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On analytic arcs of inner functions

Wen-Hui Ai, Xin-Han Dong* and Yue-Ping Jiang

Abstract. Let $I = (e^{i\alpha}, e^{i\beta})$ be an analytic arc of the infinite Blaschke product $B(z)$. We find some equivalent conditions under which the argument of $B(e^{i\alpha^+})$ or $B(e^{i\beta^-})$ is finite. As an application, we classify the analytic arcs of inner functions into four types.

Contents

1. Introduction

An *inner function* $\Theta(z)$ is a function analytic in $|z| < 1$, having the properties $|\Theta(z)| \leq 1$ and $|\Theta(e^{i\theta})| = 1$ a.e. By the canonical factorization theorem (see [Du70, p.24]), an inner function can be factorized into the product of a (finite or infinite) Blaschke product and a singular inner function

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu_s(t) \right\}, \tag{1}$$

where μ_s is a singular positive measure on $[0, 2\pi]$. Recall that a sequence of points $\{a_k\}_{k=1}^{\infty}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is said to satisfy

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* Corresponding author.

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the *Blaschke condition* if $\sum_k (1-|a_k|^2) < \infty$. For a given sequence $\{a_k\}_{k=1}^{\infty}$ obeying the Blaschke condition, the *infinite Blaschke product* is defined by

$$B(z) = \prod_{k=1}^{\infty} b(z, a_k), \quad \text{where} \quad b(z, a_k) = \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a}_k z}. \tag{2}$$

Here we use the interpretation that $\frac{|a_k|}{a_k} = 1$ if $a_k = 0$.

In recent years, there have been many articles on the study of the group of invariants of inner functions (see [CaC00, CaG07, BaG09, ChGP12]). Since an inner function $\Theta(z)$ is undefined at its singular points on $\partial\mathbb{D}$, they interpret this as meaning that $\Theta(z)$ maps singular points to singular points and regular points to regular points. Hence the singular points of an inner function have special positions. These singular points are also closely related to the Cantor boundary behavior of analytic functions (CBB), which can be seen in [DoLL13].

In this paper, we want to study the analytic arcs of inner functions in more detail. Following [ChGP12], the *spectrum* $\sigma(\Theta)$ is the complement of the set of points $p \in \partial\mathbb{D}$ such that Θ has an analytic extension into a neighborhood of p. Indeed, let E be the cluster set of $\{a_k\}_{k=1}^{\infty}$, $\sigma(\Theta) = E \cup \operatorname{supp}\mu_s$. If $\sigma(\Theta) \neq \partial\mathbb{D}$, let

$$\partial\mathbb{D} \setminus \sigma(\Theta) = \cup_j I_j \tag{3}$$

be the decomposition as connected components, where

$$I_j = (e^{i\alpha_j}, e^{i\beta_j}) = \{e^{i\theta} : \alpha_j < \theta < \beta_j\}$$

with $0 \le \alpha_j < \beta_j \le 2\pi$. It is easy to see that $\Theta(z)$ is analytic in the domain

$$\Omega := \mathbb{C} \setminus (\sigma(\Theta) \cup \{\frac{1}{\overline{a}_k}, k \ge 1\}).$$

Since $I_j \subset \Omega$, we say that I_j is an *analytic arc* of $\Theta(z)$.

In [CaG07], the authors discuss the group of invariants of infinite Blaschke products with a single singular point. One of their results is as follows.

Theorem A. [CaG07, Theorem 4] *Suppose that the Blaschke sequence $\{a_k\}_{k=1}^{\infty}$ converges to $e^{i\theta_0}$. Then there are infinitely many arcs*

$$\Gamma_n = \{z = e^{i\theta} : \theta_0 + \alpha_{n-1} \le \theta < \theta_0 + \alpha_n\},$$

with

$$n \in \mathbb{Z}, \quad \alpha_{-n} = -\alpha_n, \quad 0 = \alpha_0 < \alpha_1 < \cdots, \quad \lim_{n\to\infty} \alpha_n = \pi,$$

which are mapped by the corresponding Blaschke product continuously and injectively on the unit circle. There is a continuous passage from every one of these mappings to the next one.

The theorem above means that for any $w \in \partial\mathbb{D}$, if $E = \{e^{i\theta_0}\}$, then

$$\#(B^{-1}(w) \cap (e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})) = \infty, \quad \#(B^{-1}(w) \cap (e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})) = \infty, \tag{4}$$

where # denotes the cardinality and $\varepsilon > 0$. In [BaG09, Theorem 2.1], the authors extend $E = \{e^{i\theta_0}\}$ in Theorem A to general Cantor subsets of $\partial\mathbb{D}$.

However, [CaG07, Theorem 4] and [BaG09, Theorem 2.1] are inaccurate. In fact, we can construct an infinite Blaschke product (Example 5.1) such that one equality in (4) is finite. More generally, let $I = (e^{i\alpha}, e^{i\beta})$ be an analytic arc of the infinite Blaschke product $B(z)$, we find some necessary and sufficient conditions under which one equality in (4) is finite. Before giving the theorem, let us make some notations. Denote

$$\begin{cases} \Delta_{\alpha,\delta} & = \{a_n\}_{n=1}^{\infty} \cap \{z : \alpha \leq \arg z \leq \alpha + \delta\}, \\ \Delta_{\alpha,-\delta} & = \{a_n\}_{n=1}^{\infty} \cap \{z : \alpha - \delta \leq \arg z \leq \alpha\} \end{cases} \tag{5}$$

for small $\delta > 0$ and

$$\varphi(\alpha^{\pm}) := \lim_{\theta \to \alpha^{\pm}} \arg B(e^{i\theta}), \tag{6}$$

where the argument of $B(z)$ is defined in Section 2.

Theorem 1.1. *Let $I = (e^{i\alpha}, e^{i\beta})$ be an analytic arc of the infinite Blaschke product $B(z)$. Then the following statements are equivalent:*

(i) *$\varphi(\alpha^+)$ is finite.*

(ii) *$\lim_{\theta\to\alpha^+} B(e^{i\theta}) = L$ and $|L| = 1$.*

(iii) *For any $w \in \partial\mathbb{D}$, there exists $\varepsilon > 0$ such that*

$$\#(B^{-1}(w) \cap (e^{i\alpha}, e^{i(\alpha+\varepsilon)})) < \infty.$$

(iv) *$B(e^{i\alpha})$ converges absolutely, namely,*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|a_n - e^{i\alpha}|} < \infty,$$

and $\#\Delta_{\alpha,\delta} < \infty$ for any $\delta > 0$.

In other words, let $I = (e^{i\alpha}, e^{i\beta})$, $0 \leq \alpha < \beta < 2\pi$, be an analytic arc of the infinite Blaschke product $B(z)$, then $\varphi(\alpha^+)$ and $\varphi(\beta^-)$ can be both finite, both infinite or only one finite. However, if $e^{i\theta_0}$ is an isolated point of the cluster set E, we have following corollary.

Corollary 1.2. *Let $B(z)$ be the infinite Blaschke product with zero set $\{a_k\}_{k=1}^{\infty}$. Let E be the cluster set of $\{a_k\}_{k=1}^{\infty}$. If $e^{i\theta_0} \in E$ be an isolated point of E, then at least one of $\varphi(\theta_0^+)$ and $\varphi(\theta_0^-)$ is infinite. In particular, if the analytic arc $I = (e^{i\alpha}, e^{i\beta})$ satisfies $\beta - \alpha = 2\pi$, then at least one of $\varphi(\alpha^+)$ and $\varphi(\beta^-)$ is infinite.*

The main part of our Theorem 1.1 appears in Choike [Ch73, Theorem 3], but the proof there is much more complicated and difficult to understand. Therefore, we give a new but an elementary proof. What's more, the paper [Ch73] classifies the singular points of an inner function into three types [Ch73, Theorem 1]. This classification was also presented in [ChGP12, Definition 3.1] (see our Definition 4.1). In their definition, one question is not clear. If the limit $\lim_{\theta\to\theta_0^-}\Theta(e^{i\theta})$ does not exist, the classification is incomplete. In fact, the statement (iii) in our Theorem 1.1 means that there are finitely many solutions of $B(\xi)=w$ in $(e^{i\alpha},\ e^{i(\alpha+\varepsilon)})$. Inspired by this, we obtain following theorem.

Theorem 1.3. *Let $\Theta(z)$ be an inner function with spectrum $\sigma(\Theta)$ and $e^{i\theta_0}\in\sigma(\Theta)$. If there exists $\lambda\in\partial\mathbb{D}$ such that there are finitely many solutions of $\Theta(\xi)=\lambda$ in $(e^{i(\theta_0-\varepsilon)},\ e^{i\theta_0})$, then we have*

(i) *there exists $\varepsilon_0\in(0,\varepsilon]$ such that $\Theta(z)$ is analytic in $(e^{i(\theta_0-\varepsilon_0)},\ e^{i\theta_0})$;*
(ii) $\lim_{\theta\to\theta_0^-}\Theta(e^{i\theta})=L$ *and* $|L|=1$.

Hence, the classifications in both [Ch73] and [ChGP12] are complete. At the same time, we get a new classification of analytic arcs (Corollary 4.5) which is also complete and equivalent to the classification in [ChGP12, Definition 3.2]. As a consequence of Theorem 1.3, we have following corollary.

Corollary 1.4. *Let $\Theta(z)$ be an inner function. For $\lambda_1,\ \lambda_2\in\partial\mathbb{D}$, if there are only finitely many solutions of $\Theta(\xi)=\lambda_1$ in $(e^{i(\theta_0-\varepsilon)},\ e^{i\theta_0})$ and $\Theta(\xi)=\lambda_2$ in $(e^{i\theta_0},\ e^{i(\theta_0+\varepsilon)})$, then $\Theta(z)$ is analytic at $e^{i\theta_0}$.*

This paper is organized as follows. Section 1 is the introduction and our main results. In Section 2, we present some preparatory materials. In Section 3, we discuss the endpoints of analytic arcs of infinite Blaschke products and prove Theorem 1.1 and Corollary 1.2. In Section 4, we find the classification of analytic arcs of inner functions in [ChGP12] is complete and give the proofs of Theorem 1.3 and some corollaries. At last, we construct some interesting examples to support our theorems in Section 5.

2. Preliminaries

2.1. Absolute convergence. Following [Ta63, p.410-411], we say that

$$B(z)=\prod_{n=1}^{\infty}b(z,a_n)=\prod_{n=1}^{\infty}(1+c(z,a_n))$$

converges absolutely at $e^{i\theta}$ if

$$\sum_{n=1}^{\infty}|c(e^{i\theta},a_n)|<\infty.$$

Note that

$$c(z,a_n)=b(z,a_n)-1=\frac{1-|a_n|}{|a_n|}-\frac{1-|a_n|^2}{|a_n|(1-\overline{a}_n z)},$$

and thus

$$|c(e^{i\theta},a_n)|\le\frac{1-|a_n|}{|e^{i\theta}-a_n|}\frac{1+|a_n|}{|a_n|}+\frac{1-|a_n|}{|a_n|}.$$

Combining with the Blaschke condition and some detailed analysis, we obtain that $B(e^{i\theta})$ converges absolutely if and only if

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|e^{i\theta}-a_n|} < \infty. \tag{7}$$

On the other hand, let $(\arg b(z,a_n))_0$ denote the principal argument of $b(z,a_n)$ which will be discussed later. If we write

$$c(e^{i\alpha_j},a_n) = b(e^{i\alpha_j},a_n) - 1 = e^{i(\arg b(e^{i\alpha_j},a_n))_0} - 1,$$

a routine computation gives rise to the following inequality

$$\begin{aligned}\frac{2}{\pi}\,\left|(\arg b(e^{i\alpha_j},a_n))_0\right| &\le \left|c(e^{i\alpha_j},a_n)\right| \\ &= 2\left|\sin\frac{1}{2}(\arg b(e^{i\alpha_j},a_n))_0\right| \\ &\le \left|(\arg b(e^{i\alpha_j},a_n))_0\right|.\end{aligned}$$

This shows that $B(e^{i\alpha_j})$ converges absolutely if and only if

$$\sum_{n=1}^{\infty} |(\arg b(e^{i\alpha_j},a_n))_0| < \infty. \tag{8}$$

2.2. The argument of $B(z)$. Let $a_n = \rho_n e^{i\varphi_n}$ with $\rho_n \in (0,1), \varphi_n \in [0,2\pi)$. It is easy to check that

$$b(e^{i\theta},a_n) = \frac{2\rho_n - (1+\rho_n^2)\cos(\theta-\varphi_n) - i(1-\rho_n^2)\sin(\theta-\varphi_n)}{|1-\rho_n e^{i(\theta-\varphi_n)}|^2}.$$

For simplicity, denote

$$\Delta_n(\theta) = 2\rho_n - (1+\rho_n^2)\cos(\theta-\varphi_n),$$

and

$$Q_n(\theta) = \arctan\frac{-(1-\rho_n^2)\sin(\theta-\varphi_n)}{\Delta_n(\theta)}.$$

For $\theta \in [\alpha_j,\alpha_{j+1}]$, the principal value branch of $\arg b(e^{i\theta},a_n)$ is defined as

$$(\arg b(e^{i\theta},a_n))_0 = \begin{cases} Q_n(\theta) & \text{if } \Delta_n(\theta) > 0, \\ -\frac{\pi}{2}\operatorname{sgn}(\sin(\theta-\varphi_n)) & \text{if } \Delta_n(\theta) = 0, \\ \pi + Q_n(\theta) & \text{if } \Delta_n(\theta) < 0,\ \sin(\theta-\varphi_n) \le 0, \\ -\pi + Q_n(\theta) & \text{if } \Delta_n(\theta) < 0,\ \sin(\theta-\varphi_n) > 0. \end{cases} \tag{9}$$

Let $I = (e^{i\alpha}, e^{i\beta})$ be an analytic arc of $B(z)$. There exists a simply connected domain D such that $I \subset D \subset \Omega$ and $B(z) \ne 0,\infty$ for $z \in D$. Hence, there is a single-valued analytic branch of $\log B(z)$ in D (see [Be79, p.202]), so is $\log b(z,a_n)$. Without loss of generality, for fixed $\tau_0 \in (\alpha,\beta)$, let $\tau_0 = \alpha + \frac{1}{6}(\beta-\alpha)$. Because of the fact that $B(z) = \prod_{n=1}^{\infty} b(z,a_n)$ is analytic for $z \in I$, we can choose the initial values $\log b(e^{i\tau_0}, a_n)$ and $\log B(e^{i\tau_0})$ satisfying

$$\log B(e^{i\tau_0}) := \sum_{n=1}^{\infty} \log b(e^{i\tau_0}, a_n), \text{ or } (\arg B(e^{i\tau_0}))_0 := \sum_{n=1}^{\infty} (\arg b(e^{i\tau_0}, a_n))_0. \quad (10)$$

Lemma 2.1. *Let $I = (e^{i\alpha}, e^{i\beta})$ be an analytic arc of the infinite Blaschke product $B(z)$. For fixed $\tau_0 \in (\alpha, \beta)$, let $(\arg B(e^{i\tau_0}))_0$ be defined by (10), then $\arg B(e^{i\theta})$ can be obtained from*

$$\arg B(e^{i\theta}) = (\arg B(e^{i\tau_0}))_0 + \sum_{n=1}^{\infty} \int_{\tau_0}^{\theta} \frac{1 - |a_n|^2}{|a_n - e^{ix}|^2} dx, \quad \theta \in (\alpha, \beta).$$

Besides, $\varphi(\alpha^+)$ is finite if and only if

$$\sum_{n=1}^{\infty} \int_{\alpha}^{\tau_0} \frac{1 - |a_n|^2}{|a_n - e^{ix}|^2} dx < \infty.$$

Proof. First, let us prove the series in (10) is convergent. Choose a small $\eta > 0$ satisfying $\tau_0 \in (\alpha + \eta, \beta - \eta)$. It is clear that there exists N_0 such that

$$a_n \notin \{z : \alpha + \frac{1}{2}\eta \leq \arg z \leq \beta - \frac{1}{2}\eta\}$$

for $n \geq N_0$. This yields that there exists $N_1 > N_0$ such that for $n \geq N_1$,

$$\Delta_n(\theta) \geq 2\rho_n - (1 + \rho_n^2) \cos \frac{1}{2}\eta \geq 1 - \cos \frac{1}{2}\eta > 0, \quad \theta \in [\alpha + \eta, \beta - \eta].$$

Hence

$$|(\arg b(e^{i\theta}, a_n))_0| = |\arctan \frac{(1 - \rho_n^2) \sin(\theta - \varphi_n)}{\Delta_n(\theta)}| \leq C(1 - \rho_n), \quad n \geq N_1,$$

which ensures that

$$\sum_{n=1}^{\infty} (\arg b(e^{i\theta}, a_n))_0$$

converges uniformly on $[\alpha + \eta, \beta - \eta]$. In particular, the series in (10) is convergent.

Let $\gamma_{z_0,z} \subset D$ be a simple curve with starting point $z_0 = e^{i\tau_0}$ and end point z. The value of $\log B(z)$ is decided by continuous change of $\log B(\xi)$ when ξ changes from z_0 to z along $\gamma_{z_0,z}$ and so is $\log b(z, a_n)$. Since

$$\frac{d}{d\xi} \left(\sum_{n=1}^{\infty} \log b(\xi, a_n) \right) = \sum_{n=1}^{\infty} \frac{b'(\xi, a_n)}{b(\xi, a_n)}$$

is a single-valued analytic function in D, we can define

$$\log B(z) := \log B(z_0) + \int_{\gamma_{z_0,z}} \sum_{n=1}^{\infty} \frac{b'(\xi, a_n)}{b(\xi, a_n)} d\xi. \quad (11)$$

If we choose $\gamma_{z_0,z} = \{e^{it} : t \in [\tau_0, \theta]\}$ or $\gamma_{z_0,z} = \{e^{it} : t \in [\theta, \tau_0]\}$, we can obtain

$$\log B(e^{i\theta}) = \log B(e^{i\tau_0}) + i \sum_{n=1}^{\infty} \int_{\tau_0}^{\theta} \frac{1-|a_n|^2}{|a_n - e^{ix}|^2}\, dx,$$

that is,

$$\arg B(e^{i\theta}) = \arg B(e^{i\tau_0}) + \sum_{n=1}^{\infty} \int_{\tau_0}^{\theta} \frac{1-|a_n|^2}{|a_n - e^{ix}|^2}\, dx, \tag{12}$$

$$\arg b(e^{i\theta}, a_n) = (\arg b(e^{i\tau_0}, a_n))_0 + \int_{\tau_0}^{\theta} \frac{1-|a_n|^2}{|a_n - e^{ix}|^2}\, dx. \tag{13}$$

The contents in the above show that the series in (12) converges uniformly on each compact subset of (α, β). We now prove that $\arg b(e^{i\theta}, a_n)$ in (13) is consistent with $(\arg b(e^{i\theta}, a_n))_0$ for large n. For $\theta \in [\alpha + \eta, \beta - \eta]$, we can find $n \geq N_0$ such that

$$a_n \notin \left\{ z : \alpha + \frac{1}{2}\eta \leq \arg z \leq \beta - \frac{1}{2}\eta \right\}.$$

By the Blaschke condition, there exists $M > 0$ such that

$$\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|e^{i\theta} - a_n|^2} \leq \sum_{n=1}^{N_0} \frac{1-|a_n|^2}{|e^{i\theta} - a_n|^2} + C \sum_{N_0+1}^{\infty} (1 - |a_n|^2) \leq M.$$

This along with (12)-(13) shows that

$$\sum_{n=1}^{\infty} \left| \arg b(e^{i\theta}, a_n) \right| \leq \sum_{n=1}^{\infty} \left| (\arg b(e^{i\tau_0}, a_n))_0 \right| + M|\theta - \tau_0| < \infty.$$

Hence $\arg b(e^{i\theta}, a_n)$ converges uniformly to 0 on $[\alpha + \eta, \beta - \eta]$ as $n \to \infty$. Consequently, $\arg B(e^{i\theta})$ is strictly increasing on (α, β).

Hence $\varphi(\alpha^+)$ and $\varphi(\beta^-)$ both exist (may be infinite). Our theorem then follows from (12). ☐

3. The end-points of analytic arcs

In this section, we will prove Theorem 1.1 and Corollary 1.2.

Let $a_n = \rho_n e^{i\varphi_n}$ with $\rho_n \in (0,1), \varphi_n \in [0, 2\pi)$. For simplicity, we always use $c_1, c_2, \cdots$ to express absolute constants. If $I = (e^{i\alpha}, e^{i\beta})$ is an analytic arc of the infinite Blaschke product $B(z)$, let

$$J_n = \int_{\alpha}^{\tau_0} \frac{1-|a_n|^2}{|a_n - e^{ix}|^2} dx, \quad \tau_0 \in (\alpha, \beta). \tag{14}$$

Besides, we classify the zero set $\{a_n\}_{n=1}^{\infty}$ into four categories:

$$\begin{cases} \Lambda_1 = \{a_n\}_{n=1}^{\infty} \cap \{z : \alpha < \arg z \le \alpha + \delta\}, \\ \Lambda_2 = \{a_n\}_{n=1}^{\infty} \cap \{z : \arg z = \alpha\}, \\ \Lambda_3 = \{a_n\}_{n=1}^{\infty} \cap \{z : \alpha - \delta \le \arg z < \alpha\}, \\ \Lambda_4 = \{a_n\}_{n=1}^{\infty} \setminus (\cup_{i=1}^{3} \Lambda_i), \end{cases} \tag{15}$$

where $\delta \in (0, (\beta - \alpha)/6)$ is small. From Lemma 2.1, in order to prove Theorem 1.1, we need to prove that the condition

$$\sum_{n=1}^{\infty} J_n = \sum_{i=1}^{4} \sum_{a_n \in \Lambda_i} J_n < \infty$$

is equivalent to the conditions

$$\#\Delta_{\alpha,\delta} < \infty, \quad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|a_n - e^{i\alpha}|} < \infty.$$

For $x \neq \varphi_n$, note that

$$\frac{1 - |a_n|^2}{|a_n - e^{ix}|^2} = \frac{1 + \rho_n}{1 - \rho_n} \frac{\left(\frac{1-\rho_n}{\sin \frac{x - \varphi_n}{2}}\right)^2}{\left(\frac{1-\rho_n}{\sin \frac{x-\varphi_n}{2}}\right)^2 + 4\rho_n}.$$

Changing the variable of integration by letting

$$z(x) = \frac{1 - \rho_n}{\sin \frac{x - \varphi_n}{2}},$$

we get

$$J_n = \int_{z(\alpha)}^{z(\tau_0)} \frac{2(1 + \rho_n) z \, dz}{(z^2 + 4\rho_n)\sqrt{z^2 - (1 - \rho_n)^2}}. \tag{16}$$

Since the case $x = \varphi_n$ is trivial, we just need to estimate (16). Now, let's prove some lemmas.

Lemma 3.1. *If $\#\Lambda_1 = \infty$, then there exists an absolute constant $c > 0$ such that*

$$\left| J_n - (1 + \rho_n)(\pi - \arctan \frac{1 - \rho_n}{2 \sin \frac{1}{2}(\varphi_n - \alpha)}) \right| \le c(1 - \rho_n), \quad a_n = \rho_n e^{i\varphi_n} \in \Lambda_1.$$

Furthermore, $\sum_{a_n \in \Lambda_1} J_n < \infty$ if and only if $\#\Lambda_1 < \infty$.

Proof. For $a_n \in \Lambda_1$, we have $\alpha < \varphi_n < \tau_0$ and

$$z(\alpha) = \frac{1 - \rho_n}{\sin \frac{\alpha - \varphi_n}{2}} < 0.$$

Let us calculate (16) further.

$$\begin{aligned} J_n &= \int_{-z(\alpha)}^{\infty} + \int_{z(\tau_0)}^{\infty} \frac{2(1+\rho_n)z\,dz}{(z^2+4\rho_n)\sqrt{z^2-(1-\rho_n)^2}} \\ &= (1+\rho_n)\left(\pi - \arctan\frac{-z(\alpha)}{2} - \arctan\frac{z(\tau_0)}{2}\right) \\ &\quad + 2(1+\rho_n)\int_{-z(\alpha)}^{\infty} + \int_{z(\tau_0)}^{\infty} \delta_n(z)\,dz, \end{aligned} \tag{17}$$

where

$$\delta_n(z) = \frac{z}{4\rho_n + z^2}\ \frac{1}{\sqrt{z^2-(1-\rho_n)^2}} - \frac{1}{4+z^2}.$$

After a careful calculation, we can get

$$0 < \delta_n(z) \le \frac{c_1(1-\rho_n)\{z^4+z^2+(1-\rho_n)\}}{(1+z^2)^3\sqrt{z^2-(1-\rho_n)^2}\big(z+\sqrt{z^2-(1-\rho_n)^2}\,\big)}.$$

If $0 < -z(\alpha) < 1$, then

$$\begin{aligned} &\frac{1}{1-\rho_n}\int_{-z(\alpha)}^{\infty} \delta_n(z)\,dz \\ &\le c_4\left(1 + \log\frac{\rho_n}{2-\rho_n}\frac{1+\sin\frac{1}{2}(\varphi_n-\alpha)}{1-\sin\frac{1}{2}(\varphi_n-\alpha)}\right) + T_1 \\ &\le c_5, \end{aligned} \tag{18}$$

where

$$0 < T_1 = \frac{1}{1-\rho_n}\int_1^{\infty} \delta_n(z)\,dz \le c_2$$

and $\varphi_n \to \alpha$ as $n \to \infty$. In the same way,

$$\frac{1}{1-\rho_n}\int_{z(\tau_0)}^{\infty} \delta_n(z)\,dz \le c_6\left(1+\log\frac{1+\sin\frac{1}{2}(\tau_0-\varphi_n)}{1-\sin\frac{1}{2}(\tau_0-\varphi_n)}\right) + T_1 \le c_7. \tag{19}$$

By (17)-(19), for $\rho_n e^{i\varphi_n} \in \Lambda_1$, we obtain

$$0 < J_n - (1+\rho_n)\left(\pi - \arctan\frac{-z(\alpha)}{2} - \arctan\frac{z(\tau_0)}{2}\right) \le c_8(1-\rho_n).$$

The lemma now follows from

$$0 < \arctan\frac{z(\tau_0)}{2} \le c_9(1-\rho_n).$$

□

Lemma 3.2. *If $\#\Lambda_2 = \infty$, then there exists an absolute constant $c > 0$ such that*

$$\left|J_n - (1+\rho_n)\frac{\pi}{2}\right| \le c(1-\rho_n), \quad a_n = \rho_n e^{i\varphi_n} \in \Lambda_2.$$

Furthermore, $\sum_{a_n\in\Lambda_2} J_n < \infty$ if and only if $\#\Lambda_2 < \infty$.

Proof. For $a_n \in \Lambda_2$, we have $\varphi_n = \alpha < \tau_0$. Then

$$J_n = \int_{z(\tau_0)}^{\infty} \frac{2(1+\rho_n)z\,dz}{(z^2+4\rho_n)\sqrt{z^2-(1-\rho_n)^2}}.$$

From the proof of Lemma 3.1, we have

$$0 < \int_{z(\tau_0)}^{\infty} \delta_n(z)dz \le c_{13}(1-\rho_n).$$

Since

$$0 < \arctan\frac{z(\tau_0)}{2} \le c_9(1-\rho_n),$$

we complete the proof. □

Lemma 3.3. *If $\#\Lambda_3 = \infty$, then there exists an absolute constant $c > 0$ such that*

$$\left| J_n - (1+\rho_n)|\arctan\frac{1-\rho_n}{2\sin\frac{1}{2}(\alpha-\varphi_n)}| \right| \le c(1-\rho_n), \qquad a_n = \rho_n e^{i\varphi_n} \in \Lambda_3. \tag{20}$$

Furthermore, $\sum_{a_n\in\Lambda_3} J_n < \infty$ if and only if

$$\sum_{a_n\in\Lambda_3} \frac{1-\rho_n}{|a_n - e^{i\alpha}|} < \infty.$$

Proof. Without lose of generality, suppose that $\alpha = 0$. As $\varphi_n \in [0, 2\pi)$, if $a_n \in \Lambda_3$, we have

$$\psi_n := \varphi_n - 2\pi \in [-\frac{\pi}{3}, 0)$$

and

$$\frac{1-\rho_n}{\sin\frac{x-\psi_n}{2}} = z(x+2\pi).$$

Consequently,

$$\begin{aligned} J_n &= \int_{z(\tau_0+2\pi)}^{z(\alpha+2\pi)} \frac{2(1+\rho_n)z\,dz}{(z^2+4\rho_n)\sqrt{z^2-(1-\rho_n)^2}} \\ &= (1+\rho_n)\left(\arctan\frac{z(\alpha+2\pi)}{2} - \arctan\frac{z(\tau_0+2\pi)}{2}\right) \\ &\quad + 2(1+\rho_n)\int_{z(\tau_0+2\pi)}^{z(\alpha+2\pi)} \delta_n(z)\,dz. \end{aligned}$$

Similar to the calculation in (18) and (19), we can obtain that

$$\frac{1}{1-\rho_n}\int_{z(\tau_0+2\pi)}^{z(\alpha+2\pi)} \delta_n(z)\,dz \le c_{11}.$$

The inequality (20) follows from that

$$\left|\arctan\frac{z(\tau_0+2\pi)}{2}\right| \le c_{12}(1-\rho_n)$$

and

$$\arctan \frac{z(\alpha+2\pi)}{2} = -\arctan \frac{z(\alpha)}{2} > 0.$$

By the Blaschke condition and (20), we get $\sum_{a_n\in\Lambda_3} J_n < \infty$ if and only if

$$\sum_{a_n\in\Lambda_3} \left|\arctan \frac{z(\alpha)}{2}\right| < \infty,$$

which is equivalent to $\sum_{a_n\in\Lambda_3} |z(\alpha)| < \infty$. After some manipulations, we can get $\sum_{a_n\in\Lambda_3} J_n < \infty$ if and only if

$$\sum_{a_n\in\Lambda_3} \frac{1-\rho_n}{|a_n - e^{i\alpha}|} = \sum_{a_n\in\Lambda_3} \frac{|z(\alpha)|}{\sqrt{z(\alpha)^2+4\rho_n}} < \infty,$$

as $\rho_n \to 1$ $(n\to\infty)$. The proof is complete. □

For $a_n \in \Lambda_4$, there exists $\varepsilon > 0$ such that $|a_n - e^{i\alpha}| > \varepsilon$. It is easy to get following lemma.

Lemma 3.4. *Let*

$$\Lambda_4 = \{a_n\}_{n=1}^{\infty} \setminus \left\{\cup_{i=1}^{3}\Lambda_i\right\}.$$

Then

$$\sum_{a_n\in\Lambda_4} J_n < \infty, \quad \sum_{a_n\in\Lambda_4} \frac{1-|a_n|}{|a_n - e^{i\alpha}|} < \infty.$$

With the help of the preceding four lemmas we can now prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. It is obvious that $(i) \Leftrightarrow (ii)$. Note that $B(z)$ is continuous on the analytic arc $I = (e^{i\alpha}, e^{i\beta})$ and $\arg B(e^{i\theta})$ increases monotonously on (α,β), it is easy to see that $(i) \Leftrightarrow (iii)$. The equivalence $(ii) \Leftrightarrow (iv)$ appears in [Ch73, Theorem 3], where they proved that

$$\lim_{\theta\to\alpha^+} B(e^{i\theta}) = L(|L|=1)$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|a_n - e^{i\alpha}|} < \infty$$

and there are no zeros $\{a_k\}$ in the region

$$\triangle = \{z : 1-\varepsilon < |z| < 1, \alpha < \arg z < \alpha+\delta\},$$

where the positive numbers δ and ε are small. Our theorem is a supplement to their result.

Let us now prove that $(i) \Leftrightarrow (iv)$.

$(i) \Rightarrow (iv)$. Assume that $\varphi(\alpha^+)$ is finite. By Lemma 2.1, we have

$$\sum_{n=1}^{\infty} J_n = \sum_{i=1}^{4} \sum_{a_n\in\Lambda_i} J_n < \infty.$$

Note that $\Delta_{\alpha,\delta} = \Lambda_1 \cup \Lambda_2$. It follows from Lemma 3.1 and Lemma 3.2 that $\#\Delta_{\alpha,\delta} < \infty$. Combining with Lemma 3.3 and 3.4, we have

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|a_n - e^{i\alpha}|} = \left(\sum_{a_n \in \Delta_{\alpha,\delta}} + \sum_{a_n \in \Lambda_3} + \sum_{a_n \in \Lambda_4} \right) \frac{1-|a_n|}{|a_n - e^{i\alpha}|} < \infty,$$

i.e., $B(e^{i\alpha})$ converges absolutely (by (7)).

$(iv) \Rightarrow (i)$. Conversely, assume that $B(e^{i\alpha})$ converges absolutely and $\#\Delta_{\alpha,\delta} < \infty$. By (7), we have

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|a_n - e^{i\alpha}|} < \infty.$$

Then Lemma 3.3 implies that $\sum_{a_n \in \Lambda_3} J_n < \infty$. Since $\Delta_{\alpha,\delta} = \Lambda_1 \cup \Lambda_2$, it follows from Lemma 3.1, Lemma 3.2 and Lemma 3.4 that

$$\sum_{a_n \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_4} J_n < \infty.$$

Hence

$$\sum_{n=1}^{\infty} J_n = \sum_{i=1}^{4} \sum_{a_n \in \Lambda_i} J_n < \infty.$$

Lemma 2.1 shows that $\varphi(\alpha^+)$ is finite.

In fact, similar to $(i) \Leftrightarrow (iv)$, we can get $\varphi(\beta^-)$ is finite if and only if $B(e^{i\beta})$ converges absolutely and $\#\Delta_{\beta,-\delta} < \infty$. $\square$

Proof of Corollary 1.2. By using reduction to absurdity, we can get Corollary 1.2 immediately. Suppose that the limits $\varphi(\theta_0^+)$ and $\varphi(\theta_0^-)$ are both finite, since $e^{i\theta_0} \in E$ is an isolated point of E, then by Theorem 1.1, we have $\#(\Lambda_1 \cup \Lambda_2 \cup \Lambda_3) < \infty$. This is a contradiction to the fact that $e^{i\theta_0}$ is an accumulation point of $\{a_k\}_{k=1}^{\infty}$. $\square$

4. The classification of analytic arcs

In [ChGP12], for inner functions $\Theta(z)$ with finite spectrum, the authors classify analytic arcs of $\Theta(z)$ into four types and the endpoints of analytic arcs are classified into three types. For convenience of the reader, we present the classification of the endpoints of analytic arcs.

Definition 4.1. [ChGP12, Definition 3.1] *Let $\Theta(z)$ be an inner function with finite spectrum. Let $\xi_0 = e^{i\theta_0} \in \sigma(\Theta)$. We say that*

(i) *ξ_0 is of type $1_{a,L}$ if for $\varepsilon > 0$ sufficiently small, there are infinitely many solutions of $\Theta(\xi) = 1$ in the open interval (i.e., arc of the circle) $(e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})$, finitely many solutions in $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$, and $\lim_{\theta\to\theta_0^-} \Theta(e^{i\theta}) = L$.*

(ii) ξ_0 *is of type* $1_{b,L}$ *if for* $\varepsilon > 0$ *sufficiently small, there are infinitely many solutions of* $\Theta(\xi) = 1$ *in the open interval* $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$, *finitely many solutions in* $(e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})$, *and* $\lim_{\theta\to\theta_0^+}\Theta(e^{i\theta}) = L$.

(iii) ξ_0 *is of type 2 if for all* $\varepsilon > 0$ *there are infinitely many solutions to* $\Theta(\xi) = 1$ *in both of the intervals* $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$ *and* $(e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})$.

In their classification, one question is not clear: Is there any connection between the condition that there are finitely many solutions of $\Theta(\xi) = 1$ in $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$ and the condition that $\lim_{\theta\to\theta_0^-}\Theta(e^{i\theta}) = L$? We find that if there are finitely many solutions of $\Theta(\xi) = 1$ in $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$ then $\lim_{\theta\to\theta_0^-}\Theta(e^{i\theta}) = L$ and $|L| = 1$ (refer to Theorem 1.3). Hence, the classification in [ChGP12] is complete.

Proof of Theorem 1.3. Since there are finitely many solutions of $\Theta(\xi) = \lambda$, we can take $\varepsilon_0 \in (0, \varepsilon]$ such that $\Theta(\xi) \neq \lambda$ in $(e^{i(\theta_0-\varepsilon_0)},\ e^{i\theta_0})$. The Mobius transformation $\xi = L(w) = \frac{\lambda+w}{\lambda-w}$ maps $\mathbb{D}_w$ onto the right-half plane such that $L(\lambda) = \infty$. Hence $\xi = L(\Theta(z))$ is analytic with positive real part in $\mathbb{D}$. It follows from Theorem 2.4 in [Po75, p.40] that there exists an increasing function $\nu(t)$ on $[0, 2\pi]$ such that

$$L(\Theta(z)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\nu(t) + i\gamma,$$

where γ is a real constant. In particular,

$$u(r,\theta) = \mathrm{Re} L(\Theta(re^{i\theta})) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, d\nu(t).$$

We claim that $\nu(t)$ is absolutely continuous on $(\theta_0 - \varepsilon_0, \theta_0)$.

1). First, let's prove $\nu(t)$ is continuous on $(\theta_0 - \varepsilon_0, \theta_0)$ by contradiction. Let $t_0 \in (\theta_0 - \varepsilon_0, \theta_0)$ be a point of discontinuity, then by the lemma in [Lo52, p.244], we have $\lim_{r\to 1^-} u(r, t_0) = +\infty$. Hence $\Theta(e^{it_0}) = \lim_{r\to 1^-}\Theta(re^{it_0}) = \lambda$, it is a contradiction.

2). If $\nu(t)$ is continuous but not absolutely continuous on $(\theta_0 - \varepsilon_0, \theta_0)$, it follows from [Sa64, p.128] that $\nu(t)$ has an infinite derivative on a non-enumerable set of $(\theta_0-\varepsilon_0, \theta_0)$. Take $t_0 \in (\theta_0-\varepsilon_0, \theta_0)$ be such a point. Similar to the proof of Theorem 1.2 in [Du70, p.4], we can prove $\lim_{r\to 1^-} u(r, t_0) = +\infty$, hence $\Theta(e^{it_0}) = \lambda$, a contradiction.

Thus, $\nu(t)$ has to be absolutely continuous on $(\theta_0 - \varepsilon_0, \theta_0)$. Hence

$$\nu(x_2) - \nu(x_1) = \int_{x_1}^{x_2} \nu'(t)dt, \qquad x_1, x_2 \in (\theta_0 - \varepsilon_0, \theta_0). \tag{21}$$

Since $|\Theta(e^{i\theta})| = 1$ a.e. on $[0, 2\pi]$, we have $\lim_{r\to 1^-} u(r,\theta) = 0$ almost everywhere on $[0, 2\pi]$. On the other hand, by Theorem 1.2 in [Du70, p.4], if $\nu'(\theta)$ exists, then $\lim_{r\to 1^-} u(r,\theta) = \nu'(\theta)$. Hence $\nu'(t) = 0$ almost everywhere on $[0, 2\pi]$, since $\nu'(t)$ exists almost everywhere on $[0, 2\pi]$. By (21), we have

$\nu(t) \equiv c$ for $t \in (\theta_0 - \varepsilon_0, \theta_0)$ (for plane measure, see [DoL03, p.72]). Then $\nu'(t) \equiv 0$ for $t \in (\theta_0 - \varepsilon_0, \theta_0)$, which yields

$$\frac{\lambda + \Theta(z)}{\lambda - \Theta(z)} = \frac{1}{2\pi} \int_0^{\theta_0 - \varepsilon_0} + \int_{\theta_0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\nu(t) + i\gamma. \tag{22}$$

Thus $\Theta(z)$ is analytic in $(e^{i(\theta_0 - \varepsilon_0)}, e^{i\theta_0})$, and (i) follows.

Choose a simply connected domain D such that $(e^{i(\theta_0-\varepsilon_0)}, e^{i\theta_0}) \subset D$, and $\Theta(z)$ is analytic in D. Let $\Theta(e^{i\theta}) = e^{i\phi(\theta)}$, where $\phi(\theta_0 - \frac{1}{2}\varepsilon_0) \in [0, 2\pi)$, and the value of $\phi(\theta)$ is decided by continuous change of $\arg \Theta(\xi)$ when ξ changes from $z_0 = e^{i(\theta_0 - \frac{1}{2}\varepsilon_0)}$ to $z = e^{i\theta}$ along the simple curve $\gamma_{z_0,z} \subset D$. Let $\lambda = e^{i\alpha} \neq e^{i\phi(\theta)}$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0)$, from (22), we have

$$\cot \frac{\phi(\theta) - \alpha}{2} = \frac{1}{2\pi} \int_0^{\theta_0 - \varepsilon_0} + \int_{\theta_0}^{2\pi} \cot \frac{\theta - t}{2} \, d\nu(t) + \gamma.$$

Hence $\phi(\theta) = \arg \Theta(e^{i\theta})$ is strictly increasing on $(\theta_0 - \varepsilon_0, \theta_0)$. Note that $\phi(\theta)$ is continuous and that $\phi(\theta) \neq 2k\pi + \alpha$ in $(\theta_0 - \varepsilon_0, \theta_0)$ for any $k \in \mathbb{Z}$, there exists k_0 such that $\alpha + 2k_0\pi < \phi(\theta) < \alpha + 2(k_0 + 1)\pi$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0)$. These show the limit $\lim_{\theta \to \theta_0^-} \phi(\theta)$ is finite, and (ii) follows. □

Proof of Corollary 1.4. Let the inner function $\Theta(z) = B(z)S(z)$. Recall that we say $\Theta(z)$ is analytic at $e^{i\theta_0}$ if $e^{i\theta_0} \notin \sigma(S)$ and

$$\left(e^{i(\theta_0 - \varepsilon_0)}, e^{i(\theta_0 + \varepsilon_0)}\right) \cap E = \emptyset$$

for small $\varepsilon_0 > 0$.

By Theorem 1.3, $\Theta(z)$ is analytic in $(e^{i(\theta_0 - \varepsilon_0)}, e^{i(\theta_0 + \varepsilon_0)})$ except for the point $e^{i\theta_0}$ and $\lim_{\theta \to \theta_0} \arg \Theta(e^{i\theta})$ is finite. Since $\arg \Theta(e^{i\theta})$, $\arg B(e^{i\theta})$ and $\arg S(e^{i\theta})$ increase monotonously on $(\theta_0 - \varepsilon_0, \theta_0)$ and $(\theta_0, \theta_0 + \varepsilon_0)$ separately, we obtain that $\lim_{\theta \to \theta_0} \arg B(e^{i\theta})$ and $\lim_{\theta \to \theta_0} \arg S(e^{i\theta})$ are finite. Combining with Corollary 1.2, we have

$$\left(e^{i(\theta_0 - \varepsilon_0)}, e^{i(\theta_0 + \varepsilon_0)}\right) \cap E = \emptyset.$$

Because $S(z)$ is analytic in

$$\left(e^{i(\theta_0 - \varepsilon_0)}, e^{i(\theta_0 + \varepsilon_0)}\right) \setminus \{e^{i\theta_0}\},$$

we have $\mu_s(t) \equiv c_1$ for $t \in (\theta_0 - \varepsilon_0, \theta_0)$ and $\mu_s(t) \equiv c_2$ for $t \in (\theta_0, \theta_0 + \varepsilon_0)$. Hence

$$-\log S(z) = \left(\int_0^{\theta_0 - \varepsilon_0} + \int_{\theta_0 + \varepsilon_0}^{2\pi}\right) \frac{e^{it} + z}{e^{it} - z} \, d\mu_s(t) + \left(\mu_s(\theta_0^+) - \mu_s(\theta_0^-)\right) \frac{e^{i\theta_0} + z}{e^{i\theta_0} - z}.$$

As $\lim_{\theta \to \theta_0} \arg S(e^{i\theta})$ is finite, we have $\mu_s(\theta_0^+) - \mu_s(\theta_0^-) = 0$. Therefore, $\mu_s(t) \equiv c$ for $|t - \theta_0| < \varepsilon_0$ and $e^{i\theta_0} \notin \sigma(S)$. □

Contrary to Corollary 1.4, we can get following corollaries immediately.

Corollary 4.2. *Let $B(z)$ be a Blaschke product with the property that the cluster set of $\{a_k\}_{k=1}^{\infty}$ is the whole circle $\partial\mathbb{D}$. For $\lambda \in \partial\mathbb{D}$ we have*

$$\#\left(B^{-1}(\lambda) \cap (e^{ia}, e^{ib})\right) = \infty$$

for any arc $(e^{ia}, e^{ib}) \subset \partial\mathbb{D}$ with $0 \le a < b < 2\pi$.

Corollary 4.3. *Let $\Theta(z)$ be an inner function. If $\sigma(\Theta) \cap (e^{ia}, e^{ib}) \neq \emptyset$ for any $(a, b) \subset [0, 2\pi)$, then $\#(\Theta^{-1}(\lambda) \cap (e^{ia}, e^{ib})) = \infty$ for any $\lambda \in \partial\mathbb{D}$.*

Let $\Delta_{\theta_0,\delta}, \Delta_{\theta_0,-\delta}$ be defined in (5). Combining with Theorem 1.1, Theorem 1.3 and Corollary 1.4, we get following corollary.

Corollary 4.4. *Let $\Theta(z)$ be an inner function with finitely many singularities and let $e^{i\theta_0}$ be a singularity. If $\#\Delta_{\theta_0,\delta} = \infty$ and $\#\Delta_{\theta_0,-\delta} = \infty$ for small $\delta > 0$, then by the classification in Definition 4.1, $e^{i\theta_0}$ is of type 2.*

Therefore, we can classify analytic arcs of the inner function $\Theta(z)$ into four types as follows. This classification is complete and equivalent to the classification in [ChGP12, Definition 3.2].

Corollary 4.5. *Let B be the Blaschke product whose sequence of zeros is $\{a_k\}_{k=1}^{\infty}$ and let $\Theta(z) = BS$ be an inner function. An interval $(e^{i\alpha}, e^{i\beta})$ whose endpoints are consecutive accumulation points of $\{a_k\}_{k=1}^{\infty}$*

- (i) *is of type 0 if and only if both $B(e^{i\alpha})$ and $B(e^{i\beta})$ converges absolutely and $\#\Delta_{\alpha,\delta}$, $\#\Delta_{\beta,-\delta} < \infty$;*
- (ii) *is of type 1_a if and only if the following conditions hold simultaneously: (1) $B(e^{i\alpha})$ does not converge absolutely or $\#\Delta_{\alpha,\delta} = \infty$, (2) $B(e^{i\beta})$ converges absolutely and $\#\Delta_{\beta,-\delta} < \infty$;*
- (iii) *is of type 1_b if and only if the following conditions hold simultaneously: (1) $B(e^{i\alpha})$ converges absolutely and $\#\Delta_{\alpha,\delta} < \infty$, (2) $B(e^{i\beta})$ does not converge absolutely or $\#\Delta_{\beta,-\delta} = \infty$;*
- (iv) *is of type 2 if and only if the following conditions hold simultaneously: (1) $B(e^{i\alpha})$ does not converge absolutely or $\#\Delta_{\alpha,\delta} = \infty$, (2) $B(e^{i\beta})$ does not converge absolutely or $\#\Delta_{\beta,-\delta} = \infty$.*

5. Examples

In this section, we construct an infinite Blaschke product such that one equality in (4) is actually finite and give more examples of interesting infinite Blaschke products to support our theorems.

Example 5.1. *For $s > 2$, set $\rho_k = 1 - \frac{1}{(2k)^s}$ and $a_k = \rho_k e^{i\frac{1}{k}\pi}$. The sequence $\{a_k\}_{k=1}^{\infty}$ satisfies the Blaschke condition. Let B be the Blaschke product whose sequence of zeros is $\{a_k\}_{k=1}^{\infty}$. Then the following statements hold.*

- (i) *$B(1)$ is absolutely convergent, and $\#\Delta_{0,-\delta} = 0, \#\Delta_{0,\delta} = \infty$;*
- (ii) *$\varphi(0^-)$ is finite, $\varphi(0^+)$ is infinite. Then by the classification in Definition 4.1, the singular point 1 is of type $1_{a,L}$.*

Proof. It is obvious that the cluster set of $\{a_k\}_{k=1}^{\infty}$ is $E = \{1\}$. Since all $\arg a_k > 0$, we have $\#\Delta_{0,-\delta} = 0$ and $\#\Delta_{0,\delta} = \infty$. By Theorem 1.1, we only need to prove that

$$\sum_{k=2}^{\infty} \frac{1-\rho_k}{\left|1-\rho_k e^{i\frac{1}{k}\pi}\right|} < \infty.$$

In fact,

$$\frac{1-\rho_k}{|1-\rho_k e^{i\frac{1}{k}\pi}|} = \frac{1-\rho_k}{\sqrt{(1-\rho_k)^2 + 4\rho_k \sin^2\frac{\pi}{2k}}} \leq \frac{c}{k^{s-1}}, \quad s > 2.$$

Hence $\varphi(0^-)$ is finite. Then, by Corollary 1.2, $\varphi(0^+)$ is infinite. The proof is complete. □

Example 5.2. *For $s > 3$, set $\rho_k = 1 - \frac{1}{(2k)^s}$ and $a_{k,m} = \rho_k e^{i2\pi\frac{m}{2k}}$. Let*

$$\begin{aligned}
\Omega_{k,0} &= \{1, \cdots, k-1\},\\
\Omega_{k,1} &= \{1, \cdots, k-1, k+1\},\\
\Omega_{k,2} &= \{-1, 1, \cdots, k-1, k+1\}.
\end{aligned}$$

For $i = 0, 1, 2$, let

$$B_i(z) = \prod_{k=2}^{\infty} \prod_{m\in\Omega_{k,i}} b(z, a_{k,m}).$$

Then the following statements hold.

(i) *For $i = 0, 1, 2$, the cluster set of the zero set of $B_i(z)$ is $E = \{e^{i\theta} : \theta \in [0, \pi]\}$, and $B_i(z)$ is absolutely convergent at $z = 1, -1$.*

(ii) *The limits $\lim_{\theta\to\pi^+} \arg B_0(e^{i\theta})$ and $\lim_{\theta\to 0^-} \arg B_0(e^{i\theta})$ are both finite. Then by the classification in Corollary 4.5, for $B_0(z)$, the interval $(e^{i\pi}, e^{i2\pi})$ is of type 0;*

(iii) *The limit $\lim_{\theta\to\pi^+} B_1(e^{i\theta})$ is infinite but $\lim_{\theta\to 0^-} B_1(e^{i\theta})$ is finite. Then by the classification in Corollary 4.5, for $B_1(z)$, the interval $(e^{i\pi}, e^{i2\pi})$ is of type 1_a;*

(iv) *The limits $\lim_{\theta\to\pi^+} B_2(e^{i\theta})$ and $\lim_{\theta\to 0^-} B_2(e^{i\theta})$ are both infinite. Then by the classification in Corollary 4.5, for $B_2(z)$, the interval $(e^{i\pi}, e^{i2\pi})$ is of type 2.*

Proof. For $s > 2$, it is easy to see

$$\sum_{k=2}^{\infty} \sum_{m\in\Omega_{k,2}} (1 - |a_{k,m}|) = \sum_{k=2}^{\infty} \frac{k+1}{(2k)^s} < \infty.$$

Since $\Omega_{k,0} \subset \Omega_{k,1} \subset \Omega_{k,2}$, our zero set of $B_i(z)$ satisfies the Blaschke condition. On the other hand, for $s > 3$,

$$\sum_{k=2}^{\infty} \sum_{m\in\Omega_{k,2}} \frac{1-\rho_k}{|1-a_{k,m}|} = \sum_{k=2}^{\infty} \sum_{m\in\Omega_{k,2}} \frac{1-\rho_k}{\sqrt{(1-\rho_k)^2 + 4\rho_k \sin^2 \pi \frac{m}{2k}}}$$
$$\leq c \sum_{k=2}^{\infty} k^{2-s} < \infty,$$

and

$$\sum_{k=2}^{\infty} \sum_{m\in\Omega_{k,2}} \frac{1-\rho_k}{|1+a_{k,m}|} = \sum_{k=2}^{\infty} \sum_{m\in\Omega_{k,2}} \frac{(1-\rho_k)}{\sqrt{(1-\rho_k)^2 + 4\rho_k \cos^2 \frac{m\pi}{2k}}}$$
$$\leq c \sum_{k=2}^{\infty} k^{2-s} < \infty.$$

Hence $B_i(z)$ is absolutely convergent at $z = 1,\ -1$. Obviously,

$$\{a_{k,m} : m \in \Omega_{k,0}, k \geq 2\} \cap \{\pi \leq \arg z \leq 2\pi\} = \emptyset,$$

so we obtain (i). By Theorem 1.1, the limits

$$\lim_{\theta\to\pi^+} \arg B_0(e^{i\theta}), \quad \lim_{\theta\to 2\pi^-} \arg B_0(e^{i\theta})$$

are both finite, so (ii) follows. (iii) and (iv) follow from Theorem 1.1, (i) and

$$\#\big(\{a_{k,m} : m \in \Omega_{k,1},\ k \geq 2\} \cap \{\pi \leq \arg z \leq \frac{6}{5}\pi\}\big) = \infty,$$
$$\#\big(\{a_{k,m} : m \in \Omega_{k,1},\ k \geq 2\} \cap \{\frac{9}{5}\pi \leq \arg z \leq 2\pi\}\big) < \infty,$$
$$\#\big(\{a_{k,m} : m \in \Omega_{k,2},\ k \geq 2\} \cap \{\pi \leq \arg z \leq \frac{6}{5}\pi\}\big) = \infty,$$
$$\#\big(\{a_{k,m} : m \in \Omega_{k,2},\ k \geq 2\} \cap \{\frac{9}{5}\pi \leq \arg z \leq 2\pi\}\big) = \infty.$$

This completes the proof. □

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(Wen-Hui Ai) College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China
awhxyz123@163.com

(Xin-Han Dong) College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China
xhdonghnsd@163.com

(Yue-Ping Jiang) College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China
ypjiang@hnu.edu.cn

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