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Product Hardy spaces associated with para-accretive functions and Tb theorem

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Abstract. We introduce the Hardy spaces associated with para-accretive functions in product domains and demonstrate their atomic decompositions as well as duality. Then, we establish the endpoint version of product Tb theorem with respect to our Hardy spaces and the dual spaces.

Contents

1. Introduction

Calderón and Zygmund [2] introduced a class of convolution singular integral operators that generalize the Hilbert transform and Riesz transforms. The L^2-boundedness of such convolution operators follows from the Plancherel theorem. For non-convolution singular integral operators, David and Journé [5] gave a general criterion for the L^2-boundedness that is the remarkable $T1$ theorem. Unfortunately, the $T1$ theorem cannot be applied

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to the Cauchy integral on a Lipschitz curve defined by

$$C(f)(x) = \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{(x-y) + i(a(x) - a(y))} dy,$$

where the function $a(x)$ satisfies the Lipschitz condition. Meyer first observed that $C(b) = 0$ provided $b(x) = 1 + ia'(x)$. Therefore, if the function 1 in the $T1$ theorem is allowed to be replaced by an accretive function b which is a bounded complex-valued function satisfying $\operatorname{Re} b(x) \geq \delta > 0$ almost everywhere, then this result would imply the L^2-boundedness of the Cauchy integrals on all Lipschitz curves. Replacing the function 1 by an accretive function b, McIntosh and Meyer [25] obtained a Tb theorem. Finally, David, Journé and Semmes [6] proved a new Tb theorem by replacing the function 1 by the so-called para-accretive functions b. To extend the Tb theorem to Hardy spaces, Han, Lee and Lin [13] introduced a new class of Hardy spaces associated to a para-accretive function b, denoted by H^p_b, which was given by those distributions such that their Littlewood-Paley g-functions associated to b belong to L^p, and showed the H^p_b-boundedness of singular integer operators.

By taking the space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ along with two-parameter family of dilations $(x, y) \mapsto (\delta_1 x, \delta_2 y), x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}, \delta_i > 0, i = 1, 2,$ instead of the classical one-parameter dilation, R. Fefferman and Stein [10] studied the product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on $\mathbb{R} \times \mathbb{R}$. Journé [22] generalized the product convolution singular integral operators to the product non-convolution singular integral operators and introduced a class of singular integral operators which coincides with the product convolution singular integral operators on product spaces. Moreover, Journé [22] proved the product $T1$ theorem. Suppose $b(x_1, x_2) = b_1(x_1)b_2(x_2)$, where b_1 and b_2 are para-accretive functions on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. A *generalized singular integral operator* is a continuous linear operator T from $bC^\eta_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ into $(bC^\eta_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))'$ for all $\eta > 0$ if the kernel of T is a singular integral kernel and, for $f_1, g_1 \in C^\eta_0(\mathbb{R}^{n_1})$ with $\operatorname{supp}(f_1) \cap \operatorname{supp}(g_1) = \varnothing$ and $f_2, g_2 \in C^\eta_0(\mathbb{R}^{n_2})$ with $\operatorname{supp}(f_2) \cap \operatorname{supp}(g_2) = \varnothing$,

$$\begin{aligned}
&\langle M_b T M_b f_1 \otimes f_2, g_1 \otimes g_2 \rangle \\
&= \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} b_2(x_2) b_1(x_1) g_1(x_1) g_2(x_2) K(x_1, x_2, y_1, y_2) \\
&\qquad \times b_2(y_2) b_1(y_1) f_1(y_1) f_2(y_2) dx_1 dx_2 dy_1 dy_2,
\end{aligned}$$

where M_b denotes the multiplication operator by b; that is, $M_b f(x) = b(x)f(x)$. Han, Lee and Lin [15] obtained the following product Tb theorem.

Theorem A **([15]).** *Suppose that b_1 and b_2 are para-accretive functions on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, $b(x_1, x_2) = b_1(x_1)b_2(x_2)$, and T is a generalized* *singular integral operator. Then T and $\widetilde{T}$ are bounded on $L^2(\mathbb{R}^{n_1+n_2})$ if and only if Tb, ${}^tTb$, $\widetilde{T}b$, ${}^t\widetilde{T}b \in BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $M_bTM_b \in WBP$.*

Here ${}^tTb$, $\widetilde{T}b$ and ${}^t\widetilde{T}b$ denote the three types of adjoint and partial adjoint operators of T. For simplicity we do not repeat all the definitions here, for the detail of the definition we refer to [15]. We also note that a similar result of the product Tb theorem was obtained by Ou [28] as well.

In 2013, Han, Li and Lu [18] developed a satisfactory theory of multiparameter Hardy spaces in the framework of spaces of homogeneous type under only the doubling condition and some regularity assumption of the underlying metric spaces. Later Han, Li and Ward [19] established the theory of multiparameter Hardy spaces on space of homogeneous type in the sense of Coifman and Weiss, without assuming any extra conditions. Such a metric space of homogeneous type includes the model case of Carnot-Carathédory spaces intrinsic to a family of vector fields satisfying Hörmander's condition of finite rank. Recently, Hart [21] presented a bilinear Tb theorem for singular operators, and proved the product Lebesgue space bounds for bilinear Riesz transform defined on Lipschitz curve as an application of his Tb theorem.

Consider the Cauchy integrals on product domain $\mathbb{R}\times\mathbb{R}$

$$C_{\rm pro}f(x_1,x_2) = \text{p.v.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(y_1,y_2)}{((x_1-y_1)+i(a_1(x_1)-a_1(y_1)))} \times\frac{1}{((x_2-y_2)+i(a_2(x_2)-a_2(y_2)))}dy_1dy_2,$$

which is a particular testing example operator for product Tb theorem. It is well-known that $C_{\rm pro}$ is not bounded on Chang-Fefferman product Hardy space $H^1(\mathbb{R}\times\mathbb{R})$.

Thus, a natural question arises. What is the right version of product Hardy spaces associated to the para-accretive $b_1(x)=1+ia_1'(x)$ and $b_2(x)=1+ia_2'(x)$ such that the operator $C_{\rm pro}$ is bounded on these product Hardy spaces? In this paper we focus on the following

Question 1: Can one develop the product Hardy spaces $H^p_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ associated to para-accretive functions b_1, b_2 for $p_0<p<\infty$, $p_0<1$?

Question 2: Motivated by the duality results between product Hardy space H^1 and product BMO on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ of Chang and R. Fefferman, can we establish the duality theory for $H^p_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $p_0<p\le 1$?

Question 3: What is the analogous endpoint version of product Tb theorem for singular integral operators T when $p=1$ and $p=\infty$ or more generally for $p_0<p\le 1$?

These questions will be answered affirmatively. We will employ a unified approach to answer these questions. This approach is achieved by the following steps:

1. We first introduce new Banach spaces which are spaces of product test functions and distributions in our framework. These spaces on the one-parameter space $\mathbb{R}^n$ associated with para-accretive function were introduced in [12], and on the spaces of test functions and distributions on product spaces of homogeneous type in the sense of Coifman and Weiss only satisfying the doubling condition and the regularity assumption were introduced in [18]. In this paper, we introduce spaces of test functions and distributions on product spaces associated with para-accretive functions.

2. We then establish discrete Calderón's identity on such product test function spaces. The classical Calderón's identity was first used by Calderón in [1]. Such an identity is a very powerful tool, in particular, in the theory of wavelet analysis. See [26] for more details. Using Coifman's decomposition of the identity operator, David, Journé and Semmes [6] provided a Calderón-type identity which is a key tool to prove the $T1$ theorems on space of homogeneous type and the Tb theorem on $\mathbb{R}^n$. The continuous and discrete versions of Calderón's identities associated with para-accretive function were developed in [12], [13], [14] and [24]. In this paper, we provide discrete Caldeonón's identity on the product spaces associated with para-accretive functions. This identity will be the main tool for us to establish the whole product theory.

3. We next demonstrate the Plancherel-Pôlya-type (or sup-inf) inequality in the multiparameter setting. The classical Plancherel-Pôlya inequality says that the L^p norm of f whose Fourier transform has compact support is equivalent to the ℓ^p norm of the restrictions of f at appropriate lattices. This kind of inequality was first proved in [13] on one-parameter space associated with para-accretive function. In this paper we prove such inequalities on product spaces associated with para-accretive functions. As an immediate consequence of the Plancherel-Pôlya inequality, the product Hardy space $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is well defined.

4. We then consider the the atomic decomposition theory on product domains. The atomic decomposition for one variable space H^p_b is given by [13] and [14]. For product Hardy space, Chang and R. Fefferman provided the atomic decomposition of $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Using the atomic decomposition, Han, Lee and Lin [15] proved the $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedness of Journé's product singular integrals. Lee [23] also obtained the theory in weighted product Hardy space. In this paper, we obtain the atomic decomposition in $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1+n_2})$.

5. We develop the dual spaces of $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Coifman and Meyer [27] introduced a new Hardy space H^1_b that can be defined by all functions f such that $bf \in H^1$. Similarly, BMO_b, the dual of H^1_b, is defined as follows: a function $f \in BMO_b$ if and only if $f = bg$, where $g \in BMO$. For $p < 1$, the dual space of H^p_b is defined similarly via the dual space of the classical Hardy spaces H^p. Han, Li and Lu [17] established and studied the product type Carleson measure space CMO^p and proved that it is the dual of the product

H^p for $0 < p_0 < p \le 1$ for some p_0 on product spaces of homogeneous type. In this paper, we introduce the Carleson measure space $CMO^p_{b_1b_2}$ associated with the para-accretive functions b_1 and b_2, and prove that the dual of the product Hardy space $H^p_{b_1b_2}$ studied in this paper can be characterized by $CMO^p_{b_1b_2}$. In particular, $CMO^1_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, the dual of $H^1_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. For the definition of para-accretive functions, we refer to Section 2.

6. We finally establish endpoint version of Tb theorem for singular integral operators T on product spaces (see Definition 5.1). In this paper we apply vector-valued singular integral, Calderón's identity, Littlewood-Paley theory and the almost orthogonality together with Fefferman's rectangle atomic decomposition and Journé's covering lemma to show that Journé's product singular integrals are bounded on $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ under some conditions. To be more specific, we have the following

Theorem 1.1. *Suppose that b_i are para-accretive functions on $\mathbb{R}^{n_i}$ for $i = 1, 2$ and T is a singular integral operator in Journé's class with regularity exponent ε (see Definition 5.1). Let $\max\limits_{i=1,2}\left\{\frac{n_i}{n_i+\varepsilon}\right\} < p \le 1$.*

(1) *$T_1^*(b_1) = T_2^*(b_2) = 0$ if and only if T is bounded from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*
(2) *$T_1^*(1) = T_2^*(1) = 0$ if and only if $TM_{b_1b_2}$ is bounded from $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*
(3) *$T_1^*(b_1) = T_2^*(b_2) = 0$ if and only if $TM_{b_1b_2}$ is bounded from $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

For the details of definition of $T_1^*(1)$, $T_2^*(1)$, $T_1^*(b_1)$ and $T_2^*(b_2)$, we refer to Section 5.

By the duality of $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and the duality of $H^1_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we also have

Theorem 1.2. *Suppose that b_i are para-accretive functions on $\mathbb{R}^{n_i}$ for $i = 1, 2$ and T is a singular integral operator in Journé's class with regularity exponent ε.*

(1) *$T_1(b_1) = T_2(b_2) = 0$ if and only if T admits a bounded extension from $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*
(2) *$T_1(1) = T_2(1) = 0$ if and only if $M_{b_1b_2}T$ admits a bounded extension from the space $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*
(3) *$T_1(b_1) = T_2(b_2) = 0$ if and only if $M_{b_1b_2}T$ admits a bounded extension from the space $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

For the details of definition of $T_1(1)$, $T_2(1)$, $T_1(b_1)$ and $T_2(b_2)$, we refer to Section 5.

This paper is organized as follows. In Section 2, we develop the Littlewood-Paley-Stein theory on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. To do this, we first recall some basic definitions, notion and known results established in the one parameter case, and then introduce the spaces of test functions and distributions on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We prove discrete Calderón's identity and the Plancherel-Pôlya inequalities (Theorems 2.8 and 2.13). We introduce the Littlewood-Paley-Stein square function and define the Hardy space $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Some important properties of $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are also proved in this section. In Section 3, we establish a new atomic decomposition for $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (Theorem 3.1). As an application, we show that the Hardy space $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ can be defined by $b_1b_2f \in H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (Corollary 3.2). Section 4 deals with the dual of $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. We introduce the generalized Carleson measure space $CMO^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and show that $CMO^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is the dual space of $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (Theorem 4.1). Finally, we show the boundedness and endpoint estimates of Journé's product singular integrals (Theorems 1.1 and 1.2).

2. The Littlewood-Paley-Stein theory on $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

2.1. The Littlewood-Paley-Stein theory on $H^p_b(\mathbb{R}^n)$. We begin by recalling the definition of para-accretive functions (see for example [6, 12]).

Definition 2.1. *A bounded complex-valued function b defined on $\mathbb{R}^n$ is said to be para-accretive if there exist constants $C, \gamma > 0$ such that for each cube $Q \subset \mathbb{R}^n$, there is a $Q' \subset Q$ with $\gamma|Q| \le |Q'|$ and satisfies*

$$\frac{1}{|Q|}\left|\int_{Q'} b(x)\,dx\right| \ge C.$$

By the Lebesgue differentiation theorem, it is easy to show that $|b(x)| \ge C > 0$ almost everywhere.

The following class of "test functions" associated to a para-accretive function was introduced in [12].

Definition 2.2. Fix two exponents $0 < \beta \le 1$ and $\gamma > 0$. Let b be a para-accretive function. A function f defined on $\mathbb{R}^n$ is said to be a *test function of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$* if f satisfies

$$|f(x)| \le C\frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}}, \tag{1}$$

$$|f(x)-f(x')| \le C\Big(\frac{|x-x'|}{d+|x-x_0|}\Big)^\beta \frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}} \tag{2}$$

for $|x-x'| \le \frac{d+|x-x_0|}{2}$, and

$$\int_{\mathbb{R}^n} f(x)b(x)\,dx = 0.$$

We write $\mathcal{M}^{(\beta,\gamma)}(x_0, d)$ for the collection of all test functions of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. For $f \in \mathcal{M}^{(\beta,\gamma)}(x_0, d)$, the norm of f

in $\mathcal{M}^{(\beta,\gamma)}(x_0,d)$ is defined by

$$\|f\|_{\mathcal{M}^{(\beta,\gamma)}(x_0,d)} = \inf\big\{C : (2.1) \text{ and } (2.2) \text{ hold}\big\}.$$

We denote $\mathcal{M}^{(\beta,\gamma)}(0,1)$ simply by $\mathcal{M}^{(\beta,\gamma)}$. Then $\mathcal{M}^{(\beta,\gamma)}$ is a Banach space under the norm $\|f\|_{\mathcal{M}^{(\beta,\gamma)}}$. It is easy to check that for any $x_0 \in \mathbb{R}^n$ and $d > 0$, $\mathcal{M}^{(\beta,\gamma)}(x_0,d) = \mathcal{M}^{(\beta,\gamma)}$ with equivalent norms. As usual, we write

$$b\mathcal{M}^{(\beta,\gamma)} = \big\{bg \,|\, g \in \mathcal{M}^{(\beta,\gamma)}\big\}.$$

If $f \in b\mathcal{M}^{(\beta,\gamma)}$ and $f = bg$ for $g \in \mathcal{M}^{(\beta,\gamma)}$, then the norm of f is defined by $\|f\|_{b\mathcal{M}^{(\beta,\gamma)}} = \|g\|_{\mathcal{M}^{(\beta,\gamma)}}$. The dual space $\big(b\mathcal{M}^{(\beta,\gamma)}\big)'$ consists of all linear functionals $\mathcal{L}$ from $b\mathcal{M}^{(\beta,\gamma)}$ to $\mathbb{C}$ satisfying

$$|\mathcal{L}(f)| \le C\|f\|_{b\mathcal{M}^{(\beta,\gamma)}} \qquad \text{for all } \ f \in b\mathcal{M}^{(\beta,\gamma)}.$$

We also need the definition of an approximation to the identity associated to a para-accretive function b.

Definition 2.3 ([12])**.** Let b be a para-accretive function. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is called an *approximation to the identity associated to* b if $S_k(x,y)$, the kernels of S_k, are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ and there exist a constant C and some $0 < \varepsilon \le 1$ such that, for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathbb{R}^n$,

(i) $|S_k(x,y)| = 0$ if $|x-y| \ge C2^{-k}$ and $|S_k(x,y)| \le C2^{kn}$,

(ii) $|S_k(x,y) - S_k(x',y)| \le C2^{k(1+\varepsilon)}|x-x'|^{\varepsilon}$,

(iii) $|S_k(x,y) - S_k(x,y')| \le C2^{k(n+\varepsilon)}|y-y'|^{\varepsilon}$,

(iv) $\big|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]\big|$
$\le C2^{2k(n+\varepsilon)}|x-x'|^{\varepsilon}|y-y'|^{\varepsilon}$,

(v) $\displaystyle\int_{\mathbb{R}^n} S_k(x,y)b(y)dy = 1 \quad$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

(vi) $\displaystyle\int_{\mathbb{R}^n} S_k(x,y)b(x)dx = 1 \quad$ for all $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$.

Remark 2.1. *Let $D_k(x,y) := S_k(x,y) - S_{k-1}(x,y)$. It is clear that $D_k(x,\cdot) \in \mathcal{M}^{(\varepsilon,\varepsilon)}(x,2^{-k})$ and hence $D_k(x,\cdot) \in \mathcal{M}^{(\varepsilon,\varepsilon)}$. Similarly, $D_k(\cdot,y) \in \mathcal{M}^{(\varepsilon,\varepsilon)}$. By definition, it is clear to see that $bD_k(x,\cdot) \in b\dot{\mathcal{M}}^{(\beta,\gamma)}$ for $0 < \beta,\gamma < \varepsilon$.*

We now recall the definition of H^p_b.

Definition 2.4 ([13])**.** Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity associated to a para-accretive function b with regularity exponent ε. Set $D_k = S_k - S_{k-1}$. For $0 < \beta,\gamma < \varepsilon$, denote by $\dot{\mathcal{M}}^{(\beta,\gamma)}$ the closure of $\mathcal{M}^{(\varepsilon,\varepsilon)}$ with respect to the norm $\|\cdot\|_{\mathcal{M}^{(\beta,\gamma)}}$. The Hardy space H^p_b, $\frac{n}{n+\varepsilon} < p \le 1$, is the collection of $f \in \big(b\dot{\mathcal{M}}^{(\beta,\gamma)}\big)'$ satisfying

$$\|f\|_{H^p_b} := \|g(f)\|_p < \infty,$$

where $g(f)$, the *Littlewood-Paley g-function* of f, is defined by

$$g(f)(x) := \Big\{ \sum_k |D_k(bf)(x)|^2 \Big\}^{1/2}.$$

We would like to point out that one of main results in [13] is the following Plancherel-Pôlya type inequality.

Proposition 2.5 ([13]). *Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ and $\{R_k\}_{k\in\mathbb{Z}}$ are approximations to the identity associated to b with regularity exponent ε, and $\frac{n}{n+\varepsilon} < p < \infty$. Set $D_k = S_k - S_{k-1}$ and $E_k = R_k - R_{k-1}$. Then, for $f \in \big(b\dot{\mathcal{M}}^{(\beta,\gamma)}\big)'$,*

$$\Big\| \Big\{ \sum_k \sum_{Q_k} \Big(\sup_{z\in Q_k} |E_k(bf)(z)| \Big)^2 \chi_{Q_k} \Big\}^{1/2} \Big\|_p \approx \Big\| \Big\{ \sum_k \sum_{Q_k} \Big(\inf_{z\in Q_k} |D_k(bf)(z)| \Big)^2 \chi_{Q_k} \Big\}^{1/2} \Big\|_p,$$

where Q_k are dyadic cubes with side length $\ell(Q_k) = 2^{-k-N}$ for some fixed positive large integer N and χ_{Q_k} are characteristic functions of cubes Q_k.

We have the continuous and discrete versions of the Calderón reproducing formula.

Proposition 2.6 ([12], [14] and [24]). *Let b be a para-accretive function and $\{S_k\}$ be an approximation to the identity associated to b with regularity exponent ε. Set $D_k = S_k - S_{k-1}$. Then there exists a family of operators $\{\widetilde{D}_k\}$ such that, for $f \in \mathcal{M}^{(\varepsilon,\varepsilon)}$,*

$$f(x) = \sum_k D_k M_b \widetilde{D}_k(bf)(x)$$

and

$$f(x) = \sum_k \sum_{Q_k} D_k(x, x_{Q_k}) \widetilde{D}_k(bf)(x_{Q_k}) \int_{Q_k} b(y)dy,$$

where $\sum_{Q_k}$ runs over all dyadic cubes Q_k's with side length $\ell(Q_k) = 2^{-k-N}$ for some fixed positive large integer N, x_{Q_k} is any fixed point in Q_k, and the series converges in $\mathcal{M}^{(\beta,\gamma)}$ for $0 < \beta, \gamma < \varepsilon$ and in $L^q, 1 < q < \infty$. Moreover, $\widetilde{D}_k(x,y)$, the kernels of $\widetilde{D}_k$, satisfy the following estimates: for $0 < \varepsilon' < \varepsilon$, there exists a constant $C > 0$ such that

$$|\widetilde{D}_k(x,y)| \le C \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}},$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x,y')| \le C \Big(\frac{|y-y'|}{(2^{-k} + |x-y|)} \Big)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}}$$

$$\textit{for } |y-y'| \le (2^{-k}+|x-y|)/2,$$

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)b(y)\,dy = 0 \qquad \textit{for all } k\in\mathbb{Z} \textit{ and } x\in\mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)b(x)\,dx = 0 \qquad \textit{for all } k\in\mathbb{Z} \textit{ and } y\in\mathbb{R}^n.$$

2.2. Test functions and distributions associated to para-accretive functions. We first introduce spaces of test functions and distributions on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ associated to para-accretive functions.

Definition 2.7. Fix four exponents $0<\beta_1,\beta_2\le 1$ and $\gamma_1,\gamma_2>0$. Let b_1 and b_2 be para-accretive functions on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. A function f defined on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ is said to be a *test function of type* $(\beta_1,\beta_2,\gamma_1,\gamma_2)$ *centered at* $(x_0,y_0)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ *with width* $d_1,d_2>0$ if f satisfies

(i) $\|f(\cdot,y)\|_{\mathcal{M}^{(\beta_1,\gamma_1)}(x_0,d_1)} \le C\dfrac{d_2^{\gamma_2}}{(d_2+|y-y_0|)^{n_2+\gamma_2}}$;

(ii) $\|f(\cdot,y)-f(\cdot,y')\|_{\mathcal{M}^{(\beta_1,\gamma_1)}(x_0,d_1)}$
$\le C\Big(\dfrac{|y-y'|}{d_2+|y-y_0|}\Big)^{\beta_2}\dfrac{d_2^{\gamma_2}}{(d_2+|y-y_0|)^{n_2+\gamma_2}}$ for $|y-y'|\le\dfrac{d_2+|y-y_0|}{2}$;

(iii) properties (i) and (ii) also hold with x and y interchanged;

(iv) $\displaystyle\int_{\mathbb{R}^{n_1}} f(x,y)b_1(x)dx = \int_{\mathbb{R}^{n_2}} f(x,y)b_2(y)dy = 0$ for all $x\in\mathbb{R}^{n_1}$ and $y\in\mathbb{R}^{n_2}$.

If f is a test function of type $(\beta_1,\beta_2,\gamma_1,\gamma_2)$ centered at $(x_0,y_0)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with width $d_1,d_2>0$, we write

$$f\in\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}(x_0,y_0,d_1,d_2),$$

and the norm of f is defined by

$$\|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}(x_0,y_0,d_1,d_2)} = \inf\big\{C : \text{(i), (ii) and (iii) hold}\big\}.$$

Similarly, we denote by $\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ the class of $\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}(0,0,1,1)$. We can check that $\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}=\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}(x_0,y_0,d_1,d_2)$ with equivalent norms for all $(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Let $\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ denote the completion of the space $\mathcal{M}^{(\varepsilon_1,\varepsilon_2,\varepsilon_1,\varepsilon_2)}$ in $\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ with $0<\beta_i,\gamma_i<\varepsilon_i$, $i=1,2$. For $f\in\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$, we define $\|f\|_{\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}=\|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}$. As usual, we write

$$b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}=\big\{b_1b_2g \mid g\in\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big\}.$$

If $f\in b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ and $f=b_1b_2g$ for $g\in\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$, then the norm of f is defined by

$$\|f\|_{b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}=\|g\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}.$$

The dual space $\big(b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$ consists of all linear functionals $\mathcal{L}$ from $b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ to $\mathbb{C}$ satisfying

$$|\mathcal{L}(f)| \le C\|f\|_{b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}} \qquad \text{for all} \ \ f \in b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}.$$

2.3. Discrete Calderón identity on test function spaces. Through the paper, we always denote b_{iQ_i} by

$$\frac{1}{|Q_i|}\int_{Q_i} b_i(x_i)dx_i, \qquad i=1,2.$$

Theorem 2.8. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be the operators given in Proposition 2.6 on $\mathbb{R}^{n_i}$, $i=1,2$. Then*

$$\begin{aligned} f(x_1,x_2) = \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} &|Q_{k_1}||Q_{k_2}|D_{k_1}(x_1,x_{Q_{k_1}})D_{k_2}(x_2,y_{Q_{k_2}}) \\ &\times \widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}}, \end{aligned}$$

where the series converges in $\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ for $0<\beta_i,\gamma_i<\varepsilon_i$, $i=1,2$ and in L^p, $1<p<\infty$.

Proof. The proof of this theorem is based on the method of iteration and some known estimates on $\mathbb{R}^n$. We first show the $L^p, 1<p<\infty$, convergence. Denote

$$\begin{aligned} g(x_1,x_2) = f(x_1,x_2) - \sum_{|k_1|\le L_1}\sum_{|k_2|\le L_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} &|Q_{k_1}||Q_{k_2}|D_{k_1}(x_1,x_{Q_{k_1}}) \\ \times D_{k_2}(x_2,y_{Q_{k_2}})&\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}} \\ =: g_1(x_1,x_2)+g_2(x_1,x_2),& \end{aligned}$$

where

$$\begin{aligned} g_1(x_1,x_2) = &\sum_{|k_2|\le L_2}\sum_{Q_{k_2}}|Q_{k_2}|D_{k_2}(x_2,y_{Q_{k_2}})\widetilde{D}_{k_2}M_{b_2}f(x_1,y_{Q_{k_2}})b_{2Q_{k_2}} \\ &- \sum_{|k_1|\le L_1}\sum_{Q_{k_1}}|Q_{k_1}|D_{k_1}(x_1,x_{Q_{k_1}})b_{1Q_{k_1}}\widetilde{D}_{k_1}M_{b_1} \\ &\times\Big(\sum_{|k_2|\le L_2}\sum_{Q_{k_2}}|Q_{k_2}|D_{k_2}(x_2,y_{Q_{k_2}})\widetilde{D}_{k_2}M_{b_2}f(\cdot,y_{Q_{k_2}})b_{2Q_{k_2}}\Big)(x_{Q_{k_1}}) \end{aligned}$$

and

$$\begin{aligned} &g_2(x_1,x_2) \\ &= f(x_1,x_2) - \sum_{|k_2|\le L_2}\sum_{Q_{k_2}}|Q_{k_2}|D_{k_2}(x_2,y_{Q_{k_2}})\widetilde{D}_{k_2}M_{b_2}f(x_1,y_{Q_{k_2}})b_{2Q_{k_2}}. \end{aligned}$$

We now need the following two estimates on $\mathbb{R}^n$, which was proved in [13] (the original idea and version is due to [12], see also [7] for a systematical introduction): there exists a constant C such that for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and any integers L,

$$\Big\| \sum_{|k|\le L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \widetilde{D}_k(bf)(y_{Q_k}) b_{Q_k} \Big\|_p \le C\|f\|_p \tag{3}$$

and

$$\begin{aligned} &\Big\| \sum_{|k|\le L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \widetilde{D}_k(bf)(y_{Q_k}) b_{Q_k} - f \Big\|_p \\ &\qquad \le C \Big\| \Big\{ \sum_{|k|\ge L} \sum_{Q_k} |D_k(f)|^2 \Big\}^{1/2} \Big\|_p . \end{aligned} \tag{4}$$

Using (3) first and then (4) yields

$$\|g_1\|_{L^p} \le C \Big\| \Big\{ \sum_{|k_1|\ge L_1} \sum_{|k_2|\le L_2} |D_{k_1} D_{k_2}(f)|^2 \Big\}^{1/2} \Big\|_p \to 0 \qquad \text{as } L_1 \to \infty.$$

Similarly,

$$\|g_2\|_{L^p} \to 0 \qquad \text{as } L_2 \to \infty.$$

Hence, the L^p convergence follows.

To see the convergence in the space of test functions, we need the following estimates on $\mathbb{R}^n$ which, again, was proved in [13] with the original idea and version in [12], see also [7]): for $f \in \mathcal{M}^{(\beta,\gamma)}$ and any intrgers L,

$$\Big\| \sum_{|k|\le L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \widetilde{D}_k(M_b f)(y_{Q_k}) b_{Q_k} \Big\|_{\mathcal{M}^{(\beta,\gamma)}} \le C\|f\|_{\mathcal{M}^{(\beta,\gamma)}} \tag{5}$$

and

$$\begin{aligned} &\Big\| \sum_{|k|\le L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \widetilde{D}_k(M_b f)(y_{Q_k}) b_{Q_k} - f \Big\|_{\mathcal{M}^{(\beta',\gamma')}} \\ &\qquad \le C 2^{-L\delta} \|f\|_{\mathcal{M}^{(\beta,\gamma)}}, \end{aligned} \tag{6}$$

where C is a constant, $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$.

We observe that if $f \in \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$, then $\|f(\cdot, x_2)\|_{\mathcal{M}^{(\beta_1,\gamma_1)}}$, as a function of the variable x_2, is in $\mathcal{M}^{(\beta_2,\gamma_2)}$ and

$$\big\| \|f(\cdot,\cdot)\|_{\mathcal{M}^{(\beta_1,\gamma_1)}} \big\|_{\mathcal{M}^{(\beta_2,\gamma_2)}} \le \|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}.$$

Similarly,

$$\big\| \|f(\cdot,\cdot)\|_{\mathcal{M}^{(\beta_2,\gamma_2)}} \big\|_{\mathcal{M}^{(\beta_1,\gamma_1)}} \le \|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}.$$

Therefore, we obtain

$$\|g_1(\cdot, x_2)\|_{\mathcal{M}^{(\beta_1',\gamma_1')}}$$

$$\le C2^{-L_1\delta}\Big\| \sum_{|k_2|\le L_2}\sum_{Q_{k_2}} |Q_{k_2}| D_{k_2}(x_2, y_{Q_{k_2}})\widetilde{D}_{k_2} M_{b_2} f(x_1, y_{Q_{k_2}}) b_{2Q_{k_2}} \Big\|_{\mathcal{M}^{(\beta_1,\gamma_1)}}$$
$$\le C2^{-L_1\delta}\Big\| \|f(\cdot,\cdot)\|_{\mathcal{M}^{(\beta_2,\gamma_2)}} \frac{d_2^{\gamma_2}}{(d_2+|x_2-y_0|)^{n_2+\gamma_2}} \Big\|_{\mathcal{M}^{(\beta_1,\gamma_1)}}$$
$$\le C2^{-L_1\delta}\|f(\cdot,\cdot)\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}} \frac{d_2^{\gamma_2}}{(d_2+|x_2-y_0|)^{n_2+\gamma_2}}$$

and, similarly,

$$\|g_2(\cdot, x_2)\|_{\mathcal{M}^{(\beta_1',\gamma_1')}} \le C2^{-L_2\delta}\,\|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}} \frac{d_2^{\gamma_2}}{(d_2+|x_2-y_0|)^{n_2+\gamma_2}}.$$

Noting that $g(x_1,x_2)-g(x_1,x_2') = [g_1(x_1,x_2)-g_1(x_1,x_2')]+[g_2(x_1,x_2)-g_2(x_1,x_2')]$ and repeating the same estimates imply

$$\|g(\cdot,x_2)-g(\cdot,x_2')\|_{\mathcal{M}^{(\beta_1',\gamma_1')}}$$
$$\le C(2^{-L_1\delta}+2^{-L_2\delta})\|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}\Big(\frac{|x_2-x_2'|}{d_2+|x_2-y_0|}\Big)^{\beta_2}\frac{d_2^{\gamma_2}}{(d_2+|x_2-y_0|)^{n_2+\gamma_2}},$$

where $|x_2-x_2'| \le \frac{d_2+|x_2-y_0|}{2}$. The same proof can be carried out to the estimates if we interchange the roles of x and y. Hence,

$$\|g\|_{\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}} \le C(2^{-L_1\delta}+2^{-L_2\delta})\,\|f\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}\,, \tag{7}$$

which yields the convergence in $\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$. □

Using the same argument, we also have continuous Calderón identity on test function spaces as follows.

Theorem 2.9. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be given in Proposition 2.6 on $\mathbb{R}^{n_i}$, $i=1,2$. Then*

$$f = \sum_{k_1}\sum_{k_2} D_{k_1}M_{b_1}D_{k_2}M_{b_2}\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f,$$

where the series converges in $\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ for $0<\beta_i,\gamma_i<\varepsilon_i$, $i=1,2$ and in L^q, $1<q<\infty$.

Theorem 2.10. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be the operators given in Proposition 2.6 on $\mathbb{R}^{n_i}$, $i=1,2$. Then for $f\in b_1b_2\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$*

$$f(x_1,x_2) = \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |Q_{k_1}||Q_{k_2}| b_1(x_1)D_{k_1}(x_1,x_{Q_{k_1}})b_2(x_2)D_{k_2}(x_2,y_{Q_{k_2}})$$
$$\times \widetilde{D}_{k_1}\widetilde{D}_{k_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}},$$

where the series converges in $b_1b_2\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ for $0<\beta_i,\gamma_i<\varepsilon_i$, $i=1,2$ and in L^p, $1<p<\infty$.

Proof. Let $f \in b_1b_2\dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$. Then there exists a $g \in \dot{\mathcal{M}}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ such that $f = b_1b_2g$. Since $\|f\|_{b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}} = \|g\|_{\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}}$ and Theorem 2.8, the proof is completed. □

By the duality, we have the following theorem.

Theorem 2.11. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be the operators given in Proposition 2.6 on $\mathbb{R}^{n_i}$, $i = 1, 2$. Then for $f \in \big(b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$ with $0 < \beta_i, \gamma_i < \varepsilon_i$, $i = 1, 2$*

$$f(x_1, x_2) = \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |Q_{k_1}||Q_{k_2}|\widetilde{D}^*_{k_1}(x_1, x_{Q_{k_1}})\widetilde{D}^*_{k_2}(x_2, y_{Q_{k_2}}) \times D^*_{k_1}D^*_{k_2}f(x_{Q_{k_1}}, y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}},$$

where the series converges in the sense of distribution, where $D^*_{k_i}, i = 1, 2$ denote the adjoint operator of D_{k_i}.

By [12] and [14], we also have the following continuous Calderón reproducing formula. For $f \in \mathcal{M}^{(\varepsilon,\varepsilon)}$,

$$f(x) = \sum_k \widetilde{\widetilde{D}}_k M_b D_k(bf)(x),$$

where $\widetilde{\widetilde{D}}_k(x, y)$, the kernels of $\widetilde{\widetilde{D}}_k$ satisfy the same conditions of $\widetilde{D}_k(y, x)$. By the above argument, we also have

Theorem 2.12. *Let D_{k_i} and $\widetilde{D}_{k_i}$ be the operators given in Proposition 2.6 on $\mathbb{R}^{n_i}$, $i = 1, 2$. Then for $f \in \big(b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$ with $0 < \beta_i, \gamma_i < \varepsilon_i$, $i = 1, 2$*

$$f(x_1, x_2) = \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |Q_{k_1}||Q_{k_2}|D^*_{k_1}(x_1, x_{Q_{k_1}})D^*_{k_2}(x_2, y_{Q_{k_2}}) \times \widetilde{D}_{k_1}\widetilde{D}_{k_2}f(x_{Q_{k_1}}, y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}},$$

where the series converges in the sense of distribution.

2.4. Plancherel-Pôlya inequality on test function spaces. Using the discrete Calderón identity we prove the following Plancherel-Pôlya inequality on product domains associated with para-accretive functions.

Theorem 2.13. *Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ and $\{R_{k_i}\}_{k_i\in\mathbb{Z}}$ be two approximations to the identity associated to b_i with regularity exponent ε on $\mathbb{R}^{n_i}$ and $\frac{n_i}{n_i+\varepsilon} < p < \infty$, $i = 1, 2$. Set $D_{k_i} = S_{k_i} - S_{k_i-1}$ and $E_{k_i} = R_{k_i} - R_{k_i-1}$. Then, for $f \in \big(b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$,*

$$\bigg\|\bigg\{\sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}}\Big(\sup_{z_1\in Q_{k_1}}\sup_{z_2\in Q_{k_2}}|E_{k_1}M_{b_1}E_{k_2}M_{b_2}(f)(z_1, z_2)|\Big)^2\chi_{Q_{k_1}}\chi_{Q_{k_2}}\bigg\}^{\frac{1}{2}}\bigg\|_p$$

$$\approx \Big\|\Big\{\sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}}\Big(\inf_{z_1\in Q_{k_1}}\inf_{z_2\in Q_{k_2}}|D_{k_1}M_{b_1}D_{k_2}M_{b_2}(f)(z_1,z_2)|\Big)^2\chi_{Q_{k_1}}\chi_{Q_{k_2}}\Big\}^{\frac{1}{2}}\Big\|_p,$$

where Q_{k_i} are dyadic cubes with side length $\ell(Q_{k_i})=2^{-k_i-N}$ for some fixed positive large integer N and $\chi_{Q_{k_i}}$ are characteristic functions of cubes Q_{k_i}, $i=1,2$.

Proof. Given $f\in\big(b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$, Theorem 2.12 shows

$$\langle f,g\rangle=\Big\langle\sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}}|Q_{k_1}||Q_{k_2}|D_{k_1}(\cdot,x_{Q_{k_1}})D_{k_2}(\cdot,y_{Q_{k_2}})$$
$$\times\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}},g\Big\rangle,$$

where $g\in b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$. Hence, for any $j_1,j_2\in\mathbb{Z}$, we have

$$\begin{aligned}&E_{j_1}M_{b_1}E_{j_2}M_{b_2}(f)(x_1,x_2)\\&=\sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}}|Q_{k_1}||Q_{k_2}|E_{j_1}M_{b_1}E_{j_2}M_{b_2}D_{k_1}(x_1,x_{Q_{k_1}})D_{k_2}(x_2,y_{Q_{k_2}})\\&\qquad\times\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}}.\end{aligned}$$

For $0<\varepsilon'<\varepsilon$ and $x_1\in Q_{j_1}$, the almost orthogonality estimate (see [20, Lemma 4.3] and [24, p. 10]) gives

$$|E_{j_1}M_{b_1}D_{k_1}(x_1,x_{Q_{k_1}})|\le C|2^{-|j_1-k_1|\varepsilon'}\frac{2^{-(j_1\wedge k_1)\varepsilon'}}{(2^{-(j_1\wedge k_1)}+|x_1-x_{Q_{k_1}}|)^{n_1+\varepsilon'}}.$$

Therefore, for every $z_1\in Q_{j_1}$ and $z_2\in Q_{j_2}$, we have

$$\begin{aligned}&|E_{j_1}M_{b_1}E_{j_2}M_{b_2}(f)(z_1,z_2)|\\&\le C\sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}}|Q_{k_1}||Q_{k_2}|2^{-|j_1-k_1|\varepsilon'}\frac{2^{-(j_1\wedge k_1)\varepsilon'}}{(2^{-(j_1\wedge k_1)}+|z_1-x_{Q_{k_1}}|)^{n_1+\varepsilon'}}\\&\quad\times 2^{-|j_2-k_2|\varepsilon'}\frac{2^{-(j_2\wedge k_2)\varepsilon'}}{(2^{-(j_2\wedge k_2)}+|z_2-y_{Q_{k_2}}|)^{n_2+\varepsilon'}}|\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})|,\end{aligned}$$

where $\varepsilon'<\varepsilon$. Thus,

$$\begin{aligned}&\sup_{z_1\in Q_{j_1}}\sup_{z_2\in Q_{j_2}}|E_{j_1}M_{b_1}E_{j_2}M_{b_2}(f)(z_1,z_2)|\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2)\\&\quad\le C\sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}}|Q_{k_1}||Q_{k_2}|2^{-|j_1-k_1|\varepsilon'}\frac{2^{-(j_1\wedge k_1)\varepsilon'}}{(2^{-(j_1\wedge k_1)}+|x_1-x_{Q_{k_1}}|)^{n_1+\varepsilon'}}\\&\qquad\times 2^{-|j_2-k_2|\varepsilon'}\frac{2^{-(j_2\wedge k_2)\varepsilon'}}{(2^{-(j_2\wedge k_2)}+|x_2-y_{Q_{k_2}}|)^{n_2+\varepsilon'}}|\end{aligned}$$

$$\times \widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2).$$

By an estimate in [10, pp. 147-148],

$$\sum_{Q_{k_1}} \frac{2^{-(j_1\wedge k_1)\varepsilon'}}{(2^{-(j_1\wedge k_1)} + |x_1 - x_{Q_{k_1}}|)^{n_1+\varepsilon'}} |\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|$$
$$\leq C 2^{(k_1\wedge j_1)n_1} 2^{[k_1-(k_1\wedge j_1)]n_1/r}$$
$$\times \Big\{ M_1 \Big(\sum_{Q_{k_1}} |\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|\chi_{Q_{k_1}} \Big)^r \Big\}^{1/r}(x_1),$$

where M_1 is the Hardy-Littlewood maximal function on $\mathbb{R}^{n_1}$ and $\max_{i=1,2} \frac{n_i}{n_i+\varepsilon''} < r < p$. Therefore,

$$\sup_{z_1\in Q_{j_1}} \sup_{z_2\in Q_{j_2}} |E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1,z_2)|\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2)$$
$$\leq C \sum_{k_1}\sum_{k_2} |Q_{k_1}||Q_{k_2}| 2^{-|j_1-k_1|\varepsilon'} 2^{(k_1\wedge j_1)n_1} 2^{[k_1-(k_1\wedge j_1)]n_1/r}$$
$$\times 2^{-|j_2-k_2|\varepsilon'} 2^{(k_2\wedge j_2)n_2} 2^{[k_2-(k_2\wedge j_2)]n_2/r}$$
$$\times \Big\{ M_2 \Big(\sum_{Q_{k_1}} \Big\{ M_1 \Big(\sum_{Q_{k_1}} |\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|\chi_{Q_{k_1}} \Big)^r \Big\}(x_1)$$
$$\times \chi_{Q_{k_2}} \Big) \Big\}^{1/r}(x_2)\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2),$$

where M_2 is the Hardy-Littlewood maximal function on $\mathbb{R}^{n_2}$. By Cauchy-Schwartz inequality,

$$\sup_{z_1\in Q_{j_1}} \sup_{z_2\in Q_{j_2}} |E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1,z_2)|^2\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2)$$
$$\leq C \sum_{k_1}\sum_{k_2} 2^{-k_1 n_1} 2^{-|j_1-k_1|\varepsilon'} 2^{(k_1\wedge j_1)n_1} 2^{[k_1-(k_1\wedge j_1)]n_1/r}$$
$$\times 2^{-k_2 n_2} 2^{-|j_2-k_2|\varepsilon'} 2^{(k_2\wedge j_2)n_2} 2^{[k_2-(k_2\wedge j_2)]n_2/r}$$
$$\times \Big\{ M_2 \Big(\sum_{Q_{k_1}} \Big\{ M_1 \Big(\sum_{Q_{k_1}} |\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|\chi_{Q_{k_1}} \Big)^r \Big\}(x_1)$$
$$\times \chi_{Q_{k_2}} \Big) \Big\}^{2/r}(x_2)\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2)$$

since

$$\sup_{j_i} \sum_{k_i} 2^{-k_i n_i} 2^{-|j_i-k_i|\varepsilon'} 2^{(k_i\wedge j_i)n_i} 2^{[k_i-(k_i\wedge j_i)]n_i/r} < \infty, \qquad i = 1, 2.$$

This yields

$$\begin{aligned}
&\Big\{\sum_{j_1}\sum_{j_2}\sum_{Q_{j_1}}\sum_{Q_{j_2}}\sup_{z_1\in Q_{j_1}}\sup_{z_2\in Q_{j_2}}|E_{j_1}M_{b_1}E_{j_2}M_{b_2}(f)(z_1,z_2)|^2\chi_{Q_{j_1}}(x_1)\chi_{Q_{j_2}}(x_2)\Big\}^{1/2}\\
&\le C\Big\{\sum_{j_1}\sum_{j_2}\sum_{k_1}\sum_{k_2}2^{-k_1n_1}2^{-|j_1-k_1|\varepsilon'}2^{(k_1\wedge j_1)n_1}2^{[k_1-(k_1\wedge j_1)]n_1/r}\\
&\quad\times 2^{-k_2n_2}2^{-|j_2-k_2|\varepsilon'}2^{(k_2\wedge j_2)n_2}2^{[k_2-(k_2\wedge j_2)]n_2/r}\\
&\quad\times\Big\{M_2\Big(\sum_{Q_{k_1}}\Big\{M_1\Big(\sum_{Q_{k_1}}|\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})|\chi_{Q_{k_1}}\Big)^r\Big\}(x_1)\\
&\qquad\qquad\times\chi_{Q_{k_2}}\Big)\Big\}^{2/r}(x_2)\Big\}^{1/2}\\
&\le C\Big\{\sum_{k_1,k_2}\Big\{M_2\Big(\sum_{Q_{k_1}}\Big\{M_1\Big(\sum_{Q_{k_1}}|\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})|\chi_{Q_{k_1}}\Big)^r\Big\}(x_1)\\
&\qquad\qquad\times\chi_{Q_{k_2}}\Big)\Big\}^{\frac{2}{r}}(x_2)\Big\}^{\frac12}.
\end{aligned}$$

Since $x_{Q_{k_1}}$ and $y_{Q_{k_2}}$ are any fixed point in Q_{k_1} and Q_{k_2}, respectively, by the Fefferman-Stein vector-valusd maximal function inequality twice with $r<p$, the proof is completed. □

2.5. The Hardy spaces on product domains.

Definition 2.14. Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be approximations to the identity associated to b_i with regularity exponent ε on $\mathbb{R}^{n_i}$ and $\frac{n_i}{n_i+\varepsilon}<p<\infty$, $i=1,2$. Set $D_{k_i}=S_{k_i}-S_{k_i-1}$. For $0<\beta_i,\gamma_i<\varepsilon$, $f\in\big(b_1b_2\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$, the *Littlewood-Paley $\mathcal{G}$-function* of f is defined by

$$\mathcal{G}(f)(x_1,x_2):=\Big\{\sum_{k_1}\sum_{k_2}|D_{k_1}M_{b_1}D_{k_2}M_{b_2}(f)(x_1,x_2)|^2\Big\}^{1/2}.$$

We point out that due the Remark 2.1, this discrete Littlewood-Paley square function is well-defined. Strictly speaking we shall denote it by $\mathcal{G}_{b_1b_2}(f)(x_1,x_2)$, since it is associated with the functions b_1 and b_2. However, for the sake of simplicity, we drop the subscript b_1b_2.

Applying the result of one parameter and using the iteration as given in [10], we immediately obtain

Theorem 2.15. *If $f\in L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $1<p<\infty$, then $\|\mathcal{G}(f)\|_p\approx\|f\|_p$.*

We point out that the following discrete Littlewood-Paley function is more convenient for the study of the Hardy space $H^p_{b_1b_2}$ when $p\le 1$.

Definition 2.16. Let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be approximations to the identity associated to b_i with regularity exponent ε on $\mathbb{R}^{n_i}$ and $\frac{n_i}{n_i+\varepsilon} < p < \infty$, $i = 1, 2$. Set $D_{k_i} = S_{k_i} - S_{k_i-1}$. For $0 < \beta_i, \gamma_i < \varepsilon$, $f \in \big(b_1 b_2 \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$, the discrete *Littlewood-Paley $\mathcal{G}_d$-function* of f is defined by

$$\mathcal{G}_d(f)(x_1, x_2)$$
$$:= \Big\{ \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |D_{k_1} M_{b_1} D_{k_2} M_{b_2}(f)(x_1, x_2)|^2 \chi_{Q_{k_1}}(x_1)\chi_{Q_{k_2}}(x_2)\Big\}^{1/2}.$$

By the Plancherel-Pôlya inequality, the L^p norm of these two kinds of Littlewood-Paley functions are equivalent.

Theorem 2.17. *Let $0 < \beta_i, \gamma_i < \varepsilon$ and $\frac{n_i}{n_i+\varepsilon} < p < \infty$, $i = 1, 2$. If $f \in \big(b_1 b_2 \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$, then $\|\mathcal{G}_d(f)\|_p \approx \|\mathcal{G}(f)\|_p$.*

We now define the Hardy spaces as follows.

Definition 2.18. Suppose that $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ are approximations to the identity associated to b_i with regularity exponent ε on $\mathbb{R}^{n_i}$ and $\frac{n_i}{n_i+\varepsilon} < p \le 1$, $i = 1, 2$. Set $D_{k_i} = S_{k_i} - S_{k_i-1}$. For $0 < \beta_i, \gamma_i < \varepsilon$, the Hardy space $H^p_{b_1b_2}$ is the collection of $f \in \big(b_1 b_2 \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$ satisfying

$$\|f\|_{H^p_{b_1b_2}} := \|\mathcal{G}_d(f)\|_p < \infty.$$

Remark 2.2. *Let $f \in H^p_{b_1b_2} \subseteq \big(b_1 b_2 \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}\big)'$ and denote by*

$$f_L(x_1, x_2) = \sum_{|k_1|\le L}\sum_{|k_2|\le L}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |Q_{k_1}||Q_{k_2}| D^*_{k_1}(x_1, x_{Q_{k_1}}) D^*_{k_2}(x_2, y_{Q_{k_2}})$$
$$\times \widetilde{D}_{k_1}\widetilde{D}_{k_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}.$$

Using Theorem 2.12 and applying the same argument for the proof of Plancherel-Pôlya inequality, we obtain that

$$\|\mathcal{G}_d(f - f_L)\|_p \le C\Big\|\Big\{ \sum_{|k_1|>L}\sum_{|k_2|>L}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |\widetilde{D}_{k_1} M_{b_1}\widetilde{D}_{k_2} M_{b_2}(f)(x_1, x_2)|^2$$
$$\times \chi_{Q_{k_1}}(x_1)\chi_{Q_{k_2}}(x_2)\Big\}^{1/2}\Big\|_p.$$

Since $\|\mathcal{G}_d(f)\|_p$ and

$$\Big\|\Big\{ \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |\widetilde{D}_{k_1} M_{b_1}\widetilde{D}_{k_2} M_{b_2}(f)(x_1, x_2)|^2 \chi_{Q_{k_1}}(x_1)\chi_{Q_{k_2}}(x_2)\Big\}^{1/2}\Big\|_p$$

are equivalent (see [14, p. 75] *for one variable) and $\|\mathcal{G}_d(f)\|_p < \infty$, we have $\|\mathcal{G}_d(f - f_L)\|_p$ tend to zero as $L \to \infty$. It is clear that*

$$D_{k_1}(\cdot, x_{Q_{k_1}}) D_{k_2}(\cdot, x_{Q_{k_1}}) \in \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$$

and hence $f_L \in \mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$. *We obtain that* $\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)}$ *is dense in* $H^p_{b_1b_2}$. *Since* $\mathcal{M}^{(\beta_1,\beta_2,\gamma_1,\gamma_2)} \subseteq H^p_{b_1b_2} \cap L^2$, *the subset* $H^p_{b_1b_2} \cap L^2$ *is dense in* $H^p_{b_1b_2}$.

3. Atomic decomposition for $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

Let $\max\left\{\frac{n_1}{n_1+\varepsilon_1}, \frac{n_2}{n_2+\varepsilon_2}\right\} < p \le 1$. A $(p,2)$-atoms of $H^p_{b_1b_2}$ is a function $a(x_1,x_2)$ defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ whose support is contained in some open set Ω of finite measure such that

(1) $\|a\|_{L^2} \le |\Omega|^{1/2-1/p}$,

(2) a can be further decomposed into p *elementary particles* a_R as follows:

(i) $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, where $\mathcal{M}(\Omega)$ denotes the class of all maximal dyadic subrectangles of Ω and a_R is supported in $5R$ where $R \subset \Omega$ (say $R = I \times J$),

(ii) $\displaystyle\int_I a_R(x_1,\tilde{x}_2)b_1(x_1)dx_1 = \int_J a_R(\tilde{x}_1,x_2)b_2(x_2)dx_2 = 0$ for each $\tilde{x}_1 \in I$, $\tilde{x}_2 \in J$,

(iii) $\displaystyle\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2} \le |\Omega|^{1-2/p}$.

We establish a new atomic decomposition for $H^p_{b_1b_2} \cap L^2$ as follows.

Theorem 3.1. *Let* b_i *be para-accretive functions on* $\mathbb{R}^{n_i}, i = 1,2$ *and* $\max\left\{\frac{n_1}{n_1+\varepsilon_1}, \frac{n_2}{n_2+\varepsilon_2}\right\} < p \le 1$. *For* $f \in H^p_{b_1b_2} \cap L^2$, *there exist a sequence of* $(p,2)$*-atoms* $\{a_i\}$ *of* $H^p_{b_1b_2}$ *and a sequence of scales* $\{\lambda_i\}$ *such that* $f = \sum \lambda_i a_i$ *and* $\sum |\lambda_i|^p \le C\|f\|^p_{H^p_{b_1b_2}}$. *Moreover, the series converges in both* $H^p_{b_1b_2}$ *and* L^2 *norms.*

Proof. For $k \in \mathbb{Z}$, let

$$\Omega_k = \left\{x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \mathcal{G}_d(f)(x_1,x_2) > 2^k\right\}$$

and

$$\mathcal{R}_k = \left\{\text{dyadic rectangle } R :\ |R \cap \Omega_k| \ge \frac{1}{2}|R| \text{ and } |R \cap \Omega_{k+1}| < \frac{1}{2}|R|\right\}.$$

By Calderón reproducing formula (Theorem 2.8),

$$\begin{aligned} f(x_1,x_2) &= \sum_{k_1}\sum_{k_2}\sum_{Q_{k_1}}\sum_{Q_{k_2}} |Q_{k_1}||Q_{k_2}|D_{k_1}(x_1,x_{Q_{k_1}})D_{k_2}(x_2,y_{Q_{k_2}}) \\ &\qquad \times \widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}} \\ &= \sum_{k\in\mathbb{Z}}\sum_{R_{k_1k_2}\in\mathcal{R}_k} |R_{k_1k_2}|D_{k_1}(x_1,x_{Q_{k_1}})D_{k_2}(x_2,y_{Q_{k_2}}) \\ &\qquad \times \widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})b_{1Q_{k_1}}b_{2Q_{k_2}}, \end{aligned}$$

where $R_{k_1k_2} = Q_{k_1} \times Q_{k_2}$. We rewrite the above decomposition as

$$f(x_1, x_2) = \sum_k \lambda_k a_k(x_1, x_2), \tag{8}$$

where

$$a_k(x_1, x_2) = \frac{1}{\lambda_k} \sum_{R_{k_1k_2} \in \mathcal{R}_k} |R_{k_1k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \times \widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}},$$

and

$$\lambda_k = C \Big\| \Big\{ \sum_{R_{k_1k_2} \in \mathcal{R}_k} |\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1k_2}} \Big\}^{1/2} \Big\|_2 |\widetilde{\Omega}_k|^{1/p-1/2}$$

Let $\widetilde{\Omega}_k = \{(x_1, x_2) : M_s(\chi_{\Omega_k})(x_1, x_2) > \frac{1}{100}\}$. Then, for each $R \in \mathcal{R}_k$, there exists a maximal dyadic subrectangle $\widetilde{R} \in \mathcal{M}(\widetilde{\Omega}_k)$ such that $R \subset \widetilde{R}$. For each $S \in \mathcal{M}(\widetilde{\Omega}_k)$, set

$$a_S(x_1, x_2) = \frac{1}{\lambda_k} \sum_{\widetilde{R_{k_1k_2}} = S} |R_{k_1k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \times \widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}.$$

Then $a_k(x_1, x_2) = \sum_{S \in \mathcal{M}(\widetilde{\Omega}_k)} a_S(x_1, x_2)$. Since $R_{k_1k_2} \in \mathcal{R}_k$ and $5R_{k_1k_2} \subset \widetilde{\Omega}_k$, we have

$$\bigcup_{R_{k_1k_2} \in \mathcal{R}_k} 5R_{k_1k_2} \subseteq \widetilde{\Omega}_k.$$

This implies that a_k is supported in an open set $\widetilde{\Omega}_k$ and a_S is supported on $5S$. The vanishing moment conditions of a_S follow from the assumption of D_{k_1} and D_{k_2}. To verify the size conditions of atom, the duality and the discrete Littlewood-Paley square function estimates on L^2 show

$$\begin{aligned} &\Big\| \sum_{R_{k_1k_2} \in \mathcal{R}_k} |R_{k_1k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ &\qquad \times \widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}} \Big\|_2 \\ &= \sup_{\|g\|_2 \le 1} \sum_{R_{k_1k_2} \in \mathcal{R}_k} |R_{k_1k_2}| D_{k_1} D_{k_2} g(x_{Q_{k_1}}, y_{Q_{k_2}}) \\ &\qquad \times \widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}} \\ &\le C \sup_{\|g\|_2 \le 1} \Big\| \Big\{ \sum_{R_{k_1k_2} \in \mathcal{R}_k} |\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1k_2}} \Big\}^{1/2} \Big\|_2 \end{aligned}$$

$$\times \left\|\left\{\sum_{R_{k_1k_2}\in\mathcal{R}_k} |D_{k_1}D_{k_2}g(x_{Q_{k_1}},y_{Q_{k_2}})|^2\chi_{R_{k_1k_2}}\right\}^{1/2}\right\|_2$$
$$\le C\left\|\left\{\sum_{R_{k_1k_2}\in\mathcal{R}_k} |\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})|^2\chi_{R_{k_1k_2}}\right\}^{1/2}\right\|_2.$$

This yields

$$\|a_k\|_2 \le |\widetilde{\Omega}_k|^{1/2-1/p}.$$

Following the same proof, we have

$$\sum_{S\in\mathcal{M}(\widetilde{\Omega}_k)} \|a_S\|^2 \le |\widetilde{\Omega}_k|^{1-2/p}.$$

Note that $|\widetilde{\Omega}_k| \le C|\Omega_k|$ due to the maximal theorem. Since $(x_1,x_2)\in R_{k_1k_2}\in\mathcal{R}_k$ implies $M_s(\chi_{R_{k_1k_2}\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}})(x_1,x_2) > \frac{1}{2}$, we have $\chi_{R_{k_1k_2}} \le 2M_s(\chi_{R_{k_1k_2}\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}})$ and then $\chi_{R_{k_1k_2}} \le 4M_s^2(\chi_{R_{k_1k_2}\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}})$. Thus, by the Fefferman-Stein vector valued inequality,

$$\left\|\left\{\sum_{R_{k_1k_2}\in\mathcal{R}_k} |\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})|^2\chi_{R_{k_1k_2}}\right\}^{1/2}\right\|_2^2$$
$$\le C\iint \sum_{R_{k_1k_2}\in\mathcal{R}_k} |\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})$$
$$\times M_s(\chi_{R_{k_1k_2}\cap\widetilde{\Omega}_k\setminus\Omega_{k+1}})(x_1,x_2)|^2dx_1dx_2$$
$$\le C\iint_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} \sum_{R_{k_1k_2}\in\mathcal{R}_k} |\widetilde{D}_{k_1}M_{b_1}\widetilde{D}_{k_2}M_{b_2}f(x_{Q_{k_1}},y_{Q_{k_2}})\chi_{R_{k_1k_2}}(x_1,x_2)|^2dx_1dx_2$$
$$\le C2^{2k}|\widetilde{\Omega}_k|.$$

Therefore,

$$\begin{aligned}\sum_k |\lambda_k|^p &\le C\sum_k 2^{kp}|\widetilde{\Omega}_k|^{p/2}|\widetilde{\Omega}_k|^{1-p/2}\\ &= C\sum_k 2^{kp}|\widetilde{\Omega}_k|\\ &\le C\|\mathcal{G}_d\|_p^p \le C\|f\|^p_{H^p_{b_1b_2}}.\end{aligned}$$

This ends the proof of Theorem 3.1. □

As an application of Theorem 3.1, we have

Corollary 3.2. *Let b_i be para-accretive functions on $\mathbb{R}^{n_i}$, $i=1,2$, and*

$$\max\left\{\frac{n_1}{n_1+\varepsilon_1},\frac{n_2}{n_2+\varepsilon_2}\right\} < p \le 1.$$

Then $f \in H^p_{b_1b_2} \cap L^2$ if and only if $b_1b_2f \in H^p \cap L^2$, and the norm of f on $H^p_{b_1b_2}$ and the norm of b_1b_2f on H^p are equivalent.

The definition of product H^p is the same with $H^p_{b_1b_2}$ where $b_1 = b_2 = 1$ and the classical product atom is the same with $(p,2)$-atoms of $H^p_{b_1b_2}$ for $b_1 = b_2 = 1$.

4. The dual spaces of $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

We now consider the dual spaces of $H^p_{b_1b_2}$. For one variable, the dual of H^1_b is $BMO_b = \{bf : f \in BMO\}$ given by Meyer and Coifman [27]. As a consequence of Corollary 3.2,

the proof of dual space of $H^p_{b_1b_2}$ is just a copy of the classical product Hardy space H^p. More precisely, let $\mathcal{CMO}^p_{b_1b_2} = \{f = b_1b_2h : h \in CMO^p \cap L^2\}$ with $\|f\|_{\mathcal{CMO}^p_{b_1b_2}} = \|h\|_{CMO^p}$, where the product type Carleson measure space CMO^p was given in [17, pages 647 & 653] and [18, pages 340–341]. Denote

$$CMO^p_{b_1b_2} = \Big\{ f : \text{there is a sequence } \{f_m\} \text{ in } \mathcal{CMO}^p_{b_1b_2} \\ \text{such that } \langle f, g\rangle = \lim_{m\to\infty} \langle f_m, g\rangle \text{ for all } g \in H^p_{b_1b_2} \cap L^2 \Big\}$$

and $\|f\|_{CMO^p_{b_1b_2}} = \lim_{m\to\infty} \|f_m\|_{\mathcal{CMO}^p_{b_1b_2}}$.

Theorem 4.1. *Suppose that b_i are para-accretive functions on $\mathbb{R}^{n_i}$ for $i = 1, 2$. Let $\max_{i=1,2}\big\{\frac{n_i}{n_i+\varepsilon_i/2}\big\} < p \le 1$. The dual space of $H^p_{b_1b_2}$ is $CMO^p_{b_1b_2}$ in the following sense.*

(a) *For each $g \in CMO^p_{b_1b_2}$, the linear functional $\ell_g : f \mapsto \langle f, g\rangle$, defined initially on $H^p_{b_1b_2} \cap L^2$, has a continuous extension to $H^p_{b_1b_2}$ and $\|\ell_g\| \le C\|g\|_{CMO^p_{b_1b_2}}$.*

(b) *Conversely, every continuous linear functional ℓ on $H^p_{b_1b_2}$ can be realized as $\ell = \ell_g$, defined initially on $H^p_{b_1b_2} \cap L^2$, for some $g \in CMO^p_{b_1b_2}$ and $\|g\|_{CMO^p_{b_1b_2}} \le C\|\ell\|$.*

In particular, when $p = 1$, we obtain $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{b_1b_2g : g \in BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\}$. We refer the reader to [17, pages 647 & 653] *or* [18, pages 340–341] *for the definition and details about $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

Proof. For each $g \in CMO^p_{b_1b_2}$, there exists a sequence $\{g_m\}$ in $\mathcal{CMO}^p_{b_1b_2}$ such that $\langle f, g\rangle = \lim_{m\to\infty} \langle f, g_m\rangle$ for $f \in H^p_{b_1b_2} \cap L^2$. By the definition, $g_m = b_1b_2h_m$ with $h_m \in CMO^p \cap L^2$, and $\|g_m\|_{\mathcal{CMO}^p_{b_1b_2}} = \|h_m\|_{CMO^p}$. For $f \in H^p_{b_1b_2} \cap L^2$, Corollary 3.2 gives $b_1b_2f \in H^p \cap L^2$ and $\|f\|_{H^p_{b_1b_2}} \approx \|b_1b_2f\|_{H^p}$.

The dual of H^p (cf. [17]) yields

$$\begin{aligned}|\ell_g(f)| &= |\langle f, g\rangle| = \big|\lim_{m\to\infty}\langle f, b_1b_2h_m\rangle\big| = \big|\lim_{m\to\infty}\langle b_1b_2f, h_m\rangle\big| \\ &\le \lim_{m\to\infty}\|b_1b_2f\|_{H^p}\|h_m\|_{CMO^p} \\ &\le C\|f\|_{H^p_{b_1b_2}}\|g\|_{CMO^p_{b_1b_2}}.\end{aligned}$$

Since $H^p_{b_1b_2}\cap L^2$ is dense in $H^p_{b_1b_2}$, the map ℓ_g can be extended to a continuous linear functional on $H^p_{b_1b_2}$ satisfying $\|\ell_g\| \le C\|g\|_{CMO^p_{b_1b_2}}$.

Conversely, let $\ell \in (H^p_{b_1b_2})'$ and define ℓ_1 by $\ell_1(b_1b_2f) = \ell(f)$ for $f \in H^p_{b_1b_2}\cap L^2$. It follows from Corollary 3.2 that ℓ_1 is a linear functional on $H^p \cap L^2$. By the duality argument between H^p and CMO^p, there exists $h \in CMO^p$ such that

$$\begin{aligned}\ell(f) &= \ell_1(b_1b_2f) = \langle b_1b_2f, h\rangle \\ &= \lim_{m\to\infty}\langle b_1b_2f, h_m\rangle = \lim_{m\to\infty}\langle f, b_1b_2h_m\rangle \qquad \text{for } f \in H^p_{b_1b_2}\cap L^2,\end{aligned}$$

where $h_m \in CMO^p \cap L^2$ and $\|b_1b_2h_m\|_{CMO^p_{b_1b_2}} = \|h_m\|_{CMO^p}$. Let $\langle f, g\rangle = \lim_{m\to\infty}\langle f, b_1b_2h_m\rangle$. Then $\|g\|_{CMO^p_{b_1b_2}} = \lim_{m\to\infty}\|b_1b_2h_m\|_{CMO^p_{b_1b_2}} \le C\|\ell_1\| \le C\|\ell\|$. □

5. The boundedness and endpoint estimates of singular integral operators on product spaces

We start with recalling the definition of a Calderón-Zygmund kernel. A continuous complex-valued function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n\backslash\{(x,y) : x = y\}$ is called a *Calderón-Zygmund kernel* if there exist constant $C > 0$ and a regularity exponent $\varepsilon \in (0,1]$ such that

(i) $|K(x,y)| \le C|x-y|^{-n}$
(ii) $|K(x,y)-K(x',y)| \le C|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon}$ if $|x-x'| \le |x-y|/2$
(iii) $|K(x,y)-K(x,y')| \le C|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon}$ if $|y-y'| \le |x-y|/2$.

The smallest such constant C is denoted by $|K|_{CZ}$.

We say that an operator T is a *Calderón-Zygmund operator* if T is a continuous linear operator from $C^\infty_0(\mathbb{R}^n)$ into its dual associated with a Calderón-Zygmund kernel $K(x,y)$ given by

$$\langle Tf, g\rangle = \iint g(x)K(x,y)f(y)dydx$$

for all test functions f and g with disjoint supports and T is bounded on $L^2(\mathbb{R}^n)$. If T is a Calderón-Zygmund operator associated with a kernel K, its Calderón-Zygmund operator norm is defined by $\|T\|_{CZ} = \|T\|_{L^2\mapsto L^2} + |K|_{CZ}$. Of course, in general, one cannot conclude that a singular integral operator T is bounded on $L^2(\mathbb{R}^n)$ because Plancherel's theorem doesn't work for non-convolution operators. However, if one assumes that T is bounded on $L^2(\mathbb{R}^n)$, then the $L^p, 1 < p < \infty$, boundedness follows from Caldernón-Zygmund's real variable method. The characterization of the $L^2(\mathbb{R}^n)$ boundedness of non-convolution singular integral operators was finally proved by the remarkable $T1$ theorem by David and Journé [5], in which they gave a general criterion for the L^2-boundedness of singular integral operators. Let T be a singular integral operator defined for functions on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ by

$$Tf(x_1, x_2) = \iint_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) f(y_1, y_2)\, dy_1\, dy_2,$$

for (x_1, x_2) is outside the support of f. For each $x_1, y_1 \in \mathbb{R}^{n_1}$, set $\widetilde{K}^1(x_1, y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^{n_2}$ with the kernel $\widetilde{K}^1(x_1, y_1)(x_2, y_2) = K(x_1, x_2, y_1, y_2)$ and, similarly, $\widetilde{K}^2(x_2, y_2)(x_1, y_1) = K(x_1, x_2, y_1, y_2)$.

Definition 5.1. A singular integral operator T is said to be in *Journé's class* if the associated kernel $K(x_1, x_2, y_1, y_2)$ satisfies the following conditions. There exist constants $C > 0$ and $\varepsilon \in (0, 1]$ such that

(A_1) T is bounded on $L^2(\mathbb{R}^{n_1+n_2})$,

(A_2) $\big\|\widetilde{K}^1(x_1, y_1)\big\|_{CZ} \le C|x_1 - y_1|^{-n_1}$,

$$\big\|\widetilde{K}^1(x_1, y_1) - \widetilde{K}^1(x_1, y_1')\big\|_{CZ} \le C|y_1 - y_1'|^{\varepsilon}|x_1 - y_1|^{-(n_1+\varepsilon)} \quad \text{for } |y_1 - y_1'| \le |x_1 - y_1|/2,$$

$$\big\|\widetilde{K}^1(x_1, y_1) - \widetilde{K}^1(x_1', y_1)\big\|_{CZ} \le C|x_1 - x_1'|^{\varepsilon}|x_1 - y_1|^{-(n_1+\varepsilon)} \quad \text{for } |x_1 - x_1'| \le |x_1 - y_1|/2,$$

(A_3) $\big\|\widetilde{K}^2(x_2, y_2)\big\|_{CZ} \le C|x_2 - y_2|^{-n_2}$,

$$\big\|\widetilde{K}^2(x_2, y_2) - \widetilde{K}^2(x_2, y_2')\big\|_{CZ} \le C|y_2 - y_2'|^{\varepsilon}|x_2 - y_2|^{-(n_2+\varepsilon)} \quad \text{for } |y_2 - y_2'| \le |x_2 - y_2|/2,$$

$$\big\|\widetilde{K}^2(x_2, y_2) - \widetilde{K}^2(x_2', y_2)\big\|_{CZ} \le C|x_2 - x_2'|^{\varepsilon}|x_2 - y_2|^{-(n_2+\varepsilon)} \quad \text{for } |x_2 - x_2'| \le |x_2 - y_2|/2.$$

Let $L^2_{0,0}(\mathbb{R}^n) = \big\{f \in L^2(\mathbb{R}^n)\colon f \text{ has compact support with} \int f(x)dx = 0\big\}$. Suppose that T is a singular integral in Journé's class. By [8, p. 840], T is bounded from $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^1(\mathbb{R}^{n_1+n_2})$. Note that if $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$ and $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$, then $\varphi^1(y_1)\varphi^2(y_2) \in H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Thus, $T(\varphi^1\varphi^2)(x_1, x_2) \in L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. This implies that $T(\varphi^1\varphi^2)(x_1, x_2)$, as a function of x_1, is a integrable function on $\mathbb{R}^{n_1}$. Similarly, $T(\varphi^1\varphi^2)(x_1, x_2)$, as a function of x_2, is a integrable function on $\mathbb{R}^{n_2}$. Now we say that $T_1^*(1) = 0$ if

$$\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2)\varphi^1(y_1)\varphi^2(y_2)\, dy_1\, dy_2\, dx_1 = 0$$

for all $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$, $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$, and $x_2 \in \mathbb{R}^{n_2}$. Similarly, $T_2^*(1) = 0$ if

$$\int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2)\varphi^1(y_1)\varphi^2(y_2)\, dy_1\, dy_2\, dx_2 = 0$$

for all $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$, $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$, and $x_1 \in \mathbb{R}^{n_1}$.

We also say that $T_1^*(b_1) = 0$ if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) b_1(x_1) \varphi^1(y_1) \varphi^2(y_2)\, dy_1\, dy_2\, dx_1 = 0$$

for all $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$, $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$, and $x_2 \in \mathbb{R}^{n_2}$. Similarly, $T_1^*(b_2) = 0$ if

$$\int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) b_2(x_2) \varphi^1(y_1) \varphi^2(y_2)\, dy_1\, dy_2\, dx_2 = 0$$

for all $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$, $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$, and $x_1 \in \mathbb{R}^{n_1}$.

We now prove the main result of this article.

Proof of the "if part" of Theorem 1.1. We prove (1) only since the proof of (2) and (3) are similar. We use the skill in the proof of [16, Theorem 1]. We define the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \left\{ \{a_{k,j}\}_{k,j\in\mathbb{Z}} : \big\|\{a_{k,j}\}\big\|_{\mathcal{H}} = \left(\sum_{k,j\in\mathbb{Z}} |a_{k,j}|^2 \right)^{1/2} < \infty \right\}.$$

For simplicity, we use $D_{k_1,k_2} M_b$ and $\widetilde{D}_{k_1,k_2} M_b$ to express $D_{k_1} M_{b_1} D_{k_2} M_{b_2}$ and $\widetilde{D}_{k_1} M_{b_1} \widetilde{D}_{k_2} M_{b_2}$, respectively. We also denote $\int dv$ by $\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} dv_1 dv_2$ and similarly for other variables. Set $T_{k_1,k_2}(f) = D_{k_1,k_2} M_b T(f)$. For $f \in L^2(\mathbb{R}^{n_1+n_2}) \cap H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, by the continuous Calderón identity,

$$T_{k_1,k_2}(f)(x_1, x_2) = D_{k_1,k_2} M_b T\bigg(\sum_{j_1} \sum_{j_2} M_b D_{j_1,j_2} M_b \widetilde{D}_{j_1,j_2}(f)(x_1, x_2) \bigg).$$

Hence, the kernel $T_{k_1,k_2}(x_1, x_2, y_1, y_2)$ of T_{k_1,k_2} is given by

$$\begin{aligned} &T_{k_1,k_2}(x_1, x_2, y_1, y_2) \\ &\quad = \sum_{j_1} \sum_{j_2} \iint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ &\qquad \times M_b D_{j_1,j_2} M_b \widetilde{D}_{j_1}(v_1, y_1) \widetilde{D}_{j_2}(v_2, y_2) du dv. \end{aligned}$$

By the definition of $H^p_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, the $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) - H^p_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedness of T is equivalent to the $H^p - L^p_{\mathcal{H}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedness of the $\mathcal{H}$-valued operator $\mathcal{L}$ which maps f into $\{T_{k_1,k_2}(f)\}_{k_1,k_2\in\mathbb{Z}}$. Note that the $L^2(\mathbb{R}^{n_1+n_2})$ boundedness of T and the product Littlewood-Paley estimate imply that $\mathcal{L}$ is bounded from $L^2(\mathbb{R}^{n_1+n_2})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$. Let ε be the regularity exponent satisfying (A_2) and (A_3). We will prove that $\{T_{k_1,k_2}(x_1, x_2, y_1, y_2)\}_{k_1,k_2\in\mathbb{Z}}$ satisfies the following estimates:

(B_1) $\big\|\{T_{k_1,k_2}(x_1, x_2, y_1, y_2)\}\big\|_{\mathcal{H}} \le C|x_1 - y_1|^{-n_1}|x_2 - y_2|^{-n_2}$,

(B_2) for $\varepsilon' < \varepsilon$,

(i) $\big\|\{T_{k_1,k_2}(x_1,x_2,y_1,y_2)-T_{k_1,k_2}(x_1,x_2,y_1',y_2)\}\big\|_{\mathcal{H}}$
$\le C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n_1+\varepsilon'}}|x_2-y_2|^{-n_2}$ if $|y_1-y_1'|\le |x_1-y_1|/2,$

(ii) $\big\|\{T_{k_1,k_2}(x_1,x_2,y_1,y_2)-T_{k_1,k_2}(x_1,x_2,y_1,y_2')\}\big\|_{\mathcal{H}}$
$\le C\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{n_2+\varepsilon'}}|x_1-y_1|^{-n_1}$ if $|y_2-y_2'|\le |x_2-y_2|/2,$

(B_3) for $\varepsilon'<\varepsilon$,

$$\begin{aligned}&\big\|\{[T_{k_1,k_2}(x_1,x_2,y_1,y_2)-T_{k_1,k_2}(x_1,x_2,y_1',y_2)]\\&\qquad-[T_{k_1,k_2}(x_1,x_2,y_1,y_2')-T_{k_1,k_2}(x_1,x_2,y_1',y_2')]\}\big\|_{\mathcal{H}}\\&\le C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n_1+\varepsilon'}}\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{n_2+\varepsilon'}}\\&\qquad\text{if}\quad |y_1-y_1'|\le|x_1-y_1|/2\ \text{ and }\ |y_2-y_2'|\le|x_2-y_2|/2.\end{aligned}$$

To this end, according to the almost orthogonal estimates, we decompose the kernel $T_{k_1,k_2}(x_1,x_2,y_1,y_2)$ as follows.

$$\begin{aligned}&T_{k_1,k_2}(x_1,x_2,y_1,y_2)\\&=\sum_{j_1}\sum_{j_2}\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)\\&\qquad\times M_bD_{j_1,j_2}M_b\widetilde{D}_{j_1}(v_1,y_1)\widetilde{D}_{j_2}(v_2,y_2)dudv\\&=\Big(\sum_{j_1\ge k_1}\sum_{j_2\ge k_2}+\sum_{j_1\ge k_1}\sum_{j_2<k_2}+\sum_{j_1<k_1}\sum_{j_2\ge k_2}+\sum_{j_1<k_1}\sum_{j_2<k_2}\Big)\iint D_{k_1}(x_1,u_1)b_1(u_1)\\&\quad\times D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)M_bD_{j_1,j_2}M_b\widetilde{D}_{j_1}(v_1,y_1)\widetilde{D}_{j_2}(v_2,y_2)dudv\\&:=T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)+T^2_{k_1,k_2}(x_1,x_2,y_1,y_2)\\&\quad+T^3_{k_1,k_2}(x_1,x_2,y_1,y_2)+T^4_{k_1,k_2}(x_1,x_2,y_1,y_2).\end{aligned}$$

The estimates of $(B_1)-(B_3)$ for $\{T_{k_1,k_2}(x_1,x_2,y_1,y_2)\}_{k_1,k_2\in\mathbb{Z}}$ will follow easily by the following lemma.

Lemma 5.2. *For $1\le j\le 4$ and $k_1,k_2\in\mathbb{Z}$, there exists a constant C such that*

(D_1) *for $\varepsilon'<\varepsilon$,*

$$\big|T^j_{k_1,k_2}(x_1,x_2,y_1,y_2)\big|\le C\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-k_2\varepsilon'}}{(2^{-k_2}+|x_2-y_2|)^{n_2+\varepsilon'}},$$

(D_2) *for $\varepsilon''<\varepsilon'$,*

(i) $\big|T^j_{k_1,k_2}(x_1,x_2,y_1,y_2)-T^j_{k_1,k_2}(x_1,x_2,y_1',y_2)\big|$

$$\le C\Big(\frac{|y_1-y_1'|}{2^{-k_1}}\Big)^{\varepsilon''}\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-k_2\varepsilon'}}{(2^{-k_2}+|x_2-y_2|)^{n_2+\varepsilon'}}$$
if $|y_1-y_1'|\le 2^{-k_1-1}$,

(ii) $$\left|T^j_{k_1,k_2}(x_1,x_2,y_1,y_2)-T^j_{k_1,k_2}(x_1,x_2,y_1,y_2')\right|$$
$$\le C\Big(\frac{|y_2-y_2'|}{2^{-k_2}}\Big)^{\varepsilon''}\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-k_2\varepsilon'}}{(2^{-k_2}+|x_2-y_2|)^{n_2+\varepsilon'}}$$
if $|y_2-y_2'|\le 2^{-k_2-1}$,

(D_3) *for* $\varepsilon''<\varepsilon'$,
$$\big\|\{[T_{k_1,k_2}(x_1,x_2,y_1,y_2)-T_{k_1,k_2}(x_1,x_2,y_1',y_2)] -[T_{k_1,k_2}(x_1,x_2,y_1,y_2')-T_{k_1,k_2}(x_1,x_2,y_1',y_2')]\}\big\|_{\mathcal{H}}$$
$$\le C\Big(\frac{|y_1-y_1'|}{2^{-k_1}}\Big)^{\varepsilon''}\Big(\frac{|y_2-y_2'|}{2^{-k_2}}\Big)^{\varepsilon''}\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}} \times\frac{2^{-k_2\varepsilon'}}{(2^{-k_2}+|x_2-y_2|)^{n_2+\varepsilon'}}$$
if $|y_1-y_1'|\le 2^{-k_1-1}$ *and* $|y_2-y_2'|\le 2^{-k_2-1}$.

Proof. We will use the iteration method which reduces the product case to the classical case. We first check that $T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)$ satisfies the estimates $(D_1)-(D_3)$. For fixed k_1,x_1 and y_1, set

$$\mathcal{K}_2(u_2,v_2) =\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}}D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,u_2,v_1,v_2)b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)du_1dv_1.$$

Since $T_1^*(b_1)=0$, we have

$$\mathcal{K}_2(u_2,v_2)=\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}}\big(D_{k_1}(x_1,u_1)-D_{k_1}(x_1,y_1)\big)b_1(u_1)K(u_1,u_2,v_1,v_2) \times b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)du_1dv_1.$$

By [15, Theorem 3.1] with $j_1\ge k_1$, the operator S associated with the kernel $\mathcal{K}_2(u_2,v_2)$ is a Calderón-Zygmund operator and satisfies, for $\varepsilon'<\varepsilon$,

$$\|S\|_{CZ}\le C2^{-(j_1-k_1)\varepsilon'}\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}}. \tag{9}$$

Note that the condition $T_2^*(b_2)=0$ implies $S^*(b_2)=0$. Therefore, first writing

$$T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)=\sum_{j_1\ge k_1}\sum_{j_2\ge k_2}\int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_2}}D_{k_2}(x_2,u_2)b_2(u_2)\mathcal{K}_2(u_2,v_2) \times b_2(v_2)D_{j_2}M_{b_2}\widetilde{D}_{j_2}(v_2,y_2)du_2dv_2$$

and then applying the almost orthogonal estimate for $\mathcal{K}_2(u_2,v_2)$ with the norm estimate in (9) imply

$$|T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)|\le C\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-k_2\varepsilon'}}{(2^{-k_2}+|x_2-y_2|)^{n_2+\varepsilon'}}.$$

This shows that $T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)$ satisfies the estimate (D_1). To check (D_2)(i), we write

$$\begin{aligned}
&T^1_{k_1,k_2}(x_1,x_2,y_1,y_2) - T^1_{k_1,k_2}(x_1,x_2,y_1',y_2)\\
&\quad= \sum_{j_1\ge k_1}\sum_{j_2\ge k_2}\int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_2}} D_{k_2}(x_2,u_2)b_2(u_2)\mathcal{K}_{2,2}(u_2,v_2)\\
&\qquad\qquad \times b_2(v_2)D_{j_2}M_{b_2}\widetilde{D}_{j_2}(v_2,y_2)du_2dv_2,
\end{aligned}$$

where for fixed k_1, x_1, y_1, y_1',

$$\begin{aligned}
\mathcal{K}_{2,2}(u_2,v_2) = \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_1}} &D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,u_2,v_1,v_2)\\
&\times b_1(v_1)D_{j_1}(v_1,z_1)b_1(z_1)\big(\widetilde{D}_{j_1}(z_1,y_1)-\widetilde{D}_{j_1}(z_1,y_1')\big)du_1dv_1.
\end{aligned}$$

By the similar argument of the proof in [15, Theorem 3.1] (but simpler since T is bounded on L^2), we obtain that $\mathcal{K}_{2,2}(u_2,v_2)$ is a Calderón-Zygmund kernel and

$$|\mathcal{K}_{2,2}|_{CZ} \le C\Big(\frac{|y_1-y_1'|}{2^{-k_1}}\Big)^{\varepsilon''}\frac{2^{-(j_1-k_1)\varepsilon'}2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}} \quad \text{for} \quad |y_1-y_1'|\le 2^{-k_1-1}. \tag{10}$$

The almost orthogonal estimate together with the estimate of (10) yields

$$\begin{aligned}
&\Big|\sum_{j_1\ge k_1}\sum_{j_2\ge k_2}\int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_2}} D_{k_2}(x_2,u_2)b_2(u_2)\mathcal{K}_{2,2}(u_2,v_2)\\
&\qquad\qquad\qquad \times b_2(v_2)D_{j_2}M_{b_2}\widetilde{D}_{j_2}(v_2,y_2)du_2dv_2\Big|\\
&\quad\le \Big(\frac{|y_1-y_1'|}{2^{-k_1}}\Big)^{\varepsilon''}\frac{2^{-k_1\varepsilon'}}{(2^{-k_1}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-k_2\varepsilon'}}{(2^{-k_2}+|x_2-y_2|)^{n_2+\varepsilon'}}
\end{aligned}$$

for $|y_1-y_1'|\le 2^{-k_1-1}$ and hence (D_2)(i) follows. The proof of (D_2)(ii) is the same. To prove (D_3) for the kernel $T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)$, we write

$$\begin{aligned}
&[T^1_{k_1,k_2}(x_1,x_2,y_1,y_2) - T^1_{k_1,k_2}(x_1,x_2,y_1',y_2)]\\
&\qquad - [T^1_{k_1,k_2}(x_1,x_2,y_1,y_2') - T^1_{k_1,k_2}(x_1,x_2,y_1',y_2')]\\
&= \sum_{j_1\ge k_1}\sum_{j_2\ge k_2}\iiint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad \times b_1(v_1)D_{j_1}(v_1,z_1)b_1(z_1)\big(\widetilde{D}_{j_1}(z_1,y_1)-\widetilde{D}_{j_1}(z_1,y_1')\big)\\
&\qquad \times b_2(v_2)D_{j_2}(v_2,z_2)b_2(z_2)\big(\widetilde{D}_{j_2}(z_2,y_2)-\widetilde{D}_{j_2}(z_2,y_2')\big)dudvdz\\
&= \sum_{j_1\ge k_1}\sum_{j_2\ge k_2}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} D_{k_2}(x_2,u_2)b_2(u_2)\mathcal{K}_{2,2}(u_2,v_2)\\
&\qquad \times b_2(v_2)D_{j_2}(v_2,z_2)b_2(z_2)\big(\widetilde{D}_{j_2}(z_2,y_2)-\widetilde{D}_{j_2}(z_2,y_2')\big)du_2dv_2dz_2.
\end{aligned}$$

By the almost orthogonal estimate together with the estimate of (10), we obtain that the kernel $T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)$ satisfies (D_3).

Since the proofs for $T^j_{k_1,k_2}(x_1,x_2,y_1,y_2), j=2,3$ are similar, we estimate the kernel $T^2_{k_1,k_2}(x_1,x_2,y_1,y_2)$ only. We rewrite

$$\begin{aligned}
&T^2_{k_1,k_2}(x_1,x_2,y_1,y_2)\\
&= \sum_{j_1\geq k_1}\sum_{j_2<k_2}\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad \times M_b D_{j_1,j_2} M_b \widetilde{D}_{j_1}(v_1,y_1)\widetilde{D}_{j_2}(v_2,y_2)dudv\\
&= \sum_{j_1\geq k_1}\sum_{j_2<k_2}\iint\int_{\mathbb{R}^{n_2}} D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad \times b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)b_2(v_2)D_{j_2}(v_2,z_2)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz_2\\
&= \sum_{j_1\geq k_1}\sum_{j_2<k_2}\iint\int_{\mathbb{R}^{n_2}} D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad \times b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)b_2(v_2)\big(D_{j_2}(v_2,z_2)-D_{j_2}(x_2,z_2)\big)\\
&\qquad \times b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz_2\\
&\quad + \sum_{j_1\geq k_1}\sum_{j_2<k_2}\iint\int_{\mathbb{R}^{n_2}} D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad \times b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)b_2(v_2)D_{j_2}(x_2,z_2)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz_2\\
&:= T^{2,1}_{k_1,k_2}(x_1,x_2,y_1,y_2)+T^{2,2}_{k_1,k_2}(x_1,x_2,y_1,y_2).
\end{aligned}$$

The proof for $T^{2,1}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ is similar to $T^1_{k_1,k_2}(x_1,x_2,y_1,y_2)$, so we leave details to the reader. Write

$$P_{k_2}(x_2,y_2)=\sum_{j_2<k_2}\int_{\mathbb{R}^{n_2}} D_{j_2}(x_2,z_2)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dz_2.$$

Note that $T_1^*(b_1)=0$. We have

$$\begin{aligned}
&T^{2,2}_{k_1,k_2}(x_1,x_2,y_1,y_2)\\
&\quad = \sum_{j_1\geq k_1}\iint \big(D_{k_1}(x_1,u_1)-D_{k_1}(x_1,v_1)\big)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)\\
&\qquad \times K(u_1,u_2,v_1,v_2)b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)b_2(v_2)P_{k_2}(x_2,y_2)dudv.
\end{aligned}$$

For fixed x_2, set

$$\begin{aligned}
\mathcal{K}_1(u_1,v_1) &= \int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)b_2(v_2)du_2dv_2\\
&= \int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} D_{k_2}(x_2,u_2)b_2(u_2)\widetilde{K}^1(u_1,v_1)(u_2,v_2)b_2(v_2)du_2dv_2.
\end{aligned}$$

Note that for fixed (u_1, v_1), $\int_{\mathbb{R}^{n_2}} \widetilde{K}^1(u_1, v_1)(u_2, v_2)b_2(v_2)dv_2$, as a function of the variable u_2, is a BMO function and $D_{k_2}(x_2, \cdot)b_2$ is a function in $H^1(\mathbb{R}^{n_2})$ with $H^1(\mathbb{R}^{n_2})$-norm uniformly bounded for all x_2 and k_2. Moreover,

$$\begin{aligned}\Big\| \int_{\mathbb{R}^{n_2}} \widetilde{K}^1(u_1, v_1)(\cdot, v_2)b_2(v_2)dv_2\Big\|_{BMO(\mathbb{R}^{n_2})} &\le C\big\|\widetilde{K}^1(u_1, v_1)\big\|_{CZ} \\ &\le C|u_1 - v_1|^{-n_1},\end{aligned}$$

which implies

$$|K_1(u_1, v_1)| \le C|u_1 - v_1|^{-n_1}.$$

Similarly, for $|u_1 - u_1'| \le \frac{1}{2}|u_1 - v_1|$, we have

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^{n_2}} \big[\widetilde{K}^1(u_1, v_1)(\cdot, v_2) - \widetilde{K}^1(u_1', v_1)(\cdot, v_2)\big]b_2(v_2)dv_2\Big\|_{BMO(\mathbb{R}^{n_2})} \\ &\le C\big\|\widetilde{K}^1(u_1, v_1) - \widetilde{K}^1(u_1', v_1)\big\|_{CZ} \\ &\le C|u_1 - u_1'|^{\varepsilon}|u_1 - v_1|^{-n_1-\varepsilon},\end{aligned}$$

and hence

$$\begin{aligned}&|K_1(u_1, v_1) - K_1(u_1', v_1)| \\ &= \Big| \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} D_{k_2}(x_2, u_2)b_2(u_2) \\ &\qquad \times \big[\widetilde{K}^1(u_1, v_1)(u_2, v_2) - \widetilde{K}^1(u_1', v_1)(u_2, v_2)\big]b_2(v_2)du_2dv_2\Big| \\ &\le C|u_1 - u_1'|^{\varepsilon}|u_1 - v_1|^{-n_1-\varepsilon}.\end{aligned}$$

The estimate $|K_1(u_1, v_1) - K_1(u_1, v_1')|$ can be obtained by the same approach. Thus, $K_1(u_1, v_1)$ is a Calderón-Zygmund kernel and $|K_1|_{CZ} \le C$. Note that

$$\begin{aligned}&T^{2,2}_{k_1,k_2}(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1 \ge k_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \big(D_{k_1}(x_1, u_1) - D_{k_1}(x_1, v_1)\big)b_1(u_1)K_1(u_1, v_1) \\ &\qquad \times b_1(v_1)D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1, y_1)du_1dv_1P_{k_2}(x_2, y_2).\end{aligned}$$

Note that the kernel $P_{k_2}(x_2, y_2)$ satisfies the same size and smoothness conditions as a Poisson kernel by using the method in [15, page 133]. Applying the almost orthogonal estimate to $K_1(u_1, v_1)$ together with the size condition on P_{k_2}, we get (D_1) for $T^{2,2}_{k_1,k_2}(x_1, x_2, y_1, y_2)$. The estimates of (D_2) and (D_3) can be proved by the same way.

Finally, we rewrite

$$\begin{aligned}&T^4_{k_1,k_2}(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1<k_1} \sum_{j_2<k_2} \iint D_{k_1}(x_1, u_1)b_1(u_1)D_{k_2}(x_2, u_2)b_2(u_2)K(u_1, u_2, v_1, v_2)b_1(v_1)\end{aligned}$$

$$\times D_{j_1,j_2}M_b\widetilde{D}_{j_1}(v_1,y_1)b_2(v_2)\widetilde{D}_{j_2}(v_2,y_2)dudv$$

$$= \sum_{j_1<k_1}\sum_{j_2<k_2}\iiint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)b_1(v_1)$$
$$\times D_{j_1}(v_1,z_1)b_1(z_1)\widetilde{D}_{j_1}(z_1,y_1)b_2(v_2)D_{j_2}(v_2,z_2)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz$$
$$= \sum_{j_1<k_1}\sum_{j_2<k_2}\iiint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_1(v_1)\big(D_{j_1}(v_1,z_1)-D_{j_1}(x_1,z_1)\big)b_1(z_1)\widetilde{D}_{j_1}(z_1,y_1)$$
$$\times b_2(v_2)\big(D_{j_2}(v_2,z_2)-D_{j_2}(x_2,z_2)\big)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz$$
$$+ \sum_{j_1<k_1}\sum_{j_2<k_2}\iiint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_1(v_1)D_{j_1}(x_1,z_1)b_1(z_1)\widetilde{D}_{j_1}(z_1,y_1)$$
$$\times b_2(v_2)\big(D_{j_2}(v_2,z_2)-D_{j_2}(x_2,z_2)\big)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz$$
$$+ \sum_{j_1<k_1}\sum_{j_2<k_2}\iiint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_2(v_2)D_{j_2}(x_2,z_2)b_1(v_1)\big(D_{j_1}(v_1,z_1)-D_{j_1}(x_1,z_1)\big)$$
$$\times b_1(z_1)\widetilde{D}_{j_1}(z_1,y_1)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz$$
$$+ \sum_{j_1<k_1}\sum_{j_2<k_2}\iiint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)b_1(v_1)$$
$$\times D_{j_1}(x_1,z_1)b_2(v_2)D_{j_2}(x_2,z_2)b_1(z_1)\widetilde{D}_{j_1}(z_1,y_1)b_2(z_2)\widetilde{D}_{j_2}(z_2,y_2)dudvdz$$
$$= T^{4,1}_{k_1,k_2}(x_1,x_2,y_1,y_2)+T^{4,2}_{k_1,k_2}(x_1,x_2,y_1,y_2)$$
$$+ T^{4,3}_{k_1,k_2}(x_1,x_2,y_1,y_2)+T^{4,4}_{k_1,k_2}(x_1,x_2,y_1,y_2).$$

Since the proof for $T^{4,1}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ is similar to $T^{1}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ and the proofs of $T^{4,2}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ and $T^{4,3}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ are similar to $T^{2,2}_{k_1,k_2}(x_1,x_2,y_1,y_2)$, we estimate $T^{4,4}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ only. Let

$$P_{k_i}(x_i,y_i)=\sum_{j_i<k_i}\int_{\mathbb{R}^{n_i}} D_{j_i}(x_i,z_i)b_i(z_i)\widetilde{D}_{j_i}(z_i,y_i)dz_i, \qquad i=1,2.$$

Then

$$T^{4,4}_{k_1,k_2}(x_1,x_2,y_1,y_2)=\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_1(v_1)b_2(v_2)dudvP_{k_1}(x_1,y_1)P_{k_2}(x_2,y_2).$$

Note that $\int K(u_1,u_2,v_1,v_2)b_1(v_1)b_2(v_2)dv_1dv_2$, as a function of variables u_1 and u_2, belongs to $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, and $D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)$, as function of (u_1, u_2), is in $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with the bounded norm uniformly for all k_1, k_2 and x_1, x_2. Thus,

$$\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)b_1(v_1)b_2(v_2)dudv$$

is uniformly bounded for all k_1, k_2 and x_1, x_2. Therefore, estimates $(D_1)-(D_3)$ for $T^{4,4}_{k_1,k_2}(x_1,x_2,y_1,y_2)$ are the same as those for $P_{k_1}(x_1,y_1)P_{k_2}(x_2,y_2)$. By the similar method in [15, page 133], the kernels P_{k_1} and P_{k_1} satisfy the same size and smoothness conditions as a Poisson kernel, and hence $P_{k_1}(x_1,y_1)P_{k_2}(x_2,y_2)$ satisfies estimates $(D_1)-(D_3)$. Thus, the proof of Lemma 2 is completed. □

Now we demonstrate the regularity of the operator T_{k_1,k_2} mapping from L^2 into $L^2_{\mathcal{H}}$.

Lemma 5.3. *Let T_{k_1,k_2} be defined above and ε be the regularity exponent of T. For $\varepsilon' < \varepsilon$,*

(i) *if $|y_1 - x_I| \le |x_1 - x_I|/2$, then*

$$\Big\|\Big\{\int_{\mathbb{R}^{n_2}}\big[T_{k_1,k_2}(x_1,\cdot,y_1,y_2)-T_{k_1,k_2}(x_1,\cdot,x_I,y_2)\big]f(y_2)dy_2\Big\}\Big\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_2})}$$
$$\le C\frac{|y_1-x_I|^{\varepsilon'}}{|x_1-x_I|^{n_1+\varepsilon'}}\|f\|_2;$$

(ii) *if $|y_2 - y_J| \le |x_2 - y_J|/2$, then*

$$\Big\|\Big\{\int_{\mathbb{R}^{n_1}}\big[T_{k_1,k_2}(\cdot,x_2,y_1,y_2)-T_{k_1,k_2}(\cdot,x_2,y_1,y_J)\big]f(y_1)dy_1\Big\}\Big\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_1})}$$
$$\le C\frac{|y_2-y_J|^{\varepsilon'}}{|x_2-y_J|^{n_2+\varepsilon'}}\|f\|_2.$$

Proof. The proofs of (i) and (ii) are the same, so we show the case (i) only. We will use $0 < \varepsilon''' < \varepsilon'' < \varepsilon' < \varepsilon$ through the proof. Note that

$$\Big\|\Big\{\int_{\mathbb{R}^{n_2}}\big[T_{k_1,k_2}(x_1,\cdot,y_1,y_2)-T_{k_1,k_2}(x_1,\cdot,x_I,y_2)\big]f(y_2)dy_2\Big\}\Big\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_2})}$$
$$=\int_{\mathbb{R}^{n_2}}\Big\|\Big\{\int_{\mathbb{R}^{n_2}}\big[T_{k_1,k_2}(x_1,x_2,y_1,y_2)-T_{k_1,k_2}(x_1,x_2,x_I,y_2)\big]f(y_2)dy_2\Big\}\Big\|^2_{\mathcal{H}}dx_2.$$

We write

$$\int_{\mathbb{R}^{n_2}}\big[T_{k_1,k_2}(x_1,x_2,y_1,y_2)-T_{k_1,k_2}(x_1,x_2,x_I,y_2)\big]f(y_2)dy_2$$
$$=\int_{\mathbb{R}^{n_2}}\sum_{j_1}\sum_{j_2}\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_1(v_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1)-D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)$$

$$\times b_2(v_2)D_{j_2}M_{b_2}\widetilde{D}_{j_2}(v_2,y_2)f(y_2)dudvdy_2$$
$$= \sum_{j_1}\sum_{j_2}\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_1(v_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1) - D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)$$
$$\times b_2(v_2)D_{j_2}M_{b_2}\widetilde{D}_{j_2}M_{b_2}f(v_2)dudv$$
$$= \sum_{j_1}\iint D_{k_1}(x_1,u_1)b_1(u_1)D_{k_2}(x_2,u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)$$
$$\times b_1(v_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1) - D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)b_2(v_2)f(v_2)dudv$$
$$= D_{k_2}M_{b_2}\Bigg(\sum_{j_1}\int\int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,\cdot,v_1,v_2)$$
$$\times b_1(v_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1) - D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)b_2(v_2)f(v_2)du_1dv\Bigg)(x_2),$$

where we first write

$$\int_{\mathbb{R}^{n_2}} D_{j_2}M_{b_2}\widetilde{D}_{j_2}(v_2,y_2)b_2(y_2)f(y_2)dy_2 = D_{j_2}M_{b_2}\widetilde{D}_{j_2}M_{b_2}f(v_2)$$

and then use the Calderón identity $\sum_{j_2} D_{j_2}M_{b_2}\widetilde{D}_{j_2}M_{b_2}f(v_2) = f(v_2)$. The Littlewood-Paley estimate gives

$$\int_{\mathbb{R}^{n_2}}\left\|\left\{\int_{\mathbb{R}^{n_2}}\big[T_{k_1,k_2}(x_1,x_2,y_1,y_2) - T_{k_1,k_2}(x_1,x_2,x_I,y_2)\big]f(y_2)dy_2\right\}\right\|^2_{\mathcal{H}}dx_2$$
$$\le C\sum_{k_1}\int_{\mathbb{R}^{n_2}}\Bigg|\sum_{j_1}\int\int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,x_2,v_1,v_2)$$
$$\times b_1(y_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1) - D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)b_2(v_2)f(v_2)du_1dv\Bigg|^2dx_2. \tag{11}$$

Divide the sum $\sum_{j_1}$ into three parts as follows:

$$\int_{\mathbb{R}^{n_2}}\Bigg|\sum_{j_1}\int\int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,x_2,v_1,v_2)$$
$$\times b_1(y_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1) - D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)b_2(v_2)f(v_2)du_1dv\Bigg|^2dx_2$$
$$\le C\int_{\mathbb{R}^{n_2}}\Bigg|\sum_{j_1\ge k_1}\int_{\mathbb{R}^{n_1}}\int\int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,x_2,v_1,v_2)b_1(v_1)$$
$$\times D_{j_1}(v_1,z_1)b_1(z_1)\big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big)b_2(v_2)f(v_2)du_1dvdz_1\Bigg|^2dx_2$$

$$
\begin{aligned}
&+ C \int_{\mathbb{R}^{n_2}} \Big| \sum_{j_1<k_1} \int_{\mathbb{R}^{n_1}} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1) b_1(u_1) K(u_1,x_2,v_1,v_2) b_1(v_1) \\
&\quad \times \big(D_{j_1}(v_1,z_1) - D_{j_1}(x_1,z_1)\big) b_1(z_1) \big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big) \\
&\quad \times b_2(v_2) f(v_2) du_1 dv dz_1 \Big|^2 dx_2 \\
&+ C \int_{\mathbb{R}^{n_2}} \Big| \sum_{j_1<k_1} \int_{\mathbb{R}^{n_1}} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1) b_1(u_1) K(u_1,x_2,v_1,v_2) b_1(v_1) \\
&\quad \times D_{j_1}(x_1,z_1) b_1(z_1) \big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big) b_2(v_2) f(v_2) du_1 dv dz_1 \Big|^2 dx_2 \\
&:= E + F + G.
\end{aligned}
$$

We first consider the item G and write

$$
\begin{aligned}
G &= C \int_{\mathbb{R}^{n_2}} \Big| \sum_{j_1<k_1} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1) b_1(u_1) K(u_1,x_2,v_1,v_2) b_1(v_1) \\
&\quad \times \big(D_{j_1} M_{b_1} \widetilde{D}_{j_1}(v_1,y_1) - D_{j_1} M_{b_1} \widetilde{D}_{j_1}(v_1,x_I)\big) b_2(v_2) f(v_2) du_1 dv \Big|^2 dx_2 \\
&= C |P_{k_1}(v_1,y_1) - P_{k_1}(v_1,x_I)|^2 \int_{\mathbb{R}^{n_2}} \Big| \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1) b_1(u_1) \\
&\quad \times K(u_1,x_2,v_1,v_2) b_1(v_1) b_2(v_2) f(v_2) du_1 dv \Big|^2 dx_2 \\
&= C \sup_{\|g\|_2 \le 1} \bigg(\iint D_{k_1}(x_1,u_1) b_1(u_1) K(u_1,x_2,v_1,v_2) \\
&\qquad \times b_1(v_1) b_2(v_2) f(v_2) g(x_2) du_1 dv dx_2 \bigg)^2 |P_{k_1}(v_1,y_1) - P_{k_1}(v_1,x_I)|^2,
\end{aligned}
$$

where $P_{k_1}(v_1,\cdot) = \sum_{j_1<k_1} D_{j_1} M_{b_1} \widetilde{D}_{j_1}(v_1,\cdot)$.

For fixed u_1 and v_1, set

$$
\overline{K}(u_1,v_1) = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} K(u_1,x_2,v_1,v_2) b_2(v_2) f(v_2) g(x_2) dv_2 dx_2.
$$

Then the operator associated to the kernel $\overline{K}(u_1,v_1)$ is a Calderón-Zygmund operator with operator norm $C\|f\|_2\|g\|_2$. Since $\int_{\mathbb{R}^{n_1}} \overline{K}(u_1,v_1) dv_1$ is a BMO function for u_1,

$$
\Big| \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1) b_1(u_1) \int_{\mathbb{R}^{n_1}} \overline{K}(u_1,v_1) b_1(v_1) dv_1 du_1 \Big| \le C\|f\|_2\|g\|_2
$$

uniformly for x_1. Hence, for $|y_1 - x_I| \le t/2$,

$$\begin{aligned} G &\le C|P_{k_1}(v_1, y_1) - P_{k_1}(v_1, x_I)|^2 \|f\|_2^2 \\ &\le C(2^{k_1}|y_1 - x_I|)^{2\varepsilon} \frac{2^{-k_1 2\varepsilon}}{(2^{-k_1} + |x_1 - x_I|)^{2(n_1+\varepsilon)}} \|f\|_2^2. \end{aligned} \tag{12}$$

To estimate E, we consider two cases $\{|x_1 - z_1| > 8c2^{-k_1}\}$ and $\{|x_1 - z_1| \le 8c2^{-k_1}\}$, where the constant c satisfies item (ii) in the definition of $\{S_{k_i}\}, i = 1, 2$. By duality, we get

$$\begin{aligned} E^{1/2} &= C \sup_{\|h\|_2 \le 1} \int_{\mathbb{R}^{n_2}} h(x_2) \sum_{j_1 \ge k_1} \int_{\mathbb{R}^{n_1}} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) \\ &\quad \times K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\ &\quad \times \big(\widetilde{D}_{j_1}(z_1, y_1) - \widetilde{D}_{j_1}(z_1, x_I)\big) b_2(v_2) f(v_2) du_1 dv dz_1 dx_2 \\ &= C \sup_{\|h\|_2 \le 1} \left(\int_{|x_1 - z_1| > 8c2^{-k_1}} + \int_{|x_1 - z_1| \le 8c2^{-k_1}} \right) \sum_{j_1 \ge k_1} \iint h(x_2) \\ &\quad \times D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\ &\quad \times \big(\widetilde{D}_{j_1}(z_1, y_1) - \widetilde{D}_{j_1}(z_1, x_I)\big) b_2(v_2) f(v_2) du_1 dx_2 dv dz_1 \\ &:= E_1 + E_2. \end{aligned}$$

For E_1, we use the cancellation properties of D_{j_1} to get

$$\begin{aligned} E_1 &= C \sup_{\|h\|_2 \le 1} \int_{|x_1 - z_1| > 8c2^{-k_1}} \sum_{j_1 \ge k_1} \iint h(x_2) D_{k_1}(x_1, u_1) b_1(u_1) \\ &\quad \times \big(K(u_1, x_2, v_1, v_2) - K(u_1, x_2, z_1, v_2)\big) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\ &\quad \times \big(\widetilde{D}_{j_1}(z_1, y_1) - \widetilde{D}_{j_1}(z_1, x_I)\big) b_2(v_2) f(v_2) du_1 dx_2 dv dz_1. \end{aligned}$$

Note that the facts $|x_1 - z_1| > 8c2^{-k_1}$, $|x_1 - u_1| < c2^{-k_1}$ and $|v_1 - z_1| < c2^{-j_1} \le c2^{-k_1}$ easily imply $|v_1 - z_1| \le |u_1 - v_1|/2$ and $|u_1 - v_1| > |x_1 - z_1|/2$.

We apply (A_2) to obtain that, for $|y_1 - x_I| \le 2^{-k_1-1}$,

$$\begin{aligned}
E_1 &\le C \sup_{\|h\|_2\le 1} \|h\|_2 \|f\|_2 \sum_{j_1\ge k_1} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}} |D_{k_1}(x_1,u_1)| \frac{|v_1-z_1|^{\varepsilon}}{|u_1-v_1|^{n_1+\varepsilon}} \\
&\qquad \times |D_{j_1}(v_1,z_1)| \big|\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big| du_1 dv_1 dz_1 \\
&\le C\|f\|_2 \sum_{j_1\ge k_1} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}} 2^{-(j_1-k_1)\varepsilon} |D_{k_1}(x_1,u_1)| \frac{2^{-k_1\varepsilon}}{(2^{-k_1}+|u_1-v_1|)^{n_1+\varepsilon}} \\
&\qquad \times |D_{j_1}(v_1,z_1)| \big|\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big| du_1 dv_1 dz_1 \\
&\le C\|f\|_2 \sum_{j_1\ge k_1} \int_{\mathbb{R}^{n_1}} 2^{-(j_1-k_1)(\varepsilon-\varepsilon')} \frac{2^{-k_1\varepsilon}}{(2^{-k_1}+|x_1-z_1|)^{n_1+\varepsilon}} \Big(\frac{|y_1-x_I|}{2^{-k_1}}\Big)^{\varepsilon'} \\
&\qquad \times \bigg(\frac{2^{-j_1\varepsilon}}{(2^{-j_1}+|z_1-y_1|)^{n_1+\varepsilon}} + \frac{2^{-j_1\varepsilon}}{(2^{-j_1}+|z_1-x_I|)^{n_1+\varepsilon}}\bigg) dz_1 \\
&\le C\Big(\frac{|y_1-x_I|}{2^{-k_1}}\Big)^{\varepsilon'} \frac{2^{-k_1\varepsilon}}{(2^{-k_1}+|x_1-x_I|)^{n_1+\varepsilon}} \|f\|_2.
\end{aligned} \tag{13}$$

By the condition on the support of D_{k_1},

$$\begin{aligned}
E_2 = C &\sup_{\|h\|_2\le 1} \int_{|x_1-z_1|\le 8c2^{-k_1}} \sum_{j_1\ge k_1} \iint_{|u_1-z_1|\le 9c2^{-k_1}} h(x_2) D_{k_1}(x_1,u_1) b_1(u_1) \\
&\times K(u_1,x_2,v_1,v_2)\big) b_1(v_1) D_{j_1}(v_1,z_1) b_1(z_1) \\
&\times \big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big) b_2(v_2) f(v_2) du_1 dx_2 dv dz_1.
\end{aligned}$$

Note that $E_2 = 0$ if $|z_1 - y_1| > c2^{-k_1}$ and $|z_1 - x_I| > c2^{-k_1}$. It implies $|x_1 - x_I| \le 10c2^{-k_1}$ provided $|x_1 - z_1| \le 8c2^{-k_1}$ and $|y_1 - x_I| \le c2^{-k_1-1}$. This fact will be used later.

Now let $\eta_0 \in C^\infty(\mathbb{R}^{n_1})$ be 1 on the unit ball and 0 outside the ball $B(0,2)$. Set $\eta_1 = 1 - \eta_0$. We use $T_1^*(b_1) = 0$ to obtain

$$\begin{aligned}
E_2 = C &\sup_{\|h\|_2\le 1} \int_{|x_1-z_1|\le 8c2^{-k_1}} \sum_{j_1\ge k_1} \iint_{|u_1-z_1|\le 9c2^{-k_1}} h(x_2) \\
&\times \big(D_{k_1}(x_1,u_1) - D_{k_1}(x_1,z_1)\big) b_1(u_1) K(u_1,x_2,v_1,v_2) b_1(v_1) D_{j_1}(v_1,z_1) \\
&\times b_1(z_1) \big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big) b_2(v_2) f(v_2) du_1 dx_2 dv dz_1 \\
= C &\sup_{\|h\|_2\le 1} \int_{|x_1-z_1|\le 8c2^{-k_1}} \sum_{j_1\ge k_1} \iint_{|u_1-z_1|\le 9c2^{-k_1}} h(x_2) \\
&\times \bigg(\eta_0\Big(\frac{u_1-z_1}{2^{-j_1-2}}\Big) + \eta_1\Big(\frac{u_1-z_1}{2^{-j_1-2}}\Big)\bigg) \big(D_{k_1}(x_1,u_1) - D_{k_1}(x_1,z_1)\big) b_1(u_1) \\
&\times K(u_1,x_2,v_1,v_2) b_1(v_1) D_{j_1}(v_1,z_1) b_1(z_1) \big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big) \\
&\times b_2(v_2) f(v_2) du_1 dx_2 dv dz_1
\end{aligned}$$

$$:= E_{21} + E_{22}.$$

Let

$$f_{j_1,z_1}(u_1) = \eta_0\Big(\frac{u_1 - z_1}{2^{-j_1-2}}\Big)\big(D_{k_1}(x_1,u_1) - D_{k_1}(x_1,z_1)\big)$$

and

$$g_{j_1,z_1}(v_1) = D_{j_1}(v_1,z_1)b_1(z_1)\big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big).$$

The $L^2(\mathbb{R}^{n_1+n_2})$ boundedness of T yields, for $|y_1 - x_I| \le 2^{-k_1-1}$,

$$\begin{aligned} E_{21} &\le C \sup_{\|h\|_2\le 1} \int \sum_{j_1\ge k_1} \|h\|_2 \|f_{t',z_1}\|_2 \|g_{t',z_1}\|_2 \|f\|_2 dz_1 \\ &\le C \sum_{j_1\ge k_1} \frac{2^{-j_1}}{2^{-k_1(n_1+1)}} 2^{-j_1n_1/2}\Big(\frac{|y_1-x_I|}{2^{-j_1}}\Big)^{\varepsilon'} 2^{j_1n_1/2}\|f\|_2 \\ &\le C|y_1 - x_I|^{\varepsilon'} 2^{k_1(n_1+\varepsilon')}\|f\|_2. \end{aligned}$$

To estimate E_{22}, we use the cancellation property of D_{j_1} and write

$$\begin{aligned} E_{22} = C &\sup_{\|h\|_2\le 1} \int_{|x_1-z_1|\le 8c2^{-k_1}} \sum_{j_1\ge k_1} \iint_{|u_1-z_1|\le 9c2^{-k_1}} h(x_2)\eta_1\Big(\frac{u_1-z_1}{2^{-j_1-2}}\Big) \\ &\times \big(D_{k_1}(x_1,u_1) - D_{k_1}(x_1,z_1)\big)b_1(u_1) \\ &\times \big(K(u_1,x_2,v_1,v_2) - K(u_1,x_2,z_1,v_2)\big)b_1(v_1)D_{j_1}(v_1,z_1) \\ &\times b_1(z_1)\big(\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)\big)b_2(v_2)f(v_2)du_1dx_2dvdz_1. \end{aligned}$$

By the conditions on the supports of η_1 and D_{j_1}, we have $|u_1 - z_1| \ge 4c2^{-j_1}$ and $|v_1 - z_1| < c2^{-j_1}$. This gives $|v_1 - z_1| \le |u_1 - z_1|/2$. Applying (A_2) with the estimate

$$|D_{k_1}(x_1,u_1) - D_{k_1}(x_1,z_1| \le C\frac{|u_1-z_1|}{2^{-k_1(n_1+1)}},$$

we obtain that, for $|y_1 - x_I| \le 2^{-k_1-1}$,

$$\begin{aligned} E_{22} \le C &\sum_{j_1\ge k_1} \int_{\mathbb{R}^{n_1}} \int_{2^{-j_1-2}\le|u_1-z_1|\le 92^{-k_1}} \int_{\mathbb{R}^{n_1}} \frac{|u_1-z_1|}{2^{-k_1(n_1+1)}} \frac{|v_1-z_1|^{\varepsilon}}{|u_1-z_1|^{n_1+\varepsilon}}\|f\|_2 \\ &\times |D_{j_1}(v_1,z_1)||\widetilde{D}_{j_1}(z_1,y_1) - \widetilde{D}_{j_1}(z_1,x_I)|dv_1du_1dz_1 \\ \le\ & C|y_1-x_I|^{\varepsilon'}2^{k_1(n_1+\varepsilon')}\|f\|_2. \end{aligned}$$

Thus, $E_2 \le C|y_1 - x_I|^{\varepsilon'}2^{k_1(n_1+\varepsilon')}\|f\|_2$. By the fact $|x_1 - x_I| \le 10c2^{-k_1}$ as mentioned before, we have

$$E_2 \le C\Big(\frac{|y_1-x_I|}{2^{-k_1}}\Big)^{\varepsilon'} \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + |x_1 - x_I|)^{n_1+\varepsilon}}\|f\|_2. \tag{14}$$

The estimate of F is the same as the estimate of E. It follows from (12)−(14) that, for $|y_1 - x_I| \le 2^{-k_1-1}$,

$$\int_{\mathbb{R}^{n_2}} \Big| \sum_{j_1} \int\int_{\mathbb{R}^{n_1}} D_{k_1}(x_1,u_1)b_1(u_1)K(u_1,x_2,v_1,v_2) \times b_1(y_1)\big(D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,y_1) - D_{j_1}M_{b_1}\widetilde{D}_{j_1}(v_1,x_I)\big)b_2(v_2)f(v_2)du_1dv\Big|^2 dx_2$$
$$\le C(2^{k_1}|y_1-x_I|)^{2\varepsilon}\frac{2^{-k_1 2\varepsilon}}{(2^{-k_1}+|x_1-x_I|)^{2(n_1+\varepsilon)}}\|f\|_2^2. \tag{15}$$

Inserting (15) into (11), we obtain the desired result (i) of Lemma 5.3. □

To finish the proof of Theorem 1.1, we need the following general result that follows from Theorem 3.1 and [16, Proposition 4].

Proposition 5.4. *Let $\mathcal{L}$ be a bounded operator from $L^2(\mathbb{R}^{n_1+n_2})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$. Then, for $0 < p \le 1$, $\mathcal{L}$ extends to be a bounded operator from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$ if and only if $\|\mathcal{L}(a)\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})} \le C$ for all $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$-atoms a, where the constant C is independent of a.*

It ie known that the $L^2(\mathbb{R}^{n_1+n_2})$ boundedness of T and the product Littlewood-Paley estimate imply that $\mathcal{L}$ is bounded from $L^2(\mathbb{R}^{n_1+n_2})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$. As mentioned before, the $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) - H^p_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedness of T is equivalent to the $H^p - L^p_{\mathcal{H}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedness of the $\mathcal{H}$-valued operator $\mathcal{L}$ which maps f into $\{T_{k_1,k_2}(f)\}_{k_1,k_2\in\mathbb{Z}}$. Hence, to show the **"if part"** of Theorem 1.1, by Proposition 5.4, it suffices to prove

$$\|\{T_{k_1,k_2}(a)\}_{k_1,k_2\in\mathbb{Z}}\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})} \le C \qquad \text{for all } H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) \text{ atoms } a,$$

where the constant C is independent of a.

To do this, we follow R. Fefferman's idea [9]. Suppose that a is an $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ atom supported on an open set $\Omega \subset \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with finite measure. Furthermore, a can be decomposed as $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, where $\mathcal{M}(\Omega)$ is the collection of all maximal dyadic subrectangles contained in Ω, each a_R is supported on $2R = 2I\times 2J$, the double of $R = I\times J$, $\int_{2I} a_R(x_1,x_2)dx_1 = 0$ for all $x_2 \in 2J$, and $\int_{2J} a_R(x_1,x_2)dx_2 = 0$ for all $x_1 \in 2I$. Here the higher order moments vanishing of a_R are not needed because we only consider $\max\{\frac{n_1}{n_1+\varepsilon},\frac{n_2}{n_2+\varepsilon}\} < p \le 1$. Moreover, $\|a\|_2 \le |\Omega|^{\frac12-\frac1p}$ and $\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|_2^2 \le |\Omega|^{1-\frac2p}$. Let $\widetilde{\Omega} = \{(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : M_s(\chi_\Omega)(x_1,x_2) > 4^{-n_1-n_2}n_1^{-n_1/2}n_2^{-n_2/2}\}$, where M_s is the strong maximal function defined by

$$M_s(f)(x_1,x_2) = \sup_{(x_1,x_2)\in P}\frac{1}{|P|}\int_P |f(y_1,y_2)|dy_1dy_2,$$

where the supremum is taken over all rectangles P (a product of a cube in $\mathbb{R}^{n_1}$ with a cube in $\mathbb{R}^{n_2}$) containing (x_1,x_2). It follows from the strong maximal theorem that $|\widetilde{\Omega}| \le C|\Omega|$.

We now estimate $\|\{T_{k_1,k_2}(a)\}_{k_1,k_2\in\mathbb{Z}}\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})}$ as follows. Write $\widetilde{\widetilde{\Omega}} = \{(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : M_s(\chi_{\widetilde{\Omega}})(x_1,x_2) > 4^{-n_1-n_2} n_1^{-n_1/2} n_2^{-n_2/2}\}$ and similarly for $\widetilde{\widetilde{\widetilde{\Omega}}}$. Then

$$\int \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2$$
$$= \int_{\widetilde{\widetilde{\widetilde{\Omega}}}} \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2 + \int_{\big(\widetilde{\widetilde{\widetilde{\Omega}}}\big)^c} \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2.$$

By Hölder's inequality, the $L^2 - L^2_{\mathcal{H}}$ boundedness of $\mathcal{L}$, and the size condition of a,

$$\int_{\widetilde{\widetilde{\widetilde{\Omega}}}} \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2$$
$$\le \bigg(\int_{\widetilde{\widetilde{\widetilde{\Omega}}}} \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^2_{\mathcal{H}} dx_1dx_2\bigg)^{\frac{p}{2}} |\widetilde{\widetilde{\widetilde{\Omega}}}|^{1-\frac{p}{2}}$$
$$\le C\|a\|_2^p |\Omega|^{1-\frac{p}{2}} \le C.$$

Therefore it remains to deal with

$$\int_{\big(\widetilde{\widetilde{\widetilde{\Omega}}}\big)^c} \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2$$
$$\le \sum_{R\in\mathcal{M}(\Omega)} \int_{\big(\widetilde{\widetilde{\widetilde{\Omega}}}\big)^c} \big\|\{T_{k_1,k_2}(a)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2,$$

where we use the inequality $(\alpha+\beta)^p \le \alpha^p + \beta^p$ for $p \le 1$.

For each $R = I\times J \in \mathcal{M}(\Omega)$, we set a larger rectangle $\widetilde{R} = \widetilde{I}\times J$ such that $\widetilde{I}$ is the largest dyadic cube containing I and $\widetilde{I}\times J \subset \widetilde{\Omega}$. Similarly, $\widetilde{\widetilde{R}} = \widetilde{I}\times\widetilde{J}$ where $\widetilde{J}$ is the largest dyadic cube containing J and $\widetilde{I}\times\widetilde{J} \subset \widetilde{\widetilde{\Omega}}$. Let $\mathcal{M}_1(\Omega)$ denote the collection of all dyadic subrectangles $R\subset\Omega$, $R = I\times J$ that are maximal in the x_1 direction. It is clear that $R\in\mathcal{M}(\Omega)$ implies $R\in\mathcal{M}_2(\Omega)$ and $\widetilde{R}\in\mathcal{M}_1(\widetilde{\Omega})$. Define $\mathcal{M}_2(\Omega)$ similarly. Also note that $4\sqrt{n_1}\widetilde{I}\times 4\sqrt{n_2}\widetilde{J} \subset \widetilde{\widetilde{\widetilde{\Omega}}}$. Then

$$\int_{\big(\widetilde{\widetilde{\widetilde{\Omega}}}\big)^c} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2$$
$$\le \int_{(4\sqrt{n_1}\widetilde{I})^c\times\mathbb{R}^{n_2}} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|^p_{\mathcal{H}} dx_1dx_2$$
$$+ \int_{\mathbb{R}^{n_1}\times(4\sqrt{n_2}\widetilde{J})^c} \|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\|^p_{\mathcal{H}} dx_1dx_2$$

$$:= U(R) + V(R).$$

We define $\gamma_1(R) = \gamma_1(R,\Omega) = \frac{\ell(\widetilde{I})}{\ell(I)}$ and $\gamma_2(\widetilde{R}) = \gamma_2(\widetilde{R},\widetilde{\Omega}) = \frac{\ell(\widetilde{J})}{\ell(J)}$, where $\ell(I)$ denotes the side length of I. To estimate $U(R)$, we write

$$\begin{aligned} U(R) &= \int_{(4\sqrt{n_1}\widetilde{I})^c \times 4\sqrt{n_2}J} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^p dx_1dx_2 \\ &\quad + \int_{(4\sqrt{n_1}\widetilde{I})^c \times (4\sqrt{n_2}J)^c} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^p dx_1dx_2 \\ &:= U_1(R) + U_2(R). \end{aligned}$$

By Hölder's inequality and Minkowski's inequality,

$$U_1(R) \le C|J|^{1-\frac{p}{2}} \int_{(4\sqrt{n_1}\widetilde{I})^c} \bigg(\int_{\mathbb{R}^{n_2}} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^2 dx_2 \bigg)^{\frac{p}{2}} dx_1. \quad (16)$$

The cancellation condition of a_R yields

$$\begin{aligned} &T_{k_1,k_2}(a_R)(x_1,x_2) \\ &= \int T_{k_1,k_2}(x_1,x_2,y_1,y_2)a_R(y_1,y_2)dy_1dy_2 \\ &= \int \big[T_{k_1,k_2}(x_1,x_2,y_1,y_2) - T_{t,s}(x_1,x_2,x_I,y_2)\big]a_R(y_1,y_2)dy_1dy_2, \end{aligned}$$

where x_I denotes the center of I. Now we apply Schwarz's inequality to get

$$\begin{aligned} &\big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^2 \\ &\le C|I| \int_{2I} \bigg\| \bigg\{ \int_{2J} \big[T_{k_1,k_2}(x_1,x_2,y_1,y_2) \\ &\qquad\qquad - T_{k_1,k_2}(x_1,x_2,x_I,y_2)\big]a_R(y_1,y_2)dy_2 \bigg\} \bigg\|_{\mathcal{H}}^2 dy_1. \end{aligned}$$

This estimate and Lemma 5.3 imply that, for $x_1 \in (4\sqrt{n_1}\widetilde{I})^c$ and $y_1 \in 2I$,

$$\begin{aligned} &\int_{\mathbb{R}^m} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^2 dx_2 \\ &\le C|I| \int_{2I}\int_{\mathbb{R}^{n_2}} \bigg\| \bigg\{ \int_{2J} \big[T_{k_1,k_2}(x_1,x_2,y_1,y_2) \\ &\qquad\qquad - T_{k_1,k_2}(x_1,x_2,x_I,y_2)\big]a_R(y_1,y_2)dy_2 \bigg\} \bigg\|_{\mathcal{H}}^2 dx_2dy_1 \\ &\le C|I| \Big(\frac{\ell(I)^{\varepsilon'}}{|x_1-x_I|^{n+\varepsilon'}} \Big)^2 \|a_R\|_2^2. \end{aligned}$$

Inserting the estimate above into (16) shows

$$U_1(R) \le C\big(\gamma_1(R)\big)^{n_1-(n_1+\varepsilon')p} |R|^{1-\frac{p}{2}} \|a_R\|_2^p. \quad (17)$$

To estimate $U_2(R)$, we use the cancellation conditions of a_R to write

$$\begin{aligned}
&T_{k_1,k_2}(a_R)(x_1,x_2)\\
&= \int \Big(T_{k_1,k_2}(x_1,x_2,y_1,y_2) - T_{k_1,k_2}(x_1,x_2,x_I,y_2)\\
&\qquad - T_{k_1,k_2}(x_1,x_2,y_1,x_J) + T_{k_1,k_2}(x_1,x_2,x_I,x_J)\Big) a_R(y_1,y_2)dy_1dy_2,
\end{aligned}$$

where x_J is the center of J. For $x_1 \in (4\sqrt{n_1}\widetilde{I})^c$, $x_2 \in (4\sqrt{n_2}J)^c$, $y_1 \in 2I$, and $y_2 \in 2J$, we have $|y_1 - x_I| \le \frac{1}{2}|x_1 - x_I|$ and $|y_2 - x_J| \le \frac{1}{2}|x_2 - x_J|$. Thus, the estimate (B_3) gives

$$\begin{aligned}
&\big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}\\
&\le C\bigg(\int |R|\Big(\frac{|y_1-x_I|^{\varepsilon'}}{|x_1-x_I|^{n_1+\varepsilon'}}\frac{|y_2-x_J|^{\varepsilon'}}{|x_2-x_J|^{n_2+\varepsilon'}}|a_R(y_1,y_2)|\Big)^2 dy_1dy_2\bigg)^{1/2}.
\end{aligned}$$

Hence,

$$U_2(R) \le C\big(\gamma_1(R)\big)^{n_1-(n_1+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p. \tag{18}$$

Both estimates (17) and (18) give

$$U(R) \le C\big(\gamma_1(R)\big)^{n_1-(n_1+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p.$$

The estimate for $V(R)$, though slightly different from $U(R)$, can be handled in much the same manner so that

$$V(R) \le C\big(\gamma_2(\widetilde{R})\big)^{n_2-(n_2+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p.$$

Summing over R gives

$$\begin{aligned}
&\sum_{R\in\mathcal{M}(\Omega)} \int_{(\widetilde{\widetilde{\Omega}})^c} \big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^p dx_1dx_2\\
&\le C\sum_{R\in\mathcal{M}(\Omega)} \big(\gamma_1(R)\big)^{n_1-(n_1+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p\\
&\qquad + C\sum_{R\in\mathcal{M}(\Omega)} \big(\gamma_2(\widetilde{R})\big)^{n_2-(n_2+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p\\
&\le C\Bigg\{\bigg(\sum_{R\in\mathcal{M}_2(\Omega)} |R|\big(\gamma_1(R)\big)^{-\delta_1}\bigg)^{1-\frac{p}{2}}\\
&\qquad + \bigg(\sum_{\widetilde{R}\in\mathcal{M}_1(\widetilde{\Omega})} |\widetilde{R}|\big(\gamma_2(\widetilde{R})\big)^{-\delta_2}\bigg)^{1-\frac{p}{2}}\Bigg\}\bigg(\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_2^2\bigg)^{\frac{p}{2}},
\end{aligned}$$

where $\delta_1 = \frac{2[n_1-(n_1+\varepsilon')p]}{p-2} > 0$ and $\delta_2 = \frac{2[n_2-(n_2+\varepsilon')p]}{p-2} > 0$.

To estimate the last part above, we use the following

Journé's lemma. Let $\mathcal{M}_1(\Omega)$ and $\mathcal{M}_2(\Omega)$ be defined as above. Then $\sum_{R\in\mathcal{M}_2(\Omega)}|R|\big(\gamma_1(R)\big)^{-\delta}\le C_\delta|\Omega|$ and $\sum_{R\in\mathcal{M}_1(\Omega)}|R|\big(\gamma_2(R)\big)^{-\delta}\le C_\delta|\Omega|$ for any $\delta>0$, where C_δ is a constant depending on δ only.

Journé's lemma and the size condition of a_R imply

$$\sum_{R\in\mathcal{M}(\Omega)}\int_{\big(\tilde{\tilde{\Omega}}\big)^c}\big\|\{T_{k_1,k_2}(a_R)\}(x_1,x_2)\big\|_{\mathcal{H}}^p dx_1dx_2\le C|\Omega|^{1-\frac{p}{2}}|\Omega|^{\frac{p}{2}-1}\le C.$$

We complete the proof of the **"if part"** of Theorem 1.1. □

Proof of Theorem 1.2. "if" part:

We point out that the "if" part follows directly from duality of $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. We provide the details as follows.

(1) Suppose $T_1(b_1)=T_2(b_2)=0$.

Then for the adjoint operator T^* of T, it is clear to see that T^* satisfies

$$(T^*)_1^*(b_1)=(T^*)_2^*(b_2)=0.$$

Hence, from (1) in Theorem 1.1 we obtain that T^* is bounded from $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

Then for every $f\in BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $g\in H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have

$$\begin{aligned}|\langle Tf,g\rangle|=|\langle f,T^*g\rangle|&\le C\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|T^*g\|_{H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\le C\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|g\|_{H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\end{aligned}$$

By density argument we can obtain that Tf is in $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

(2) Suppose $T_1(1)=T_2(1)=0$.

Then for the adjoint operator T^* of T, it is clear to see that T^* satisfies

$$(T^*)_1^*(1)=(T^*)_2^*(1)=0.$$

Hence, from (2) in Theorem 1.1 we obtain that $T^*M_{b_1b_2}$ is bounded from $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

Then for every $f\in BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $g\in H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have

$$\begin{aligned}|\langle M_{b_1b_2}Tf,g\rangle|&=|\langle f,T^*b_1b_2g\rangle|\\&\le C\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|T^*M_{b_1b_2}g\|_{H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\le C\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|g\|_{H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\end{aligned}$$

By density argument we can obtain that $M_{b_1b_2}Tf$ is in $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

(3) Suppose $T_1(b_1)=T_2(b_2)=0$.

Then for the adjoint operator T^* of T, it is clear to see that T^* satisfies

$$(T^*)_1^*(b_1)=(T^*)_2^*(b_2)=0.$$

Hence, from (3) in Theorem 1.1 we obtain that $T^*M_{b_1b_2}$ is bounded from $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

Then for every $f\in BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $g\in H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have

$$\begin{aligned}|\langle M_{b_1b_2}Tf,g\rangle| &= |\langle f,T^*b_1b_2g\rangle|\\ &\le C\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|T^*M_{b_1b_2}g\|_{H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\ &\le C\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|g\|_{H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\end{aligned}$$

By density argument we can obtain that $M_{b_1b_2}Tf$ is in $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

"only if" part:

We now prove the "only if" part.

(1) Suppose T admits a bounded extension from $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

We now consider the function $f(x_1,x_2)=\chi_1(x_1)f_2(x_2)b_2^{-1}(x_2)$, where $\chi_1(x_1)\equiv 1$ on $\mathbb{R}^{n_1}$, and $f_2(x_2)\in C_0^\infty(\mathbb{R}^{n_2})$. Then it is clear that this $f(x_1,x_2)$ is in $L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, and hence it is in $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|f\|_{BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Then we also have $g(x_1,x_2)=b_1(x_1)b_2(x_2)f(x_1,x_2)$ is in $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|g\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Moreover, since Tg is in $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|Tg\|_{BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le\|g\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Hence, we obtain that for all $\psi_1\in C_0^\infty(\mathbb{R}^{n_1})$ with $\int_{\mathbb{R}^{n_1}}\psi_1(x_1)dx_1=0$ and all $\psi_2\in C_0^\infty(\mathbb{R}^{n_2})$ with $\int_{\mathbb{R}^{n_2}}\psi_2(x_2)dx_2=0$,

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\psi_1(x_1)\psi_2(x_2)Tg(x_1,x_2)dx_1dx_2=0,$$

since $\psi_1(x_1)\psi_2(x_2)$ is in $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. This yields

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)b_2(y_2)f(y_1,y_2)dy_1dy_2=0,$$

which, together with the definition of f, gives

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)f_2(y_2)dy_1dy_2=0,$$

Since $\psi_1(x_1)\psi_2(x_2)$ is in $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have that $T^*(\psi_1\psi_2)$ is in $L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Moreover, since $f_2(x_2)\in C_0^\infty(\mathbb{R}^{n_2})$ and b_2 is bounded, we see that the set $\{b_2f_2:\ |b_2|\le C,\ f_2\in C_0^\infty(\mathbb{R}^{n_2})\}$ is dense in $L^2(\mathbb{R}^{n_2})$. We get that

$$\int_{\mathbb{R}^{n_1}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)dy_1=0,$$

which is

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}}\psi_1(x_1)\psi_2(x_2)K(x_1,x_2,y_1,y_2)b_1(y_1)dy_1dx_1dx_2=0.$$

Based on the definition, this shows that

$$T_1(b_1)=0.$$

Similarly we can obtain that

$$T_2(b_2)=0.$$

(2) Suppose $M_{b_1b_2}T$ admits a bounded extension from $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

We now consider the function $f(x_1,x_2)=\chi_1(x_1)f_2(x_2)$, where $\chi_1(x_1)\equiv 1$ on $\mathbb{R}^{n_1}$, and $f_2(x_2)\in C_0^\infty(\mathbb{R}^{n_2})$. Then it is clear that this $f(x_1,x_2)$ is in $L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, and hence it is in $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|f\|_{BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Therefore, $M_{b_1b_2}Tf$ is in $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|M_{b_1b_2}Tf\|_{BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le\|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

For all $\psi_1\in C_0^\infty(\mathbb{R}^{n_1})$ with $\int_{\mathbb{R}^{n_1}}\psi_1(x_1)dx_1=0$ and all $\psi_2\in C_0^\infty(\mathbb{R}^{n_2})$ with $\int_{\mathbb{R}^{n_2}}\psi_2(x_2)dx_2=0$, we have $\psi_1(x_1)\psi_2(x_2)$ is in $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and then $\psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)$ is in $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Hence

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)M_{b_1b_2}Tf(x_1,x_2)dx_1dx_2=0,$$

This yields

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}T^*(\psi_1\psi_2)(y_1,y_2)f(y_1,y_2)dy_1dy_2=0,$$

which, together with the definition of f, gives

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}T^*(\psi_1\psi_2)(y_1,y_2)f_2(y_2)dy_1dy_2=0,$$

Since $\psi_1(x_1)\psi_2(x_2)$ is in $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have that $T^*(\psi_1\psi_2)$ is in $L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Moreover, since $f_2(x_2)\in C_0^\infty(\mathbb{R}^{n_2})$, we see that the set $\{f_2:\ f_2\in C_0^\infty(\mathbb{R}^{n_2})\}$ is dense in $L^2(\mathbb{R}^{n_2})$. We get that

$$\int_{\mathbb{R}^{n_1}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)dy_1=0,$$

which is

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}}\psi_1(x_1)\psi_2(x_2)K(x_1,x_2,y_1,y_2)dy_1dx_1dx_2=0.$$

Based on the definition, this shows that $T_1(1)=0$. Similarly we can obtain that $T_2(2)=0$.

(3) Suppose T admits a bounded extension from $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

We now consider the function $f(x_1,x_2)=\chi_1(x_1)f_2(x_2)b_2^{-1}(x_2)$, where $\chi_1(x_1)\equiv 1$ on $\mathbb{R}^{n_1}$, and $f_2(x_2)\in C_0^\infty(\mathbb{R}^{n_2})$. Then it is clear that this $f(x_1,x_2)$ is in $L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, and hence it is in $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|f\|_{BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Then we also have $g(x_1,x_2)=b_1(x_1)b_2(x_2)f(x_1,x_2)$ is in $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|g\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Moreover, since $M_{b_1b_2}Tg$ is in $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with

$$\|M_{b_1b_2}Tg\|_{BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le\|g\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=0.$$

Hence, we obtain that for all $\psi_1\in C_0^\infty(\mathbb{R}^{n_1})$ with $\int_{\mathbb{R}^{n_1}}\psi_1(x_1)dx_1=0$ and all $\psi_2\in C_0^\infty(\mathbb{R}^{n_2})$ with $\int_{\mathbb{R}^{n_2}}\psi_2(x_2)dx_2=0$,

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)M_{b_1b_2}Tg(x_1,x_2)dx_1dx_2=0,$$

since $\psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)$ is in $H^1_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. This yields

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)b_2(y_2)f(y_1,y_2)dy_1dy_2=0,$$

which, together with the definition of f, gives

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)f_2(y_2)dy_1dy_2=0,$$

Since $\psi_1(x_1)\psi_2(x_2)$ is in $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we have that $T^*(\psi_1\psi_2)$ is in $L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\cap L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Moreover, since $f_2(x_2)\in C_0^\infty(\mathbb{R}^{n_2})$ and b_2 is bounded, we see that the set $\{b_2f_2:\ |b_2|\le C,\ f_2\in C_0^\infty(\mathbb{R}^{n_2})\}$ is dense in $L^2(\mathbb{R}^{n_2})$. We get that

$$\int_{\mathbb{R}^{n_1}}T^*(\psi_1\psi_2)(y_1,y_2)b_1(y_1)dy_1=0,$$

which is

$$\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}}\psi_1(x_1)\psi_2(x_2)K(x_1,x_2,y_1,y_2)b_1(y_1)dy_1dx_1dx_2=0.$$

Based on the definition, this shows that $T_1(b_1)=0$. Similarly we can obtain that $T_2(b_2)=0$.

The proof of Theorem 1.2 is complete. □

We now finalize the proof of Theorem 1.1.

Proof of the "only if part" of Theorem 1.1. (1) Suppose T is bounded from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $H^p_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Then we see that the adjoint T^* extends to a bounded operator from $BMO_{b_1b_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $BMO(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Then apply the "only if" part for (2) of Theorem 1.2, we see that $T_1^*(b_1)=T_2^*(b_2)=0$.

Similarly we can verify the "only if" part of (2) and (3).

The proof of Theorem 1.1 is complete. □

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