then  $f$  might not be a group homomorphism, but  $\|f(gh) - f(g)f(h)\| \leq c(g, h)\epsilon$ , where  $c(g, h) > 0$  is a constant depending only on  $g$  and  $h$ , but not on the bundle  $E$ . Such data constitute a *quasi-representation* of  $M$ , and in fact, almost flat bundles and quasi-representations of the fundamental group turn out to be two sides of the same coin. This relationship between almost flat bundles and quasi-representations was already noted by Connes, Gromov and Moscovici [CGM90] and was made precise by Carrión and Dadarlat [CD18]. However, their exposition seems to be a bit ad hoc, while similar results will be natural consequences of the results presented in this paper.

There is another important consequence of the fact that parallel transport along contractible curves is close to the identity. Namely, suppose that  $M$  is smoothly triangulated, and  $\sigma$  is a simplex of  $M$ . Then, after a choice of basis for the fibre over the barycentre of  $\sigma$ , we can trivialize the bundle  $E$  over  $\sigma$  by parallel transporting this basis from the barycenter outwards. Now if  $\rho$  is another simplex then the transition functions between those two trivializations turn out to be Lipschitz functions, where the Lipschitz constant is small if  $\epsilon$  is small.

This idea enabled Mishchenko and Teleman [MT05] to show that every  $\epsilon$ -flat bundle can be pulled back from a bundle over  $B\pi_1(M)$  along the classifying map of the universal cover of  $M$  if  $\epsilon$  is sufficiently small. In the course of the proof of this statement, they introduced the concept of small bundles over a simplicial complex: An  $\epsilon$ -flat bundle is a vector bundle such that the transition functions with respect to some family of trivializations over the simplices are  $\epsilon$ -Lipschitz. We will adapt to this definition in this paper and show that it is, in fact, more or less equivalent to the old one if we consider bundles over a triangulated manifold. Also, our results will prove a generalization of the theorem of Mishchenko and Teleman, namely that every  $\epsilon$ -flat bundle is the pull-back of a  $c\epsilon$ -flat bundle over a finite subcomplex of  $B\pi_1(M)$  if  $\epsilon$  is small enough.

Dual to the concept of almost flat K-theory classes is the notion of *infinite K-area* introduced by Gromov [Gro96]. Namely, a Riemannian manifold  $M$  has infinite K-area if, for every  $\epsilon > 0$ , there is an  $\epsilon$ -flat bundle  $(E, \nabla)$  over  $M$  with at least one non-vanishing Chern number. Infinite K-area is one of several important largeness properties of Riemannian manifolds introduced by Gromov and Lawson [Gro86, GL80]. Brunnbauer and Hanke [BH10] showed that other largeness properties, including enlargeability, are *homologically invariant* in the sense that they only depend on the image of the fundamental class under the classifying map of the universal cover. Their

proof proceeds as follows: First they define enlargeability of an arbitrary homology class of a simplicial complex in such a way that a closed Riemannian manifold  $M$  is enlargeable if and only if its fundamental class  $[M]$  is enlargeable. Then they use an extension lemma to show that if a map  $f: X \rightarrow Y$  induces an isomorphism on fundamental groups, then a class  $\eta \in H_*(X)$  is enlargeable if and only if  $f_*\eta \in H_*(Y)$  is. Since, by definition, the classifying map of the universal cover  $\Phi: M \rightarrow B\pi_1(M)$  induces an isomorphism of fundamental groups, this implies homological invariance of enlargeability.

Motivated by this scheme, we will define a homology class to have infinite K-area if, for every  $\epsilon > 0$ , there is an  $\epsilon$ -flat bundle whose Chern classes detect the given homology class. An extension result for  $\epsilon$ -flat bundles will then be used to show that infinite K-area is homologically invariant. This also implies that the infinite K-area is invariant under  $p$ -surgery with  $p \neq 1$ , a fact which has been proven directly by Fukumoto [Fuk15].

Hanke and Schick [Han12, HS08] used a notion of almost flat bundles of Hilbert- $A$ -modules for arbitrary  $C^*$ -algebra to prove a special case of the Novikov conjecture. It turns out that one needs precisely the Lipschitz condition on the transition functions in order to prove their results. Therefore, it makes sense not only to consider Hermitian vector bundles, but also bundles of Hilbert  $A$ -modules for arbitrary  $C^*$ -algebras  $A$ .

We conclude the introduction by giving an outline of the following sections and the main results.

In section 2 we will give the precise definition of an  $\epsilon$ -flat bundle of Hilbert  $A$ -modules, and show that examples are given by Hilbert module bundles with compatible connection having small curvature.

Section 3 provides the most important technical result of this paper, the *trivialization lemma* 3.6. This states that every  $\epsilon$ -flat bundle over a simply-connected space is trivial if  $\epsilon$  is small enough. As a corollary, one can extend almost flat bundles defined on the boundary of a disk  $D^k$  to the whole disk, since they are trivial on the boundary. The main ingredients in the proof of the trivialization lemma are an extension statement for unitary-valued Lipschitz functions (lemma 3.5) and a combinatorial version of the statement that parallel transport along boundaries of small disks is close to the identity (theorem 3.4).

In section 4, we will give first applications of the trivialization lemma: Firstly, we show that almost flat bundles on the barycentric subdivision of a finite-dimensional complex are almost flat with respect to the original complex. Secondly, we give conditions under which an almost flat bundle can be extended to an almost flat bundle over a larger subcomplex.

The rest of this paper will consist of rather easy applications of the trivialization lemma, beginning with section 8, where we relate the concepts of almost representations and quasi-representations to the concept of almost flat K-theory classes.

In section 5 we will use this to show that our definition of an almost flat bundle corresponds to the definition via smooth connections of Connes, Gromov and Moscovici [CGM90]. This will make use of an extension theorem for connections with small curvature which is mainly due to Fukumoto [Fuk15] in his proof of invariance of infinite K-area under certain surgeries. This will be used later in order to show that our definition of infinite K-area is a generalization of Gromov's [Gro96] definition.

Next, in section 6, we use our results on extensions of almost flat bundles in order to show the following functoriality result: Given a map  $f: X \rightarrow Y$ , almost flat bundles over  $Y$  pull back to almost flat bundles over  $X$ , and if the map  $f$  induces an isomorphism on fundamental groups, the pull-back map is in fact surjective (in a certain sense) on almost flat bundles. This gives the generalization of the theorem of Mishchenko and Teleman [MT05] cited before. This section is independent of section 5.

We will put all those results together in section 7 to define a notion of infinite K-area for arbitrary homology classes of simplicial complexes, and to prove homological invariance, i. e. a class has infinite K-area if and only if its image under the classifying map of the universal cover has infinite K-area. This will directly imply that, for a Riemannian manifold, having infinite K-area only depends on the image of the fundamental class under the classifying map of the universal cover. We will show how to use this to regather the theorem of Fukumoto [Fuk15] about the invariance of infinite K-area under surgeries in codimension not equal to one.

Finally, in section 8, we relate the notions of almost and asymptotic representations [MM01], of almost flat bundles, and of asymptotically flat K-theory. This section only uses material from sections 3 and 4, and may be read independently of the rest of this paper.

This work is based on the author's Master's thesis at Universität Augsburg. The author would like to thank his thesis advisor Bernhard Hanke for his invaluable help and support, and the referee for their helpful remarks.

## 2. Almost flat bundles

**2.1. Preliminaries and main definition.** The principal aim of this section is to give the definition of an almost flat bundle over an arbitrary simplicial complex. It seems to be useful to consider not only Hermitian bundles, but rather bundles of Hilbert  $C^*$ -modules [Han12, Sch05]. We will work in this more general setting, since it does not require any more work.

Let  $A$  be a  $C^*$ -algebra. Recall that a *Hilbert  $A$ -module* is a right  $A$ -module  $V$  together with an inner product  $V \times V \rightarrow A$ ,  $(v, w) \mapsto \langle v, w \rangle$ , satisfying certain conditions [JT91], for instance that  $\|v\| = \sqrt{\|\langle v, v \rangle\|}$  defines a complete norm on  $V$ . For example, a Hilbert  $\mathbb{C}$ -module is the same thing as a complex Hilbert space.

Given two Hilbert  $A$ -modules  $V$  and  $W$ , a map  $f: V \rightarrow W$  is called *adjointable* if there is another map  $f^*: W \rightarrow V$  (the *adjoint* of  $f$ ) such that

$\langle f(v), w \rangle = \langle v, f^*(w) \rangle$  for all  $v \in V$ ,  $w \in W$ . It follows from the axioms of a Hilbert  $A$ -module that an adjointable map is a bounded linear operator, and that it commutes with the action of  $A$ . We write  $\mathcal{L}_A(V, W)$  for the set of all adjointable maps  $V \rightarrow W$ . In particular,  $\mathcal{L}_A(V, W)$  is a normed vector space, equipped with the operator norm. It turns out that  $\mathcal{L}_A(V) = \mathcal{L}_A(V, V)$  is a  $C^*$ -algebra with involution  $f \mapsto f^*$ .

An *isometry* of Hilbert  $A$ -modules  $V$  and  $W$  is an adjointable bijection  $f \in \mathcal{L}_A(V, W)$ , satisfying  $\langle f(v), f(v') \rangle = \langle v, v' \rangle$  for all  $v, v' \in V$ . Obviously, a map  $f \in \mathcal{L}_A(V, W)$  is an isometry if and only if  $f^*f = \text{id}$  and  $ff^* = \text{id}$ . In particular, for any Hilbert  $A$ -module  $V$ , the *isometries*  $V \rightarrow V$  are precisely the unitary elements of  $\mathcal{L}_A(V)$ . More generally, we write  $U(A) = \{u \in A : u^*u = uu^* = 1\}$  for the set of *unitaries* in an arbitrary  $C^*$ -algebra  $A$ .

**Example 2.1.**

- Every  $C^*$ -algebra  $A$  is a Hilbert  $A$ -module with respect to the inner product  $\langle x, y \rangle = x^*y$ .
- If  $V, W$  are Hilbert  $A$ -modules, then also  $V \oplus W$  is a Hilbert  $A$ -module, with inner product  $\langle v + w, v' + w' \rangle = \langle v, v' \rangle + \langle w, w' \rangle$  for  $v, v' \in V$ ,  $w, w' \in W$ .
- If  $V$  is a Hilbert  $A$ -module, and  $p \in \mathcal{L}_A(V)$  is a *projection*, i. e.  $p^2 = p = p^*$ , then also  $pV$  is a Hilbert  $A$ -module, and  $V \cong pV \oplus (1 - p)V$ .

Now a *finitely generated projective* Hilbert  $A$ -module is a Hilbert  $A$ -module which is isomorphic to  $pA^k$ , where  $A^k = A \oplus \cdots \oplus A$  and where  $p \in \mathcal{L}_A(A^k)$  is a projection. Of course, finitely generated projective Hilbert  $\mathbb{C}$ -modules are nothing but finite-dimensional complex vector spaces with Hermitian inner product.

**Definition 2.2.** [Sch05] A Hilbert  $A$ -module bundle over a space  $X$  is a fibre bundle  $E \rightarrow X$  with typical fibre a finitely generated projective Hilbert  $A$ -module  $V$ , and with structure group  $U(\mathcal{L}_A(V))$ .

In particular, such a bundle may be described by local trivializations such that the transition functions take values in  $U(\mathcal{L}_A(V))$ . This gives a well-defined  $A$ -valued inner product on every fibre, such that every fibre is a Hilbert  $A$ -module isomorphic to  $V$ . A Hilbert  $\mathbb{C}$ -module bundle is the same thing as a Hermitian vector bundle, since  $U(\mathcal{L}_{\mathbb{C}}(V)) \cong U(n)$  is the classical group of unitary matrices.

In order to fix notations, recall that an (abstract) simplicial complex consists of a set  $V_X$ , the *vertices*, and a set  $X$  of non-empty finite subsets of  $V_X$ , the *simplices*, such that every one-element set  $\{p\}$  ( $p \in V_X$ ) is contained in  $X$ , and such that  $\emptyset \neq \rho \subset \sigma \in X$  implies that also  $\rho \in X$ , i. e.,  $X$  is closed under taking non-empty subsets. By abuse of notation, we will refer to these data as the simplicial complex  $X$ . The *dimension* of a simplex  $\sigma \in X$  is  $\dim(\sigma) = \#\sigma - 1 \in \mathbb{N}$ . We denote by  $X_n$  the set of all simplices of dimension  $n$ , called the  *$n$ -simplices*.

If  $X$  is a simplicial complex and  $k \geq 0$  is a natural number, then  $X^{(k)}$  is the simplicial complex having the same set of vertices as  $X$ , and the simplices of  $X^{(k)}$  are precisely the simplices of  $X$  which have dimension at most  $k$ .

The *geometric realization* of a simplicial complex  $X$  is the topological space whose underlying set  $|X|$  is the set of all real linear combinations  $\sum_{p \in V_X} \lambda_p \cdot p$ , such that

- the set of those  $p \in V_X$  with  $\lambda_p \neq 0$  is a simplex of  $X$  (and in particular, there are only finitely many non-zero  $\lambda_p$ ), and
- $\sum_{p \in V_X} \lambda_p = 1$ ,
- $\lambda_p \geq 0$  for all  $p \in V_X$ .

For every simplex  $\sigma \in X_n$ , after a choice of ordering  $\sigma = \{p_0, \dots, p_n\}$  of its vertices, there is an injective map

$$j_\sigma: \Delta^n \rightarrow |X|, \quad (\lambda_0, \dots, \lambda_n) \mapsto \sum_{i=0}^n \lambda_i \cdot p_i.$$

Here the *standard  $n$ -simplex*  $\Delta^n \subset \mathbb{R}^{n+1}$  is the convex hull of the standard unit vectors in  $\mathbb{R}^{n+1}$ , i. e. the set of all tuples  $(\lambda_0, \dots, \lambda_n)$  such that  $\sum_{i=0}^n \lambda_i = 1$  and  $\lambda_i \geq 0$  for all  $i$ . Now  $|X|$  is equipped with the weakest topology such that all  $j_\sigma$  are continuous. This means that a set  $U \subset |X|$  is open if and only if all  $j_\sigma^{-1}U$  are open. In particular, the maps  $j_\sigma$  are embeddings of topological subspaces. We denote by  $|\sigma| = j_\sigma(\Delta^n)$  the geometric realization of the simplex  $\sigma \in X_n$ . Thus, the elements of  $|\sigma|$  are convex combinations of the vertices of  $\sigma$ .

Now let  $X$  be a simplicial complex, and let  $E \rightarrow |X|$  be a Hilbert  $A$ -module bundle modelled on the finitely generated projective Hilbert  $A$ -module  $V$ , for instance  $V = \mathbb{C}^n$ . Suppose that for each simplex  $\sigma \in X_n$ , we have a trivialization  $\Phi_\sigma: |\sigma| \times V \rightarrow j_\sigma^*E$ , i. e.  $\Phi_\sigma|_{\{x\} \times V}$  is an isometry of Hilbert  $A$ -modules for each  $x \in \Delta^n$ . For ordinary Hermitian bundles, this simply means that  $\Phi_\sigma$  respects the inner product in every fibre.

Now consider a simplex  $\sigma \in X_n$  and some sub-simplex  $\rho \subset \sigma \in X_k$ . We define the transition function

$$\Psi_{\rho \subset \sigma}: |\rho| \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi_\rho(x, \cdot)^{-1} \circ \Phi_\sigma(x, \cdot).$$

**Definition 2.3.** An  $\epsilon$ -flat family of trivializations (where  $\epsilon > 0$  is a number) of a Hilbert  $A$ -module bundle  $E \rightarrow |X|$  consists of trivializations  $\Phi_\sigma: |\sigma| \times V \rightarrow j_\sigma^*E$ , such that the transition functions  $\Psi_{\rho \subset \sigma}: |\rho| \rightarrow U(\mathcal{L}_A(V))$  are Lipschitz functions with Lipschitz constant at most  $\epsilon$ . Here  $|\rho|$  carries the metric such that  $j_\rho: \Delta^k \rightarrow |\rho|$  is an isometry. An  $\epsilon$ -flat bundle is a Hilbert  $A$ -module bundle together with an  $\epsilon$ -flat family of trivializations. An *almost flat bundle* is an  $\epsilon$ -flat bundle for some  $\epsilon$ .

Naturally, the isomorphism class of  $E$  is uniquely determined by the transition functions. Therefore, an equivalent formulation of an  $\epsilon$ -flat bundle

could simply specify a family of  $\epsilon$ -Lipschitz transition functions satisfying appropriate cocycle conditions.

**2.2. Hilbert module bundles with connections.** An important class of examples for  $\epsilon$ -flat bundles comes from Riemannian geometry. Namely, let  $E \rightarrow M$  be a *smooth* Hilbert  $A$ -module bundle over a Riemannian manifold  $M$ . This means that  $M$  can be covered by open sets  $U_i$  such that  $E$  can be trivialized over each  $U_i$ , and such that the transition functions  $U_i \cap U_j \rightarrow U(\mathcal{L}_A(V))$  are smooth. A *connection* on  $E$  [Sch05] is a linear map  $\nabla: \mathcal{C}^\infty(TM) \otimes \mathcal{C}^\infty(E) \rightarrow \mathcal{C}^\infty(E)$ ,  $X \otimes s \mapsto \nabla_X s$ , such that

- $\nabla_X(s \cdot f) = s \cdot (Xf) + (\nabla_X s) \cdot f$ , and
- $\nabla_{gX} s = g \nabla_X s$

for every  $X \in \mathcal{C}^\infty(TM)$ ,  $f \in \mathcal{C}^\infty(M; A)$ ,  $g \in \mathcal{C}^\infty(M)$  and  $s \in \mathcal{C}^\infty(E)$ . Such a connection is called *compatible* (with the metric) if  $X \langle s, s' \rangle = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle$  for all  $X \in \mathcal{C}^\infty(TM)$ ,  $s, s' \in \mathcal{C}^\infty(E)$ .

**Example 2.4.** If  $E = M \times V$  is trivial, then a compatible connection  $\partial$  on  $E$  is given by partial derivative  $\partial_X s = \frac{d}{dt} \Big|_{t=0} s(\gamma(t))$  where  $\gamma$  is a curve with  $\gamma'(0) = X$ .

For any  $C^*$ -algebra  $A$ , we consider the subspace of *skew-adjoint* elements  $\mathfrak{u}(A) = \{x \in A : x^* + x = 0\}$ . Then there is the following analogue of a classical statement from Riemannian geometry:

**Proposition 2.5** ([Sch05, Lemma 4.3]). *Every compatible connection  $\nabla$  on a trivial bundle  $E = M \times V$  is of the form  $\nabla_X s = \partial_X s + \Gamma(X)s$  where  $\Gamma \in \mathcal{C}^\infty(T^*M \otimes \mathfrak{u}(\mathcal{L}_A(V)))$  is a smooth  $\mathfrak{u}(\mathcal{L}_A(V))$ -valued 1-form.  $\square$*

Now the *curvature* induced by  $\nabla$  is defined by the formula

$$\mathcal{R}^\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s$$

for  $X, Y \in \mathcal{C}^\infty(TM)$ ,  $s \in \mathcal{C}^\infty(E)$ . As in the case of vector bundles  $E$ , one immediately sees that  $\mathcal{R}^\nabla$  is tensorial in all three entries, i. e.  $\mathcal{R}^\nabla(gX, Y) = g\mathcal{R}^\nabla(X, Y) = \mathcal{R}^\nabla(X, gY)$  and  $\mathcal{R}^\nabla(X, Y)(s \cdot f) = (\mathcal{R}^\nabla(X, Y)s) \cdot f$  for  $X, Y \in \mathcal{C}^\infty(TM)$ ,  $s \in \mathcal{C}^\infty(E)$ ,  $f \in \mathcal{C}^\infty(M; A)$  and  $g \in \mathcal{C}^\infty(M)$ . This implies that  $\mathcal{R}^\nabla$  defines a  $\mathcal{L}_A(E)$ -valued 2-form.

Given a smooth Hilbert module bundle  $E \rightarrow M$  and a smooth map  $f: N \rightarrow M$ , the bundle  $f^*E$  is obviously also a smooth Hilbert module bundle over  $N$ . We denote the canonical bundle map by  $\hat{f}: f^*E \rightarrow E$ . Of course, every section  $s \in \mathcal{C}^\infty(E)$  induces a section  $f^*s \in \mathcal{C}^\infty(f^*E)$  which is determined uniquely by the property that  $\hat{f} \circ f^*s = s \circ f$ . Now if  $E$  is equipped with a connection  $\nabla$  then a connection  $f^*\nabla$  on  $f^*E$  is determined uniquely by the property that  $(f^*\nabla)_X(f^*s) = f^*(\nabla_{f_*X} s)$ .

Let  $\gamma: (a, b) \rightarrow M$  be a smooth curve. A section  $s$  of  $E$  along  $\gamma$  is a section of the pullback bundle  $\gamma^*E$ , and such a section is called *parallel* if  $(\gamma^*\nabla)_{\partial_t} s = 0$  on  $(a, b)$ . As for ordinary vector bundles, for every  $t \in (a, b)$  and every  $s(t) \in (\gamma^*E)_t$ , there exists a parallel section  $s$  along  $\gamma$  which



coincides with  $s(t)$  at point  $t$ . Thus, if  $\gamma: [0, 1] \rightarrow M$  is a smooth curve connecting  $p = \gamma(0)$  and  $q = \gamma(1)$ , we may define a parallel transport map  $T_\gamma: E_p \rightarrow E_q$  by mapping an element  $v \in E_p$  to  $\hat{\gamma}(s(1))$  where  $s: [0, 1] \rightarrow \gamma^*E$  is the unique parallel section along  $\gamma$  which satisfies  $\hat{\gamma}(s(0)) = v$ .

Now if  $M$  is triangulated (in this paper, a triangulated manifold will always be smoothly triangulated in the sense that the simplices are smoothly embedded) and if  $|\sigma| \subset M$  is the embedding of a simplex, then we may trivialize  $E|_{|\sigma|}$  over  $|\sigma|$  by choosing an isometry of the Hilbert module  $V$  with the fibre over the barycenter  $b_\sigma \in |\sigma|$  and composing this isometry with the parallel transport outwards along curves of the form  $t \mapsto tx + (1-t)b_\sigma$  for  $x \in |\partial\sigma|$ . Here the barycenter  $b_\sigma \in |\sigma|$  is the point  $b_\sigma = \sum_{v \in \sigma} \frac{1}{\#\sigma} \cdot v$ . This procedure then gives a trivialization  $\Phi_\sigma: |\sigma| \times V \rightarrow j_\sigma^*E$ , and it is easy to see that this trivialization via parallel transport preserves the inner product if  $\nabla$  is compatible and if the isomorphism at  $b$  preserves the inner product. Thus, in this case the transition functions take their values in  $U(\mathcal{L}_A(V))$ .

**Theorem 2.6.** *Let  $M$  be a closed triangulated Riemannian manifold. Then there is a constant  $c(M) > 0$  such that the following holds. Let  $E \rightarrow M$  be a Hilbert  $A$ -module bundle over an arbitrary  $C^*$ -algebra  $A$ . Assume that  $E$  is equipped with a compatible connection, and let  $\Phi_\sigma$  be trivializations via parallel transport as defined above. Assume that*

$$\|\mathcal{R}^\nabla\|_g = \sup_{x \in M} \sup_{\substack{X \wedge Y \in \Lambda^2 T_x M \\ \|X \wedge Y\|_g \leq 1}} \|\mathcal{R}^\nabla(X, Y)\|_{op} \leq \epsilon.$$

*Then the  $\Phi_\sigma$  constitute an  $c(M)\epsilon$ -flat family of trivializations on  $E$ .*

For the proof of theorem 2.6, we will need that parallel transport along loops which bound a small area is close to the identity. The proof of this statement is not so easily found in the literature, in particular not for Hilbert module bundles, so I will give it in appendix A. The precise formulation is the following.

**Proposition 2.7.** *Let  $f: [0, 1] \times [0, 1] \rightarrow M$  be a smooth map. We denote parallel transport in  $E$  along the curve  $f(\partial([0, 1] \times [0, 1]))$  by the symbol  $T_{\partial f}$ . Further, we consider a Hilbert  $A$ -module bundle  $E \rightarrow M$  with compatible connection  $\nabla$ , and the associated curvature tensor  $\mathcal{R}^\nabla$ . Then*

$$\|T_{\partial f} - \text{id}\| \leq \int_0^1 \int_0^1 \|\mathcal{R}^\nabla(\partial_s f(s, t) \wedge \partial_t f(s, t))\| ds dt.$$

Using this, we can prove theorem 2.6.

**Proof of theorem 2.6.** Let  $\rho \subset \sigma$  be simplices of  $M$ . We want to show that the transition function  $\Psi_{\rho \subset \sigma}: |\rho| \rightarrow U(\mathcal{L}_A(V))$  is Lipschitz with Lipschitz constant bounded by a multiple of  $\epsilon$ . Since multiplication with a constant unitary does not change the Lipschitz constant of a map, we may assume that  $\Phi_\rho(b_\rho, v)$  is given by parallel transport of  $\Phi_\sigma(b_\sigma, v)$  along the curve  $t \mapsto tb_\rho + (1-t)b_\sigma$ .

For every pair of points  $a, b \in |\sigma|$  we denote by  $T_{a,b}: E_a \rightarrow E_b$  the parallel transport map along the straight line segment  $t \mapsto tb + (1-t)a$ . Note that  $T_{a,b}$  preserves the inner product since the connection is compatible, and that  $T_{a,b}^{-1} = T_{b,a}$ . Then we have that

$$\begin{aligned}\Phi_\sigma(p, v) &= T_{b_\sigma, p} \Phi_\sigma(b_\sigma, v), \\ \Phi_\rho(p, v) &= T_{b_\rho, p} T_{b_\sigma, b_\rho} \Phi_\sigma(b_\sigma, v)\end{aligned}$$

for every point  $p \in |\rho|$ . This immediately implies that

$$\Psi_{\rho \subset \sigma}(p) = \Phi_\rho(p, \cdot)^{-1} \Phi_\sigma(p, \cdot) = \Phi_\sigma(b_\sigma, \cdot)^{-1} T_{b_\rho, b_\sigma} T_{p, b_\rho} T_{b_\sigma, p} \Phi_\sigma(b_\sigma, \cdot).$$

Now consider arbitrary points  $x, y \in |\rho|$ . The above equations imply that

$$\begin{aligned}\|\Psi_{\rho \subset \sigma}(x) - \Psi_{\rho \subset \sigma}(y)\| &= \|\Psi_{\rho \subset \sigma}(x) \Psi_{\rho \subset \sigma}(y)^{-1} - \text{id}\| \\ &= \|\Phi_\sigma(b_\sigma, \cdot)^{-1} T_{b_\rho, b_\sigma} T_{x, b_\rho} T_{b_\sigma, x} T_{y, b_\sigma} T_{b_\rho, y} T_{b_\sigma, b_\rho} \Phi_\sigma(b_\sigma, \cdot) - \text{id}\| \\ &= \|T_{x, b_\rho} T_{b_\sigma, x} T_{y, b_\sigma} T_{b_\rho, y} - \text{id}\| \\ &= \|T_{b_\rho, y} T_{x, b_\rho} T_{b_\sigma, x} T_{y, b_\sigma} - \text{id}\| \\ &= \|(T_{b_\rho, y} T_{x, b_\rho} T_{y, x})(T_{x, y} T_{b_\sigma, x} T_{y, b_\sigma}) - \text{id}\| \\ &= \|T_{b_\rho, y} T_{x, b_\rho} T_{y, x} - T_{b_\sigma, y} T_{x, b_\sigma} T_{y, x}\| \\ &\leq \|T_{b_\rho, y} T_{x, b_\rho} T_{y, x} - \text{id}\| + \|T_{b_\sigma, y} T_{x, b_\sigma} T_{y, x} - \text{id}\|\end{aligned}$$

since  $T_{b_\rho, b_\sigma}$  and  $\Phi_\sigma(b_\sigma, \cdot)$  preserve the norm. Thus, we have to show that transport along triangles of the form  $\Delta(y, x, b_\rho)$  is close to the identity whenever  $x$  and  $y$  are close. This is true because those triangles obviously bound disks of a small area, so we may use proposition 2.7 to obtain the result.

To make this precise, let  $\iota: \Delta^{\dim \sigma} \rightarrow |\sigma|$  be the embedding of the simplex  $|\sigma|$ , where  $\Delta^{\dim \sigma}$  is equipped with the standard metric as a subset of Euclidean space. Since the simplex  $|\sigma|$  is compact and  $\iota$  is a diffeomorphism, there is a constant  $C_1(\sigma) > 0$  such that  $\iota$  is  $C_1(\sigma)$ -Lipschitz and  $\iota^{-1}$  expands areas by a factor of at most  $C_1(\sigma)$ . Since  $M$  is compact, and hence has only finitely many simplices, we may choose  $C_1(M) = C_1(\sigma)$  independently of  $\sigma$ . Now the filled triangle  $\bar{\Delta}(y, x, b_\rho)$  (which may be defined as the image, under  $\iota^{-1}$ , of the convex hull  $\text{conv}(\iota(y), \iota(x), \iota(b_\rho))$ ) satisfies

$$\begin{aligned}\text{area}(\bar{\Delta}(y, x, b_\rho)) &\leq C_1(M) \cdot \frac{1}{2} d(\iota(x), \iota(y)) \cdot \text{diam}(\Delta^{\dim \sigma}) \\ &\leq \frac{1}{2} \sqrt{2} \cdot C_1(M)^2 \cdot d(x, y).\end{aligned}$$

Now proposition 2.7 implies that

$$\begin{aligned}\|T_{b_\sigma, y} T_{x, b_\sigma} T_{y, x} - \text{id}\| &\leq \text{area}(\bar{\Delta}(y, x, b_\rho)) \cdot \|\mathcal{R}^\nabla\|_g \\ &\leq \frac{1}{2} \sqrt{2} \cdot C_1(M)^2 \cdot \|\mathcal{R}^\nabla\|_g \cdot d(x, y).\end{aligned}$$

Since the same calculation holds for  $\rho$  as well, the calculation above proves that  $\Psi_{\rho \subset \sigma}$  is Lipschitz with constant at most  $\sqrt{2} \cdot C_1(M)^2 \cdot \|\mathcal{R}^\nabla\|_g$ . Thus, we may take  $c(M) = \sqrt{2} \cdot C_1(M)^2$ .  $\square$

### 3. The trivialization lemma

The goal of this section is to prove the trivialization lemma which states that an  $\epsilon$ -flat bundle over a simply connected space is trivial if  $\epsilon$  is small enough. This is the basic result which enables us to extend  $\epsilon$ -flat bundles to larger  $\Delta$ -sets under certain conditions. Namely, one can apply the trivialization lemma to the sphere  $S^{n-1}$  if  $n > 2$  to trivialize the bundle over the boundary  $\partial\Delta^n$  of a simplex and thus to extend the bundle over the whole simplex  $\Delta^n$ .

**3.1. Transport in the 1-skeleton.** We first want to show that for  $\epsilon$ -flat bundles, transport along contractible simplicial loops is close to the identity, in analogy with proposition 2.7. This will be one of the main ingredients in the proof of the trivialization lemma.

For any simplicial complex  $X$ , we define the *simplicial path category*  $\mathcal{P}_X$  as follows: Objects of  $\mathcal{P}_X$  are the vertices of  $X$ , and morphisms from  $v_0$  to  $v_k$  are *simplicial paths*, i. e. tuples  $(v_0, \dots, v_k)$  such that  $\{v_i, v_{i+1}\} \in X_1$ . One should imagine simplicial paths as concatenations of the paths  $t \mapsto (1-t)v_i + tv_{i+1}$ . The composition of two simplicial paths  $\Gamma = (v_0, \dots, v_k)$  and  $\Gamma' = (v_k, \dots, v_{k+l})$  is to be the path  $\Gamma * \Gamma' = (v_0, \dots, v_{k+l})$ .

Let  $E \rightarrow |X|$  be an  $\epsilon$ -flat bundle, and let  $\sigma = \{v_0, v_1\} \in X_1$  be an edge. Then *transport along*  $(v_0, v_1)$  is the isometry of Hilbert  $A$ -modules

$$T_{(v_0, v_1)} = \Phi_\sigma(v_1, \cdot) \Phi_\sigma(v_0, \cdot)^{-1}: E_{v_0} \rightarrow E_{v_1}.$$

If  $\Gamma = (v_0, \dots, v_k)$  is a simplicial path then *transport along*  $\Gamma$  is defined as

$$T_\Gamma = T_{(v_{k-1}, v_k)} \cdots T_{(v_0, v_1)}: E_{v_0} \rightarrow E_{v_k}.$$

Of course, if  $\Gamma = \Gamma_1 * \Gamma_2$  then  $T_\Gamma = T_{\Gamma_2} T_{\Gamma_1}$ , so the associations  $v \mapsto E_v$ ,  $\Gamma \mapsto T_\Gamma$  define a functor  $\mathcal{P}_X \rightarrow \mathcal{M}_A$  into the category of Hilbert  $A$ -modules and Hilbert  $A$ -module isometries.

We want to analyse the transport along contractible simplicial loops  $\Gamma$ . We first consider the special case that  $\Gamma$  is the boundary curve of a 2-simplex.

**Proposition 3.1.** *Let  $\Gamma = (v_0, v_1, v_2, v_0)$  be the simplicial loop along the boundary of a 2-simplex  $\sigma = \{v_0, v_1, v_2\} \in X_2$ . If  $E$  is an  $\epsilon$ -flat bundle over  $X$  with  $\epsilon \leq 1/\sqrt{2}$ , then*

$$\|T_\Gamma - \text{id}\| \leq \epsilon \cdot 7\sqrt{2}.$$

**Proof.** By definition of the transition function we have that

$$\Phi_\rho(x, \cdot) = \Phi_\sigma(x, \cdot) \circ \Psi_{\rho \subset \sigma}(x, \cdot)^{-1}$$

for  $\rho = \{v_i, v_j\} \subset \sigma$ . It follows that

$$T_{(v_i, v_j)} = \Phi_\sigma(v_j, \cdot) \Psi_{\rho \subset \sigma}(v_j)^{-1} \Psi_{\rho \subset \sigma}(v_i) \Phi_\sigma(v_i, \cdot)^{-1}.$$

By definition we have that  $T_\Gamma = T_{(v_2, v_0)} T_{(v_1, v_2)} T_{(v_0, v_1)}$ . Together it follows that

$$\begin{aligned} T_\Gamma &= \Phi_\sigma(v_0, \cdot) \cdot (\Psi_{\{v_2, v_0\} \subset \sigma}(v_0)^{-1} \Psi_{\{v_2, v_0\} \subset \sigma}(v_2)) \cdot \\ &\quad \cdot (\Psi_{\{v_1, v_2\} \subset \sigma}(v_2)^{-1} \Psi_{\{v_1, v_2\} \subset \sigma}(v_1)) \cdot \\ &\quad \cdot (\Psi_{\{v_0, v_1\} \subset \sigma}(v_1)^{-1} \Psi_{\{v_0, v_1\} \subset \sigma}(v_0)) \cdot \Phi_\sigma(v_0, \cdot)^{-1} \end{aligned}$$

holds. Since the  $\Psi_{\rho \subset \sigma}$  are  $\epsilon$ -Lipschitz maps and the vertices have distance  $\sqrt{2}$  we conclude that

$$\|\Psi_{\rho \subset \sigma}(v_j)^{-1} \Psi_{\rho \subset \sigma}(v_i) - \text{id}\| \leq \epsilon\sqrt{2}.$$

Now we apply the following very useful lemma.

**Lemma 3.2.** *Let  $0 < \epsilon \leq 1$ . Let  $V_1, \dots, V_n$  be Hilbert  $A$ -modules. Let furthermore*

$$P = A_n B_n A_{n-1} B_{n-1} \cdots A_1 B_1 : V_1 \rightarrow V_1$$

where all  $A_i : V_i \rightarrow V_{i+1}$  are isometries of Hilbert  $A$ -modules,  $V_{n+1} = V_1$ ,  $A_n \cdots A_1 = \text{id}$ , and where all the  $B_i : V_i \rightarrow V_i$  are linear with  $\|B_i - \text{id}\| < \epsilon$ . Then also  $\|P - \text{id}\| < c\epsilon$ , and  $c = c(n) = 2^n - 1$  depends only on  $n$ .

**Proof.** Replace  $B_i$  by  $(B_i - \text{id}) + \text{id}$  in the definition of  $P$  and expand. Then one sees that  $P - \text{id}$  is the sum of  $2^n - 1$  linear maps having norm bounded by  $\epsilon$ , since  $\epsilon \leq 1$ . The claim follows using the triangle inequality.  $\square$

To finish the proof of proposition 3.1 we can use the lemma with  $n = 3$ , applied with  $V_1 = V_2 = V_3 = V$  (the fibre of  $E$ ), all  $A_i = \text{id}$ , and

$$B_i = \Psi_{\{v_{i-1}, v_i\} \subset \sigma}(v_i)^{-1} \Psi_{\{v_{i-1}, v_i\} \subset \sigma}(v_{i-1})$$

(with indices modulo 3). Thus we get  $\|T_\Gamma - \text{id}\| \leq c\epsilon\sqrt{2}$  with  $c = 2^3 - 1 = 7$ .  $\square$

Of course, the constants used in lemma 3.2 and proposition 3.1 are somewhat arbitrary: If we require other bounds for  $\epsilon$ , then we get other constants. However, since we will only consider  $\epsilon$ -flat bundles when  $\epsilon$  is small, this extra flexibility is not important here.

We recall the following well-known description of the fundamental group of  $X$ . The *homotopy simplicial path category*  $\mathcal{P}'_X$  is the quotient category of  $\mathcal{P}_X$  modulo the congruence relation generated by identifying  $(v_0, v_1, v_0)$  and  $(v_0)$  for every  $\{v_0, v_1\} \in X_1$ , and by identifying  $(v_0, v_1, v_2, v_0)$  and  $(v_0)$  for every  $\{v_0, v_1, v_2\} \in X_2$ .

The *path groupoid* of a space  $X$  is the category  $\Pi_X$  which has as objects the points in  $X$  and as morphisms homotopy classes of paths  $[0, 1] \rightarrow X$  relative to the endpoints, where composition is given as concatenation. In particular,  $\pi_1(X, x_0) = \Pi_X(x_0, x_0)$ .

**Proposition 3.3.** *If  $X$  is a simplicial complex, the natural functor  $\mathcal{P}_X \rightarrow \Pi_X$  descends to a fully faithful functor  $\mathcal{P}'_X \rightarrow \Pi_X$ . In particular,  $\pi_1(X, x_0) \cong \mathcal{P}'_X(x_0, x_0)$  for every vertex  $x_0$ .  $\square$*

This implies that if  $\Gamma \in \mathcal{P}_X(x_0, x_0)$  is a contractible simplicial loop then  $\Gamma$  arises by a finite sequence of omissions or insertions of pieces of the form  $(v_0, v_1, v_0)$  where  $\{v_0, v_1\} \in X_1$ , or of pieces of the form  $(v_0, v_1, v_2, v_0)$  where  $\{v_0, v_1, v_2\} \in X_2$ . Note that omissions or insertions of the first type do not change the transport map. The *homotopical complexity*  $\text{hc}(\Gamma)$  of such a contractible simplicial loop  $\Gamma$  is the minimum number of insertions or omissions of the second form which is needed to obtain  $\Gamma$  from the empty loop.

**Theorem 3.4.** *Let  $\Gamma$  be a contractible simplicial loop in  $X$  and  $n = \text{hc}(\Gamma)$  its homotopical complexity as defined above. Then there exist constants  $c(n), \delta(n) > 0$ , depending only on  $n$ , such that for every  $\epsilon$ -flat bundle over  $X$  with  $\epsilon \leq \delta(n)$  the transport along  $\Gamma$  satisfies the inequality*

$$\|T_\Gamma - \text{id}\| \leq c(n)\epsilon.$$

**Proof.** We prove the claim by induction over  $n$ . If  $n = 0$  there is nothing to do, since insertions or omissions of the first type do not alterate the transport map. Thus we assume that  $\Gamma = \Gamma_1 * \hat{\Gamma} * \Gamma_2$  with  $\hat{\Gamma} = (v_0, v_1, v_2, v_0)$  and  $\text{hc}(\Gamma_1, \Gamma_2) = n - 1$ . Since the transport maps are isometries of Hilbert  $A$ -modules, we get that

$$\|T_\Gamma - \text{id}\| = \|T_{\Gamma_2} T_{\hat{\Gamma}} T_{\Gamma_1} - \text{id}\| = \|T_{\hat{\Gamma}} T_{\Gamma_1} T_{\Gamma_2} - \text{id}\| = \|T_{\hat{\Gamma}} T_{\Gamma_2 * \Gamma_1} - \text{id}\|.$$

By proposition 3.1,  $\|T_{\hat{\Gamma}} - \text{id}\| \leq c(1)\epsilon$  where  $c(1) = 7\sqrt{2}$ , and by induction we may assume that  $\|T_{\Gamma_2 * \Gamma_1} - \text{id}\| \leq c(n-1)\epsilon$  if  $\epsilon \leq \min\{1/\sqrt{2}, \delta(n-1)\}$ . If we now let  $\delta(n) = \min\{c(1)^{-1}, c(n-1)^{-1}, \delta(n-1)\}$  then  $\max\{c(1), c(n-1)\}\epsilon \leq 1$ , so we may use lemma 3.2 to show that

$$\|T_\Gamma - \text{id}\| \leq 3 \max\{c(1), c(n-1)\}\epsilon.$$

If, on the other hand,  $\Gamma = \Gamma_1 * \Gamma_2$  and  $\text{hc}(\Gamma_1, \hat{\Gamma}, \Gamma_2) \leq n - 1$ , then transport along  $\Gamma$  is the same thing as transport along the curve  $\Gamma_1 * (v_0, v_2, v_1, v_0) * \hat{\Gamma} * \Gamma_2$  and we may use the same proof to show that  $\|T_\Gamma - \text{id}\| \leq 3 \max\{c(1), c(n-1)\}\epsilon$ . The claim of the theorem follows with  $c(n) = 3 \max\{c(1), c(n-1)\}$ .  $\square$

**3.2. The trivialization lemma.** Now we can use the previous results to prove the trivialization lemma. This states that  $\epsilon$ -flat bundles over simply connected spaces are trivial if  $\epsilon$  is small enough. We can further achieve that the transition functions from the  $\epsilon$ -flat family to the global trivialization are Lipschitz with small Lipschitz constant.

This is to say that if  $E \rightarrow |X|$  is an almost flat Hilbert  $A$ -module bundle with global trivialization  $\Phi_X: |X| \times V \rightarrow E$ , then, as before, we obtain the transition function

$$\Psi_{\rho \in X}: |\rho| \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi_\rho(x, \cdot)^{-1} \circ \Phi_X(x, \cdot).$$

Now  $\Phi_X$  is a *global  $\epsilon$ -trivialization* if the maps  $\Psi_{\rho \in X}$  are  $\epsilon$ -Lipschitz for every simplex  $\rho \in X$ .

The proof of the trivialization lemma relies heavily on the following extension property for maps into the unitaries of a  $C^*$ -algebra, for which we give a proof in appendix B.

**Lemma 3.5.** *There is a number  $C > 0$  with the following property. Let  $A$  be a  $C^*$ -algebra with unit, and denote  $U(A)$  the set of unitary elements of  $A$ . Let further  $\alpha_0: S^{n-1} \rightarrow U(A)$  be a  $\lambda$ -Lipschitz map. If  $\text{diam } \alpha_0(S^{n-1}) \leq \frac{1}{2}$ , there exists an extension  $\alpha: D^n \rightarrow U(A)$  with Lipschitz constant at most  $C\lambda$ . Here, the constant  $C$  is independent of  $A$ .*

For the statement of the trivialization lemma, recall that a *tree* in a simplicial complex  $X$  is a contractible subcomplex of the 1-skeleton of  $X$ . Every simplicial complex contains a maximal tree, and it is a basic fact that a tree is maximal in  $X$  if and only if it contains all the vertices of  $X$ .

**Theorem 3.6** (Trivialization lemma). *Let  $X$  be a finite-dimensional simplicial complex. Then there are constants  $C(X), \delta(X) > 0$  such that the following holds: Let  $T \subset X$  be a maximal tree in  $X$ . Let  $E \rightarrow X$  be an  $\epsilon$ -flat bundle where  $\epsilon \leq \delta(X)$ . Suppose further that for every simplex  $\sigma = \{x, y\} \in X_1$ , there is a simplicial loop  $\Gamma_\sigma = (x, y) * \Gamma_\sigma^0$  such that  $\Gamma_\sigma^0$  is a simplicial path in  $T$  and such that  $\|T_{\Gamma_\sigma} - \text{id}\| \leq \epsilon$ . Then  $E$  admits a global  $C(X)\epsilon$ -trivialization.*

**Proof.** Firstly, we want to prove that such a trivialization exists in the case that  $X$  is a graph, i. e. 1-dimensional. First note that it is trivial to construct a global 0-trivialization over the tree  $T$ . Now let  $\sigma = \{x, y\} \in X_1$  be a simplex, and let  $\Gamma_\sigma = (x, y) * \Gamma_\sigma^0$  be a simplicial loop as in the assumption of the theorem. Consider the transition function

$$\Psi_{\sigma, T}: \{x, y\} \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi_\sigma(x, \cdot)^{-1} \circ \Phi_T(x, \cdot).$$

Then we have that

$$\begin{aligned} \|\Psi_{\sigma, T}(x) - \Psi_{\sigma, T}(y)\| &= \|\Phi_T(y, \cdot)^{-1} T_{(x, y)} \Phi_T(x, \cdot) - \text{id}\| \\ &= \|\Phi_T(y, \cdot)^{-1} T_{\Gamma_\sigma^0}^{-1} \Phi_T(x, \cdot) \Phi_T(x, \cdot)^{-1} T_{\Gamma_\sigma} \Phi_T(x, \cdot) - \text{id}\| \\ &= \|T_{\Gamma_\sigma} - \text{id}\| \leq \epsilon \end{aligned}$$

by assumption and using that  $\Phi_T(y, \cdot)^{-1} T_{\Gamma_\sigma^0}^{-1} \Phi_T(x, \cdot) = \text{id}$  because  $\Phi_T$  is a 0-trivialization. Now by lemma 3.5, the map  $\Psi_{\sigma, T}$  possesses a  $C\epsilon$ -Lipschitz extension  $\Psi'_{\sigma, T}$ . Of course, we have that  $\Phi_T(p, \cdot) = \Phi_\sigma(p, \cdot) \circ \Psi'_{\sigma, T}(p)$  for  $p \in \{x, y\}$ , so we can extend  $\Phi_T$  by the same formula onto  $|\sigma|$ , and the transition function for  $\sigma$  will be  $C\epsilon$ -Lipschitz.

In the general case, we proceed by induction on the dimension of  $Y$ . Thus, we may assume that we already have a global  $\epsilon$ -trivialization over the  $k$ -skeleton of  $Y$ , where  $k \geq 1$ , and we want to extend it to the  $(k+1)$ -skeleton. Let  $S = Y^{(k)} = \bigcup_{i \leq k} Y_i$ , and let  $\Phi_S$  be the corresponding global

trivialization. Given a  $(k + 1)$ -simplex  $\rho$ , the map

$$\Psi_{\rho,S}: |\partial\rho| \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi_\sigma(x, \cdot)^{-1} \circ \Phi_S(x, \cdot)$$

is  $\epsilon$ -Lipschitz on every simplex of  $|\partial\rho|$ . By connectedness of  $|\partial\rho|$ , it is globally  $\epsilon$ -Lipschitz, and we may use lemma 3.5 again to find a Lipschitz extension on the whole of  $|\rho|$ . As above, this completes the proof in the general case.  $\square$

**Corollary 3.7.** *Let  $X$  be a finite simply connected simplicial complex. Then there are constants  $C(X), \delta(X) > 0$  such that the following holds: Let  $E \rightarrow X$  be an  $\epsilon$ -flat bundle where  $\epsilon \leq \delta(X)$ . Then  $E$  admits a global  $C(X)\epsilon$ -trivialization.*

**Proof.** Choose a tree  $T \subset X$ , and arbitrary paths  $\Gamma_\sigma = (x, y) * \Gamma_\sigma^0$  for every simplex  $\sigma = \{x, y\} \in X_1 - T$ , such that  $\Gamma_\sigma^0$  is a simplicial path in  $T$ . Then these curves satisfy the assumptions from theorem 3.6 because of theorem 3.4.  $\square$

### 4. Applications of the trivialization lemma

**4.1. Subdivision.** As a first application of the trivialization lemma, we show that almost flatness is invariant under barycentric subdivision.

The *barycentric subdivision* of a simplicial complex  $X$  is the simplicial complex  $\mathcal{S}(X)$  whose vertices are  $V_{\mathcal{S}(X)} = X$  and whose simplices are  $\mathcal{S}(X) = \{\{\sigma_0, \dots, \sigma_k\} : \sigma_i \subset \sigma_{i+1}\}$ . This deserves the name subdivision, as the following shows.

**Lemma 4.1** ([Spa12]). *For any simplicial complex  $X$ , there is a natural homeomorphism  $\Xi_X: |\mathcal{S}(X)| \rightarrow |X|$  which is given on vertices by  $\Xi_X(\sigma) = \sum_{v \in \sigma} \frac{1}{\#\sigma} v$ , and which is affine linear on every simplex of  $\mathcal{S}(X)$ .*  $\square$

Thus, we may identify  $|\mathcal{S}(X)|$  and  $|X|$ , and in particular, a bundle  $E \rightarrow |X|$  is the same thing as a bundle  $E \rightarrow |\mathcal{S}(X)|$ . Now suppose that we have an  $\epsilon$ -flat bundle  $E \rightarrow |X|$  (with respect to the triangulation  $X$ ). If  $\rho$  is a simplex of  $\mathcal{S}(X)$  then the image of  $|\rho|$  in  $|X|$  is contained in the realization of a simplex  $\bigcup \rho \in X$ ,<sup>2</sup> and the embedding  $\Xi_X: |\rho| \rightarrow |\bigcup \rho|$  induces a trivialization

$$\Phi_\rho = \Phi_{\bigcup \rho} \circ (\Xi_X \times \text{id}): |\rho| \times V \rightarrow E|_{|\rho|}.$$

Now if  $\rho' \subset \rho$ , then  $\Psi_{\rho' \subset \rho} = \Psi_{\bigcup \rho' \subset \bigcup \rho} \circ \Xi_X$ .

This shows that almost flat bundles over  $X$  induce almost flat bundles over  $\mathcal{S}(X)$ . The opposite statement is also true, as we will see momentarily. For the proof of this statement, we will need the following useful observation.

---

<sup>2</sup>The reason for this notation is the following: If  $\rho \in \mathcal{S}(X)$  is any simplex then  $\rho = \{\sigma_0, \dots, \sigma_k\}$  with  $\sigma_i \subset \sigma_{i+1}$ . Thus,  $\bigcup \rho = \bigcup_{\sigma \in \rho} \sigma = \sigma_k$ . Now it follows from the formula for  $\Xi_X$  that indeed  $\Xi_X(|\rho|) \subset |\sigma_k|$ . However, it is not important for the argument to know what exactly  $\bigcup \rho$  is. It suffices that  $\Xi_X(|\rho|) \subset |\bigcup \rho|$ .

**Lemma 4.2.** *Let  $X$  be a metric space and let  $f, g: X \rightarrow U(\mathcal{L}_A(V))$  be  $\epsilon$ -Lipschitz maps for some  $\epsilon > 0$ . Then the map  $X \rightarrow U(\mathcal{L}_A(V))$ ,  $x \mapsto f(x) \circ g(x)$ , is  $3\epsilon$ -Lipschitz.*

**Proof.** If  $x, y \in X$ , then

$$\|f(x)g(x) - f(y)g(y)\| = \|f(y)^{-1}f(x)g(x)g(y)^{-1} - \text{id}\| \leq 3\epsilon$$

by lemma 3.2. □

**Proposition 4.3.** *Consider a finite-dimensional simplicial complex  $X$  and its barycentric subdivision  $\mathcal{S}(X)$ . Then there are constants  $C_1, C_2, \delta > 0$ , depending only on the dimension of  $X$ , such that:*

- every  $\epsilon$ -flat bundle over  $X$  is canonically a  $C_1\epsilon$ -flat bundle over  $\mathcal{S}(X)$ , and
- if  $\epsilon \leq \delta$ , then every  $\epsilon$ -flat bundle over  $\mathcal{S}(X)$  admits a  $C_2\epsilon$ -flat family of trivializations with respect to the triangulation  $X$ .

**Proof.** The first assertion follows from the discussion above since  $\Xi_X|_{|\rho|}$  is  $C_1\epsilon$ -Lipschitz with some constant  $C_1$  which depends only on the dimension of  $\rho$ . In fact,  $C_1$  is the Lipschitz constant of the map  $\Xi_X$  defined above.

For the second assertion, let  $E \rightarrow \mathcal{S}(X)$  be an  $\epsilon$ -flat bundle, and let  $\rho$  be a simplex of  $X$ . Then  $|\rho|$  is the geometric realization of a contractible sub-complex of  $\mathcal{S}(X)$ . Thus, by the trivialization lemma, there is a global  $C(\rho)\epsilon$ -trivialization  $\Phi_\rho$  on  $|\rho|$  whenever  $\epsilon \leq \delta(\rho)$ . Note that the constants  $C_n = C(\rho)$  and  $\delta_n = \delta(\rho)$  in fact only depend on the dimension  $n = \dim(\rho)$  of the simplex, so that there are constants  $C' = \max\{C_n : n \leq \dim X\}$  and  $\delta = \min\{\delta_n : n \leq \dim X\}$  which only depends on the dimension of  $X$ , such that  $C(\rho) \leq C'$  and  $\delta(\rho) \leq \delta$  for every simplex  $\rho$ .

Now consider another simplex  $\sigma \subset \rho$ , and simplices  $\sigma' \subset \rho'$  of  $\mathcal{S}(X)$  with  $|\sigma'| \subset |\sigma|$  and  $|\rho'| \subset |\rho|$ . Then, for every  $x \in |\rho'|$ , we have that

$$\Psi_{\sigma \subset \rho}(x) = [\Phi_\sigma(x, \cdot)^{-1} \Phi_{\sigma'}(x, \cdot)] \cdot [\Phi_{\sigma'}(x, \cdot)^{-1} \Phi_{\rho'}(x, \cdot)] \cdot [\Phi_{\rho'}(x, \cdot)^{-1} \Phi_\rho(x, \cdot)].$$

This means that  $\Psi_{\sigma \subset \rho}$  is (locally) the product of three maps

$$|\sigma'| \rightarrow U(\mathcal{L}_A(V))$$

which are  $\max(C', 1)\epsilon$ -Lipschitz. Now a two-fold application of lemma 4.2 shows that  $\Psi_{\sigma \subset \rho}$  is  $9 \max(C', 1)\epsilon$ -Lipschitz. Thus, we may take  $C_2 = 9 \max(C', 1)$ . □

**4.2. Extensions of almost flat bundles.** We give another application of the trivialization lemma, which will be the key observation for most of this paper. Namely, we show that one may extend  $\epsilon$ -flat bundles in an essentially unique way.

Consider a simplicial complex  $X$ , and an  $\epsilon$ -flat bundle  $E \rightarrow |X^{(k)}|$  over the  $k$ -skeleton of  $X$ . We want to extend  $E$  to a  $C\epsilon$ -small bundle  $E \rightarrow |X^{(k+1)}|$  over the  $(k+1)$ -skeleton of  $X$ . Such an extension is the same thing as a global  $C\epsilon$ -trivialization over the boundary  $|\partial\rho|$  of every simplex  $\rho \in X_{k+1}$ .



Now if  $k \geq 2$ , then  $|\partial\rho| \cong S^k$  is simply connected, so that the trivialization lemma (or rather: corollary 3.7) provides a constant  $C$  such that global  $C\epsilon$ -trivializations over all  $|\partial\rho|$  exist if  $\epsilon$  is small enough. If  $k = 0$ , then there are trivially global 0-trivializations over all  $|\partial\rho|$ . Thus, we have:

**Theorem 4.4.** *For every  $k \in \mathbb{N} - \{1\}$ , there are constants  $C = C(k), \delta = \delta(k) > 0$  with the property that for every simplicial complex  $X$  and every  $\epsilon$ -flat bundle  $E \rightarrow |X^{(k)}|$  with  $\epsilon \leq \delta$ , there exists an extension of  $E$  to a  $C\epsilon$ -flat bundle over  $|X^{(k+1)}|$ .  $\square$*

For  $k = 1$ , the existence of such an extension implies that parallel transport along the boundary of every simplex is small. In turn, this is also a sufficient condition for the existence of an extension by the trivialization lemma. This gives the following statement.

**Theorem 4.5.** *There are constants  $C, \delta > 0$  such that the following holds: Let  $E \rightarrow |X^{(1)}|$  be a bundle with trivializations  $\Phi_\sigma: |\sigma| \times V \rightarrow E$ . Assume that transport along the boundary of every 2-simplex of  $X$  is  $\epsilon$ -close to the identity where  $\epsilon \leq \delta$ . Then there is an extension of  $E$  to a  $C\epsilon$ -flat bundle over  $|X^{(2)}|$ .  $\square$*

On the other hand, such extensions are unique in the following sense.

**Theorem 4.6.** *Let  $X$  be a simplicial complex of dimension  $n$ . Then there are constants  $C = C(n), \delta = \delta(n) > 0$ , depending only on the dimension of  $X$ , such that the following holds: Let  $E, E' \rightarrow |X|$  be two  $\epsilon$ -small bundles modeled on the same projective Hilbert  $A$ -module  $A$ , where  $\epsilon \leq \delta$ . We denote the trivializations by  $\Phi_\rho, \Phi'_\rho$ , respectively. Assume that for every 1-simplex  $\{p, q\} \in X_1$ , we have that*

$$\|\Phi_q(q, \cdot)^{-1}T_{[p,q]}\Phi_p(p, \cdot) - \Phi'_q(q, \cdot)^{-1}T'_{[p,q]}\Phi'_p(p, \cdot)\| < \epsilon.$$

*Then there is an isometry of bundles  $\Xi: E \rightarrow E'$  with the property that the map*

$$|\sigma| \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi'_\rho(x, \cdot)^{-1}\Xi\Phi_\rho(x, \cdot)$$

*is  $C\epsilon$ -Lipschitz for every simplex  $\sigma \in X$ .*

**Proof.** We prove the statement by induction on the dimension  $n$ . We begin with the base case  $n = 1$ . For every vertex  $v \in X_0$ , we let

$$\Xi_v = \Phi'_v(v, \cdot)\Phi_v(v, \cdot)^{-1}: E_v \rightarrow E'_v.$$

If  $\rho = \{p, q\} \in X_1$  is an edge, then

$$\begin{aligned} & \|\Phi'_\rho(p, \cdot)^{-1}\Xi_p\Phi_\rho(p, \cdot) - \Phi'_\rho(q, \cdot)^{-1}\Xi_q\Phi_\rho(q, \cdot)\| \\ &= \|\Phi'_q(q, \cdot)^{-1}\Phi'_\rho(q, \cdot)\Phi'_\rho(p, \cdot)^{-1}\Phi'_p(p, \cdot) - \Phi_q(q, \cdot)^{-1}\Phi_\rho(q, \cdot)\Phi_\rho(p, \cdot)^{-1}\Phi_p(p, \cdot)\| \\ &= \|\Phi'_q(q, \cdot)^{-1}T'_{[p,q]}\Phi'_p(p, \cdot) - \Phi_q(q, \cdot)^{-1}T_{[p,q]}\Phi_p(p, \cdot)\| < \epsilon \end{aligned}$$

by assumption.

Now we may use lemma 3.5 to construct a  $C(1)\epsilon$ -Lipschitz map  $f: |\rho| \rightarrow U(\mathcal{L}_A(V))$  extending the map  $|\partial\rho| \rightarrow U(\mathcal{L}_A(V))$ ,  $x \mapsto \Phi'_\rho(x, \cdot)^{-1}\Xi_x\Phi_\rho(x, \cdot)$ , provided that  $\epsilon$  is small enough. Obviously, the map

$$\Xi_x = \Phi'_\rho(x, \cdot)f(x)\Phi_\rho(x, \cdot)^{-1}: E_x \rightarrow E'_x$$

extends the maps  $\Xi_v$  defined before, and they fit together to an isometry of bundles  $\Xi_1: E|_{X^{(1)}} \rightarrow E'|_{X^{(1)}}$  which satisfies the Lipschitz property stated in the theorem.

Thus, consider  $n \geq 2$ . We assume that we already have a bundle isometry  $\Xi_{n-1}: E|_{X^{(n-1)}} \rightarrow E'|_{X^{(n-1)}}$  satisfying the demanded Lipschitz property. Let  $\rho \in X_n$  be a simplex. By assumption, the map

$$|\partial\rho| \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi'_\rho(x, \cdot)^{-1}\Xi\Phi_\rho(x, \cdot)$$

is locally (thus, globally)  $C(n-1)\epsilon$ -Lipschitz, and if  $C(n-1)\epsilon$  is small enough, we may use lemma 3.5 again to produce a  $C(n)\epsilon$ -Lipschitz extension  $f: |\rho| \rightarrow U(\mathcal{L}_A(V))$ . As before, we may put  $\Xi_x = \Phi'_\rho(x, \cdot)f(x)\Phi_\rho(x, \cdot)^{-1}$ , which gives the desired extension of  $\Xi_{n-1}$  to a bundle isometry  $\Xi_n$ .  $\square$

## 5. Almost flat bundles over Riemannian manifolds

In this section, let  $X$  be a triangulated closed Riemannian manifold, and let  $E \rightarrow X$  be an  $\epsilon$ -flat bundle. We want to show that  $E$  may be equipped with a smooth structure, a smooth Hermitian metric, and a compatible connection  $\nabla$ , such that the curvature  $\mathcal{R}^\nabla$  satisfies  $\|\mathcal{R}^\nabla\| \leq C\epsilon$  for some constant  $C > 0$  which depends only on  $X$  and on the choice of triangulation. Thus, for triangulated Riemannian manifolds, our definition of an almost flat bundle strongly corresponds to the definition via the curvature tensor.

Here the idea is that one may define the connection inductively over neighborhoods of the skeleta of  $X$ . Those neighborhoods will be constructed as subsets of the union of the open stars of the skeleta in the barycentric subdivision of  $X$ . The trivialization lemma will give a trivialization over those open stars, by the following lemma.

**Lemma 5.1.** *Let  $X$  be a simplicial complex, let  $\sigma \in X_k$  be a simplex, and consider the barycentric subdivision  $\mathcal{S}(X)$  of  $X$ . Then there is a contractible subcomplex  $S \subset \mathcal{S}(X)$ , such that  $|S|$  is a neighborhood of  $|\sigma|$ .*

**Proof.** Let  $S_0 \subset X$  be the set of those simplices  $\rho$  which intersect  $\sigma$ . Since the vertices of  $\mathcal{S}(X)$  are in a bijective correspondence with the simplices of  $X$ , we may define  $S \subset \mathcal{S}(X)$  to be the subcomplex consisting of those simplices whose vertices lie in  $S_0$ .

Then  $|S|$  deformation retracts onto  $|\sigma|$ : Namely, if  $\rho \in S_0$  is a vertex of  $S$ , then the mapping  $t \mapsto t(\rho \cap \sigma) + (1-t)\rho$  is a well-defined linear curve in  $S$ , since by definition of  $S_0$  we have that  $\rho \cap \sigma \neq \emptyset$ . This map may be extended linearly to a homotopy  $|S| \times I \rightarrow |S|$  from the identity to a retraction onto  $|\sigma|$  because the realization of a simplex  $\{\sigma_0, \dots, \sigma_k\}$  is contained in  $|\sigma|$  if and only if all  $\sigma_i \subset \sigma$ . This implies that  $|S|$  is contractible.

It is clear that  $|S|$  is a neighborhood of  $|\sigma|$ , since for every vertex  $\rho \in S_0$  with  $|\rho| \in |\sigma|$ , the open star of  $\rho$  is also contained in  $S_0$ .  $\square$

Now the first step in constructing the smooth bundle with connection is the following:

**Proposition 5.2.** *Let  $X$  be a closed triangulated Riemannian manifold. Then there is a constant  $C > 0$ , depending only on  $X$ , and a family of open sets  $U_\sigma$ , one for every simplex  $\sigma \in X$ , such that  $|\sigma| \subset U_\sigma$ , with the following property:*

*Let  $E \rightarrow |X|$  be an  $\epsilon$ -flat bundle. Then there are trivializations  $\Theta_\sigma: U_\sigma \times V \rightarrow E|_{U_\sigma}$ , such that the transition functions*

$$\Psi_{\sigma,\rho}: U_\sigma \cap U_\rho \rightarrow U(\mathcal{L}_A(V)), \quad x \mapsto \Phi_\sigma(x, \cdot)^{-1} \Phi_\rho(x, \cdot)$$

*are all smooth and  $C\epsilon$ -Lipschitz.*

**Proof.** By proposition 4.3, we may assume that  $E$  is an  $C_1\epsilon$ -flat bundle with respect to  $\mathcal{S}(X)$ . Let  $\sigma$  be any simplex, and consider the subcomplex  $S \subset \mathcal{S}(X)$  of the barycentric subdivision which was described in lemma 5.1. Then, over  $|S|$ , we have a global  $C_2\epsilon$ -trivialization  $\Theta_\sigma$  of  $E$  by theorem 3.6. Thus, we may define  $U_\sigma$  to be the interior of  $|S|$ . The Lipschitzness is a consequence of lemma 4.2.

Up to now, there is no reason for the transition functions to be smooth. However, if we replace the sets  $U_\sigma$  by smaller open subsets, it is possible to smoothen the transition functions using the following lemma.

**Lemma 5.3.** *Let  $U_0, U_1, \dots, U_k \subset X$  be open subsets, and let  $\Phi_i: U_i \times V \rightarrow E|_{U_i}$  be trivializations such that the transition functions  $\Psi_{i,j}: U_i \cap U_j \rightarrow U(\mathcal{L}_A(V))$ ,  $x \mapsto \Phi_i(x, \cdot)^{-1} \Phi_j(x, \cdot)$  are all  $\lambda$ -Lipschitz. Let  $K_i \subset U_i$  be compact subsets which are completely contained in the image of some chart. Suppose further that all  $\Psi_{i,j}$  are smooth if  $i, j \geq 1$ .*

*Then there is a constant  $C > 0$ , depending on the sets  $U_i$  but not on  $E$  nor  $V$ , and open subsets  $V_0, \dots, V_k \subset X$  satisfying  $K_i \subset V_i \subset U_i$ , and a trivialization  $\tilde{\Phi}_0: V_0 \times V \rightarrow E|_{V_0}$  such that the transition functions  $\tilde{\Psi}_{0,j}: V_0 \cap V_j \rightarrow U(\mathcal{L}_A(V))$ ,  $x \mapsto \tilde{\Phi}_0(x, \cdot)^{-1} \Phi_j(x, \cdot)$  are all smooth and  $C\lambda$ -Lipschitz.*

Now we may complete the proof of proposition 5.2. Namely, since the simplices of  $X$  are smoothly embedded, they are all contained in a coordinate chart, so we may apply the lemma iteratively to get smooth transition functions which are still Lipschitz with controlled Lipschitz constant.  $\square$

**Proof of lemma 5.3.** Inductively, we may assume that  $\Psi_{0,i}$  is smooth if  $i < j$ . Restricting  $U_0$  to the image of a bi-Lipschitz chart around  $K_0$ , we may consider  $U_0$  to be a subset of  $\mathbb{R}^n$  with the induced metric. Let  $V_0$  be an open neighborhood of  $K_0$  in  $U_0$  such that  $\bar{V}_0 \subset U_0$ . For every natural number  $k \in \mathbb{N}$ , consider smooth functions  $\phi_k: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  having support

in the  $k^{-1}$ -ball around the origin, such that  $\int_{\mathbb{R}^k} \phi_k = 1$ . Then, if  $k$  is large enough, the map

$$\Psi'_{0,j}: V_0 \cap U_j \rightarrow \mathcal{L}_A(V), \quad x \mapsto \int_{\mathbb{R}^n} \Psi_{0,j}(y) \phi_k(y-x) dy$$

is a well-defined smooth map, and it is easily seen to have the same Lipschitz constant  $\lambda$  as  $\Psi_{0,j}$ . Furthermore,  $\|\Psi'_{0,j} - \Psi_{0,j}\|_{\text{sup}} \leq \frac{\lambda}{k}$ . We choose  $k$  so large that  $\frac{\lambda}{k} < \frac{1}{3}$ .

Now choose an open neighborhood  $V_j \subset U_j$  of  $K_j$  such that  $\bar{V}_j \subset U_j$ , and let  $\chi: V_0 \rightarrow [0, 1]$  be a smooth map satisfying  $\chi|_{V_0 \cap V_j} = 1$  and  $\chi = 0$  on a neighborhood of  $V_0 - U_j$ . Let

$$\Psi''_{0,j}: V_0 \cap V_j \rightarrow \mathcal{L}_A(V), \quad x \mapsto \chi(x) \Psi'_{0,j}(x) + (1 - \chi(x)) \Psi_{0,j}(x).$$

Then we have that

$$\begin{aligned} \Psi''_{0,j}(x) - \Psi''_{0,j}(y) &= \chi(x) \Psi'_{0,j}(x) + (1 - \chi(x)) \Psi_{0,j}(x) \\ &\quad - \chi(y) \Psi'_{0,j}(y) - (1 - \chi(y)) \Psi_{0,j}(y) \\ &= \chi(x) (\Psi'_{0,j}(x) - \Psi'_{0,j}(y)) + (\chi(x) - \chi(y)) \Psi'_{0,j}(y) \\ &\quad + (1 - \chi(x)) (\Psi_{0,j}(x) - \Psi_{0,j}(y)) - (\chi(x) - \chi(y)) \Psi_{0,j}(y). \end{aligned}$$

Now, using that  $0 \leq \chi \leq 1$ , the triangle inequality shows that

$$\begin{aligned} \|\Psi''_{0,j}(x) - \Psi''_{0,j}(y)\| &\leq \|\Psi'_{0,j}(x) - \Psi'_{0,j}(y)\| + \|\Psi_{0,j}(x) - \Psi_{0,j}(y)\| \\ &\quad + \|\chi(x) - \chi(y)\| \|\Psi'_{0,j}(y) - \Psi_{0,j}(y)\|, \end{aligned}$$

so we see that  $\Psi''_{0,j}$  is  $C_1 \lambda$ -Lipschitz for  $C_1 = 2 + L(\chi)k^{-1}$ . Finally, we set  $\tilde{\Psi}_{0,j}(x) = (\Psi''_{0,j}(x) \Psi''_{0,j}(x)^*)^{-1/2} \Psi''_{0,j}(x)$ . Because of lemma B.4,  $\tilde{\Psi}_{0,j}$  is  $C\lambda$ -Lipschitz for some constant which does not depend on  $E$  nor on  $V$ .

Now let  $\tilde{\Phi}_0(x, \cdot) := \tilde{\Psi}_{0,j}(x)^{-1} \Phi_j(x, \cdot)$  for every  $x \in V_0 \cap U_j$ , and  $\tilde{\Phi}_0(x, \cdot) := \Phi_0(x, \cdot)$  outside of  $U_j$ .  $\square$

Now the idea is to construct a connection with small curvature inductively on open neighborhoods of the skeleta of  $X$ . In order to extend this connection, we will need the following extension lemma for connections mainly due to Fukumoto [Fuk15].

**Lemma 5.4.** *Let  $B$  be a (not necessarily closed) Riemannian manifold without boundary. Write  $X = B \times [0, 5]$  equipped with the product metric. Then there is a constant  $C > 0$  with the following property:*

*Let  $E = X \times V$  be a trivial Hilbert module bundle over  $X$ , and let  $E$  be equipped with a connection  $\nabla$  which is compatible with the canonical Hermitian metric on  $E$ . We may write  $\nabla = \partial + \Gamma$ , where  $\partial$  denotes taking directional derivatives and where  $\Gamma$  is a section of  $T^*X \otimes \text{End}(E)$ . Assume that  $\|\Gamma_x\| \leq \epsilon \|x\|$  for every  $x \in TX$ , and that  $\|\mathcal{R}^\nabla\| \leq \epsilon$ , where  $\epsilon \leq 1$ .*

*Then there exists another compatible connection  $\tilde{\nabla}$  on  $E$  with the following properties:*

- $\|\tilde{\Gamma}_x\| \leq C\epsilon\|x\|$  for every  $x \in TX$  if  $\tilde{\nabla} = \partial + \tilde{\Gamma}$ .
- $\|\mathcal{R}^{\tilde{\nabla}}\| \leq C\epsilon$ ,
- $\tilde{\nabla} = \nabla$  over  $B \times [0, 1]$ , and
- $\tilde{\nabla} = \partial$  over  $B \times [4, 5]$ .

**Proof.** Let  $\chi_1: [0, 5] \rightarrow [0, 5]$  be a smooth map satisfying  $\chi_1|_{[0,1]} = \mathbb{1}$  and  $\chi_1|_{[2,5]} = 5$ . Consider the map  $\Phi = \text{id} \times \chi_1: X \rightarrow X$ . We define a new connection  $\nabla' = \partial + \Gamma'$  by  $\Gamma'_v = \Gamma_{\Phi_*v}$  for every  $v \in TX$ . In particular,  $\|\Gamma'_v\| = \|\Gamma_{\Phi_*v}\| \leq \epsilon\|\Phi_*v\| \leq c\epsilon\|v\|$  where  $c = \|(\text{id} \times \chi_1)_*\|_{\text{sup}} \leq \sqrt{1 + \|\dot{\chi}_1\|_{\text{sup}}^2}$ . Of course,  $\nabla' = \nabla$  over  $B \times [0, 1]$ , but now  $\Gamma'_{(x,\tau)} = \Gamma_{(x,0)}$  if  $x \in TB$ ,  $\tau \in T_t[2, 5]$ .

It is easy to calculate that

$$\mathcal{R}^{\nabla}(x, y) = (\partial_x \Gamma_y - \partial_y \Gamma_x) + [\Gamma_x, \Gamma_y] \quad (1)$$

if  $x$  and  $y$  are vector fields on  $X$  such that  $[x, y] = 0$ . In particular, by a calculation in local coordinates (or alternatively by the fact that  $\nabla'$  equals the pullback connection  $\Phi^*\nabla$ ), this easily implies that  $\mathcal{R}^{\nabla'} = \Phi^*\mathcal{R}^{\nabla}$ , and therefore  $\|\mathcal{R}^{\nabla'}\| \leq c'\epsilon$  for another constant  $c'$ . Thus, we may replace  $\nabla$  by  $\nabla'$ , and in particular assume that  $\Gamma_{(x,\tau)} = \Gamma_{(x,0)}$  over  $B \times [2, 5]$ .

Let  $\chi_2: [2, 5] \rightarrow [0, 1]$  be a smooth map which satisfies  $\chi_2|_{[2,3]} = 1$  and  $\chi_2|_{[4,5]} = 0$ . We let

$$\tilde{\Gamma}_{(x,\tau)} = \chi_2(t)\Gamma_{(x,0)}$$

if  $\tau \in T_t[2, 5]$ . Then  $\tilde{\Gamma} = 0$  on  $B \times [4, 5]$  (and therefore  $\tilde{\nabla} = \partial$  over  $B \times [4, 5]$  if we let  $\tilde{\nabla} = \partial + \tilde{\Gamma}$ ), and  $\tilde{\Gamma}_{(x,\tau)} = \Gamma_{(x,\tau)}$  if  $\tau \in T_t[2, 3]$ , so  $\tilde{\Gamma}$  may be extended to  $B \times [0, 5]$  by letting  $\tilde{\Gamma} = \Gamma$  on  $B \times [0, 2]$ . Obviously, still  $\|\tilde{\Gamma}_x\| \leq \epsilon\|x\|$ . It only remains to show that  $\|\mathcal{R}^{\tilde{\nabla}}\| \leq C\epsilon$ .

Now let  $q = (t, p)$ ,  $x = (v_1, \tau_1)$  and  $y = (v_2, \tau_2)$  with  $v_1, v_2 \in T_pB$  and  $\tau_1, \tau_2 \in T_t[0, 5]$ . Assume in addition that  $\|x\|, \|y\| \leq 1$ . Then (1) becomes

$$\mathcal{R}^{\tilde{\nabla}}(x, y) = (\partial_{\tau_1} \chi_2) \Gamma_y - (\partial_{\tau_2} \chi_2) \Gamma_x + \chi_2 \mathcal{R}^{\nabla}(x, y) + (\chi_2^2 - \chi_2) [\Gamma_x, \Gamma_y].$$

We have  $\|\Gamma_x\|, \|\Gamma_y\| \leq \epsilon$ ,  $\chi_2 \leq 1$ ,  $|\chi_2^2 - \chi_2| \leq \frac{1}{4}$ ,  $\|\mathcal{R}^{\nabla}(x, y)\| \leq \epsilon$  and  $\partial_{\tau_i} \chi_2 \leq \|\dot{\chi}_2\|_{\text{sup}}$ . Therefore,

$$\|\mathcal{R}^{\tilde{\nabla}}(x, y)\| \leq 2\|\dot{\chi}_2\|_{\text{sup}}\epsilon + \epsilon + \frac{1}{2}\epsilon^2 \leq C\epsilon$$

if  $C = 2\|\dot{\chi}_2\|_{\text{sup}} + \frac{3}{2}$ . Then  $\|\mathcal{R}^{\tilde{\nabla}}\| \leq C\epsilon$  because every vector  $\alpha \in \Lambda^2 TM$  with  $\|\alpha\| \leq 1$  can be written as  $\alpha = x \wedge y$  with  $\|x\|, \|y\| \leq 1$ .  $\square$

We have gathered all the technical details to prove the main theorem of this section.

**Theorem 5.5.** *Let  $X$  be a smoothly triangulated closed Riemannian manifold. Then there are constants  $C, \delta > 0$  such that for every  $\epsilon$ -flat bundle  $E \rightarrow X$  with  $\epsilon \leq \delta$  there exists a compatible connection  $\nabla$  on  $E$  satisfying  $\|\mathcal{R}^{\nabla}\| \leq C\epsilon$ .*

**Proof.** Let  $U_\sigma$ ,  $\Theta_\sigma$  and  $\Psi_{\sigma,\rho}$  be as in proposition 5.2. Assume that  $\nabla$  has already been constructed on a neighborhood  $N$  of the  $(k-1)$ -skeleton of  $X$ . We will also assume that if we write  $\nabla = \partial + \Gamma$  with respect to any of the trivializations  $\Theta_\sigma$ , then  $\|\Gamma_x\| \leq C_{k-1}\epsilon$  for all  $x \in TU_\sigma$ . We have to construct  $\nabla$  on a neighborhood of the  $k$ -skeleton.

Let  $X_k$  denote the union of all the interiors of  $k$ -simplices of  $X$ . This is a smooth submanifold, so it is the zero section in a tubular neighborhood  $X_k \times \mathbb{R}^{n-k} \subset X$ . Now let  $\sigma$  be a simplex of dimension  $k$ , and let  $T_\sigma := \mathring{\Delta}^\sigma \times \mathbb{R}^{n-k}$ . Note that for distinct  $\sigma, \rho$  of dimension  $k$ ,  $T_\sigma$  and  $T_\rho$  are disjoint. We may assume that  $T_\sigma \subset U_\sigma$ . Since  $N$  is a neighborhood of the  $(k-1)$ -skeleton, we can identify  $\mathring{\Delta}^\sigma$  with  $\mathbb{R}^k$  in such a way that  $\{x \in \mathbb{R}^k : \|x\| \geq 1\} \times \mathbb{R}^{n-k}$  is contained in  $N$ .

Now  $\{x \in \mathbb{R}^k : 1 \leq \|x\| \leq 2\} \times \{x \in \mathbb{R}^{n-k} : \|x\| \leq 1\}$  is canonically diffeomorphic to  $([1, 2] \times S^{k-1}) \times \{x \in \mathbb{R}^{n-k} : \|x\| \leq 1\}$  and we may assume that this space has the product metric with respect to the latter product decomposition since any two metrics are in bi-Lipschitz correspondence on a compact set. Now by lemma 5.4 we may assume that  $\nabla = \partial$  on a neighborhood of  $\{1\} \times S^{k-1} \times \{x \in \mathbb{R}^{n-k} : \|x\| < 1\}$  with respect to  $\Theta_\sigma$ , and therefore can be extended by  $\nabla = \partial$  on  $\{x \in \mathbb{R}^k : \|x\| \leq 1\} \times \{x \in \mathbb{R}^{n-k} : \|x\| < 1\}$ . By 5.4, the induction hypotheses are fulfilled for the new connection, and since the  $T_\sigma$  are all disjoint, this construction may be performed for all  $k$ -simplices  $\sigma$  simultaneously. If now  $\rho$  is another simplex such that  $U_\rho$  and  $U_\sigma$  intersect, we may write  $\nabla = \partial + \Gamma^\rho$  with respect to  $\Theta_\rho$ . Using the  $C\epsilon$ -Lipschitzness of  $\Psi_{\sigma,\rho}$  and the fact that  $\Gamma_X^\rho = \partial_X \Psi_{\sigma,\rho} + \Gamma_X \Psi_{\sigma,\rho}$  and that  $\Psi$  is unitary, it follows that  $\|\Gamma_X^\rho\| \leq C_k \epsilon \|X\|$ .  $\square$

## 6. Pullbacks of almost flat bundles

In this section, we investigate pullbacks of almost flat bundles. It is rather easy to see that pullbacks of almost flat bundles are still almost flat. However, if a map induces an isomorphism on fundamental groups, there is also a sort of converse for this statement which we will prove. The following statement asserts that almost flat bundles are pulled back to almost flat bundles.

**Proposition 6.1.** *Let  $f: X \rightarrow Y$  be a continuous map between simplicial complexes, and suppose that  $X$  is finite-dimensional. Then there are constants  $C, \delta > 0$  such that for all  $\epsilon$ -flat bundles  $E \rightarrow |Y|$  with  $\epsilon \leq \delta$  the bundle  $f^*E \rightarrow |X|$  admits trivializations making it an  $C\epsilon$ -flat bundle.*

**Proof.** By proposition 4.3, we may replace  $X$  by a repeated barycentric subdivision of itself, and therefore we may assume that  $f$  is simplicial. Now the first statement immediately follows by pulling back the trivializations over the simplices of  $Y$ , because a simplicial map is 1-Lipschitz on every simplex.  $\square$

In particular, the property of being an almost flat bundle does not depend very much on the choice of triangulation, as we see if we take  $f = \text{id}$  in proposition 6.1:

**Corollary 6.2.** *Suppose  $X$  and  $X'$  are finite-dimensional triangulations of the same space, i. e.  $|X| \cong |X'|$ . Then there are constants  $C, \delta > 0$  such that for every  $\epsilon \leq \delta$ , every  $\epsilon$ -flat bundle with respect to  $X$  admits the structure of a  $C\epsilon$ -flat bundle with respect to  $X'$ .  $\square$*

Conversely, if  $f$  induces an isomorphism of fundamental groups, and if  $X$  and  $Y$  are finite, then every K-theory class of  $|X|$  which is represented by  $\epsilon$ -flat bundles for all  $\epsilon > 0$  is the pullback of a K-theory class of  $|Y|$  with the same property. This clarifies a point left open in [MT05]. In fact, we are going to prove a slightly stronger statement.

For the formulation of this statement, we give a few auxiliary definitions. Recall from [BH10] that a  $\pi_1$ -surjective subcomplex of a simplicial complex  $Y$  is a subcomplex  $X \subset Y$  such that the inclusion induces an epimorphism  $\pi_1(X, x_0) \rightarrow \pi_1(Y, x_0)$  for every vertex  $x_0 \in X_0$ . Denote by  $\Omega_{X \subset Y}$  the set of all simplicial loops in  $X$  which are nullhomotopic as loops in  $Y$ . Consider a map  $c: \Omega_{X \subset Y} \rightarrow \mathbb{R}_{>0}$ . Now a  $(c, \epsilon)$ -flat bundle is an  $\epsilon$ -flat bundle  $E \rightarrow |X|$  such that  $\|T_\Gamma - \text{id}\| \leq c(\Gamma)\epsilon$  for every  $\Gamma \in \Omega_{X \subset Y}$ . We shall prove a special case first.

**Lemma 6.3.** *Let  $X \subset Y$  be a finite connected  $\pi_1$ -surjective subcomplex, let  $X' \subset Y$  be another subcomplex containing  $X$ , and let  $c: \Omega_{X \subset Y} \rightarrow \mathbb{R}_{>0}$  be a map as described above. Then there are constants  $C, \delta > 0$  and a map  $c': \Omega_{X' \subset Y} \rightarrow \mathbb{R}_{>0}$  such that every  $(c, \epsilon)$ -flat bundle  $E \rightarrow |X|$  is isomorphic to the restriction of a  $(c', C\epsilon)$ -flat bundle  $E' \rightarrow |X'|$  provided that  $\epsilon \leq \delta$ .*

**Proof.** Suppose first that  $X'$  arises from  $X$  by adding a single vertex  $p$  and an edge  $\sigma = \{p, q\}$  where  $q \in X_0$ . In this case, we may choose arbitrary trivialisations over  $p$  and over  $\sigma$ . If  $\Gamma \in \Omega_{X' \subset Y}$ , transport along  $\Gamma$  is the same thing as transport along the curve  $\Gamma'$  which arises from  $\Gamma$  by elimination of all occurrences of the piece  $(q, p, q)$ . Thus, we may set  $c'(\Gamma) = c(\Gamma')$ ,  $C = 1$  and  $\delta$  arbitrary.

This shows that we may assume that  $X'$  and  $X$  have the same set of vertices. By induction on the number of simplices in  $X' - X$ , we may further assume that  $X'$  arises from  $X$  by adding a single simplex  $\sigma$ .

If  $\sigma = \{p, q\}$  is a 1-simplex, consider a simplicial path  $\Gamma$  from  $p$  to  $q$  in  $X$ . Then  $\Gamma' = \Gamma * (q, p)$  is a simplicial loop in  $X'$ , based at  $p$ . Since  $X \subset Y$  is  $\pi_1$ -surjective, there is another simplicial loop  $\Gamma''$ , based at  $p$ , which is contained in  $X$  and which is homotopic to  $\Gamma'$  in  $Y$ . In particular, the path  $\bar{\Gamma}'' * \Gamma'$  is nullhomotopic in  $Y$ , where  $\bar{\Gamma}''$  is the path which traverses  $\Gamma''$  in the opposite direction. Thus, we may assume that already  $\Gamma'$  is nullhomotopic in  $Y$ . We choose an extension of the bundle  $E$ , and a trivialization over  $\sigma$  such that transport along  $\Gamma'$  equals the identity map. In particular, transport along  $\Gamma$  and along the path  $(p, q)$  are equal. Now suppose  $\tilde{\Gamma}$  is an arbitrary loop

in  $X'$  which is contractible in  $Y$ . Denote by  $\tilde{\Gamma}'$  the path which arises from  $\tilde{\Gamma}$  by substituting every occurrence of  $(p, q)$  by  $\Gamma$ , and every occurrence of  $(q, p)$  by the opposite  $\Gamma'$ . Since  $(p, q)$  and  $\Gamma$  are homotopic in  $Y$ , also  $\tilde{\Gamma}'$  is nullhomotopic in  $Y$ , and transport along  $\tilde{\Gamma}'$  and along  $\tilde{\Gamma}$  are equal. Thus, the statement of the lemma follows with  $c'(\tilde{\Gamma}) = c(\tilde{\Gamma}')$ ,  $C = 1$  and  $\delta$  arbitrary.

If the dimension of  $\sigma$  is 2, the statement follows using theorem 4.5. If the dimension of  $\sigma$  is at least 3, we may use theorem 4.4 to get the desired conclusion.  $\square$

We are now able to prove the statement in full generality.

**Theorem 6.4.** *Let  $X \subset Y$  and  $X' \subset Y'$  be finite connected  $\pi_1$ -surjective subcomplexes, and let  $c: \Omega_{X \subset Y} \rightarrow \mathbb{R}_{>0}$  be a map. Suppose  $f: Y \rightarrow Y'$  is a map which induces an isomorphism on fundamental groups, and where  $f(X) \subset X'$ . Then there are constants  $C, \delta > 0$  and a map  $c': \Omega_{X' \subset Y'} \rightarrow \mathbb{R}_{>0}$  such that every  $(c, \epsilon)$ -flat bundle  $E \rightarrow |X|$  is isomorphic to the pullback of a  $(c', C\epsilon)$ -flat bundle  $E' \rightarrow |X'|$  along  $f|_X$  provided that  $\epsilon \leq \delta$ .*

**Proof.** Passing to a subdivision of  $X$ , we may assume that  $f|_X$  is simplicial. Consider the mapping cylinder  $M = X \times [0, 1] \sqcup Y' / \sim$ , where  $\sim$  is the equivalence relation generated by the identification  $(x, 1) \sim f(x)$ . Now  $M$  contains  $Y'$  as a deformation retract, and the composition  $X \subset M \rightarrow Y'$  equals  $f|_X$ . This implies that  $X \times \{0\}$  is a  $\pi_1$ -surjective subcomplex of  $M$ , and that  $\Omega_{X \subset Y} = \Omega_{X \subset M}$ . On the other hand, also  $M' = X \times [0, 1] \sqcup X' / \sim$  is a finite connected  $\pi_1$ -surjective subcomplex of  $M$  which contains both  $X \times \{0\}$  and  $X'$ . Thus, the statement of the theorem follows from lemma 6.3, applied to  $X \times \{0\}$  and  $M'$  as subcomplexes of  $M$ .  $\square$

## 7. Homological invariance of infinite K-area

In this chapter, we will explain how to generalize the concept of infinite K-area of closed Riemannian manifolds to homology classes of simplicial complexes. The obvious definition is that a homology class  $\eta \in H_*(X; G)$  is to be of infinite K-area if for every  $\epsilon > 0$  there are  $\epsilon$ -flat bundles whose Chern classes detect  $\eta$ . However, in view of the finiteness assumption of theorem 6.4 it turns out that it is more useful to consider bundles which are defined only on finite  $\pi_1$ -surjective subcomplexes.

**Definition 7.1.** Let  $X$  be any simplicial complex with finitely generated fundamental group, and let  $\eta \in H_{2*}(X; G)$ . Consider a finite connected  $\pi_1$ -surjective subcomplex  $S \subset X$ . Suppose that there is a class  $\eta_S \in H_{2*}(S; G)$  such that  $\eta = \iota_* \eta_S$ . Now  $\eta$  is said to have *infinite K-area* if there is a function  $c: \Omega_{S \subset X} \rightarrow \mathbb{R}_{>0}$  such that for every  $\epsilon > 0$  there is an  $(c, \epsilon)$ -flat Hermitian bundle  $E \rightarrow S$  such that if  $f: S \rightarrow BU$  classifies the bundle  $E$ , then  $f_* \eta_S \neq 0 \in H_n(BU; G)$ .

Suppose  $S$  and  $S'$  are two different finite connected  $\pi_1$ -surjective subcomplexes and that there are classes  $\eta_S \in H_{2*}(S; G)$  and  $\eta_{S'} \in H_{2*}(S'; G)$  both



mapping to  $\eta$ . Then there is a larger finite connected  $\pi_1$ -surjective subcomplex  $T$  containing both  $S$  and  $S'$ , such that  $\eta_S$  and  $\eta_{S'}$  map to the same class in  $H_{2*}(T; G)$ . Thus, lemma 6.3 immediately implies that the definition of infinite K-area is independent of the choice of  $S$ .

We now have the following immediate consequences of proposition 6.1 and theorem 6.4.

**Theorem 7.2.** *Let  $f: X \rightarrow Y$  be a map between simplicial complexes, and consider  $\eta \in H_{2*}(X; G)$ .*

- *If  $f_*\eta$  has infinite K-area, then so has  $\eta$ ,*
- *If  $\eta$  has infinite K-area and  $f_*: \pi_1 X \rightarrow \pi_1 Y$  is an isomorphism, then also  $f_*\eta$  has infinite K-area.*  $\square$

**Theorem 7.3.** *An even-dimensional closed oriented manifold  $M^{2n}$  has infinite K-area if and only if its fundamental class  $[M] \in H_{2n}(M; \mathbb{Q})$  has infinite K-area.*

**Proof.** This follows directly from theorem 2.6 and theorem 5.5.  $\square$

**Corollary 7.4.** *A closed oriented Riemannian manifold  $M$  has infinite K-area if and only if  $\phi_*[M] \in H_{2n}(B\pi_1(M); \mathbb{Q})$  has infinite K-area, where  $\phi: M \rightarrow \pi_1(M)$  is the classifying map of the universal bundle.*  $\square$

**Definition 7.5.** A closed oriented manifold  $M$  is called *essential* if  $\phi_*[M] \neq 0$ , where  $\phi: M \rightarrow \pi_1 M$  classifies the universal bundle.

**Corollary 7.6.** *A closed oriented manifold of infinite K-area is essential.*  $\square$

Finally, we may reprove the theorem of Fukumoto on the invariance of infinite K-area under surgery. Let  $M^n$  be a differentiable manifold. If  $S^p \times D^q \subset M$  is an embedding, we consider the manifold

$$M^\# = (M - S^p \times \text{int}(D^q)) \cup_{S^p \times S^{q-1}} D^{p+1} \times S^{q-1}.$$

We say that  $M^\#$  is obtained from  $M$  by *p-surgery*.

**Theorem 7.7** ([Fuk15]). *Let  $M^{2n}$  be a closed oriented manifold with infinite K-area, and let  $M^\#$  be obtained from  $M$  by surgery of index  $p \neq 1$ . Then also  $M^\#$  has infinite K-area.*

**Proof.** Consider the trace

$$B = M \times I \cup_{S^p \times D^q \times \{1\}} D^{p+1} \times D^q.$$

This is a bordism between  $M$  and  $M^\#$ , so  $[M]$  and  $[M^\#]$  define the same class in  $B$ . Now let  $f: M \rightarrow B\pi_1(M)$  be the classifying map of the universal cover of  $M$ . By theorem 7.2,  $f_*[M]$  has infinite K-area. However, since  $p \neq 1$ , we have that  $f|_{S^p \times D^q}$  is null-homotopic because  $\pi_p(B\pi_1(M)) = 0$ . Thus,  $f$  can be extended to  $B$ , and  $f_*[M^\#] = f_*[M]$  has infinite K-area. Thus, by theorem 7.2, also  $[M^\#]$  has infinite K-area.  $\square$

In [Lis13] Listing gave the following definition of infinite K-area for homology classes of manifolds  $M$ . A class  $\eta \in H_{2*}(M; G)$  has infinite K-area if for every  $\epsilon > 0$  there exists a smooth Hermitian vector bundle  $E \rightarrow M$  with compatible connection  $\nabla$  such that  $\|\mathcal{R}^\nabla\| \leq \epsilon$  and  $f_*\eta \neq 0$  if  $f: M \rightarrow BU$  classifies the bundle  $E$ . It is clear that our definition generalizes the definition of Listing. In the case  $G = \mathbb{Q}$  the condition  $f_*\eta \neq 0$  simply means that some polynomial in the Chern classes of  $E$  detects  $\eta$ .

One could obviously change this definition by demanding that a particular polynomial in the Chern classes, for instance the Chern character, should detect  $\eta$ , which corresponds to  $f_*\eta$  lying in a particular vector subspace of  $H_{2*}(BU; \mathbb{Q})$ . All statements in this section hold equally well for this kind of definition.

On the other hand, one could also consider K-homology classes  $\eta \in K_0(M)$ , as was done by Hanke [Han12]. Here the condition on the bundles would simply be that their class pairs non-trivially with  $\eta$ . Furthermore, in this case one could consider arbitrary Hilbert  $A$ -module bundles  $E \rightarrow M$ . In this case, their index  $\langle [E], \eta \rangle$  would be an element of  $K_0(A)$ , and one could still demand it to be nonzero. All theorems in this section hold equally well for this definition of infinite K-area.

Finally, one could consider classes of *finite K-area* as in [Lis13]. Here, the K-area of a class  $\eta$  would be the largest number  $a \in [0, \infty]$  such that there is a function  $c$  as above with the property that for every  $\epsilon > a^{-1}$  there is a  $(c, \epsilon)$ -flat bundle detecting  $\eta$ . Of course, proposition 6.1 and theorem 6.4 imply that there are appropriate generalizations of theorem 7.2. However, since this notion of K-area strongly depends on the choice of triangulation, it is not clear how this might be of any use.

## 8. Almost representations and quasi-representations

Let  $X$  be a simplicial complex with finitely presented fundamental group. In this section, we will exhibit the relation between so-called almost representations of  $\pi_1 X$  and almost flat bundles over  $X$ . Specifically, we will show that an  $\epsilon$ -almost representation of  $\pi_1 X$  gives a  $C\epsilon$ -flat bundle over  $X$  and vice versa. While similar statements have already been shown in [CD18], and this relation has already been suggested in [CGM90], it will follow easily from the ideas developed in this paper.

**8.1. Almost representations of finitely presented groups.** A group  $\Pi$  is *generated* by a set  $L \subset \Pi$  if every element of  $\Pi$  can be written as a product of elements of  $L$  and their inverses. Here we view the identity element of  $\Pi$  as the empty product. Given such a *generating set*  $L \subset \Pi$ , we may form the free group  $\text{Fr}(L)$  generated by the elements of  $L$ . Then there is a natural surjective group homomorphism  $\pi: \text{Fr}(L) \rightarrow \Pi$  induced by the inclusion map  $L \subset \Pi$ .

Now a set of *relations* is a subset  $R \subset \text{Fr}(L)$  such that the kernel of  $\pi$  is the smallest normal subgroup of  $\text{Fr}(L)$  which contains  $R$ . Thus, elements of  $R$  are words in  $L \cup L^{-1}$ . A *presentation* of a group  $\Pi$  is a choice of such sets  $L$  and  $R$ . In this situation, we write  $\Pi = \langle L \mid R \rangle$ . This means that every element of  $\Pi$  may be written as a product of elements of  $L \cup L^{-1}$ . Such a presentation is called *finite* if both  $L$  and  $R$  are finite.

**Example 8.1.** Consider a simplicial complex  $X$ , and let  $T \subset X$  be a maximal tree in  $X$ . Then there is the following presentation of  $\pi_1(X)$ :

For every edge  $\sigma = \{v_0, v_1\} \in X_1 - T$ , we choose a loop  $\Gamma_\sigma = \Gamma_1 * (v_0, v_1) * \Gamma_2$ , where  $\Gamma_1$  and  $\Gamma_2$  are completely contained in  $T$ . Now the set of generators  $L$  consists of the homotopy classes of the  $\Gamma_\sigma$  for all  $\sigma \in X_1 - T$ . The set of relations is indexed by the two simplices  $\rho \in X_2$ , and implements the fact that a curve along the boundary of  $\rho$  is null-homotopic. For instance, if  $\rho = \{v_0, v_1, v_2\}$  and neither of the edges of  $\rho$  is contained in  $T$ , then the relation associated to  $\rho$  is  $[\Gamma_{\{v_0, v_1\}}][\Gamma_{\{v_1, v_2\}}][\Gamma_{\{v_2, v_0\}}]$ . Note that this is a finite presentation if  $X$  is finite.

**Definition 8.2.** A (unitary)  $\epsilon$ -almost representation [MM01] of  $\Pi$  on the Hilbert  $A$ -module  $V$  with respect to the presentation  $\Pi = \langle L \mid R \rangle$  is a group homomorphism  $\phi: \text{Fr}(L) \rightarrow U(\mathcal{L}_A(V))$  with the property that  $\|\phi(r) - \text{id}\| < \epsilon$  for every  $r \in R$ . We denote the set of such  $\epsilon$ -almost representations by  $R_\epsilon(L \mid R)$ .

Two almost representations  $\phi, \psi: \text{Fr}(L) \rightarrow U(\mathcal{L}_A(V))$  are  $\delta$ -close if

$$\|\phi(g) - \psi(g)\| \leq \delta$$

for all  $g \in L \subset \text{Fr}(L)$ .

The following proposition lists a few elementary properties of almost representations.

**Proposition 8.3.** *Let  $\Pi = \langle L \mid R \rangle$  and  $\Pi' = \langle L' \mid R' \rangle$  be two finite presentations of groups, and let  $f: \Pi' \rightarrow \Pi$  be a group homomorphism. Denote the canonical projections by  $\pi: \text{Fr}(L) \rightarrow \Pi$  and  $\pi': \text{Fr}(L') \rightarrow \Pi'$ .*

- a) *Let  $s: \text{Fr}(L') \rightarrow \text{Fr}(L)$  be a homeomorphism satisfying  $\pi \circ s = f \circ \pi'$ . Then there is a constant  $C_1 > 0$ , depending on the presentations and the choice of section, such that  $\phi \circ s \in R_{C_1\epsilon}(L' \mid R')$  whenever  $\phi \in R_\epsilon(L \mid R)$ .*
- b) *In the same situation, there is a constant  $C_2 > 0$ , such that  $\phi \circ s$  and  $\psi \circ s$  are  $C_2\delta$ -close whenever  $\phi, \psi \in R_\epsilon(L \mid R)$  are  $\delta$ -close.*
- c) *If  $s_1, s_2: L' \rightarrow \text{Fr}(L)$  satisfy  $\pi \circ s_1 = f \circ \pi' = \pi \circ s_2$ , then there is a constant  $C_3 > 0$ , such that the almost representations  $\phi \circ s_1$  and  $\phi \circ s_2$  are  $C_3\epsilon$ -close whenever  $\phi \in R_\epsilon(L \mid R)$ .*
- d) *Suppose that  $f$  is an isomorphism. If  $s: L' \rightarrow \text{Fr}(L)$  and  $s': L' \rightarrow \text{Fr}(L')$  are such that  $\pi \circ s = f \circ \pi'$  and  $\pi' \circ s = f \circ \pi$ , then there is a constant  $C_4 > 0$ , such that  $\phi$  is  $C_4\epsilon$ -close to  $\phi \circ s \circ s'$  whenever  $\phi \in R_\epsilon(L \mid R)$ .*

- Proof.** a) For the first statement, note that  $\pi \circ s = f \circ \pi'$  implies that  $s(R') \subset \ker \pi$ . Since  $R'$  is finite, there is a number  $N \in \mathbb{N}$  such that every element of  $s(R')$  can be written as a product of at most  $N$  conjugates of elements of  $R \cup R^{-1}$ . Thus, if  $r \in R'$ , there are elements  $r_1, \dots, r_k \in R \cup R^{-1}$  and  $w_1, \dots, w_k \in \text{Fr}(L)$  such that  $s(r) = (w_1^{-1}r_1w_1) \cdots (w_k^{-1}r_kw_k)$ , and therefore  $\|\phi \circ s(r) - \text{id}\| = \|(\phi(w_1)^{-1}\phi(r_1)\phi(w_1)) \cdots (\phi(w_k)^{-1}\phi(r_k)\phi(w_k)) - \text{id}\| \leq C_1\epsilon$  by lemma 3.2, where  $C_1 = C_1(N)$  depends only on the maximum number of factors needed.
- b) Consider  $g \in L'$ , and let  $s(g) = g_1 \cdots g_n$  where each  $g_i \in L \cup L^{-1}$ . Write  $\phi_i = \phi(g_i)$ ,  $\psi_i = \psi(g_i)$ . Then, by assumption,  $\|\phi_i - \psi_i\| \leq \delta$ ,  $\phi \circ s(g) = \phi_1 \cdots \phi_n$ , and  $\psi \circ s(g) = \psi_1 \cdots \psi_n$ . We have to show that  $\|\phi_1 \cdots \phi_n - \psi_1 \cdots \psi_n\| \leq C\delta$ . By induction, we only have to consider the case where  $n = 2$ , where the claim follows from lemma 3.2 because  $\|\phi_1\phi_2 - \psi_1\psi_2\| = \|(\psi_1^{-1}\phi_1)(\phi_2\psi_2^{-1}) - \text{id}\|$ .
- c) Note that  $\phi(s_1(g))^{-1}\phi(s_2(g)) \in \ker \pi$  for every  $g \in L'$ , and proceed as above to show that  $\|\phi(s_1(g)) - \phi(s_2(g))\| = \|\phi(s_1(g))^{-1}\phi(s_2(g)) - \text{id}\| \leq C_3\epsilon$ .
- d) Since  $(s \circ s'(g))^{-1}g \in \ker \pi$ , the same argument as above shows that  $\|\phi(s \circ s'(g)) - \phi(g)\| = \|\phi((s \circ s'(g))^{-1}g) - \text{id}\| \leq C_4\epsilon$  for every  $g \in L$ .  $\square$

**Definition 8.4** ([MM01]). Let  $\Pi = \langle L \mid R \rangle$  be a finitely presented group, and  $A$  a  $C^*$ -algebra. An  $A$ -asymptotic representation of  $\Pi$  with respect to this presentation is a series  $\phi = (\phi_n: \text{Fr}(L) \rightarrow U(\mathcal{L}_A(V_n)))_{n \in \mathbb{N}}$ , such that:

- every  $V_n$  is a projective sub-module of some  $A^k$ ,
- for every  $\epsilon > 0$ , there is a number  $N \in \mathbb{N}$ , such that  $\phi_n$  is an  $\epsilon$ -almost representation whenever  $n \geq N$ ,
- for every  $\delta > 0$ , there is a number  $N \in \mathbb{N}$ , such that  $\phi_n$  and  $\phi_m$  are  $\delta$ -close whenever  $n, m \geq N$ , where  $\phi_n$  and  $\phi_m$  are considered to have values in some large enough  $U(\mathcal{L}_A(A^k))$ .

Two asymptotic representations  $\phi = (\phi_n)$  and  $\psi = (\psi_n)$  are *equivalent* if for every  $\delta > 0$ , there is a number  $N \in \mathbb{N}$ , such that  $\phi_n$  and  $\psi_m$  are  $\delta$ -close whenever  $n, m \geq N$ .

We denote by  $R_{\text{as}}(L \mid R; A)$  the set of equivalence classes of  $A$ -asymptotic representations.

Now proposition 8.3 immediately implies the following:

- Proposition 8.5.** a) Suppose  $\Pi$  has two finite presentations  $\Pi = \langle L \mid R \rangle$  and  $\Pi = \langle L' \mid R' \rangle$ . Then the sets  $R_{\text{as}}(L \mid R; A)$  and  $R_{\text{as}}(L' \mid R'; A)$  are in canonical 1-to-1-correspondence. We will simply write  $R_{\text{as}}(\Pi; A)$  for any choice of finite presentation.
- b) Every group homomorphism  $f: \Pi \rightarrow \Pi'$  induces a map  $R_{\text{as}}(\Pi'; A) \rightarrow R_{\text{as}}(\Pi; A)$ .  $\square$

**8.2. Almost representations and almost flat bundles.** Let  $X$  be a simplicial complex with finitely presented fundamental group

$$\Pi = \pi_1(|X|, x_0) = \langle L \mid R \rangle.$$

Choose representing simplicial loops  $\Gamma_g$  for every element  $g = [\Gamma_g] \in L$ . Then every  $r \in R$  is a word in  $L \cup L^{-1}$ , so these choices associate to  $r$  a contractible simplicial loop  $P_r$ .

**Proposition 8.6.** *Suppose that  $E \rightarrow |X|$  is an  $\epsilon$ -flat bundle. Then transport along the curves  $\Gamma_g$  gives a  $C\epsilon$ -almost representation of  $\pi_1(|X|, x_0)$ , where  $C$  is a constant depending only on  $X$ , the presentation, and the choices of the  $\Gamma_g$ 's.*

**Proof.** Apply theorem 3.4 to the curves  $P_r$ . Since there are only finitely many of them, the constant from the theorem may be chosen for all  $P_r$  simultaneously.  $\square$

We are going to prove that the reverse also holds true, i. e. an  $\epsilon$ -almost representation of  $\pi_1(|X|, x_0)$  with respect to the given presentation induces a  $C\epsilon$ -small bundle  $E \rightarrow |X|$  (for another constant  $C$ ) such that transport along the curves  $\Gamma_g$  induces an  $\epsilon$ -almost representation which is close to the one we started with.

**Theorem 8.7.** *Let  $X$  be a finite simplicial complex, and choose a finite presentation  $\pi_1(|X|, x_0) = \langle L \mid R \rangle$  of the fundamental group of  $X$ . In addition, choose representing curves  $\Gamma_g$  for the generators  $g = [\Gamma_g] \in L$ . Then there are constants  $\delta, C > 0$ , depending on  $X$ , the presentation of the fundamental group, and the choices of the representing curves, such that the following holds:*

*Suppose  $\phi: \text{Fr}(L) \rightarrow U(\mathcal{L}_A(V))$  is an  $\epsilon$ -almost representation of the fundamental group  $\pi_1(|X|, x_0)$  where  $\epsilon \leq \delta$ . Then there exists a  $C\epsilon$ -flat bundle  $E \rightarrow |X|$  with the property that transport along the curves  $\Gamma_g$  gives an almost representation which is  $C\epsilon$ -close to  $\phi$ .*

**Proof.** We will first restrict to the case where the presentation  $\pi_1(|X|, x_0) = \langle L_0 \mid R_0 \rangle$  is the one described in example 8.1, and we assume that the representing curves are precisely the loops  $\Gamma_e$  described there. Now we take  $E|_{X^{(1)}}$  to be the trivial bundle  $X^{(1)} \times V \rightarrow X^{(1)}$ , and let the trivializations  $\Phi_\rho: |\rho| \times V \rightarrow E|_{|\rho|}$  be the identities  $(x, v) \mapsto (x, v)$  if  $\rho \in T$  is contained in the maximal tree. Now we may trivialize  $E|_{|e|}$  over every edge  $e \in X_1 - T$  such that transport along  $e$  equals  $\phi([\Gamma_e])$ . We may extend this bundle with trivializations to a  $C\epsilon$ -flat bundle  $E \rightarrow |X|$  using theorems 4.4 and 4.5. In turn, this extended bundle obviously induces the almost representation  $\phi$ .

Next we want to reduce the general case to the one described above. Thus, consider an arbitrary finite presentation  $\pi_1(|X|, x_0) = \langle L \mid R \rangle$ , and simplicial loops  $\Gamma_g$  associated to the elements  $g \in L$ . We may now choose a homomorphism  $s_0: \text{Fr}(L_0) \rightarrow \text{Fr}(L)$  such that  $\pi s_0 = \pi_0$ . This defines a

$C_0\epsilon$ -almost representation  $\phi \circ s_0: \text{Fr}(L_0) \rightarrow U(\mathcal{L}_A(V))$  by proposition 8.3. Now we may construct the bundle  $E \rightarrow |X|$  as above, for the representation  $\phi \circ \tilde{s}_0$ .

We have to show that transport along the curves  $\Gamma_g$  gives an almost representation which is close to  $\phi$ . Every  $\Gamma_g$  is of the form

$$\Gamma_g = \gamma_g^0 * e_1 * \gamma_g^1 * e_2 * \dots * e_{k_g} * \gamma_g^{k_g}$$

where each  $e_i$  is an edge in  $X_1 - T$ , and each  $\gamma_g^i$  is completely contained in  $T$ . This defines a homomorphism  $s: \text{Fr}(L) \rightarrow \text{Fr}(L_0)$  via  $g \mapsto [\Gamma_{e_{k_g}}] \cdots [\Gamma_{e_1}]$ . Here again, the  $\Gamma_{e_i}$  are the loops associated to the edges  $e_i$  in example 8.1. Now transport along  $\Gamma_g$  equals transport along the compositions of the curves  $\Gamma_{e_{k_g}}$ , i. e., it equals  $\phi \circ s_0 \circ s(g)$ . Thus, transport along the curves  $\Gamma_g$  gives the almost representation  $\phi \circ s_0 \circ s$ , which is  $C\epsilon$ -close to  $\phi$  by proposition 8.3.  $\square$

It also turns out that the isomorphism class of the bundle is uniquely determined by the almost representation induced by the transport, as the following theorem shows.

**Theorem 8.8.** *Let  $X$  be a finite simplicial complex, and let  $\pi_1(|X|, x_0) = \langle L \mid R \rangle$  be a finite presentation of the fundamental group of  $X$ . Suppose that we have two choices  $\Gamma_g, \tilde{\Gamma}_g$  for the generators in  $L$ . Then there is a constant  $\delta > 0$ , depending on  $X$ , the presentation, and the choices  $\Gamma_g$  and  $\Gamma'_g$  such that the following holds:*

*If  $E \rightarrow |X|$  and  $E' \rightarrow |X|$  are  $\delta$ -flat bundles such that transport in  $E$  along the curves  $\Gamma_g$  and transport in  $E'$  along the curves  $\Gamma'_g$  give almost representations which are  $\delta$ -close, then  $E$  and  $E'$  are isomorphic bundles.*

**Proof.** As in example 8.1, we choose a maximal tree  $T \subset X$ , and 0-trivializations of  $E$  and  $E'$  over  $T$ . Thus, parallel transport along edges in  $T$  becomes trivial with respect to these choices of trivializations.

Again, we consider the standard presentation  $\pi_1(|X|, x_0) = \langle L_0 \mid R_0 \rangle$  from example 8.1. As in the proof of theorem 8.7, parallel transport along the edges in  $X_1 - T$  gives almost representations  $\phi_0, \phi'_0: \text{Fr}(L_0) \rightarrow U(\mathcal{L}_A(V))$ . The choices of  $\Gamma_g$  and  $\Gamma'_g$  induce group homomorphisms  $s, s': \text{Fr}(L) \rightarrow \text{Fr}(L_0)$ , such that  $\phi_0 \circ s$  and  $\phi'_0 \circ s'$  are the almost representations given by parallel transporting along the curves  $\Gamma_g$  and  $\Gamma'_g$ , respectively.

Choose a homomorphism  $s: \text{Fr}(L_0) \rightarrow \text{Fr}(L)$  satisfying  $\pi s = \pi_0$ . By assumption,  $\phi \circ s$  and  $\phi \circ s'$  are  $\delta$ -close, so that  $\phi \circ \tilde{s} \circ \tilde{s}_0$  and  $\phi \circ \tilde{s}' \circ \tilde{s}_0$  are  $C_0\delta$ -close by proposition 8.3. On the other hand, these almost representations are  $C_1\delta$ -close to  $\phi_0$  and  $\phi'_0$ , respectively, again by proposition 8.3. This implies that  $\phi_0$  and  $\phi'_0$  are  $C_2\delta$ -close. However, this  $C_2\delta$ -closeness is precisely the condition for theorem 4.6 to work, so the bundles are isomorphic if  $\delta$  is small enough.  $\square$

**8.3. Asymptotically flat K-theory.** A class  $\eta \in K^0(X; A)$  can be represented as the difference  $\eta = [E_1] - [E_2]$  of two Hilbert  $A$ -module bundles  $E_i \rightarrow |X|$ . We denote by  $K_\epsilon^0(X; A) \subset K^0(X; A)$  the set of those classes such that  $E_1$  and  $E_2$  may be chosen to be  $\epsilon$ -flat. In addition, we define the subset of *asymptotically flat K-theory classes* by  $K_{\text{af}}^0(X; A) = \bigcap_{\epsilon > 0} K_\epsilon^0(X; A)$ . That is, a class  $\eta \in K^0(X; A)$  is asymptotically flat if for every  $\epsilon > 0$ , there exist  $\epsilon$ -flat Hilbert  $A$ -module bundles  $E_1, E_2 \rightarrow |X|$  such that  $\eta = [E_1] - [E_2]$ .

Note that there is an obvious notion of direct sum for asymptotic representations, which makes  $R_{\text{as}}(\pi_1 X; A)$  into a semi-group. Now theorems 8.7 and 8.8 show that there is a well-defined semi-group homomorphism  $R_{\text{as}}(\pi_1 X; A) \rightarrow K_{\text{af}}^0(X; A)$  which induces a group homomorphism

$$\alpha: \text{Gr}(R_{\text{as}}(\pi_1 X; A)) \rightarrow K_{\text{af}}^0(X; A).$$

Furthermore, one can show that  $\alpha$  is compatible with the pullback maps of asymptotic representations and asymptotically flat K-theory, so it gives a natural transformation. It is not quite clear if  $\alpha$  is surjective. Namely, it might be that an almost flat K-theory class is represented by almost representations which are not close. In any case,  $\alpha$  is certainly not an isomorphism ( $R_{\text{as}}(\pi_1 X; A)$  is not even abelian), so it would be interesting to examine the kernel and cokernel of  $\alpha$ .

## Appendix A. Parallel transport and curvature

In this section, we will prove proposition 2.7 which states that parallel transport along curves which bound a small area is small. This proof follows ideas from [Ran09] and an unpublished proof by Jost-Hinrich Eschenburg, who in turn learned the idea from Hermann Karcher.

In the course of the proof we will need the following lemma:

**Lemma A.1.** *Let  $E \rightarrow [0, 1]$  be a smooth Hilbert  $A$ -module bundle with (not necessarily compatible) connection  $\nabla$ . We denote parallel transport along  $\gamma$  by  $T_\gamma(t): E_t \rightarrow E_1$  and consider a section  $s: [0, 1] \rightarrow E$ . Then*

$$\partial_t(T_\gamma(t)s(t)) = T_\gamma(t)\nabla_{\partial_t}s(t)$$

for all  $t \in [0, 1]$ .

**Proof.** If  $E$  is modeled on a free Hilbert  $A$ -module, the statement is easily shown by writing both sides in a parallel frame. In the general case, one has to consider another bundle  $E'$  such that  $E \oplus E'$  is modeled on a free Hilbert  $A$ -module. It is easily possible to extend the connection on  $E$  to a connection on  $E \oplus E'$ , so the general case follows from the free case.  $\square$

**Proof of proposition 2.7.** Take  $x \in E_{f(0,0)}$  with  $\|x\| = 1$ , and let  $x' = P_{\partial f}x$ . For  $s \in [0, 1]$ , let  $X(s, 0) \in E_{f(s,0)}$  be the parallel translate of  $x$  along the curve  $s \mapsto f(s, 0)$ , and for  $(s, t) \in [0, 1] \times [0, 1]$ , let  $X(s, t) \in E_{f(s,t)}$  be the parallel translate of  $X(s, 0)$  along the curve  $t \mapsto f(s, t)$ .

Furthermore, let  $P_{(s,t)}: E_{f(s,t)} \rightarrow E_{f(1,1)}$  be defined by first parallel translating along  $t \mapsto f(s,t)$  and then along  $s \mapsto f(s,1)$ . Now, by definition,  $P_{(0,0)}x' = X(1,1)$ , and  $P_{(0,0)}x = P_{(0,1)}X(0,1)$ , so that

$$P_{(0,0)}(x' - x) = P_{(1,1)}X(1,1) - P_{(0,1)}X(0,1) = \int_0^1 \partial_s (P_{(s,1)}X(s,1)) ds.$$

Since  $P_{(s,1)}$  is parallel transport along the curve  $s \mapsto (s,1)$ , lemma A.1 implies that  $\partial_s(P_{(s,1)}X(s,1)) = P_{(s,1)}\nabla_{\partial_s f(s,1)}X(s,1)$ . Now  $X(s,0)$  is parallel along  $s \mapsto f(s,0)$  by definition, so that  $\nabla_{\partial_s f}X(s,0) = 0$ . Again with lemma A.1, it follows that

$$\begin{aligned} P_{(s,1)}\nabla_{\partial_s f}X(s,1) &= \int_0^1 \partial_t(P_{(s,t)}\nabla_{\partial_s f}X(s,t)) dt \\ &= \int_0^1 P_{(s,t)}\nabla_{\partial_t f}\nabla_{\partial_s f}X(s,t) dt \end{aligned}$$

for all  $s \in [0,1]$ . In addition, we have that  $\nabla_{\partial_t f}X(s,t) = 0$  since  $X$  is parallel in the  $t$ -direction by definition. Therefore,  $\mathcal{R}^\nabla(\partial_t f \wedge \partial_s f)X = \nabla_{\partial_t f}\nabla_{\partial_s f}X$ .

Since  $\nabla$  is compatible with the metric, one can easily show that parallel transport preserves the metric and in particular the norm. Thus, the equations combine to give

$$\|x' - x\| \leq \int_0^1 \int_0^1 \|\mathcal{R}^\nabla(\partial_t f \wedge \partial_s f)\| dt ds. \quad \square$$

## Appendix B. Unitary elements of C\*-algebras

In this section, we will give a proof of lemma 3.5, which states that  $\epsilon$ -Lipschitz maps from the sphere  $S^{n-1}$  into the unitary elements  $U(A)$  of an arbitrary C\*-algebra  $A$  may be extended to  $C\epsilon$ -Lipschitz maps on the whole disk  $D^n$  whenever  $\epsilon$  is small enough. Here  $C$  is some universal constant which depends neither on the C\*-algebra  $A$ , nor on the dimension of the sphere  $S^{n-1}$ . The result will be important even for the classical case of Hermitian vector bundles since it shows that maps into the set of unitary matrices can be extended as above with a constant  $C$  independent of the size of the matrices.

We begin with a statement which allows the extension of Lipschitz maps  $S^{n-1} \rightarrow V$  to Lipschitz maps on  $D^n$  if  $V$  is a normed vector space.

**Lemma B.1.** *There is a universal constant  $C_0 > 0$  with the following property: Let  $\beta_0: S^{n-1} \rightarrow V$  be a  $\lambda$ -Lipschitz map into a normed vector space  $V$ . Assume additionally that  $\beta_0(S^{n-1}) \subset B_R(0)$  for a number  $R > 0$  and  $\beta_0(s_0) = 0$  for some  $s_0 \in S^{n-1}$ . Then there is an extension  $\beta: D^n \rightarrow B_R(0) \subset V$  which is Lipschitz with constant at most  $C_0\lambda$ .*

**Proof.** A first idea would be to define the extension by  $\beta(t \cdot x) = t\beta_0(x)$  for  $x \in \partial D^n$ ,  $t \in [0,1]$ . This certainly gives a continuous extension, but it turns out that the problem of calculating the Lipschitz constant for the resulting



map is not as easy as it looks. However, one can simply do the contraction on a ring and extend constantly by zero on the interior, i. e.

$$\beta(t \cdot x) = \begin{cases} (2t - 1)\beta_0(x), & t \geq \frac{1}{2}, \\ 0, & t \leq \frac{1}{2}. \end{cases}$$

Then, using that  $\{x \in \mathbb{R}^n : \frac{1}{2} \leq \|x\| \leq 1\}$  and  $S^{n-1} \times [0, 1]$  are bi-Lipschitz equivalent (and that, by an explicit calculation, the Lipschitz constants do not depend on  $n$ ), one can easily deduce the statement.  $\square$

We will need the statement that every function given by holomorphic functional calculus is Fréchet differentiable.

**Lemma B.2.** *Let  $f: U_\epsilon \rightarrow \mathbb{C}$  be a holomorphic map where  $U_\epsilon = \{z \in \mathbb{C} : \|z\| < \epsilon\}$  is a ball of radius  $\epsilon > 0$  around  $0 \in \mathbb{C}$ . Let  $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$  be the power series expansion of  $f$  around  $0$ . Furthermore, consider the power series  $\hat{f}(z) = \sum_{n=0}^{\infty} |\lambda_n| z^n$ . If  $A$  is a Banach algebra, then the map*

$$\tilde{f}: \{x \in A : \|x\| < \epsilon\} \rightarrow A, \quad x \mapsto \sum_{n=0}^{\infty} \lambda_n x^n$$

*is well-defined and Fréchet differentiable, and the operator norm of the Fréchet differential of  $\tilde{f}$  at  $x \in A$  is bounded by  $\hat{f}'(\|x\|)$ . For every  $x$  in the domain of  $\tilde{f}$ , we have that  $x\tilde{f}(x) = \tilde{f}(x)x$ . This is a special case of the so-called functional calculus.*

**Proof.** For every point  $x$  in the domain of  $\tilde{f}$ , it is easy to see that the series

$$\Delta_x: A \rightarrow A, \quad h \mapsto \sum_{n=1}^{\infty} \lambda_n \sum_{k=0}^{n-1} x^k h x^{n-k-1}$$

converges and gives a linear map with operator norm bounded by  $\hat{f}'(\|x\|)$ . Now a straightforward calculation shows that this is the Fréchet differential of  $\tilde{f}$  at  $x$ .  $\square$

Now let  $A$  be a  $C^*$ -algebra. We write  $U(A) = \{u \in A : uu^* = u^*u = 1\}$ ,  $\mathfrak{u}(A) = \{v \in A : v^* = -v\}$  and  $H_A = \{v \in A : v^* = v\}$ . Elements of  $U(A)$  are called *unitary*, elements of  $\mathfrak{u}(A)$  are called *skew-Hermitian*, and elements of  $H_A$  are called *Hermitian*. Obviously  $A = \mathfrak{u}(A) \oplus H_A$  as a vector space. We denote by  $\pi: A \rightarrow \mathfrak{u}(A)$  the projection onto the first summand with respect to this decomposition

If  $v \in \mathfrak{u}(A)$ , we have that  $v^2 \in H_A$ . Now we consider the map  $f: \{z \in \mathbb{C} : \|z\| < \frac{1}{2}\} \rightarrow \mathbb{C}$  which is given by  $f(z) = (1 + z^2)^{1/2}$ . Since  $f(z) = f(-z)$ ,  $f$  is really a power series in  $z^2$ . so if we define  $\tilde{f}$  as in lemma B.2, we see that  $\tilde{f}$  maps elements of  $\mathfrak{u}(A)$  into  $H_A$ .

**Lemma B.3.** *There is a constant  $L > 0$ , independent of  $A$ , such that the so-defined map*

$$\tilde{f}: \left\{ v \in \mathfrak{u}(A) : \|v\| < \frac{1}{2} \right\} \rightarrow H_A, \quad v \mapsto (1 + v^2)^{1/2}$$

*is Lipschitz with constant at most  $L$ .*

**Proof.** By lemma B.2, the map  $f$  is Fréchet differentiable, and the operator norm of the differential is bounded by a differentiable map  $\hat{f}': (-1, 1) \rightarrow \mathbb{R}$  which is independent of  $A$ . This map is bounded by some constant  $L > 0$  if it is restricted to the closed interval  $[0, \frac{1}{2}]$ , so the operator norm of the Fréchet differential of  $f$  is bounded by  $L$  on its whole domain. Now the result immediately follows immediately using the convexity of the domain of  $f$ .  $\square$

Now let  $v \in \mathfrak{u}(A)$  with  $\|v\| < \frac{1}{2}$  and  $w = \tilde{f}(v) \in H_A$ . Since  $w$  arose from  $v$  by functional calculus, the elements  $v$  and  $w$  commute, so

$$(v + w)(v + w)^* = (v + w)(-v + w) = -v^2 + [v, w] + w^2 = -v^2 + (1 + v^2) = 1,$$

and, similarly,  $(v + w)^*(v + w) = 1$ . Therefore,  $v + w \in U(A)$ .

On the other hand, the projection  $\pi: A \rightarrow \mathfrak{u}(A)$  is a linear map with operator norm equal to 1 because  $\pi(a) = \frac{1}{2}(a - a^*)$ . Thus,  $\|\pi(x)\| = \|\pi(x) - \pi(1)\| = \|\pi(x - 1)\| \leq \|x - 1\|$  for all  $x \in A$  where we used that  $1 \in H_A$ . Now we can prove the announced extension result for maps into  $U(A)$ .

**Proof of lemma 3.5.** We consider the map

$$g: \left\{ v \in \mathfrak{u}(A) : \|v\| < \frac{1}{2} \right\} \rightarrow U(A), \quad v \mapsto v + (1 + v^2)^{1/2}$$

and the projection  $\pi: U(A) \subset A = \mathfrak{u}(A) \oplus H_A \rightarrow \mathfrak{u}(A)$ . Using lemma B.3, one easily shows that  $g$  is  $(1 + L)$ -Lipschitz, where  $L > 0$  is independent of  $A$ . We have already noted that  $\pi$  is 1-Lipschitz.

Now let  $\alpha_0: S^{n-1} \rightarrow U(A)$  be as in the statement of the theorem. We choose  $s_0 \in S^{n-1}$  and consider the map

$$\alpha_1: S^{n-1} \rightarrow \mathfrak{u}(A), \quad x \mapsto \pi(\alpha_0(s_0)^{-1}\alpha_0(x)).$$

Since  $\alpha_0(s_0)^{-1} \in U(A)$  and multiplication by unitary elements is an isometry in  $C^*$ -algebras, we have that  $\alpha_1(S^{n-1}) \subset \{v \in \mathfrak{u}(A) : \|v\| < \frac{1}{2}\}$ . Furthermore,  $\alpha_1$  is Lipschitz with constant at most  $\lambda$ . We may now extend  $\alpha_1$  to a  $(C_0\lambda)$ -Lipschitz map  $\alpha_2: D^n \rightarrow \{v \in \mathfrak{u}(A) : \|v\| < \frac{1}{2}\}$  using lemma B.1. Now we let

$$\alpha: D^n \rightarrow U(A), \quad x \mapsto \alpha_0(s_0) \cdot g\alpha_2(x).$$

Then the Lipschitz constant of  $\alpha$  is at most  $C_0(1 + L)\lambda$ , and it is rather clear that  $\alpha|_{S^{n-1}} = \alpha_0$ . This shows the statement of the lemma with constant  $C = C_0(1 + L)$ , which is in fact independent of  $A$ .  $\square$

Note that the condition on  $\text{diam}(\alpha_0)$  is immediate if  $\lambda \leq \frac{1}{4}$ .

The methods in this chapter can also be used to prove the following statement, which is used in the proof of lemma 5.3.

**Lemma B.4.** *The map*

$$f: \left\{ x \in GL(A) : \text{dist}(x, U(A)) < \frac{1}{3} \right\} \rightarrow U(A), \quad x \mapsto (xx^*)^{-1/2}x$$

*is well-defined, equals the identity on  $U(A)$ , and is Lipschitz with some Lipschitz constant  $L$  which does not depend on  $A$ .*

**Proof.** Consider an element  $x$  in the domain of  $f$ . Then  $\|xx^* - 1\| = \|(x - u)x^* + u(x - u)^*\| \leq \|x - u\|(\|x\| + \|u\|) < \frac{1}{3}(2 + \frac{1}{3}) = \frac{7}{9}$ . Because of lemma B.2, the map  $H_A \rightarrow H_A$  sending  $h$  to  $h^{-1/2}$  is well-defined and  $L_1$ -Lipschitz on the set of all  $h \in H_A$  with  $\|h - 1\| \leq \frac{7}{9}$ . Using this, the Lipschitzness of  $f$  is straightforward. The other assertions of the lemma are clear.  $\square$

## References

- [BH10] BRUNNBAUER, MICHAEL; HANKE, BERNHARD. Large and small group homology. *J. Topol.* **3** (2010), no. 2, 463–486. MR2651368, Zbl 1196.53028, arXiv:0902.0869, doi: 10.1112/jtopol/jtq014. 689, 709
- [CD18] CARRIÓN, JOSÉ R.; DADARLAT, MARIUS. Almost flat K-theory of classifying spaces. *J. Noncommut. Geom.* **12** (2018), no. 2, 407–438. MR3825191, Zbl 1402.46048, arXiv:1503.06170, doi: 10.4171/JNCG/281. 689, 712
- [CGM90] CONNES, ALAIN; GROMOV, MIKHAIL; MOSCOVICI, HENRI. Conjecture de Novikov et fibrés presque plats. *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), no. 5, 273–277. MR1042862, Zbl 0693.53007. 687, 688, 689, 691, 712
- [Fuk15] FUKUMOTO, YOSHIYASU. Invariance of finiteness of K-area under surgery. *Geom. Dedicata* **176** (2015), 175–183. MR3347576, Zbl 1318.53038, arXiv:1305.0102, doi: 10.1007/s10711-014-9962-6. 690, 691, 706, 711
- [Gro86] GROMOV, MIKHAE L. Large Riemannian manifolds. *Curvature and topology of Riemannian manifolds* (Katata, 1985), 108–121, Lecture Notes in Math., 1201. Springer, Berlin, 1986. MR0859578, Zbl 0601.53038, doi: 10.1007/BFb0075649. 689
- [Gro96] GROMOV, MIKHAE L. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. *Functional analysis on the eve of the 21st century, Vol. II* (New Brunswick, NJ, 1993), 1–213, Progr. Math., 132. Birkhäuser Boston, Boston, MA, 1996. MR1389019, Zbl 0945.53022, doi: 10.1007/978-1-4612-4098-3. 689, 691
- [GL80] GROMOV, MIKHAE L.; LAWSON, H. BLAINE, JR. Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)* **111** (1980), no. 2, 209–230. MR0569070, Zbl 0445.53025, doi: 10.2307/1971198. 689
- [Han12] HANKE, BERNHARD. Positive scalar curvature, K-area and essentialness. *Global differential geometry*, 275–302. Springer Proc. Math., 17. Springer, Heidelberg, 2012. MR3289846, Zbl 1252.53039, arXiv:1011.3987, doi: 10.1007/978-3-642-22842-1. 690, 691, 712
- [HS08] HANKE, BERNHARD; SCHICK, THOMAS. The strong Novikov conjecture for low degree cohomology. *Geom. Dedicata* **135** (2008), 119–127. MR2413333, Zbl 1149.19006, arXiv:0705.2578, doi: 10.1007/s10711-008-9266-9. 690

- [JT91] JENSEN, KJELD KNUDSEN; THOMSEN, KLAUS. Elements of  $KK$ -theory. Mathematics: Theory & Applications. *Birkhäuser Boston, Inc., Boston, MA*, 1991. viii+202 pp. ISBN: 0-8176-3496-7. MR1124848, Zbl 1155.19300, doi: 10.1007/978-1-4612-0449-7. 691
- [Lis13] LISTING, MARIO. Homology of finite  $K$ -area. *Math. Z.* **275** (2013), no. 1–2, 91–107. MR3101798, Zbl 1284.55010, arXiv:1007.3166, doi: 10.1007/s00209-012-1124-7. 712
- [MM01] MANUIŢOV, VLADIMIR M.; MISHCHENKO, ALEKSANDR S. Almost, asymptotic and Fredholm representations of discrete groups. Noncommutative geometry and operator  $K$ -theory. *Acta Appl. Math.* **68** (2001), no. 1–3, 159–210. MR1865957, Zbl 1012.46061, doi: 10.1023/A:1012077015181. 688, 691, 713, 714
- [MT05] MISHCHENKO, ALEKSANDR S.; TELEMAN, NICOLAE. Almost flat bundles and almost flat structures. *Topol. Methods Nonlinear Anal.* **26** (2005), no. 1, 75–87. MR2179351, Zbl 1093.19005, doi: 10.12775/TMNA.2005.025. 689, 691, 709
- [Ran09] RANI, RAFFAELE. On parallel transport and curvature. Preprint, 2009. <http://www.math.ku.dk/~moller/students/rani.pdf> (visited on 07/10/2019). 717
- [Sch05] SCHICK, THOMAS.  $L^2$ -index theorems,  $KK$ -theory, and connections. *New York J. Math.* **11** (2005), 387–443. MR2188248, Zbl 1093.58009, arXiv:math/0306171. 691, 692, 694
- [Spa12] SPANIER, EDWIN H. Algebraic topology. *Springer-Verlag, New York*, 1995. xvi+528 pp. ISBN: 0-387-94426-5. MR1325242, Zbl 0810.55001, doi: 10.1007/978-1-4684-9322-1. 701

(Benedikt Hunger) INSTITUT FÜR MATHEMATIK, UNIVERSITÄT AUGSBURG, 86135 AUGSBURG, GERMANY

[benedikt.seelstrang@math.uni-augsburg.de](mailto:benedikt.seelstrang@math.uni-augsburg.de)

This paper is available via <http://nyjm.albany.edu/j/2019/25-33.html>.