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Fixed-point algebras for weakly proper Fell bundles

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Abstract. We define weakly proper Fell bundles and construct exotic fixed-point algebras for such bundles. Three alternative constructions of such algebras are given. Under a kind of freeness condition, one of our constructions implies that every exotic cross-sectional C*-algebra of a weakly proper Fell bundle is Morita equivalent to an exotic fixed-point algebra. The other constructions are used to show that ours generalizes that of Buss and Echterhoff on weakly proper actions on C*-algebras. We also generalize to Fell bundles the fact that every C*-action which is proper in Kasparov's sense is amenable.

Contents

Introduction

Green's Imprimitivity Theorem [17] implies that given a free and proper action σ of a locally compact and Hausdorff (LCH) group G on a LCH space X; the full crossed product $C_0(X) \rtimes_\sigma G$ is strongly Morita equivalent to $C_0(X/G)$, X/G being the space of $\sigma-$orbits. The "large fixed-point algebra" of σ, $C_b(X)_\sigma$, is the set of fixed points of the canonical action $\overline{\sigma}$ of G on $C_b(X)$ determined by σ, $\overline{\sigma}_t(f)(x) = f(\sigma_{t^{-1}}(x))$. Note $C_b(X)_\sigma$ is C*-isomorphic to $C_b(X/G)$, hence $C_0(X/G)$ is C*-isomorphic to a C*-subalgebra of the large fixed-point algebra of σ. That is why $C_0(X/G)$ is called a "fixed-point algebra".

Green's Theorem may be used to show that σ is amenable in the sense that $C_0(X) \rtimes_\sigma G$ agrees with the reduced cross product $C_0(X) \rtimes_{r\sigma} G$. Indeed, let I be the kernel of the regular representation $C_0(X) \rtimes_\sigma G \to C_0(X) \rtimes_{r\sigma} G$. This ideal is induced from some ideal of $C_0(X/G)$, i.e. from a $\sigma-$invariant open set $U \subset X$. But $C_0(U)$ is a $\sigma-$invariant ideal of $C_0(X)$, so using Green's Theorem once

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again we deduce that $I = C_0(U) \rtimes_{\sigma|_U} G$. The only way of having $C_0(U) \rtimes_{\sigma|_U} G$ contained in the kernel of the regular representation is if $U = \emptyset$, meaning that $I = \{0\}$.

In [13] Kasparov used proper actions on LCH spaces in order to construct fixed-point algebras for actions on $C_0(X)$−algebras. The same year Rieffel tried to give a definition of proper action on a C*-algebra without using proper actions on LCH spaces [18]. Rieffel starts with an action α of a LCH group G on a C*-algebra A and seeks a definition of proper action that allows him to establish a strong Morita equivalence between $A \rtimes_\alpha G$ and a C*-subalgebra of the (large) fixed-point algebra

$$M(A)_\alpha := \{T \in M(A) : \tilde{\alpha}_t(T) = T, \ \forall\, t \in G\},$$

where $\tilde{\alpha}$ is the natural extension of α to the multiplier algebra $M(A)$.

Rieffel notices that certain $C_c(G, A)$−valued inner products are positive definite in the reduced crossed product $A \rtimes_{\mathrm{r}\alpha} G$, but not in the full (or universal) one $A \rtimes_\alpha G \equiv A \rtimes_{\mathrm{u}\alpha} G$. Kasparov does not have this problem because his proper actions are always amenable (the full and reduced crossed products agree).

Buss and Echterhoff introduced in [6] the concept of the weakly proper action. Every Kasparov proper action is weakly proper and any weakly proper action is proper in Rieffel's sense. Even though not every weakly proper action is amenable, they are nice enough as to be able to prove that the $C_c(G, A)$−valued inner products are positive in the full crossed product. Moreover, one even has Symmetric Imprimitivity Theorems for weakly proper actions [7] (generalizing Raeburn's Symmetric Imprimitivity Theorem [16]).

An exotic crossed product for α is a strictly intermediate quotient $A \rtimes_{\mu\alpha} G$ between $A \rtimes_{\mathrm{u}\alpha} G$ and $A \rtimes_{\mathrm{r}\alpha} G$. Provided that α is weakly proper and satisfies a certain freeness condition, Buss-Echterhoff's Imprimitivity Theorem assigns a fixed-point algebra A^α_μ to every crossed product $\rtimes_\mu$ (universal, reduced or exotic) and establishes a Morita equivalence between A^α_μ and $A \rtimes_{\mu\alpha} G$.

There have been other attempts to define proper actions on C*-algebras that have lead to study integrable (or square integrable) actions, see for example [7, 15, 19] and the references therein. Our work can be seen as a generalization of Buss and Echterhoff's one to Fell bundles, which in turn generalizes Kasparov's.

With some notational effort one can extend Buss and Echterhoff's work to twisted actions and to partial actions, separately. But a natural preservation instinct should prevent everybody to try give a definition of "weakly proper twisted partial action". Twisted partial action are defined in [8], where it is shown that under certain hypotheses a Fell bundle can be described as the semidirect product bundle of a twisted partial action. This is of importance to us because it says that there is a kind of natural action associated to every Fell bundle. Then (at least in some cases) it should be possible to determine whether or not an action on a C*-algebra is weakly proper using only the semidirect product bundle of the action. A step further in this direction would be to state the definition of weakly proper action itself using the semidirect product bundle. After this one should obtain something very close to a definition of weakly proper Fell bundle. This is what we have done, and we show our results in this work.

Quite often authors working with Fell bundles assume the bundles are saturated. This is even true in some parts of [12], where Fell bundles are called C*-algebraic bundles. But we do not want to assume our Fell bundles are saturated because the semidirect product bundle of a C*-partial action (as considered in [8]) is saturated if and only if the partial action is global (i.e. just an action).

The "test case" that motivated this work was the semidirect product bundle of a partial action on a commutative C*-algebra or, equivalently, a C*-partial action on a commutative C*-algebra. This turned out to be quite important in the general theory, so we dedicate the first section of this work to study this test case.

In Section 2 we give the definition of weakly proper and Kasparov proper Fell bundles. We also show that every Kasparov proper Fell bundle is amenable. Finally, in the last section, we construct the fixed-point algebra of a Fell bundle and give three different ways of constructing these algebras.

1. Proper partial actions on LCH spaces

Let's start by recalling some equivalent definitions of proper actions on LCH spaces. Suppose σ is an action of the LCH group G on the LCH space X. Then σ is proper if satisfies any (and hence all) the equivalent conditions:

(1) For every pair of compact sets, $L, M \subset X$, the set

$$((L,M)) := \{t \in G : \sigma_t(L) \cap M \neq \emptyset\}$$

has compact closure.

(2) For every pair of compact sets, $L, M \subset X$, the set $((L,M))$ is compact.

(3) The map $G \times X \to X \times X, (t,x) \mapsto (\sigma_t(x), x)$, is proper (the preimage of every compact set is compact).

Let's translate each one of the conditions above to partial actions and compare the respective candidates for the definition of proper partial action. Assume $\sigma := (\{X_t\}_{t\in G}, \{\sigma_t\}_{t\in G})$ is a LCH partial action, that is, σ is a topological partial action of G on X in the sense of [2, Definition 1.1]; with both G and X being LCH. By definition, for all $t \in G$ the set $X_t \subset X$ is open and $\sigma_t : X_{t^{-1}} \to X_t$ is a homeomorphism. The domain and graph of σ are, respectively,

$$\Gamma_\sigma := \{(t,x) \in G \times X : x \in X_{t^{-1}}\}$$
$$\mathrm{Gr}(\sigma) := \{(t,x,y) \in G \times X \times X : x \in X_{t^{-1}},\ y = \sigma_t(x)\}.$$

Notice the definition of topological partial action implies that Γ_σ is open in $G\times X$ and also that $\Gamma_\sigma \to X,\ (t,x) \mapsto \sigma_t(x)$, is continuous. Besides, $\mathrm{Gr}(\sigma)$ determines σ.

The natural translation of conditions (1-3) above are

(P1) For every pair of compact sets, $L, M \subset X$, the set

$$((L,M)) := \{t \in G : \sigma_t(L \cap X_{t^{-1}}) \cap M \neq \emptyset\} \tag{1.1}$$

has compact closure.

(P2) For every pair of compact sets, $L, M \subset X$, the set $((L,M))$ is compact.

(P3) The map $F_\sigma : \Gamma_\sigma \to X \times X$, $(t,x) \mapsto (\sigma_t(x), x)$, is proper.

Sometimes we write $((L,M))_\sigma$ instead of $((L,M))$, specially if it is not clear which partial action must be used to compute $((L,M))$.

The following examples show the conditions above are not equivalent.

Example 1.1 ([2, Example 1.2]). Let φ be the flow of the vector field $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, f(x,y) = (1,0)$. Then φ defines a partial action of $\mathbb{R}$ on $X := \mathbb{R}^2 \setminus \{(0,0)\}$ that we now describe. Given $t \in \mathbb{R}$, let X_t be $X \setminus \{(st, 0) : s \in [0,1]\}$. Then $\varphi_t : X_{-t} \to X_t$ is given by $\varphi_t(x,y) = (x+t, y)$. We leave to the reader to verify that φ satisfies (P1). But φ does not satisfy (P2) because for the closed segments $L := [(-2,1),(-1,0)]$ and $M = [(1,1),(1,0)]$, $((L,M)) = (2,3]$ is not compact.

Example 1.2 ([2, Example 1.4]). Let $G := \mathbb{Z}_2$ act partially on $X = [0,1]$ by the partial action such that $\sigma_0 = \mathrm{id}_X$ and $\sigma_1 = \mathrm{id}_{[0,1/2)}$. Since every subset of G is compact, σ satisfies (P2). But σ does not satisfy (P3) because $\Gamma_\sigma = F_\sigma^{-1}(X \times X)$ is not compact.

With the previous examples and some extra work one can show that

Proposition 1.3. *For every LCH partial action σ, (P3)⇒(P2)⇒(P1) but none of the converses holds in general (as the previous examples have shown).*

Proof. Assume σ satisfies (P3) and take two compacts sets, $L, M \subset X$. Take a net $\{t_i\}_{i\in I} \subset ((L,M))$. Then for all $i \in I$ there exists $x_i \in L \cap X_{t_i^{-1}}$ such that $\sigma_{t_i}(x_i) \in M$. Thus $\{(t_i, x_i)\}_{i\in I} \subset F_\sigma^{-1}(L \times M)$ and there exists a subnet $\{(t_{i_j}, x_{i_j})\}_{j\in J}$ converging to some $(t,x) \in F_\sigma^{-1}(L \times M)$. Thus, $x \in X_{t^{-1}}$, $x = \lim_j x_{i_j} \in L$, $\sigma_t(x) = \lim_j \sigma_{t_{i_j}}(x_{i_j}) \in M$ and this implies $\lim_j t_{i_j} = t \in ((L,M))$. Hence (P3) implies (P2), which in turn implies (P1) because all the compact sets of G are closed. □

A key feature of proper actions is that one can construct fixed-point algebras with them. In the topological context this means that the orbit space is a LCH space. So let's define the orbit space of a topological partial action and let's try to see if any of the conditions (P1-P3) guarantees a LCH orbit space.

Definition 1.4. Let $\tau = (\{\tau_t\}_{t\in H}, \{Y_t\}_{t\in H})$ be a topological partial action. The orbit of a set $U \subset Y$ is defined as $[U]_\tau := \cup_{t\in H} \tau_t(U \cap Y_{t^{-1}})$ and the orbit of a point $y \in Y$ is $[y]_\tau := [\{y\}]_\tau$. A subset $U \subset Y$ is said to be invariant (or $\tau-$invariant) if $[U]_\tau \subset U$ (or, equivalently, $U = [U]_\tau$).

Note that $U \subset [U]_\tau$ because for $t = e$, $\tau_t(U \cap X_{t^{-1}}) = U$. Besides, $[U]_\tau$ is invariant because the properties of set theoretic partial actions imply

$$\begin{aligned}
[[U]_\tau]_\tau &= \bigcup_{t,s\in H} \sigma_t(\sigma_s(X_{s^{-1}} \cap U) \cap X_{t^{-1}}) \\
&= \bigcup_{t,s\in H} \sigma_t(\sigma_s(X_{s^{-1}} \cap U) \cap \sigma_s(X_{s^{-1}} \cap X_{t^{-1}})) \\
&= \bigcup_{t,s\in H} \sigma_t(\sigma_s(X_{s^{-1}} \cap X_{s^{-1}t^{-1}} \cap U)) = \bigcup_{t,s\in H} \sigma_{ts}(X_{s^{-1}} \cap X_{s^{-1}t^{-1}} \cap U) \\
&\subset \bigcup_{t,s\in H} \sigma_{ts}(X_{s^{-1}t^{-1}} \cap U) \subset [U]_\tau \subset [[U]_\tau]_\tau.
\end{aligned}$$

Remark 1.5. If $U \subset V \subset Y$ then $[U]_\tau \subset [V]_\tau$. This implies $[U]_\tau$ is the smallest invariant set containing U. Note also that $[U]_\tau$ is open if U is.

Remark 1.6. The whole space Y is the disjoint union of the orbits of its points. Indeed, it is clear that $y \in [y]_\tau$ for all $y \in Y$. Assume $y, z \in Y$ are such that $[y]_\tau \cap [z]_\tau \neq \emptyset$. Then there exists $r, s \in H$ such that $y \in X_{r^{-1}}$, $z \in X_{s^{-1}}$ and $\tau_r(y) = \tau_s(z)$. Hence, by the definition of partial action, $y \in X_{r^{-1}} \cap X_{r^{-1}s}$ and $\tau_{s^{-1}r}(y) = \tau_{s^{-1}}(\tau_r(y)) = z$. Thus $z \in [y]_\tau$ and this implies $[z]_\tau \subset [y]_\tau$. By symmetry we get that $[z]_\tau = [y]_\tau$.

It is evident that the relation $y \sim z \Leftrightarrow [y]_\tau = [z]_\tau$ is an equivalence relation. This relation is open in the sense that

$$[U]_\tau = \{z \in Y : [z]_\sigma \cap U \neq \emptyset\}$$

is open if U is.

Definition 1.7. The orbit space of τ is $Y/\tau := \{[y]_\tau : y \in Y\}$, the canonical projection is $\pi : Y \to Y/\tau$, $y \mapsto [y]_\tau$, and the topology of Y/τ is $\{U \subset Y/\tau : \pi^{-1}(U) \text{ is open}\}$.

The canonical projection is continuous, open and surjective. Hence the orbit space of a topological partial action on a locally compact space is always locally compact. The next example shows that condition (P2) does not guarantee a Hausdorff orbit space.

Example 1.8. Let $G = \mathbb{Z}_2$ act partially on $X = [-2, 2]$ by the partial action $\sigma = (\{\sigma_0, \sigma_1\}, \{X, (-1, 1)\})$ with $\sigma_0 := \mathrm{id}_X$ and $\sigma_1 : (-1, 1) \to (-1, 1)$ given by $\sigma_1(x) = -x$. Then σ satisfies (P2) and $[1]_\sigma \neq [-1]_\sigma$, but every open invariant subset containing 1 intersects every open invariant subset containing -1. Thus X/σ is not Hausdorff, but it is locally compact.

Now we relate the orbit space of a topological partial action with the orbit space of its topological enveloping action, as defined in [2, Definition 1.2] and whose existence and uniqueness are ensured by [2, Theorem 1.1]. The topological enveloping action of τ is, up to a conjugation by a homeomorphism, a topological (global) action τ^e of H on a topological space Y^e such that:

- Y is an open subset of Y^e.
- For all $t \in H$, $Y_t = Y \cap \tau^e_t(Y)$.
- For all $t \in H$ and $y \in Y_{t^{-1}}$, $\tau_t(y) = \tau^e_t(y)$.
- $Y^e = \bigcup_{t\in H} \sigma^e_t(Y)$ or, equivalently, $Y^e = [Y]_{\tau^e}$.

Proposition 1.9. *Let τ be a topological partial action of H on Y and τ^e the enveloping action of τ, acting on the enveloping space Y^e. Then the map $Y/\tau \to Y^e/\tau^e$, $[y]_\tau \mapsto [y]_{\tau^e}$, is defined and is a homeomorphism.*

Proof. We claim that $[y]_{\tau^e} \cap Y = [y]_\tau$, for all $y \in Y$. Indeed, it is clear that $[y]_\tau \subset [y]_{\tau^e} \cap Y$. Conversely, if $z \in [y]_{\tau^e} \cap Y$ then there exists $t \in H$ such that $\tau^e_t(y) = z \in Y$. Thus $y \in Y \cap \tau^e_{t^{-1}}(Y) = Y_{t^{-1}}$ and $\tau_t(y) = \tau^e_t(y) = z$. Hence $z \in [y]_\tau$.

The function $Y \to Y^e/\tau^e$, $y \mapsto [y]_{\tau^e}$, is continuous and constant in the $\tau-$orbits. Moreover, it is surjective because $[Y]_{\tau^e} = Y$. Then there exists a unique continuous and surjective map $h\colon Y/\tau \to Y^e/\tau^e$, $h([y]_\tau) = [y]_{\tau^e}$. Note h is injective because, if $h([y]_\tau) = h([z]_\tau)$, then $[y]_\tau = h([y]_\tau) \cap Y = h([z]_\tau) \cap Y = [z]_\tau$. We also have that h is open because if $U \subset Y$ is open, then $h([U]_\tau) = [U]_{\tau^e}$ is open. □

The next result and (its consequence) Remark 1.12 are our main reasons to adopt condition (P3) as the definition of proper partial action.

Proposition 1.10. *Let σ be a LCH partial action of G on X and let σ^e be its enveloping action, acting on the enveloping space X^e. Then the following are equivalent:*

(1) σ^e is a proper LCH action.
(2) Given a net $\{(t_i, x_i)\}_{i\in I} \subset \Gamma_\sigma$ such that $\{(\sigma_{t_i}(x_i), x_i)\}_{i\in I} \subset X\times X$ converges (to a point of $X \times X$), there exists a subnet of $\{(t_i, x_i)\}_{i\in I}$ converging to a point of Γ_σ.
(3) σ satisfies condition (P3), that is: $F_\sigma : \Gamma_\sigma \to X\times X, F_\sigma(t, x) = (\sigma_t(x), x)$, is proper.

Proof. Assume (1) and take a net $\{(t_i, x_i)\}_{i\in I} \subset \Gamma_\sigma$ such that

$$(\sigma_{t_i}(x_i), x_i) \to (y, x) \in X\times X.$$

Take compact neighborhoods of x and y, U and V, respectively. Then there exists $i_0 \in I$ such that $\{(t_i, x_i)\}_{i\geq i_0} \in F^{-1}_{\sigma^e}(U \times V)$. Hence there exists a subnet $\{(t_{i_j}, x_{i_j})\}_{j\in J}$ converging to a point $(t, z) \in G\times X$. We then have $z = \lim_j x_{i_j} = \lim_i x_i = x$ because X is Hausdorff and $\sigma^e_t(x) = \lim_j \sigma^e_{t_{i_j}}(x_{i_j}) = \lim_j \sigma_{t_{i_j}}(x_{i_j}) = y \in X$. Thus $x \in X \cap \sigma^e_{t^{-1}}(X) = X_{t^{-1}}$, meaning that $(t, x) \in \Gamma_\sigma$.

Now assume (2) holds and take a compact set $L \subset X\times X$ and a net $\{(t_i, x_i)\}_{i\in I} \subset F^{-1}_\sigma(L)$. Then $\{(\sigma_{t_i}(x_i), x_i)\}_{i\in I} \subset L$ has a converging subnet and, by passing to a subnet again, we get a subnet $\{(t_{i_j}, x_{i_j})\}_{j\in J}$ converging to a point $(t, x) \in \Gamma_\sigma$ and such that $\{(\sigma_{t_i}(x_i), x_i)\}_{i\in I}$ converges to some $(y, z) \in L$. We then have $z = x$

and, by continuity, $\sigma_t(x) = \lim_j \sigma_{t_{i_j}}(x_{i_j}) = y$. This implies $(t, x) \in F_\sigma^{-1}(L)$. Now (3) follows because the net $\{(t_{i_j}, x_{i_j})\}_{j\in J} \subset F_\sigma^{-1}(L)$ converges to $(t, x) \in F_\sigma^{-1}(L)$.

Suppose σ satisfies (3). Note that $X^e = \bigcup_{t\in G} \sigma_t^e(X)$ is locally compact because it is the union of open locally compact subsets.

To show that X^e is Hausdorff it suffices, by [2, Proposition 1.2], to show $\mathrm{Gr}(\sigma)$ is closed in $G \times X \times X$. Take a net $\{(t_i, x_i, y_i)\}_{i\in I} \subset \mathrm{Gr}(\sigma)$ converging to (t, x, y). Then $\{(t_i, x_i)\}_{i\in I} \subset \Gamma_\sigma$ and $\{(\sigma_{t_i}(x_i), x_i)\}_{i\in I} = \{(y_i, x_i)\}_{i\in I}$ converges to (y, x). By taking a compact neighborhood of (y, x), L, and considering $F_\sigma^{-1}(L)$ we get a subnet $\{(t_{i_j}, x_{i_j})\}_{j\in J}$ converging to some $(s, z) \in F_\sigma^{-1}(L) \subset \Gamma_\sigma$. Since $G \times X$ is Hausdorff, $(t, x) = (s, z) \in \Gamma_\sigma$ and by continuity we get $\sigma_t(x) = \lim_i \sigma_{t_i}(x_i) = \lim_i y_i = y$. This means that $(t, x, y) \in \mathrm{Gr}(\sigma)$. Hence $\mathrm{Gr}(\sigma)$ is closed and X^e is Hausdorff.

Take a compact set $L \subset X^e \times X^e$ and a net $\{(t_i, x_i)\}_{i\in I} \in F_{\sigma^e}^{-1}(L)$. Then there exists a subnet $\{(t_{i_j}, x_{i_j})\}_{J\in J}$ such that $\{(\sigma^e_{t_{i_j}}(x_{i_j}), x_{i_j})\}_{j\in J} \subset L$ converges to a point $(y, x) \in L$. Take $r, s \in G$ such that $\sigma_s^e(y), \sigma_r^e(x) \in X$. Note that

$$\lim_j \sigma_r^e(x_{i_j}) = \sigma_r^e(x) \in X \qquad \lim_j \sigma^e_{st_{i_j}r^{-1}}(\sigma_r^e(x_{i_j})) = \sigma_s^e(y) \in X.$$

Since X is open in X^e there exists $j_0 \in J$ such that, for all $j \geq j_0$, $\sigma_r^e(x_{i_j}) \in X$ and $\sigma^e_{st_{i_j}r^{-1}}(\sigma_r^e(x_{i_j})) \in X$.

Let $U \subset X \times X$ be a compact neighborhood of $(\sigma_s^e(y), \sigma_r^e(y))$. Then the net $\{(st_{i_j}r^{-1}, \sigma_r^e(x_{i_j}))\}_{j\geq j_0}$ is contained in $F_\sigma^{-1}(U)$ and, by passing to a subnet and relabeling, we can assume $\{(st_{i_j}r^{-1}, \sigma_r^e(x_{i_j}))\}_{j\in J}$ converges. In particular we get that $\{t_{i_j}\}_{j\in J}$ converges to some $t \in G$. This implies $\{(t_{i_j}, x_{i_j})\}_{J\in J}$ converges to (t, x) and $(t, x) \in F_{\sigma^e}^{-1}(L)$ because $(\sigma_t^e(x), x) = \lim_j (\sigma^e_{t_{i_j}}(x_{i_j}), x_{i_j}) \in L$. □

Definition 1.11. A LCH partial action σ is proper if it satisfies the equivalent conditions of Proposition 1.10.

Remark 1.12. The orbit space of every proper LCH partial action σ is LCH. Indeed, this in known to hold for global actions, in particular for σ^e. Hence the same holds for σ by Proposition 1.9.

At this point Proposition 1.10 becomes a machine to construct every possible proper LCH partial action: just take a proper LCH (global) action and restrict it to an open set. Let's be more precise about this.

Consider a proper LCH action τ of G on Y, $(t, y) \mapsto \tau_t(y)$, and let $Z \subset Y$ be a $\tau-$invariant open subset. Then the action $\tau|_Z$ of G on Z given by $(t, z) \mapsto \tau_t(z)$ is a LCH action. Notice that given compact sets $L, M \subset Z$, the set $((L, M))_{\tau|_Z} = ((L, M))_\tau$ is compact. Hence $\tau|_Z$ is proper because it is global an satisfies (P2).

Now assume Z is an arbitrary open subset of Y and let $\tau|_Z$ be the restriction of τ to Z, as in [2, Example 1.1]. That is to say $\tau|_Z := (\{Z_t\}_{t\in G}, \{\nu_t\}_{t\in G})$ is the topological partial action such that $Z_t = Z \cap \tau_t(Z)$ and $\nu_t(z) = \tau_t(z)$ for all $t \in G$ and $z \in Z_{t^{-1}}$. Then $\tau|_Z$ is LCH and it is proper because its enveloping action

$\tau|_{[Z]_\tau}$ is LCH and proper. If we now go back to Proposition 1.10, we see that $\sigma^e|_X = \sigma$ is LCH and proper if and only if σ^e is LCH and proper. We then have proved the following.

Remark 1.13. The restriction of every proper LCH action to an open set is a proper LCH partial action. Moreover, every proper LCH partial action arises in this way.

We hope to have exhibited enough reasons to adopt our definition of proper partial actions.

2. Weakly proper Fell bundles

We start this section by recalling some facts about Fell bundles, after which we construct the *primal examples* of weakly proper Fell bundles.

2.1. Fell bundles and notation. When we say $\mathcal{B} = \{B_t\}_{t\in G}$ is a Fell bundle we mean that G is a LCH group and that each set B_t is the fibre over $t \in G$ of a C*-algebraic bundle (as defined [12, VIII 16]). The modular function of G will be denoted Δ and dt will indicate integration with respect to a fixed left invariant Haar measure of G.

We denote $C_c(\mathcal{B})$ the set of continuous cross sections of $\mathcal{B}$ with compact support. This set is a normed *-algebra with pointwise vector space operations, convolution product $f * g(t) := \int_G f(s)g(s^{-1}t)\,ds$, involution $f^*(t) = \Delta(t^{-1})f(t^{-1})^*$ and norm $\|f\|_1 := \int_G \|f(t)\|\,dt$. The $\|\ \|_1-$completion of $C_c(\mathcal{B})$, $L^1(\mathcal{B})$, is the L^1-cross-sectional algebra of [12, VIII 5]. The (universal) cross-sectional C*-algebra of $\mathcal{B}$, $C^*(\mathcal{B})$, is the enveloping C*-algebra of $L^1(\mathcal{B})$. By [12, VIII 16.4], $L^1(\mathcal{B})$ may be considered as a dense *-subalgebra of $C^*(\mathcal{B})$.

The space $C_c(\mathcal{B})$ carries the so called inductive limit topology, which we describe here following [12, II 14.2]. Given a compact set $K \subset G$ we denote $C_K(\mathcal{B})$ the subset of $C_c(\mathcal{B})$ formed by all the elements whose support is contained in K. Then $C_K(\mathcal{B})$ is a Banach space with the norm $\|f\|_\infty := \sup_{t\in G}\|f(t)\|$. The inductive limit topology of $C_c(\mathcal{B})$ is the largest topology making $C_c(\mathcal{B})$ a locally convex space and $\iota_K : C_K(\mathcal{B}) \to C_c(\mathcal{B})$ continuous (for every compact $K \subset G$). This description implies $\|\ \|_1$ is inductive-limit continuous.

The unit fibre B_e is a C*-algebra with the operation inherited from $\mathcal{B}$. Besides, each fibre B_t is a $B_e - B_e-$Hilbert bimodule with action by multiplication and left and right inner products $(a,b) \mapsto a^*b$ and $(a,b) \mapsto ab^*$, respectively.

We will work with Hilbert modules and inner products quite a lot. When doing so the notation below will prove to be useful.

Notation 2.1. Given sets U and V, a topological vector space W, a binary operation $U \times V \to W$, $(u,v) \mapsto u \cdot v$, and subsets $A \subset U$ and $B \subset V$, we denote $A \cdot B$ the closed linear span of $\{a \cdot b : a \in A,\ b \in B\}$.

Notation 2.2. Given a C*-algebra A and a right $A-$Hilbert module V with inner product $\langle\ ,\ \rangle$, we denote $\mathbb{B}(V)$ the C*-algebra of adjointable operators on V. The "ket-bra" operator associated to $u,v \in V$ is $|u\rangle\langle v| \in \mathbb{B}(V)$, with

$|u\rangle\langle v|w := u\langle v,w\rangle$. The C*-algebra of generalized compact operators on V, $\mathbb{K}(V)$, is the closed linear span of the ket-bra operators. Hilbert spaces are regarded as right $\mathbb{C}-$Hilbert modules, so their inner products are linear in the second variable. We identify the multiplier algebra of A, $M(A)$, with $\mathbb{B}(A)$ and $\mathbb{B}(V) = M(\mathbb{K}(V))$.

2.1.1. Crossed product norms. The universal crossed product norm

$$\| \|_{\mathrm{u}} : C_c(\mathcal{B}) \to [0,+\infty)$$

is given by $\|f\|_{\mathrm{u}} := \|f\|_{C^*(\mathcal{B})}$ and it is the largest C*-norm on $C_c(\mathcal{B})$ dominated by $\| \|_1$. In [10, Proposition 2.6] Exel and Ng construct a *-homomorphism $\Lambda : L^1(\mathcal{B}) \to \mathbb{B}(L^2_e(\mathcal{B}))$ such that $\Lambda_f g = f * g$ for all $f, g \in C_c(\mathcal{B})$. The regular representation of $C^*(\mathcal{B})$ is the unique *-homomorphism $C^*(\mathcal{B}) \to \mathbb{B}(L^2_e(\mathcal{B}))$ extending Λ, which we also denote Λ. By definition, the reduced cross-sectional C*-algebra of $\mathcal{B}$ is $C^*_{\mathrm{r}}(\mathcal{B}) := \Lambda(C^*(\mathcal{B}))$. We canonically identify $C^*_{\mathrm{r}}(\mathcal{B})$ with the completion of $C_c(\mathcal{B})$ with respect to the reduced crossed product norm $C_c(\mathcal{B}) \to [0,+\infty)$, $\|f\|_{\mathrm{r}} := \|\Lambda_f\|$.

A *crossed product norm* on $C_c(\mathcal{B})$ is any C*-norm $\| \|$ such that $\| \|_{\mathrm{r}} \leq \| \| \leq \| \|_{\mathrm{u}}$. We say $\| \|$ is exotic if $\| \|_{\mathrm{r}} \neq \| \| \neq \| \|_{\mathrm{u}}$ and $\mathcal{B}$ is amenable if $\| \|_{\mathrm{r}} = \| \|_{\mathrm{u}}$. Given a crossed product norm μ of $C_c(\mathcal{B})$ we can form the completion $C^*_\mu(\mathcal{B})$ of $C_c(\mathcal{B})$ with respect to μ, thus obtaining a C*-algebra which we call the $\mu-$cross-sectional C*-algebra of $\mathcal{B}$.

For every *-representation $\pi : C^*_\mu(\mathcal{B}) \to \mathbb{B}(V)$ (with V a Hilbert space) the restriction $\pi|_{C_c(\mathcal{B})}$ extends to a *-representation $\pi' : L^1(\mathcal{B}) \to \mathbb{B}(V)$. By [12, VIII 13.2], π' is the integrated form of a unique *-representation $T : \mathcal{B} \to \mathbb{B}(V)$. Since $L^1(\mathcal{B})$ is a dense *-subalgebra of $C^*(\mathcal{B})$, we may consider the integrated form $\widetilde{T}$ of T as a *-representation of $C^*(\mathcal{B})$. If $q_\mu : C^*(\mathcal{B}) \to C^*_\mu(\mathcal{B})$ is the canonical quotient map, then $\pi \circ q_\mu = \widetilde{T}$. In case π is faithful, we may identify $C^*_\mu(\mathcal{B})$ with the closed linear span of $\{\widetilde{T}_f : f \in C_c(\mathcal{B})\}$, what we will do more than once.

2.2. Primal examples. Let $\sigma = (\{X_t\}_{t\in G}, \{\sigma_t\}_{t\in G})$ be a LCH partial action of G on X. The natural partial action of G on $C_0(X)$ defined by σ, $\theta = \theta(\sigma) := (\{C_0(X_t)\}_{t\in G}, \{\theta_t\}_{t\in G})$, is given by $\theta_t(f)(x) = f(\sigma_{t^{-1}}(x))$ for all $f \in C_0(X_{t^{-1}})$, $x \in X_t$.

In general, by a C*-partial action we mean a partial action on a C*-algebra as in [2, Definition 2.2]. The correspondence $\sigma \rightsquigarrow \theta(\sigma)$ is bijective between LCH partial actions and C*-partial actions on commutative C*-algebras.

Given a C*-partial action $\beta = (\{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$ of G on B, the semidirect product bundle of β, $\mathcal{B}_\beta$, is the Banach subbundle $\{(t,a) : a \in B_t,\ t \in G\}$ of the trivial Banach bundle $B \times G$ over G together with the product and involution given by

$$(s,a)^* = (s^{-1}, \beta_{s^{-1}}(a^*)) \qquad (s,a)(t,b) = (st, \beta_s(\beta_{s^{-1}}(a)b)).$$

Exel defines (in [8]) semidirect product bundles even for twisted partial actions. Although our theory will cover this kind of bundles, we will not have to deal with them explicitly. Following Exel we will write $a\delta_s$ instead of (s, a). So $B_s\delta_s$ is actually the fibre (s, B_s) of $\mathcal{B}_\beta$ over $s \in G$. This notation is convenient because we canonically identify the fibre $B_e\delta_e = B\delta_e$ (over the group's unit e) with B. Under this identification B_s is a C*-ideal of $B\delta_e$, not the fibre $B_s\delta_s$.

In case $B = C_0(X)$ and $\beta = \theta(\sigma)$, we write $\mathcal{A}_\sigma$ instead of $\mathcal{B}_{\theta(\sigma)}$. We call $\mathcal{A}_\sigma$ the semidirect product bundle of σ.

Definition 2.3. A *primal example* of a weakly proper Fell bundle is the semidirect product bundle of a proper LCH partial action.

Recall from [1] (and from [3] for non discrete groups) that every Fell bundle $\mathcal{B} = \{B_t\}_{t\in G}$ defines a topological partial action $\hat{\alpha}$ of G on the spectrum $\hat{B}_e$ of the unit fibre B_e. Given $t \in G$, the set $I^{\mathcal{B}}_t := B_tB_t^*$ is an ideal of B_e, so the spectrum $\hat{I^{\mathcal{B}}_t}$ is an open subset of $\hat{B}_e$. The homeomorphism $\hat{\alpha}_t : \widehat{I^{\mathcal{B}}_{t^{-1}}} \to \widehat{I^{\mathcal{B}}_t}$ is the Rieffel homeomorphism associated to the $I^{\mathcal{B}}_{t^{-1}} - I^{\mathcal{B}}_t$—equivalence bimodule B_t.

The partial action $\hat{\alpha}$ described before is compatible with the natural quotient map $q : \hat{B}_e \to \mathrm{Prim}(B_e)$ (from the spectrum to the primitive ideal space) given by $q([\pi]) = \ker(\pi)$, where $[\pi]$ is the unitary equivalence class of the irreducible representation π. This means that there exists a unique topological partial action $\tilde{\alpha}$ of G on $\mathrm{Prim}(B_e)$ such that $\tilde{\alpha}_t$ maps $\mathcal{O}_{t^{-1}} := \mathrm{Prim}(I^{\mathcal{B}}_{t^{-1}})$ bijectively to $\mathcal{O}_t = \mathrm{Prim}(I^{\mathcal{B}}_t)$ sending $\ker(\pi)$ to $\ker(\hat{\alpha}_t(\pi))$.

For a primal example $\mathcal{B} = \mathcal{A}_\sigma$ we have $\hat{\alpha} = \sigma$, so the identity $\mathcal{A}_\sigma = \mathcal{A}_\tau$ implies $\sigma = \tau$ and there is no possible ambiguity in our last definition above. It also follows that, for an arbitrary LCH partial action σ, $\mathcal{A}_\sigma$ is a primal example of a weakly proper Fell bundle if and only if σ is proper.

In order to motivate our definition of weakly proper Fell bundle, and to justify the term “weakly proper” we are using, let's put Buss and Echterhoff's weakly proper actions [7] in the context of Fell bundles.

A C*-action β of G on B is weakly proper if there exists a proper LCH action τ of G on Y together with a *-homomorphism $\phi : C_0(Y) \to M(B)$ such that, with $\theta = \theta(\sigma)$,

- $B = \phi(C_0(Y))B$. By Cohen-Hewitt's Theorem this is equivalent to say every $b \in B$ admits a factorization $b = \phi(f)c$.
- For all $f \in C_0(Y)$, $b \in B$ and $t \in G$, $\phi(\theta_t(f))\beta_t(b) = \beta_t(\phi(f)b)$.

In the situation above one can construct the function

$$\mathcal{A}_\tau \times \mathcal{B}_\beta \to \mathcal{B}_\beta,\ (f\delta_s, b\delta_t) \mapsto \phi(f)\beta_s(b)\delta_{st},$$

which we interpret as an action of $\mathcal{A}_\tau$ on $\mathcal{B}_\beta$. The properties satisfied by this action motivate the following.

Definition 2.4. Let $\mathcal{A} = \{A_t\}_{t\in G}$ and $\mathcal{B} = \{B_t\}_{t\in G}$ be Fell bundles over G. We say a function $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto a \cdot b$, is an action of $\mathcal{A}$ on $\mathcal{B}$ by adjointable maps if

(1) For all $a \in A_s$, $b \in B_t$ and $s, t \in G$, $a \cdot b \in B_{st}$.
(2) For all $s, t \in G$ and $b \in B_t$ the function $A_s \to B_{st},\ a \mapsto a \cdot b$ is linear.
(3) For all $a, c \in \mathcal{A}$ and $b \in \mathcal{B}$, $a \cdot (c \cdot b) = (ac) \cdot b$.
(4) For all $a \in \mathcal{A}$ and $b, d \in \mathcal{B}$, $(a \cdot b)^* d = b^*(a^* \cdot d)$.
(5) For all $b \in B_e$ the function $\mathcal{A} \to \mathcal{B},\ a \mapsto a \cdot b$, is continuous.

We say the action is nondegenerate if for all $t \in G$, $(A_t A_{t^{-1}}) \cdot B_e = B_t B_{t^{-1}}$ (recall Notation 2.1).

In the conditions above there exists a unique *-homomorphism $\phi\colon A_e \to M(B_e)$ such that $\phi(a)b = a \cdot b$. If we were working with semidirect product bundles $\mathcal{B} = \mathcal{B}_\beta$ and $\mathcal{A} = \mathcal{A}_\sigma$, then this map ϕ would be exactly the map $\phi\colon C_0(X) \to M(B)$ (after the identification $A_e = C_0(X)$ and $B_e = B$). The difference between weakly proper actions and Kasparov proper actions is that the latter assume ϕ is central (i.e. the image of ϕ is contained in the center $ZM(B_e)$).

Definition 2.5. A Fell bundle $\mathcal{B}$ over G is weakly proper if there exists a primal example of a weakly proper Fell bundle over G, $\mathcal{A}$, and a nondegenerate action of $\mathcal{A}$ on $\mathcal{B}$ by adjointable maps. If, in addition, the map $\phi\colon A_e \to M(B_e)$ described above is central, we say $\mathcal{B}$ is Kasparov proper.

Example 2.6. Every primal example of a weakly proper Fell bundle, say $\mathcal{A}$, is weakly proper. Indeed, one just needs to consider the multiplication of $\mathcal{A}$ as an action of $\mathcal{A}$ on $\mathcal{A}$.

We will show later (in Corollary 2.14) that a semidirect product bundle $\mathcal{A}_\sigma$ of a LCH partial action σ is weakly proper if and only if σ is proper. So the Example above produces all weakly proper Fell bundles coming from semidirect product bundles of LCH partial actions.

The non degeneracy requirement in Definition 2.5 is motivated by condition (C1) in the example below (and to exclude the zero action).

Example 2.7 (Weakly proper partial actions). Consider a C*-partial action $\beta = (\{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$ of G on B for which there exists a proper LCH partial action $\sigma = (\{X_t\}_{t\in G}, \{\sigma_t\}_{t\in G})$ of G on X and a *-homomorphism $\phi\colon C_0(X) \to M(B)$ such that, with $\theta = \theta(\sigma)$, the following conditions hold:

(C1) For all $t \in G$, $B_t = \phi(C_0(X_t))B$. By Cohen-Hewitt's factorization Theorem this implies for every $b \in B_t$ and $t \in G$ there exists $f \in C_0(X_t)$ and $c \in B_t$ such that $b = \phi(f)c$.
(C2) For all $t \in G$, $f \in C_0(X_{t^{-1}})$ and $b \in B_{t^{-1}}$, $\phi(\theta_t(f))\beta_t(b) = \beta_t(\phi(f)b)$.

Then the semidirect product bundle $\mathcal{B}_\beta$ is weakly proper with respect to the action

$$\mathcal{A}_\sigma \times \mathcal{B}_\beta \to \mathcal{B}_\beta,\ (f\delta_t, b\delta_s) \mapsto \beta_t(\phi(\theta_{t^{-1}}(f))b)\delta_{ts}. \tag{2.1}$$

Note that the map above is defined because $\phi(C_0(X_{t^{-1}}))B_s \in B_{t^{-1}} \cap B_s$, for all $s, t \in G$. Thus $\beta_t(\phi(C_0(X_{t^{-1}}))B_s) \subset B_t \cap B_{st}$. One can not define the action of (2.1) if in (C1) one requires, for example, $B_t = \phi(C_0(X_t))B_t$. The reader may explore other alternatives, but the author must say (C1) is the only satisfactory condition he has been able to find.

It is not straightforward to verify (2.1) defines an action by adjointable maps. After some attempts to do this one feels that the computations needed are quite similar to those necessary to show the semidirect product bundle of a C*-partial action is a Fell bundle. We leave to the reader the adaptation of Exel's method to do this, see [8] (fortunately there are no twists here).

The action of a Fell bundle on another has some extra properties that we summarize below.

Proposition 2.8. *If $\mathcal{A}\times\mathcal{B}\to\mathcal{B}$, $(a,b)\mapsto a\cdot b$, is an action of the Fell bundle $\mathcal{A}=\{A_t\}_{t\in G}$ on the Fell bundle $\mathcal{B}=\{B_t\}_{t\in G}$ by adjointable maps then:*

1. *For all $s,t\in G$ the map $A_s\times B_t\mapsto B_{st}$, $(a,b)\mapsto a\cdot b$, is bilinear.*
2. *For all $a\in\mathcal{A}$ and $b,c\in\mathcal{B}$, $a\cdot(bc)=(a\cdot b)c$.*
3. *For all $a\in\mathcal{A}$ and $b\in\mathcal{B}$, $\|ab\|\leq\|a\|\|b\|$ and $(a\cdot b)^*(a\cdot b)\leq\|a\|^2b^*b$.*
4. *The action $\mathcal{A}\times\mathcal{B}\to\mathcal{B}$, $(a,b)\mapsto a\cdot b$, is continuous.*
5. *For all $t\in G$, $A_t\cdot B_e=A_e\cdot B_t=B_t$.*

Proof. Clearly the map from (1) is linear in the first variable, to show it is linear in the second variable take $a\in A_s$, $b,c\in B_t$ and $\lambda\in\mathbb{C}$. Then, with $z:=a\cdot(b+\lambda c)-(a\cdot b+\lambda[a\cdot c])$,

$$\|z\|^2=\|z^*z\|=\|(b+\lambda c)^*(a^*\cdot z)-b^*(a^*\cdot z)-\overline{\lambda}[b^*(a^*\cdot c)]\|=0.$$

This implies $z=0$, that is $a\cdot(b+\lambda c)=a\cdot b+\lambda[a\cdot c]$.

The proof of (2) is quite similar to the previous one. If $z=a\cdot(bc)-(a\cdot b)c$, then

$$\|z\|^2=\|z^*z\|=\|(bc)^*(a^*\cdot z)-c^*b^*(a^*\cdot z)\|=0.$$

Hence (2) follows.

To prove (3) take $a\in A_s$ and $b\in B_t$ and note that $(a\cdot b)^*(a\cdot b)=b^*(a^*a\cdot b)$. For every $c\in A_e$ there exists a unique $\phi_c\in M(B_e)$ such that $\phi_c x=c\cdot x$. In fact the map $\phi\colon A_e\to M(B_e)$, $c\mapsto\phi_c$, is a *-homomorphism. Considering the fibre B_t as a right B_e-Hilbert module (with inner product $\langle x,y\rangle=x^*y$) we have a nondegenerate *-homomorphism $\varphi\colon B_e\to\mathbb{B}(B_t)$ such that $\varphi(x)y=xy$. If $\overline{\varphi}\colon M(B_e)\to\mathbb{B}(B_t)$ is the unique extension of φ, then since every *-homomorphism between C*-algebras is contractive we have

$$(a\cdot b)^*(a\cdot b)=\langle b,\overline{\varphi}(\phi_{a^*a})b\rangle\leq\|\overline{\varphi}(\phi_{a^*a})\|\langle b,b\rangle\leq\|a^*a\|b^*b=\|a\|^2b^*b,$$

where we have used that $a^*a\geq 0$ in A_e. Then $\|a\cdot b\|=\|(a\cdot b)^*(a\cdot b)\|^{1/2}\leq\|a\|\|b\|$.

To prove (4) take a net $\{(a_i,b_i)\}_{i\in I}\subset\mathcal{A}\times\mathcal{B}$ converging to $(a,b)\in\mathcal{A}\times\mathcal{B}$. Let $\{(s_i,t_i)\}_{i\in I}\subset G\times G$ and $(s,t)\in G\times G$ be such that $a_i\in A_{s_i}$, $b_i\in B_{t_i}$, $a\in A_s$ and $b\in B_t$. Then $s_i\to s$, $t_i\to t$, $a_i\cdot b_i\in B_{s_it_i}$ and $s_it_i\to st$.

Fix $\varepsilon>0$. Then, since $\mathcal{B}$ has approximate units, there exists $u_\varepsilon\in B_e$ such that $\|b-u_\varepsilon b\|<\varepsilon(1+\|a\|)^{-1}$. Take $i_\varepsilon\in I$ such that $\|b_i-u_\varepsilon b_i\|<\varepsilon(1+\|a\|)^{-1}$ and $\|a_i\|<\|a\|+1$ for all $i\geq i_\varepsilon$. Our construction implies that, for all $i\geq i_\varepsilon$,

$\|a_i \cdot b_i - a_i \cdot (u_\varepsilon b_i)\| = \|a_i \cdot (b_i - u_\varepsilon b_i)\| < \varepsilon$. Now the definition of action by adjointable maps and claim (2) imply

$$\lim_i a_i \cdot (u_i b_i) = \lim_i (a_i \cdot u_\varepsilon) b_i = (a \cdot u_\varepsilon) b = a \cdot (u_\varepsilon b).$$

Besides, $\|a \cdot b - a \cdot (u_\varepsilon b)\| < \varepsilon$. Now we can use [11, II 13.12] to deduce that $\lim_i a_i \cdot b_i = a \cdot b$, thus the proof of (4) is complete.

Regarding (5), by considering B_t as a right B_e−Hilbert module we deduce that

$$B_t = B_t B_t^* B_t = A_t A_t^* \cdot B_t \subset A_t \cdot (A_{t^{-1}} \cdot B_t) \subset A_t \cdot B_e \subset B_t$$

and also that

$$B_t = B_e B_e^* B_t = (A_e A_e^* \cdot B_e) B_t = A_e \cdot (A_e \cdot B_t) \subset A_e \cdot B_t \subset B_t.$$

Now the proof is complete. □

The next Lemma may look harmless or even unnecessary, but it is of key importance to us because it relates a LCH partial action σ with the action of $\mathcal{A}_\sigma$ on a Fell bundle.

Lemma 2.9. *Let σ be a LCH partial action of G on X, $\mathcal{B}$ a Fell bundle over G, $\mathcal{A}_\sigma \times \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto a \cdot b$, a nondegenerate action by adjointable maps and set $\theta := \theta(\sigma)$. If the map $\phi\colon A_e \to M(B_e)$, $\phi(x)y := x \cdot y$, is central, then*

$$\theta_t(f)\delta_e \cdot bc = b(f\delta_e \cdot c),\ \forall\, t \in G,\ f \in C_0(X_{t^{-1}}),\ b \in B_t,\ c \in \mathcal{B}.$$

Proof. Fix $s, t \in G$ and $f \in C_0(X_{t^{-1}})$. The maps $B_t \times B_s \to B_{ts}$ given by $(b, c) \mapsto \theta_t(f)\delta_e \cdot bc$ and $(b, c) \mapsto b(f\delta_e \cdot c)$ are continuous and bilinear. Besides $B_t = C_0(X_t)\delta_t \cdot B_e$ and $B_s = B_e B_s$, thus we may assume $b = u\delta_t \cdot v$ and $c = zw$ for some $u \in C_0(X_t)$, $v \in B_e$, $z \in B_e$ and $w \in B_s$. By considering B_s and B_t as a right $B_e - B_e$−Hilbert bimodules we deduce that

$$\begin{aligned} b(f\delta_e \cdot c) &= (u\delta_t \cdot v)([f\delta_e \cdot z]w) = u\delta_t \cdot ([v\phi(f)z]w) = u\delta_t \cdot ([\phi(f)vz]w) \\ &= u\delta_t f\delta_e \cdot (vc) = \theta_t(\theta_{t^{-1}}(u)f)\delta_t \cdot (vc) = \theta_t(f)u\delta_t \cdot (vc) \\ &= \theta_t(f)\delta_e u\delta_t \cdot (vc) = \theta_t(f)\delta_e \cdot (u\delta_t \cdot (vc)) = \theta_t(f)\delta_e \cdot bc. \end{aligned}$$

□

In the C*-algebraic context, if A acts on B and B on C by nondegenerate actions by adjointable maps, then one can define a nondegenerate action of A on C by adjointable maps. This is also true for Fell bundles.

Proposition 2.10. *Let $\mathcal{A} = \{A_t\}_{t\in G}$, $\mathcal{B} = \{B_t\}_{t\in G}$ and $\mathcal{C} = \{C_t\}_{t\in G}$ be Fell bundles over G and $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto a \cdot b$, and $\mathcal{B} \times \mathcal{C} \to \mathcal{C}$, $(b, c) \mapsto b \diamond c$, nondegenerate actions by adjointable maps. Then there exists a unique action by adjointable maps $\mathcal{A} \times \mathcal{C} \to \mathcal{C}$, $(a, c) \mapsto a \star c$, such that $a \star (b \diamond c) = (a \cdot b) \diamond c$ for all $(a, b, c) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$. Moreover, $\star$ is nondegenerate.*

Proof. Fix $s, t \in G$. In order to show there exists a unique bounded bilinear map $F_{s,t} : A_s \times C_t \to C_{st}$ such that $F_{s,t}(a, b \diamond c) = (a \cdot b) \diamond c$ for all $(a, c) \in A_s \times C_t$ and $b \in B_e$, it suffices to show that $\| \sum_{j=1}^{n} (a \cdot b_j) \diamond c_j \| \leq \|a\| \| \sum_{j=1}^{n} b_j \diamond c_j \|$, for all $n \in \mathbb{N}$, $b_1, \dots, b_n \in B_e$ and $c_1, \dots, c_n \in C_t$.

Take $b_1, \dots, b_n \in B_e$ and $c_1, \dots, c_n \in C_t$. Define $u := \sum_{j=1}^{n} b_j \diamond c_j$ and $v := \sum_{j=1}^{n} (a \cdot b_j) \diamond c_j$. Consider the map $\phi : A_e \to M(B_e)$ given by $\phi(x)y = x \cdot y$ and let $\overline{\phi} : M(A_e) \to M(B_e)$ be the unique extension of ϕ. If w is the positive square root of $\|a\|^2 - a^* a$ in $M(A_e)$, then

$$
\begin{aligned}
\|a\|^2 u^* u - v^* v &= \sum_{i,j=1}^{n} \|a\|^2 c_i^* (b_i^* b_j \diamond c_j) - c_i^* ([b_i^* (a^* a \cdot b_j)] \diamond c_j) \\
&= \sum_{i,j=1}^{n} c_i^* ([b_i^* \overline{\phi}(w^* w) b_j] \diamond c_j) \\
&= \sum_{i,j=1}^{n} (\overline{\phi}(w) b_i \diamond c_i)^* (\overline{\phi}(w) b_j \diamond c_j) \geq 0.
\end{aligned}
$$

This implies $\| \sum_{j=1}^{n} (a \cdot b_j) \diamond c_j \| \leq \|a\| \| \sum_{j=1}^{n} b_j \diamond c_j \|$ (i.e. $\|v\| \leq \|a\| \|u\|$).

Now we define $\mathcal{A} \times \mathcal{C} \to \mathcal{C}$, $(a, c) \mapsto a \star c$, in such a way that for $a \in A_s$ and $c \in C_t$, $a \star c = F_{s,t}(a, c)$. Clearly, $\star$ satisfies conditions (1) and (2) from Definition 2.4.

Take $(a, b, c) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ and $(r, s, t) \in G^3$ such that $a \in A_r$, $b \in B_s$ and $c \in C_t$. In order to compute $a \star (b \diamond c)$ we take an approximate unit of $\mathcal{B}$, $\{u_j\}_{j \in J} \subset B_e$. Then the continuity of the actions and of the maps $F_{p,q}$ $(p, q \in G)$ implies

$$
\begin{aligned}
a \star (b \diamond c) &= \lim_j a \star (u_j \diamond b \diamond c) = \lim_j (a \cdot u_j) \diamond (b \diamond c) \\
&= \lim_j (a \cdot u_j b) \diamond c = (a \cdot b) \diamond c.
\end{aligned}
$$

Now we show condition (3) from Definition 2.4 using the identity we have just proved. For all $a, d \in \mathcal{A}$ and $c \in \mathcal{C}$ we have

$$
\begin{aligned}
a \star (d \star c) &= \lim_j a \star (d \star [u_j \diamond c]) \\
&= \lim_j a \star ((d \cdot u_j) \diamond c) = \lim_j (a \cdot (d \cdot u_j)) \diamond c \\
&= \lim_j (ad \cdot u_j) \diamond c = \lim_j (ad) \star (u_j \diamond c) = (ad) \star c.
\end{aligned}
$$

To prove (4) from Definition 2.4 take $a \in \mathcal{A}$ and $c, f \in \mathcal{C}$. Then

$$\begin{aligned}(a \star c)^* f &= \lim_j (a \star (u_j \diamond c))^* f = \lim_j ((a \cdot u_j) \diamond c)^* f = \lim_j c^*((a \cdot u_j)^* \diamond f) \\ &= \lim_j \lim_k c^*((a \cdot u_j)^* u_k \diamond f) = \lim_j \lim_k c^*(u_j^*(a^* \cdot u_k) \diamond f) \\ &= \lim_j \lim_k c^*(u_j^* \diamond (a^* \cdot u_k) \diamond f) = \lim_j \lim_k (u_j \diamond c)^*((a^* \cdot u_k) \diamond f) \\ &= \lim_j \lim_k (u_j \diamond c)^*(a \star (u_k \diamond f)) = \lim_j (u_j \diamond c)^*(a \star f) = c^*(a \star f).\end{aligned}$$

Let's show (5) from Definition 2.4. Note that by construction we get $\|a \star c\| \leq \|a\|\|c\|$, for all $(a, c) \in \mathcal{A} \times \mathcal{C}$. Fix $c \in C_e$ and take a net $\{a_i\}_{i \in I} \subset \mathcal{A}$ converging to $a \in A_t$. Given $\varepsilon > 0$ take $u_\varepsilon \in B_e$ (for example one of the terms of $\{u_j\}_{j \in J}$) such that $\|c - u_\varepsilon \diamond c\| < \varepsilon(1 + \|a\|)^{-1}$. Then

$$\lim_i a_i \star (u_\varepsilon \diamond c) = \lim_i a_i \star (u_\varepsilon \diamond c) = \lim_i (a_i \cdot u_\varepsilon) \diamond c = (a \cdot u_\varepsilon) \diamond c = a \star (u_\varepsilon \diamond c).$$

and $\|a \star (u_\varepsilon \diamond c) - a \star c\| < \varepsilon$. Besides, taking $i_\varepsilon \in I$ such that $\|a_i\| < \|a\| + 1$ for all $i \geq i_\varepsilon$, we get that $\|a_i \star (u_\varepsilon \diamond c) - a_i \star c\| \leq \|a\|\|u_\varepsilon \diamond c - c\| < \varepsilon$ for all $i \geq i_\varepsilon$. Now [11, II 13.12] implies $\lim_i a_i \star c = a \star c$.

Finally, $\star$ is nondegenerate because for all $t \in G$

$$A_t A_{t^{-1}} \star C_e = A_t A_{t^{-1}} \star (B_e \cdot C_e) = (A_t A_{t^{-1}} \cdot B_e) \diamond C_e = B_t B_{t^{-1}} \diamond C_e = C_t C_{t^{-1}}.$$

Now the proof is complete. □

We have defined weakly proper Fell bundles using actions of primal examples of weakly proper Fell bundles. Then one might define "weakly weakly proper Fell bundles" using actions of weakly proper Fell bundles and so on. Fortunately "weakly^n proper Fell bundles" = "weakly proper Fell bundles". This fact and Corollary 2.14 (which we suggest to consult at this point) kind of justify our choice of the name "primal example".

Corollary 2.11. *If $\mathcal{B}$ is a weakly proper Fell bundle over G and $\mathcal{C}$ is a Fell bundle over G admitting a nondegenerate action by adjointable maps of $\mathcal{B}$, then $\mathcal{C}$ is weakly proper.*

Proof. Let σ be a proper LCH partial action such that $\mathcal{A}_\sigma$ acts on $\mathcal{B}$ by a nondegenerate action by adjointable maps. Proposition 2.10 gives a nondegenerate action by adjointable maps of $\mathcal{A}_\sigma$ on $\mathcal{C}$, thus $\mathcal{C}$ is weakly proper. □

2.3. Kasparov proper Fell bundles and amenability. The main result in this (sub)section states that every Kasparov proper Fell bundle is amenable in the sense that its full and reduced cross-sectional C*-algebras agree; what we will prove using the following.

Theorem 2.12. *Let σ be a LCH partial action of G on X, $\mathcal{B}$ a Fell bundle over G and $\mathcal{A}_\sigma \times \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto a \cdot b$, a nondegenerate action by adjointable maps. Denote $\hat{\alpha}$ the topological partial action G on* $\mathrm{Prim}(B_e)$ *defined by $\mathcal{B}$ (described after Definition 2.3). If the map $\phi\colon C_0(X) \to M(B_e)$ given by $\phi(f)b = (f\delta_e) \cdot b$ is central, then there exists a continuous function $h\colon$* $\mathrm{Prim}(B_e) \to X$ *such that*

(1) *For all* $t \in G$, $h^{-1}(X_t) = \mathcal{O}_t$ *(recall* $\mathcal{O}_t := \{P \in \mathrm{Prim}(B_e) : B_t B_t^* \nsubseteq P\}$*).*
(2) *For all* $t \in G$ *and* $P \in \mathcal{O}_{t^{-1}}$, $h(\tilde{\alpha}_t(P)) = \tilde{\alpha}_t(h(P))$.

Proof. Recall that the fibre over $t \in G$ of $\mathcal{A}_\sigma$ is $A_t := C_0(X_t)\delta_e$. As done before we denote θ (instead of $\theta(\sigma)$) the C*-partial action defined by σ on $C_0(X)$. We identify $C_0(X)$ with A_e canonically and think of $C_0(X_t)$ as an ideal of A_e. A direct computation shows that $A_t A_t^* = C_0(X_t)$.

The map ϕ is nondegenerate because $\phi(C_0(X))B_e = (C_0(X)\delta_e C_0(X)\delta_e)\cdot B_e = B_e$. By Dauns-Hoffman's Theorem there exists a continuous map $h : \mathrm{Prim}(B_e) \to X$ such that, for all $a \in C_0(X)$, $b \in B_e$ and $P \in \mathrm{Prim}(B_e)$, $\pi_P(a\delta_e \cdot b) = a(h(P))\pi_P(b)$; where $\pi_P : B_e \to B_e/P$ is the canonical quotient map.

Fix $t \in G$. Given $P \in h^{-1}(X_t)$, take $a \in C_0(X_t)$ such that $a(h(P)) = 1$ and $b \in B_e$ such that $\pi_P(b) \neq 0$. Then $\pi_P(a\delta_e \cdot c) = a(h(P))\pi_P(c) \neq 0$ and we conclude that $C_0(X_t)\delta_e \cdot B_e = A_t A_t^* \cdot B_e = B_t B_t^*$ is not contained in P, meaning that $P \in \mathcal{O}_t$. Assume, conversely, that we have $P \in \mathcal{O}_t$. Since $C_0(X_t)\delta_e \cdot B_e = B_t B_t^* \nsubseteq P$, there exists $a \in C_0(X_t)$ and $b \in B_e$ such that $0 \neq \pi_P(a\delta_e \cdot b) = a(h(P))\pi_P(b)$. Hence $a(h(P)) \neq 0$ and this implies $P \in X_t$.

It is now time to prove claim (2). Take $t \in G$ and $P \in \mathcal{O}_{t^{-1}}$. By construction (see [3]) the representation

$$\rho : B_e \to \mathbb{B}(B_t \otimes_{\pi_P} (B_e/P)),\ \rho(b)(c \otimes \pi_P(d)) = bc \otimes \pi_P(d),$$

has kernel $\tilde{\alpha}_t(P)$. Then, for all $a \in C_0(X_{t^{-1}})$ and $c \in B_e$, we have $\rho(a\delta_e \cdot c) = a(h(\tilde{\alpha}_t(P)))\rho(c)$. Take $z \otimes \pi_P(w), z' \otimes \pi_P(w') \in B_t \otimes_{\pi_P} (B_e/P)$. Recalling our inner products are linear in the second variable and using Lemma 2.9 we get

$$\begin{aligned}
a(h(\tilde{\alpha}_t(P)))\pi_P(w^* z^* c^* z' w') &= a(h(\tilde{\alpha}_t(P)))\langle c(z \otimes \pi_P(w)), z' \otimes \pi_P(w')\rangle \\
&= \langle \rho(a^*\delta_e \cdot c)(z \otimes \pi_P(w)), z' \otimes \pi_P(w')\rangle \\
&= \pi_P(w^*(a^*\delta_e \cdot cz)^* z' w') \\
&= \pi_P(\underbrace{w^* z^* c^*}_{\in B_{t^{-1}}} a\delta_e \cdot z' w') \\
&= \pi_P(\theta_{t^{-1}}(a)\delta_e \cdot \underbrace{w^* z^* c^* z' w'}_{\in B_e}) \\
&= \theta_{t^{-1}}(a)(h(P))\pi_P(w^* z^* c^* z' w').
\end{aligned}$$

Since $\pi_P(w^* z^* c^* z' w')$ can not be null for all $z \otimes \pi_P(w), z' \otimes \pi_P(w') \in B_t \otimes_{\pi_P} B_e/P$, we conclude that

$$a(h(\tilde{\alpha}_t(P))) = a(\sigma_t(h(P)))\ \forall a \in C_0(X_t).$$

Noticing that $h(\tilde{\alpha}_t(P)), \sigma_t(h(P)) \in X_t$ and recalling that $C_0(X_t)$ separates the points of X_t we deduce that $h(\tilde{\alpha}_t(P)) = \sigma_t(h(P))$. ☐

Corollary 2.13. *Every Kasparov proper Fell bundle* $\mathcal{B} = \{B_t\}_{t\in G}$ *is amenable. In case* $\mathrm{Prim}(B_e)$ *is Hausdorff, the partial action* $\tilde{\alpha}$ *of* G *on* $\mathrm{Prim}(B_e)$ *is proper.*

Proof. We can use the notation and hypotheses of Theorem 2.12, with the additional assumption that σ is proper. The transformation group (X^e, G) defined by the enveloping action σ^e is amenable in the sense of [5, Definition 2.1] by [5, Examples 2.7 (3)]. Besides, we can think of $h : \mathrm{Prim}(B_e) \to X$ as a map from $\mathrm{Prim}(B_e)$ to X^e which is $\tilde{\alpha} - \sigma^e$ equivariant. Then $\mathcal{B}$ is amenable by [3, Theorem 6.3], because condition (1) of that Theorem should be read "(X^e, G) is an amenable transformation group" in our context[1].

Now assume $\mathrm{Prim}(B_e)$ is Hausdorff (hence LCH). We will use condition (2) of Proposition 1.10 to show $\tilde{\alpha}$ is proper. Take a net $\{(t_i, P_i)\}_{i\in I} \subset \Gamma_{\tilde{\alpha}}$ such that $\{(\tilde{\alpha}_{t_i}(P_i), P_i)\}_{i\in I}$ converges to $(Q, R) \in \mathrm{Prim}(B_e)\times\mathrm{Prim}(B_e)$. Then $\{(t_i, h(P_i))\}_{i\in I} \subset \Gamma_\sigma$ and $\{(\sigma_{t_i}(h(P_i)), h(P_i))\}_{i\in I}$ converges to $(h(Q), h(R))$. There exists a subnet $\{(t_{i_j}, P_{i_j})\}_{j\in J}$ such that $\{(t_{i_j}, h(P_{i_j}))\}_{i\in I} \subset \Gamma_\sigma$ converges to a point $(t, x) \in \Gamma_\sigma$. By construction $h(R) = \lim_j h(P_{i_j}) = x \in X_{t^{-1}}$. Since $R \in h^{-1}(X_{t^{-1}}) = \mathcal{O}_{t^{-1}}$, we obtain $(t, R) \in \Gamma_{\tilde{\alpha}}$ is the limit of $\{(t_{i_j}, P_{i_j})\}_{j\in J}$. □

Corollary 2.14. *Let σ be a LCH partial action of G on $C_0(X)$. Then $\mathcal{A}_\sigma$ is a weakly proper Fell bundle if and only if σ is proper (i.e. $\mathcal{A}_\sigma$ is a primal example).*

Proof. The multiplier algebra of $C_0(X)\delta_e$ is (C*-isomorphic to) $C_b(X)$ and hence commutative. Thus the direct implication follows from Corollary 2.13 because $\tilde{\alpha} = \sigma$. For the converse just consider $\mathcal{A}_\sigma$ acting on $\mathcal{A}_\sigma$ by multiplication. □

3. Fixed-point algebras

As we mentioned in the Introduction, the analysis of the primal examples of weakly proper Fell bundles plays an important role in the general theory.

3.1. Primal examples. Let σ be a proper LCH partial action of G on X and denote θ the respective partial action on $C_0(X)$. The enveloping action and enveloping spaces of σ will be denoted σ^e and X^e, respectively. We view $C_0(X)$ as an ideal of $C_0(X^e)$ and $\theta^e := \theta(\sigma^e)$ as the enveloping action of θ (in the C*-algebraic sense [2]).

The partial crossed product $C_0(X) \rtimes_\sigma G$ is, by definition, the cross-sectional C*-algebra of $\mathcal{A}_\sigma$, $C^*(\mathcal{A}_\sigma)$. The bundle $\mathcal{A}_\sigma$ is an hereditary subbundle of $\mathcal{A}_{\sigma^e}$ and thus we can view $C_0(X) \rtimes_\sigma G$ as a full hereditary C*-subalgebra of $C_0(X^e) \rtimes_{\sigma^e} G$ [4]. Recall from Corollary 2.13 that the full and reduced cross-sectional C*-algebras are the same in the present situation.

The set of compactly supported continuous cross-sections of $\mathcal{A}_\sigma$, $C_c(\mathcal{A}_\sigma)$, is formed by functions of the form $f^\dagger : G \to \mathcal{A}_\sigma,\ t \mapsto f(t)\delta_t$, with

$$f \in C_c^\sigma(G, C_0(X)) := \{g \in C_c(G, C_0(X)) :\ g(t) \in C_0(X_t),\ \forall\, t \in G\}.$$

Let's recall the construction of a $C_0(X^e/\sigma^e) - C_0(X^e) \rtimes_{\sigma^e} G$–bimodule E_{σ^e} performed in [17]. We are not assuming σ^e is free, then we will not be able to guarantee E_{σ^e} is an equivalence bimodule.

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[1] This is so because we are using the term *amenable* following [10]. This notion of amenability is also known as *weak containment*.

The σ^e−orbit of a point $x \in X^e$ will be denoted $[x]$. Recall from Proposition 1.9 that we may think $X/\sigma = X^e/\sigma^e$ by identifying $[x] = [x]_{\sigma^e}$ with $[x]_\sigma$, for all $x \in X$.

Consider $E_{\sigma^e} := C_c(X^e)$ with the pre $C_0(X/\sigma)$−left Hilbert module structure given by

$$fg(x) = f([x])g(x) \qquad \langle g, h\rangle([x]) := \int_G (g\overline{h})(\sigma^e_{t^{-1}}(x))\,dt,$$

for all $f \in C_0(X/\sigma)$, $g, h \in C_c(X^e)$ and $x \in X$.

Routine computations show that the operations above are compatible and that one can complete $C_c(X^e)$ to get a left $C_0(X/\sigma)$−Hilbert module E_{σ^e}. In fact one can use the Stone-Weierstrass Theorem to show the ideal

$$\operatorname{span}\{\langle g, h\rangle : \ g, h \in C_c(X)\}$$

is dense in $C_c(X/\sigma)$ in the inductive limit topology (as defined in [12, II 14.2]). Thus E_{σ^e} is in fact a full left Hilbert module.

Full and reduced crossed products agree here, then one may appeal to [18] to describe the right $C_0(X^e) \rtimes_{\sigma^e} G$ structure of E_{σ^e} (even if σ^e is not free). For $g, h \in C_c(X^e)$ and $k^\dagger \in C_c(\mathcal{A}_{\sigma^e})$ (with $k \in C_c(G, C_0(X^e))$) the action and inner products are given by

$$fk^\dagger(x) = \int_G f(\sigma^e_t(x))k(t)(\sigma^e_t(x))\Delta(t)^{-1/2}\,dt \quad \forall\, x \in X^e. \tag{3.1}$$

$$\langle\langle f, g\rangle\rangle_{\sigma^e}(t) = \Delta(t)^{-1/2} f^*\theta^e_t(g)\delta_t \quad \forall\, t \in G. \tag{3.2}$$

In case σ^e is free we have E_{σ^e} is a $C_0(X/\sigma) - C_0(X^e) \rtimes_{\sigma^e} G$−equivalence bimodule.

Our goal now is to show the closure of $C_c(X)$ in E_{σ^e}, henceforth denoted E_σ, inherits a $C_0(X/\sigma) - C_0(X) \rtimes_\sigma G$−bimodule structure from E_{σ^e}.

Proposition 3.1. *E_σ is the closure of $E^0_\sigma := C_c(X^e) \cap C_0(X)$ in E_{σ^e}.*

Proof. It suffices to show that every $f \in C_c(X^e) \cap C_0(X)$ can be approximated in E_{σ^e} by elements of $C_c(X)$. Fix $f \in C_c(X^e) \cap C_0(X)$ and take an approximate unit $\{g_i\}_{i\in I}$ of $C_0(X)$ contained in $C_c(X)$. Since the projection of $\operatorname{supp}(f)$ into X^e/σ^e is compact, there exists a compact set $K \subset X^e$ such that $f(x) = 0$ if $[x] \cap K = \emptyset$. Then, for all $i \in I$,

$$\|f - g_i f\|^2_{E_{\sigma^e}} = \sup_{x\in K} \int_G |f - g_i f|^2(\sigma^e_{t^{-1}}(x))\,dt.$$

The trick now is to restrict the integral over G to a compact subset of G. To do this note that if $x \in K$ and $|f - g_i f|^2(\sigma^e_{t^{-1}}(x)) \neq 0$, then $\sigma^e_{t^{-1}}(x) \in \operatorname{supp}(f)$. Thus $t \in L := \{s \in G : \ K \cap \sigma^e_t(\operatorname{supp}(f)) \neq \emptyset\}$ and L is compact because σ^e is LCH and proper. If $\mu(L)$ is the measure of L, then for all $i \in I$

$$\|f - g_i f\|^2_{E_{\sigma^e}} \le \mu(L)\|f - g_i f\|^2_\infty$$

The proof now follows directly by taking limit in i. □

Continuing our discussion note that $C_0(X/\sigma)E^0_\sigma \subset E^0_\sigma$ because E^0_σ is an ideal of $C_c(X^e)$. Thus E_σ has a natural $C_0(X/\sigma)$–left Hilbert module structure inherited from E_{σ^e}. Note also that when we showed E_{σ^e} is left full we actually showed E_σ is left full.

Given $f \in E^0_\sigma$, $k^\dagger \in C_c(\mathcal{A}_\sigma)$ and $x \in X^e \setminus X$ we have, by (3.1), $fk^\dagger(x) = 0$. This proves $E^0_\sigma C_c(\mathcal{A}_\sigma) \subset E^0_\sigma$. Besides, for all $f, g \in E^0_\sigma$ and $t \in G$ we have $f^*\theta^e_t(g) \in C_0(X) \cap \theta^e_t(C_0(X)) = C_0(X_t)$. This implies, by (3.2), that $\langle\langle f, g\rangle\rangle_{\sigma^e} \in C_c(\mathcal{A}_\sigma)$. Then we conclude that E_σ has a natural $C_0(X/\sigma)$–$C_0(X)\rtimes_\sigma G$–bimodule structure (inherited from E_{σ^e}).

The natural choice for the fixed-point algebra of σ, or $\mathcal{A}_\sigma$, is $C_0(X/G)$. To ensure it is strongly Morita equivalent to $C^*(\mathcal{A}_\sigma) = C_0(X) \rtimes_\sigma G$ one needs to show the image of the $C_0(X)\rtimes_\sigma G$–valued inner product spans a dense subspace of $C_0(X) \rtimes_\sigma G$. As for global actions this can be done by assuming the partial action is free.

Definition 3.2. A topological partial action τ of H on Y is free if for all $t \in G \setminus \{e\}$, $\{y \in Y_{t^{-1}} : \tau_t(y) = y\} = \emptyset$.

In terms of topological freeness for partial actions, as defined in [9], a topological partial action is free if it is free considered as an action of a discrete group on a discrete space.

Proposition 3.3. *A topological partial action is free if and only if its enveloping action is free.*

Proof. Assume τ is a free topological partial action of H on Y and let τ^e be its enveloping action, with enveloping space Y^e. Take $y \in Y^e$ and $t \in H$ such that $\sigma^e_t(y) = y$. There exists $r \in H$ such that $x := \sigma^e_r(y) \in Y$. Then $\sigma^e_{rtr^{-1}}(x) = x \in Y$, this implies $x \in Y_{rt^{-1}r^{-1}}$ and $\sigma_{rtr^{-1}}(x) = x$, thus $rtr^{-1} = e$ and we get $t = e$. The converse is trivial because σ is a restriction of σ^e. □

The Morita equivalence between the fixed-point algebra and cross-sectional C*-algebra is now available, at least for the primal examples of weakly proper Fell bundles coming from free partial actions.

Theorem 3.4. *Let σ be a LCH free and proper partial action of G on X. Then the bimodule E_σ described early in this section is a $C_0(X/\sigma)$–$C_0(X)\rtimes_\sigma G$–equivalence bimodule.*

Proof. All we need to do is to show the ideal generated by the $C_0(X)\rtimes_\sigma G$–valued inner products is dense in $C_0(X) \rtimes_\sigma G$. We know, by Propositions 3.3 and 1.10, that σ^e is free and proper. Thus E_{σ^e} is a $C_0(X/\sigma) - C_0(X^e) \rtimes_{\sigma^e} G$–equivalence bimodule (see for example [17]).

Using (3.1) it is straightforward to prove that $C_c(X^e)C_c(\mathcal{A}_\sigma) \subset E^0_\sigma$. Recalling that $C_0(X) \rtimes_\sigma G$ is a full hereditary C*-subalgebra of $C_0(X^e) \rtimes_{\sigma^e} G$ we get that

$$\begin{aligned} C_0(X) \rtimes_\sigma G &= \overline{\operatorname{span}}\, C_0(X) \rtimes_\sigma G \langle\langle C_c(X^e), C_c(X^e)\rangle\rangle_{\sigma^e} C_0(X) \rtimes_\sigma G \\ &= \overline{\operatorname{span}} \langle\langle E^0_\sigma, E^0_\sigma\rangle\rangle_{\sigma^e} \subset C_0(X) \rtimes_\sigma G. \end{aligned}$$

$\square$

Notation 3.5. The $C_0(X) \rtimes_\sigma G-$valued inner product of E_σ will be denoted $\langle\langle\ ,\ \rangle\rangle_\sigma$. By construction $\langle\langle f,g\rangle\rangle_\sigma = \langle\langle f,g\rangle\rangle_{\sigma^e}$ for all $f,g \in E_\sigma$.

3.2. General weakly proper Fell bundles. Let us now take a Fell bundle $\mathcal{B} = \{B_t\}_{t\in G}$ which is weakly proper with respect to the action $\mathcal{A}_\sigma \times \mathcal{B} \to \mathcal{B}$, $(a,b) \mapsto a\cdot b$, where $\sigma = (\{X_t\}_{t\in G}, \{\sigma_t\}_{t\in G})$ is a proper LCH partial action of G on X. As usual we set $\theta := \theta(\sigma)$, σ^e is the enveloping action of σ, X^e the enveloping space and the enveloping action of θ, θ^e, is the action on $C_0(X^e)$ defined by σ^e.

To avoid repetition, whenever we write $\sigma, \mathcal{B}, \theta$ (and any other mathematical symbol appearing in the paragraph above) we will be implicitly assuming the situation is the one we described before. The same will happen for objects constructed out of $\sigma, \mathcal{B}, \theta$, etc; like the space $E^0_{\mathcal{B}}$ or the fixed-point algebras we will construct some lines below.

Unfortunately, the construction of the fixed-point algebra for $\mathcal{B}$ depends on σ, but this is no surprise because something similar happens for weakly proper actions on C*-algebras [6].

The map $\phi : C_0(X) \to M(B_e)$, $\phi(f)b = f\delta_e\cdot b$, is a nondegenerate *-homomorphism and, since $C_0(X)$ is a C*-ideal of $C_0(X^e)$, there exists a unique extension ϕ^e of ϕ to $C_0(X^e)$. Motivated by Proposition 3.1 we define

$$E^0_{\mathcal{B}} := \{\phi^e(f)b : f \in C_c(X^e),\ b \in B_e\}.$$

For future reference we set the following.

Lemma 3.6. *$E^0_{\mathcal{B}}$ is a subspace of B_e and for all $b \in E^0_{\mathcal{B}}$ there exists $f \in E^0_\sigma = C_c(X^e)\cap C_0(X)$ and $b' \in B_e$ such that $b = f\delta_e \cdot b'$.*

Proof. Given $b,c \in E^0_{\mathcal{B}}$ and $\lambda \in \mathbb{C}$ take $b',c' \in B_e$ and $f,g \in C_c(X^e)$ such that $b = \phi^e(f)b'$ and $c = \phi^e(g)c'$. Now take $h \in C_c(X^e)$ such that $hf = f$ and $hg = g$. Then $b + \lambda c = \phi^e(h)b + \lambda\phi^e(h)c = \phi^e(h)(b+\lambda c) \in E^0_{\mathcal{B}}$.

By Cohen-Hewitt's factorization Theorem there exists $k \in C_0(X)$ and $b'' \in B_e$ such that $b' = k\delta_e\cdot b'$. Then $b = \phi^e(h)b' = \phi^e(h)k\delta_e\cdot b'' = (hk)\delta_e\cdot b''$, and $hk \in E^0_\sigma$. $\square$

The goal of the Proposition below is the construction of the $C_c(\mathcal{B})$ valued inner product of $E^0_{\mathcal{B}}$ using the $C_c(\mathcal{A}_\sigma)-$valued inner product of E^0_σ.

Proposition 3.7. *There exists a unique function*

$$\langle\langle\ ,\ \rangle\rangle_{\mathcal{B}} : E^0_{\mathcal{B}} \times E^0_{\mathcal{B}} \to C_c(\mathcal{B}),\ (a,b) \mapsto \langle\langle a,b\rangle\rangle_{\mathcal{B}},$$

such that for all $f,g \in E^0_\sigma$, $a,b \in B_e$ and $t \in G$,

$$\langle\langle f\delta_e\cdot a, g\delta_e\cdot b\rangle\rangle_{\mathcal{B}}(t) = a^*(\langle\langle f,g\rangle\rangle_\sigma(t)\cdot b). \tag{3.3}$$

Moreover,

(1) $\langle\langle\ ,\ \rangle\rangle_{\mathcal{B}}$ is linear in the second variable.
(2) For all $a,b \in E^0_{\mathcal{B}}$, $\langle\langle a,b\rangle\rangle^*_{\mathcal{B}} = \langle\langle b,a\rangle\rangle_{\mathcal{B}}$.

(3) *Given $f, g \in C_0(X^e)$ and a net $\{(a_j, b_j)\}_{j\in J} \subset B_e \times B_e$ that converges to $(a, b) \in B_e \times B_e$, then the net $\{\langle\langle \phi^e(f)a_j, \phi^e(g)b_j\rangle\rangle_{\mathcal{B}}\}_{j\in J}$ converges to $\langle\langle \phi^e(f)a, \phi^e(f)b\rangle\rangle_{\mathcal{B}}$ in the inductive limit topology of $C_c(\mathcal{B})$.*

Proof. To show existence take $f, g, h, k \in E^0_\sigma$ and $a, b, c, d \in B_e$ such that $f\delta_e \cdot a = h\delta_e \cdot c$ and $g\delta_e \cdot b = k\delta_e \cdot d$. Fix $t \in G$ and take an approximate unit of $C_0(X_t)$, $\{u_i\}_{i\in I}$. Then,

$$\begin{aligned}
a^*(\langle\langle f, g\rangle\rangle_\sigma(t) \cdot b) &= \Delta(t)a^*(f^*\theta^e_t(g)\delta_t \cdot b)\\
&= \lim_i \Delta(t)a^*((u_i f)^*\theta^e_t(\theta_{t^{-1}}(u_i)g)\delta_t \cdot b)\\
&= \lim_i \Delta(t)a^*((f\delta_e)^* \cdot (u_i\delta_e)^* \cdot (u_i\delta_t) \cdot (g\delta_e) \cdot b)\\
&= \lim_i \Delta(t)(f\delta_e \cdot a)^*((u_i^2\delta_t) \cdot (g\delta_e) \cdot b)\\
&= \lim_i \Delta(t)(h\delta_e \cdot c)^*((u_i^2\delta_t) \cdot (k\delta_e) \cdot d)\\
&= c^*(\langle\langle h, k\rangle\rangle_\sigma(t) \cdot d).
\end{aligned}$$

The identities above imply formula (3.3) can actually be used as a definition and can also be used to show that $\langle\langle\ ,\ \rangle\rangle_{\mathcal{B}}$ is linear in the second variable.

The following identities prove claim (2):

$$\begin{aligned}
\langle\langle f\delta_e \cdot a, g\delta_e \cdot b\rangle\rangle^*_{\mathcal{B}}(t) &= \Delta(t)^{-1}\langle\langle f\delta_e \cdot a, g\delta_e \cdot b\rangle\rangle_{\mathcal{B}}(t^{-1})^*\\
&= \Delta(t)^{-1}[a^*(\langle\langle f, g\rangle\rangle_\sigma(t^{-1}) \cdot b)]^*\\
&= \Delta(t)^{-1}(\langle\langle f, g\rangle\rangle_\sigma(t^{-1}) \cdot b)^*a\\
&= b^*(\Delta(t)^{-1}\langle\langle f, g\rangle\rangle_\sigma(t^{-1})^* \cdot a)\\
&= b^*(\langle\langle g, f\rangle\rangle_\sigma(t) \cdot a) = \langle\langle g\delta_e \cdot b, f\delta_e \cdot a\rangle\rangle_{\mathcal{B}}(t).
\end{aligned}$$

Before proving claim (3), we will develop an alternative way of computing $\langle\langle x, y\rangle\rangle_{\mathcal{B}}(t)$, for $x, y \in E^0_{\mathcal{B}}$ and $t \in G$. Take an approximate unit of $C_0(X)$, $\{u_i\}_{i\in I}$, and factorizations $x = f \cdot a$ and $y = g \cdot b$ with $f, g \in E^0_\sigma$ and $a, b \in B_e$. Then

$$\begin{aligned}
\lim_i \langle\langle u_i \cdot x, u_i \cdot y\rangle\rangle_{\mathcal{B}}(t) &= \lim_i \langle\langle u_i f \cdot a, u_i g \cdot b\rangle\rangle_{\mathcal{B}}(t)\\
&= \lim_i \Delta(t)^{-1}a^*(\langle\langle u_i f, u_i g\rangle\rangle_\sigma(t^{-1}) \cdot b)\\
&= \lim_i \Delta(t)^{-1}a^*(f^*\theta^e_t(g)u_i\theta^e_t(u_i)\delta_t \cdot b) = \langle\langle x, y\rangle\rangle_{\mathcal{B}}(t), \quad (3.4)
\end{aligned}$$

where the last identity holds because $\{u_i\theta^e_t(u_i)\}_{i\in I}$ is an approximate unit to $C_0(X_t)$ and $f^*\theta^e_t(g) \in C_0(X_t)$.

Now take a net $\{(a_j, b_j)\}_{j\in J} \subset B_e \times B_e$ and $f, g \in C_0(X^e)$ as in claim (3). Using (3.4) we deduce that, for all $j \in J$, $\operatorname{supp}\langle\langle f\delta_e \cdot a_j, g\delta_e \cdot b_j\rangle\rangle_{\mathcal{B}} \subset \operatorname{supp}\langle\langle f, g\rangle\rangle_{\sigma^e}$. Thus it suffices to prove $\{\langle\langle f\delta_e \cdot a_j, g\delta_e \cdot b_j\rangle\rangle_{\mathcal{B}}\}_{j\in J}$ converges uniformly to $\langle\langle f\delta_e \cdot$

$a_j, g\delta_e \cdot b_j\rangle\rangle_{\mathcal{B}}$. Again by (3.4) we have, for all $t \in G$,

$$\begin{aligned}
&\|\langle\langle f\delta_e \cdot a_j, g\delta_e \cdot b_j\rangle\rangle_{\mathcal{B}}(t) - \langle\langle f\delta_e \cdot a, g\delta_e \cdot b\rangle\rangle_{\mathcal{B}}(t)\| \le \\
&\quad \le \|\langle\langle f\delta_e \cdot (a_j - a), g\delta_e \cdot b_j\rangle\rangle_{\mathcal{B}}(t)\| + \|\langle\langle f\delta_e \cdot a, g\delta_e \cdot (b_j - b)\rangle\rangle_{\mathcal{B}}(t)\| \\
&\qquad \le \|a_j - a\|\|b_j\|\|\langle\langle f, g\rangle\rangle_{\sigma^e}\|_\infty + \|a\|\|b_j - b\|\|\langle\langle f, g\rangle\rangle_{\sigma^e}\|_\infty .
\end{aligned}$$

It is then straightforward to show that

$$\lim_j \|\langle\langle f\delta_e \cdot a_j, g\delta_e \cdot b_j\rangle\rangle_{\mathcal{B}} - \langle\langle f\delta_e \cdot a, g\delta_e \cdot b\rangle\rangle_{\mathcal{B}}\|_\infty = 0$$

□

Our intention is to use $\langle\langle\ ,\ \rangle\rangle_{\mathcal{B}}$ as a $C^*(\mathcal{B})$–valued inner product and construct a Hilbert module with it. To do so we will need to show $\langle\langle\ ,\ \rangle\rangle_{\mathcal{B}}$ is positive.

Lemma 3.8. *Consider Fell bundles over G, $\mathcal{C} = \{C_t\}_{t\in G}$ and $\mathcal{D} = \{D_t\}_{t\in G}$, and a nondegenerate action by adjointable maps $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$, $(c, d) \mapsto c \cdot d$. Then for every nondegenerate *-representation $T : \mathcal{D} \to \mathbb{B}(V)$ there exists a unique *-representation $\hat{T} : \mathcal{C} \to \mathbb{B}(V)$ such that $\hat{T}_c T_d \xi = T_{c\cdot d}\xi$, for all $(c, d, \xi) \in \mathcal{C} \times \mathcal{D} \times V$. Moreover, $\hat{T}$ is nondegenerate.*

Proof. Fix $c \in \mathcal{C}$, $d_1, \dots, d_n \in D_e$ and $\xi_1, \dots, \xi_n \in V$. Let w be a square root of $\|c\|^2 - c^*c \in M(C_e)$. Using the arguments of the proof of Proposition 2.10 we get

$$\begin{aligned}
\|c\|^2\|\sum_{i=1}^n T_{d_i}\xi_i\| - \|\sum_{i=1}^n T_{c\cdot d_i}\xi_i\|^2 &= \sum_{i,j=1}^n \|c\|^2\langle T_{d_i}\xi_i, T_{d_j}\xi_j\rangle - \langle T_{c\cdot d_i}\xi_i, T_{c\cdot d_j}\xi_j\rangle \\
&= \sum_{i,j=1}^n \langle \xi_i, T_{\|c\|^2 d_j^* d_j - d_j^* c c^* d_j}\xi_j\rangle \\
&= \sum_{i,j=1}^n \langle \xi_i, T_{d_j^* w^* w d_j}\xi_j\rangle = \|\sum_{i=1}^n T_{w d_i}\xi_i\| \ge 0.
\end{aligned}$$

Since the restriction $T|_{D_e}$ is nondegenerate, the inequalities above imply there exists a unique operator $\hat{T}_c \in \mathbb{B}(V)$ such that $\hat{T}_c T_d \xi = T_{c\cdot d}\xi$, for all $d \in D_e$ and $\xi \in V$. Given any $d \in \mathcal{D}$, taking an approximate unit $\{d_j\}_{j\in J}$ of D_e, it follows that $\hat{T}_c T_d \xi = \lim_j \hat{T}_c T_{d_j} T_d \xi = \lim_j T_{c\cdot d_j\cdot d}\xi = T_{c\cdot d}\xi$.

Having defined the operators $\hat{T}_c$ (for all $a \in \mathcal{C}$) we leave the rest of the proof to the reader. □

Lemma 3.9. *For all $x \in E^0_{\mathcal{B}}$, $\langle\langle x, x\rangle\rangle_{\mathcal{B}} \ge 0$ in $C^*(\mathcal{B})$. Moreover, $\langle\langle x, x\rangle\rangle_{\mathcal{B}} = 0$ if and only if $x = 0$.*

Proof. Take a faithful and nondegenerate *-representation $T : \mathcal{B} \to \mathbb{B}(V)$ with faithful integrated form $\tilde{T} : C^*(\mathcal{B}) \to \mathbb{B}(V)$. Let $\hat{T} : \mathcal{A}_\sigma \to \mathbb{B}(V)$ be the

*-representation given by Lemma 3.8 and take $f \in E^0_\sigma$ and $b \in B_e$ such that $x = f\delta_e \cdot b$. Then, for all $\xi \in V$,

$$\begin{aligned}\langle \tilde{T}_{\langle\langle x,x\rangle\rangle_{\mathcal{B}}}\xi,\xi\rangle &= \int_G \langle T_{\langle\langle x,x\rangle\rangle_{\mathcal{B}}(t)}\xi,\xi\rangle\,dt = \int_G \langle T_{b^*(\langle\langle f,f\rangle\rangle_\sigma(t)\cdot b)}\xi,\xi\rangle\,dt \\ &= \int_G \langle \hat{T}_{\langle\langle f,f\rangle\rangle_\sigma(t)}T_b\xi,T_b\xi\rangle\,dt = \langle \tilde{\hat{T}}_{\langle\langle f,f\rangle\rangle_\sigma}T_b\xi,T_b\xi\rangle \geq 0,\end{aligned} \tag{3.5}$$

where $\tilde{\hat{T}}$ is the integrated form of $\hat{T}$ and the last inequality above holds because $\langle\langle f,f\rangle\rangle_\sigma \geq 0$ in $C^*(\mathcal{A}_\sigma)$.

In case $\langle\langle x,x\rangle\rangle_{\mathcal{B}} = 0$, $x^*x = \langle\langle x,x\rangle\rangle_{\mathcal{B}}(e) = 0$ and this implies $x = 0$. The converse is immediate. □

Remark 3.10. Equation (3.5) holds for every nondegenerate *-representation T of $\mathcal{B}$, not only for those with faithful integrated form. Besides, since both $\tilde{T}_{\langle\langle x,x\rangle\rangle_{\mathcal{B}}}$ and $T_b^*\tilde{\hat{T}}_{\langle\langle f,f\rangle\rangle_\sigma}T_b$ are positive, (3.5) implies $\tilde{T}_{\langle\langle x,x\rangle\rangle_{\mathcal{B}}} = T_b^*\tilde{\hat{T}}_{\langle\langle f,f\rangle\rangle_\sigma}T_b$. These facts will be used many times in the rest of the article.

Now we define an action $\diamond$ of $C_c(\mathcal{B})$ on $E^0_{\mathcal{B}}$ on the right.

Lemma 3.11. *For each $x \in E^0_{\mathcal{B}}$ and $f \in C_c(\mathcal{B})$ there exists a unique function $x \lhd f \in C_c(G,B_e)$ such that given any approximate unit $\{u_i\}_{i\in I}$ of*

$$C^\sigma_0(G,X) := \{f \in C_0(G,C_0(X)) : f(t) \in C_0(X_t),\ \forall\, t \in G\},$$

the net $\{r \mapsto u_i(r^{-1})\delta_{r^{-1}} \cdot xf(r)\}_{i\in I} \subset C_c(G,B_e)$ converges to $x \lhd f$ in the inductive limit topology. Besides,

$$x \diamond f := \int_G \Delta(r)^{-1/2} x \lhd f(r)\,dr \in E^0_{\mathcal{B}}. \tag{3.6}$$

Proof. The set of continuous sections of $\mathcal{B}$ vanishing at ∞, $C_0(\mathcal{B})$, is a Banach space with the norm $\|\ \|_\infty$. The function $C^\sigma_0(G,X) \times C_0(\mathcal{B}) \to C_0(\mathcal{B})$, $(f,g) \mapsto f \star g$, with $f \star g(r) = f(r)\delta_e \cdot g(r)$, is a linear action such that $\|f \star g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. We claim that $\star$ in nondegenerate in the sense that $C^\sigma_0(G,X) \star C_0(\mathcal{B}) = C_0(\mathcal{B})$. Indeed, let $S := \operatorname{span}\{f \star g : f \in C^\theta_c(G,X),\ g \in C_c(\mathcal{B})\}$. It is clear that $\{vf : v \in C_c(G),\ v \in S\} \subset S$ and, for all $t \in G$,

$$\{f(t) : f \in S\} = \operatorname{span}\{u \cdot v : u \in C_0(X_r)\delta_e,\ v \in B_r\}.$$

The non degeneracy condition of the action of $\mathcal{A}_\sigma$ on $\mathcal{B}$ implies

$$B_r = B_rB_r^*B_eB_r = C_0(X_r)\delta_e \cdot B_eB_r \subset C_0(X_r)\delta_e \cdot B_r \subset B_r.$$

Hence $\{f(t) : f \in S\}$ is dense in B_r. By [11, II 14.6] the conditions above imply S is dense in $C_c(\mathcal{B})$ in the inductive limit topology.

For all $f \in S$ and approximate unit $\{u_i\}_{i\in I}$ of $C^\sigma_0(G,X)$ we have $\lim_i \|u_i \star f - f\|_\infty = 0$ and this implies the same holds for all $f \in C_0(\mathcal{B})$. Now the Cohen-Hewitt Theorem implies for all $f \in C_0(\mathcal{B})$ there exists $g \in C^\sigma_0(G,X)$ and $f' \in C_0(\mathcal{B})$ such that $f = g \star f'$.

Fix $x \in E^0_{\mathcal{B}}$ and $f \in C_c(\mathcal{B})$. The function $xf \in C_c(\mathcal{B})$, given by $(xf)(r) = xf(r)$, admits a factorization $g \star h$ with $g \in C^\sigma_c(G,X)$ and $h \in C_c(\mathcal{B})$. Consider an approximate unit $\{u_i\}_{i\in I}$ of $C^\sigma_0(G,X)$ and define, for each $i \in I$, the function $[xf]_i \in C_c(G,B_e)$ by $[xf]_i(r) := u_i(r^{-1})\delta_{r^{-1}} \cdot xf(r)$. Clearly, $\operatorname{supp}[xf]_i \subset \operatorname{supp} f$. Thus to show $\{[xf]_i\}_{i\in I}$ converges in the inductive limit topology it suffices to prove it converges uniformly.

Define $k\colon G \to B_e$ by $k(r) := \theta_{r^{-1}}(g(r))\delta_{r^{-1}} \cdot h(r)$. Then $k \in C_c(G,B_e)$ and, for all $r \in G$,

$$\begin{aligned} \|[xf]_i(r) - k(r)\| &= \|u_i(r^{-1})\delta_{r^{-1}} \cdot xf(r) - \theta_{r^{-1}}(g(r))\delta_{r^{-1}} \cdot h(r)\| \\ &= \|u_i(r^{-1})\delta_{r^{-1}} \cdot g(r)\delta_e \cdot h(r) - \theta_{r^{-1}}(g(r))\delta_{r^{-1}} \cdot h(r)\| \\ &\le \|u_i(r^{-1})\delta_{r^{-1}} g(r)\delta_e - \theta_{r^{-1}}(g(r))\delta_{r^{-1}}\| \|h\|_\infty \\ &\le \|\theta_r(u_i(r^{-1}))g(r) - g(r)\| \|h\|_\infty \end{aligned}$$

The function $\mu\colon C^\sigma_0(G,X) \to C^\sigma_0(G,X)$ given by $\mu(z)(r) = \theta_r(z(r^{-1}))$ is an isomorphism of C*-algebras. Then $\{\mu(u_i)\}_{i\in I}$ is an approximate unit of $C^\sigma_0(G,X)$ and the inequalities above imply $\|[xf]_i - k\| \le \|\mu(u_i)g - g\|\|h\|_\infty$. Thus $\{[xf]_i\}_{i\in I}$ converges to k in the inductive limit topology.

We set, by definition, $k := x \triangleleft f$. In order to prove (3.6) choose $w \in C_c(X^e)$ such that $w\delta_e \cdot x = x$. Then $(wg) \star h(r) = wg(r)\delta_r \cdot h(r) = w\delta_e \cdot g(r)\delta_r \cdot h(r) = w\delta_e \cdot xf(r) = xf(r)$. Performing the construction of $x \triangleleft f$ using the factorization $xf = (wg) \star h$ we obtain $x \triangleleft f(r) = \theta_{r^{-1}}(wg(r))\delta_{r^{-1}} \cdot h(r)$.

For every $t \in \operatorname{supp}(h)$ we have $\operatorname{supp}(\theta_{r^{-1}}(wg(r))) \subset \sigma^e_{r^{-1}}(\operatorname{supp}(w))$. Since σ^e is proper there exist a compact subset of the enveloping space X^e containing $\bigcup\{\operatorname{supp}(\theta_{r^{-1}}(wg(r)))\colon\ t \in \operatorname{supp}(h)\}$. Thus we may find $z \in C_c(X^e)$ such that $z\theta_{r^{-1}}(wg(r)) = \theta_{r^{-1}}(wg(r))$, for all $r \in G$. This construction of z guarantees that $z\delta_e \cdot (x \triangleleft f(r)) = x \triangleleft f(r)$, for all $r \in G$. Then we have

$$x \diamond f = \int_G \Delta(r)^{-1/2} z\delta_e \cdot (x \triangleleft f(r))\,dr = z\delta_e \cdot (x \diamond f) \in E^0_{\mathcal{B}}.$$

□

We want to construct a right $C_c(\mathcal{B})$–module with inner product out of $E^0_{\mathcal{B}}$. For this we need to show the following.

Lemma 3.12. *For all $x, y \in E^0_{\mathcal{B}}$ and $f, g \in C_c(\mathcal{B})$, the identities*

$$\langle\langle x, y \diamond f\rangle\rangle_{\mathcal{B}} = \langle\langle x, y\rangle\rangle_{\mathcal{B}} * f \qquad\qquad (x \diamond f) \diamond g = x \diamond (f * g)$$

obtain, where $*$ is the convolution product in $C_c(\mathcal{B})$.

Proof. Without loss of generality we can replace x and y with $g\delta_e \cdot x$ and $h\delta_e \cdot y$, with $f, g \in E^0_\sigma$. Fix $t \in G$ and let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be approximate units of $C_0(X)$ and $C^\sigma_0(G,X)$, respectively. The construction of $\langle\langle\ ,\ \rangle\rangle_{\mathcal{B}}$ described in the proof of Proposition 3.7 together with Lemma 3.11 imply

$$
\begin{aligned}
&\langle\langle g\delta_e \cdot x, (h\delta_e \cdot y) \diamond f\rangle\rangle_{\mathcal{B}}(t) \\
&\qquad = \lim_i \Delta(t)^{-1/2}(g\delta_e \cdot x)^*(u_i\theta_t^e(u_i)\delta_t \cdot [(h\delta_e \cdot y) \diamond f]) \\
&= \lim_i \int_G \lim_j \Delta(ts)^{-1/2}(g\delta_e \cdot x)^*(u_i\theta_t^e(u_i)\delta_t \cdot v_j(s^{-1})\delta_{s^{-1}} \cdot (h\delta_e \cdot y)f(s))\,ds \\
&= \lim_i \int_G \lim_j \Delta(ts)^{-1/2}x^*(g^*u_i\theta_t^e(u_i v_j(s^{-1})\theta_{s^{-1}}^e(h))\delta_{ts^{-1}} \cdot yf(s))\,ds \\
&= \lim_i \int_G \Delta(ts)^{-1/2}x^*(g^*u_i\theta_t^e(u_i\theta_{s^{-1}}^e(h))\delta_{ts^{-1}} \cdot yf(s))\,ds \\
&= \lim_i \int_G \Delta(ts^{-1})^{-1/2}\Delta(s^{-1})x^*(g^*u_i\theta_t^e(u_i\theta_s^e(h))\delta_{ts} \cdot yf(s^{-1}))\,ds \\
&= \lim_i \int_G \Delta(s)^{-1/2}x^*(g^*u_i\theta_t^e(u_i\theta_{t^{-1}s}^e(h))\delta_s \cdot yf(s^{-1}t))\,ds \\
&= \lim_i \int_G \Delta(s)^{-1/2}x^*(u_i\theta_t^e(u_i)\delta_e \cdot g^*\theta_s^e(h)\delta_s \cdot yf(s^{-1}t))\,ds. \qquad (3.7)
\end{aligned}
$$

Note that $g^*\theta_s^e(h)\delta_s \cdot yf(s^{-1}t) \in B_t$ for all $s \in G$. Let $F_i, F \in C_c(G, B_t)$ be defined as

$$
\begin{aligned}
F_i(s) &:= u_i\theta_t^e(u_i)\delta_e \cdot g^*\theta_s^e(h)\delta_s \cdot yf(s^{-1}t) \\
F(s) &:= g^*\theta_s^e(h)\delta_s \cdot yf(s^{-1}t)
\end{aligned}
$$

The supports of both F_i and F are contained in $t\mathrm{supp}(f)^{-1}$, thus the net $\{F_i\}_{i\in I}$ converges to F in the inductive limit topology if and only if it converges uniformly. To show uniform convergence it suffices to prove that given a net $\{s_i\}_{i\in I} \subset G$ converging to $s \in G$, it follows that $\lim_i \|F_i(s_i) - F(s)\| = 0$. Since $F(s) \in B_t = B_tB_t^*B_eB_t = C_0(X_t)\delta_e \cdot B_t$, there exist $m \in C_c(X_t)$ and $z \in B_t$ such that $F(s) = m\delta_e \cdot z$. Then

$$
\begin{aligned}
0 \le \lim_i \|F_i(s_i) - F(s)\| &\le \lim_i \|F_i(s_i) - F_i(s)\| + \|F_i(s) - F(s)\| \\
&\le \lim_i \|u_i\theta_t^e(u_i)\|\|F(s_i) - F(s)\| + \|F_i(s) - F(s)\| \\
&\le \lim_i \|F_i(s) - F(s)\| \\
&= \lim_i \|u_i\theta_t^e(u_i)\delta_e \cdot F(s) - F(s)\| \\
&= \lim_i \|u_i\theta_t^e(u_i)m - m\|\|z\| = 0,
\end{aligned}
$$

where the last identity holds because $\{u_i\theta_t^e(u_i)\}_{i\in I}$ is an approximate unit of $C_0(X_t)$.

Now we can continue the computations (3.7) to get

$$\begin{aligned}\langle\langle g\delta_e \cdot x, (h\delta_e \cdot y) \diamond f\rangle\rangle_{\mathcal{B}}(t) &= \lim_i \int_G F_i(s)\, ds = \int_G F(s)\, ds\\ &= \int_G \Delta(s)^{-1/2} x^*(g^*\theta^e_s(h)\delta_s \cdot y) f(s^{-1}t)\, ds\\ &= \int_G \langle\langle g\delta_e \cdot x, h\delta_e \cdot y\rangle\rangle(s) f(s^{-1}t)\, ds\\ &= \langle\langle g\delta_e \cdot x, h\delta_e \cdot y\rangle\rangle * f(t).\end{aligned}$$

This proves $\langle\langle g\delta_e \cdot x, (h\delta_e \cdot y) \diamond f\rangle\rangle_{\mathcal{B}} = \langle\langle g\delta_e \cdot x, h\delta_e \cdot y\rangle\rangle * f$.

To show that $(x \diamond f) \diamond g = x \diamond (f * g)$ (for $x \in E^0_{\mathcal{B}}$ and $f, g \in C_c(\mathcal{B})$) it suffices to show, by Lemma 3.9, that for $y := (x \diamond f) \diamond g - x \diamond (f * g)$ one has $\langle\langle y, y\rangle\rangle_{\mathcal{B}} = 0$. But this is so because

$$\langle\langle y, y\rangle\rangle_{\mathcal{B}} = (\langle\langle y, x\rangle\rangle_{\mathcal{B}} * f) * g - \langle\langle y, x\rangle\rangle_{\mathcal{B}} * (f * g) = 0.$$

□

Definition 3.13. Let μ be a crossed product norm of $C_c(\mathcal{B})$. We define $E^\mu_{\mathcal{B}}$ as the completion of $E^0_{\mathcal{B}}$ with respect to the norm $\|\ \|_\mu \colon E^0_{\mathcal{B}} \to [0, +\infty)$, $x \mapsto \mu(\langle\langle x, x\rangle\rangle_{\mathcal{B}})^{1/2}$, and regard $E^\mu_{\mathcal{B}}$ as a right $C_\mu(\mathcal{B})$−Hilbert module with the unique inner product and action extending the operations

$$\begin{aligned} &E^0_{\mathcal{B}} \times E^0_{\mathcal{B}} \to C_c(\mathcal{B}) && (x, y) \mapsto \langle\langle x, y\rangle\rangle_{\mathcal{B}}\\ &E^0_{\mathcal{B}} \times C_c(\mathcal{B}) \to E^0_{\mathcal{B}} && (x, f) \mapsto x \diamond f.\end{aligned}$$

The μ−fixed-point algebra for $\mathcal{B}$ is $\mathbb{F}^\mu_{\mathcal{B}} := \mathbb{K}(E^\mu_{\mathcal{B}})$ (see Notation 2.2).

For future use we give a bound on $\|\ \|_\mu$.

Remark 3.14. For every $f \in E^0_\sigma$ and $b \in B_e$, $\|f\delta_e \cdot b\|_\mu \le \|f\|_\sigma \|b\|$, where $\|\ \|_\sigma$ is the norm of E_σ. Indeed, we may assume μ is the universal norm (because this change can only increase $\|\ \|_\mu$). Notice $x := f\delta_e \cdot b \in E^0_\sigma$ and that we can reuse the computations in (3.5) (with $\widetilde{T}$ faithful and nondegenerate). By Remark 3.10,

$$\begin{aligned}\|f\delta_e \cdot b\|^2_{\mathrm{u}} &= \|\widetilde{T}_{\langle\langle f\delta_e \cdot b, f\delta_e \cdot b\rangle\rangle_{\mathcal{B}}}\| = \|T^*_b \tilde{\widetilde{T}}_{\langle\langle f, f\rangle\rangle_\sigma} T_b\| \le \|T_{b^*b}\| \|\langle\langle f, f\rangle\rangle_\sigma\|\\ &\le \|b\|^2 \|f\|^2_\sigma;\end{aligned}$$

which gives the desired inequality.

The fixed-point algebras $\mathbb{F}^\mu_{\mathcal{B}}$ have a natural $C_0(X/\sigma)$−algebra structure, as we show below.

Proposition 3.15. *Let $C_b(X) = M(C_0(X))$ act on B_e by extending the action of $C_0(X) = C_0(X)\delta_e$ on B_e. Consider $C_0(X/\sigma)$ as a C*-subalgebra of $C_b(X)$ and let $C_0(X/\sigma)$ act on B_e through the action of $C_b(X)$. Then $C_0(X/\sigma)E^0_{\mathcal{B}} \subset E^0_{\mathcal{B}}$ and this gives an action*

$$C_0(X/\sigma) \times E^0_{\mathcal{B}} \to E^0_{\mathcal{B}},\ (f, x) \mapsto fx.$$

*Moreover, for every crossed product norm μ of $C_c(\mathcal{B})$, there exists a unique *-homomorphism $\phi_\mu : C_0(X/\sigma) \to \mathbb{B}(E^\mu_{\mathcal{B}}) = M(\mathbb{F}^\mu_{\mathcal{B}})$ such that $\phi_\mu(f)x = fx$, for all $f \in C_0(X/\sigma)$ and $x \in E^0_\mu$. Besides, ϕ_μ is nondegenerate.*

Proof. Take $f \in C_0(X/\sigma)$ and $x \in E^0_{\mathcal{B}}$. Consider a factorization $x = g\delta_e \cdot y$ with $g \in E^0_\sigma$ and $y \in B_e$. Then, by construction, $fx = f(g\delta_e \cdot y) = (fg)\delta_e \cdot y \in E^0_{\mathcal{B}}$.

Now let μ be a crossed product norm of $C_c(\mathcal{B})$. Take a nondegenerate *-representation $T : \mathcal{B} \to \mathbb{B}(V)$ such that the integrated form $\tilde{T} : C^*(\mathcal{B}) \to \mathbb{B}(V)$ factors through a faithful representation of $C^*_\mu(\mathcal{B})$. Let $\hat{T} : \mathcal{A}_\sigma \to \mathbb{B}(V)$ be the *-representation described in Lemma 3.8. Given $f \in C_0(X/\sigma)$ and $x \in E^0_\sigma$ take a factorization $x = g\delta_e \cdot y$ as explained in the last paragraph. We know (see Section 3.1) that $\langle\langle fg, fg\rangle\rangle_\sigma \le \|f\|^2\langle\langle g, g\rangle\rangle_\sigma$ in $C^*(\mathcal{A}_\sigma)$. As indicated in Remark 3.10, equation (3.5) can be used in the present situation to deduce that for all $\xi \in V$,

$$\begin{aligned}
\langle \tilde{T}_{\|f\|^2\langle\langle x,x\rangle\rangle_{\mathcal{B}} - \langle\langle fx,fx\rangle\rangle_{\mathcal{B}}}\xi, \xi\rangle &= \langle \tilde{T}_{\|f\|^2\langle\langle g\delta_e\cdot y, g\delta_e\cdot y\rangle\rangle_{\mathcal{B}} - \langle\langle fg\delta_e\cdot y, fg\delta_e\cdot y\rangle\rangle_{\mathcal{B}}}\xi, \xi\rangle \\
&= \langle \tilde{T}_{\|f\|^2\langle\langle g\delta_e\cdot y, g\delta_e\cdot y\rangle\rangle_{\mathcal{B}}}\xi, \xi\rangle - \langle \tilde{T}_{\langle\langle fg\delta_e\cdot y, fg\delta_e\cdot y\rangle\rangle_{\mathcal{B}}}\xi, \xi\rangle \\
&= \langle \tilde{\hat{T}}_{\|f\|^2\langle\langle g,g\rangle\rangle_\sigma} T_y\xi, T_y\xi\rangle - \langle \tilde{\hat{T}}_{\langle\langle fg,fg\rangle\rangle_\sigma} T_y\xi, T_y\xi\rangle \\
&= \langle \tilde{\hat{T}}_{\|f\|^2\langle\langle g,g\rangle\rangle_\sigma - \langle\langle fg,fg\rangle\rangle_\sigma} T_y\xi, T_y\xi\rangle \ge 0. \quad (3.8)
\end{aligned}$$

The way we chose T implies that we may regard $C^*_\mu(\mathcal{B})$ as the closed linear span of $\{\widetilde{T}_f : f \in C_c(\mathcal{B})\} \subset \mathbb{B}(V)$. When doing so, each $f \in C_c(\mathcal{B})$ gets identified with $\widetilde{T}_f$. So (3.8) shows the bound $\|f\|^2\langle\langle x, x\rangle\rangle_{\mathcal{B}} \ge \langle\langle fx, fx\rangle\rangle_{\mathcal{B}}$ holds in $C^*_\mu(\mathcal{B})$. This implies that for all $f \in C_0(X/\sigma)$ there exists a unique bounded operator $\phi_\mu(f) : E^\mu_{\mathcal{B}} \to E^\mu_{\mathcal{B}}$ such that $\phi_\mu(f)x = fx$, for all $x \in E^\mu_{\mathcal{B}}$. Moreover, $\|\phi_\mu(f)x\|_\mu \le \|f\|\|x\|_\mu$.

The operator $\phi_\mu(f)$ is adjointable with adjoint $\phi_\mu(f^*)$ because, for all $x, y \in B_e$, $g, h \in E^0_\sigma$ and $t \in G$,

$$\begin{aligned}
\langle\langle \phi_\mu(f)(g\delta_e \cdot x), h\delta_e \cdot y\rangle\rangle_{\mathcal{B}}(t) &= x^*(\langle\langle fg, h\rangle\rangle_\sigma(t) \cdot y) = x^*(\langle\langle g, f^*h\rangle\rangle_\sigma(t) \cdot y) \\
&= \langle\langle g\delta_e \cdot x, \phi_\mu(f^*)(h\delta_e \cdot y)\rangle\rangle_{\mathcal{B}}(t).
\end{aligned}$$

Now that we know the map $\phi_\mu : C_0(X/\sigma) \to M(\mathbb{F}^\mu_{\mathcal{B}})$ is defined and preserves the involution, we leave to the reader the verification of the fact that ϕ_μ is linear and multiplicative.

In order to show that ϕ_μ is nondegenerate it suffices to show that given an approximate unit $\{f_i\}_{i\in I}$ of $C_0(X/\sigma)$, $g \in E^0_\sigma$ and $x \in B_e$, we have that

$$\lim_i \|\phi_\mu(f_i)g\delta_e \cdot x - g\delta_e \cdot x\|_\mu = 0.$$

By Remark 3.14 we have

$$\|\phi_\mu(f_i)g\delta_e \cdot x - g\delta_e \cdot x\|_\mu \le \|f_i g - g\|_\sigma \|x\|.$$

The construction of E_σ in Section 3.1 implies $\lim_i \|f_i g - g\|_\sigma = 0$. Thus ϕ_μ is nondegenerate. $\square$

The next result implies there are as many fixed-point algebras as crossed product norms.

In the proof below we consider Hilbert modules as ternary C*-rings (C*-trings) [20]. More precisely, given a right $A-$Hilbert module Y we consider on Y the ternary operation $(x,y,z)_Y := x\langle y,z\rangle_A$. An homomorphism of C*-trings is a linear map $\phi\colon E\to F$ such that $\phi(x,y,z)=(\phi(x),\phi(y),\phi(z))$, for all $x,y,z\in E$. We consider the modules $E^\mu_{\mathcal{B}}$ as right $C^*_\mu(\mathcal{B})-$Hilbert modules

Proposition 3.16. *Given two crossed product norms of $C_c(\mathcal{B})$, μ and ν with $\mu\le\nu$, consider Hilbert modules $E^\mu_{\mathcal{B}}$ and $E^\nu_{\mathcal{B}}$ of Definition* 3.13 *as C*-trings. Then there exists a unique homomorphism of C*-trings $\kappa^\mu_\nu\colon\ \colon E^\nu_{\mathcal{B}}\to E^\mu_{\mathcal{B}}$ extending the natural identity map of $E^0_{\mathcal{B}}$. Moreover, κ^μ_ν is surjective. In case $\{\langle\langle x,y\rangle\rangle_{\mathcal{B}}\colon x,y\in E^0_{\mathcal{B}}\}$ spans a dense subset of $C^*_\nu(\mathcal{B})$, the following are equivalent:*

(1) κ^μ_ν is injective (and hence an isomorphism).
(2) κ^μ_ν is isometric (and hence an isomorphism).
(3) $\mu=\nu$.

Proof. For all $x\in E^0_{\mathcal{B}}$ we have

$$\|x\|_\mu=\mu(\langle\langle x,x\rangle\rangle_{\mathcal{B}})^{1/2}\le\nu(\langle\langle x,x\rangle\rangle_{\mathcal{B}})^{1/2}=\|x\|_\nu.$$

Thus the identity map of $E^0_{\mathcal{B}}$ admits a unique linear and continuous extension κ^μ_ν. This extension is a homomorphism because the identity

$$\kappa^\mu_\nu(x,y,z)=\kappa^\mu_\nu(x\diamond\langle\langle y,z\rangle\rangle_{\mathcal{B}})=x\diamond\langle\langle y,z\rangle\rangle_{\mathcal{B}}=(\kappa^\mu_\nu(x),\kappa^\mu_\nu(y),\kappa^\mu_\nu(z))$$

holds for all $x,y,z\in E^0_{\mathcal{B}}$ and hence, by continuity, for all $x,y,z\in E^\nu_{\mathcal{B}}$.

The range of κ^μ_ν is closed by [4, Corollary 4.8]. Clearly (3) implies (1) and (2) and these last two conditions are equivalent by [4, Proposition 3.11].

Assume (2) holds. Since the inner products span a dense subset of $C_\nu(\mathcal{B})$, they also span a dense subset of $C_\mu(\mathcal{B})$. Regarding $E^\nu_{\mathcal{B}}$ (respectively, $E^\mu_{\mathcal{B}}$) as full right $C^*_\nu(\mathcal{B})-$Hilbert module (respectively, $C^*_\nu(\mathcal{B})-$Hilbert module) we obtain, for all $f\in C_c(\mathcal{B})$,

$$\nu(f)=\sup\{\|x\diamond f\|_\nu\colon x\in E^0_{\mathcal{B}},\ \|x\|_\nu\le 1\}=\mu(f).$$

This completes the proof. □

As explained in [20] one can recover the $\mu-$fixed-point algebra out of the C*-tring structure of $E^\mu_{\mathcal{B}}$. In fact the maps $\kappa^\mu_\nu\colon E^\nu_{\mathcal{B}}\to E^\mu_{\mathcal{B}}$ induce surjective morphism of C*-algebras $\kappa^{\mu^r}_\nu\colon\mathbb{F}^\nu_{\mathcal{B}}\to\mathbb{F}^\mu_{\mathcal{B}}$ [2, Proposition 4.1] and the equivalence in our last Proposition also holds for these maps.

The main result of this section is the following one, in which we prove a Morita equivalence between crossed products and fixed-point algebras.

Theorem 3.17. *If σ is free then*

$$I_{\mathcal{B}} := \operatorname{span}\{\langle\langle x,y\rangle\rangle_{\mathcal{B}}\colon x,y\in E^0_{\mathcal{B}}\}$$

is dense in $C_c(\mathcal{B})$ in the inductive limit topology. In particular, for every crossed product norm μ of $C_c(\mathcal{B})$ the bimodule $E^\mu_{\mathcal{B}}$ is a $\mathbb{F}^\mu_{\mathcal{B}}-C^*_\mu(\mathcal{B})-$equivalence bimodule.

Proof. It suffices to work with the universal C*-norm $\|\ \|_{\mathrm{u}}$ of $C_c(\mathcal{B})$. The proof of Green's Symmetric Imprimitivity Theorem presented in [17], used here for the enveloping action σ^e, implies that

$$I_{\sigma_e} := \operatorname{span}\{\langle\langle f,g\rangle\rangle_{\sigma^e} :\ f,g\in E^0_{\sigma^e}\}\subset C_c(\mathcal{A}_{\sigma^e})$$

is dense in the inductive limit topology in $C_c(\mathcal{A}_{\sigma^e})$. Moreover, as shown in [17], for every $k\in C_c(\mathcal{A}_{\sigma^e})$ there exists a compact set $L\subset G$ and a net $\{k_i\}_{i\in I}\in I_{\sigma^e}$ such that $\operatorname{supp}(k_i)\subset L$ for all $i\in I$ and $\|k_i-k\|_\infty\to 0$. Since $C_c(\mathcal{A}_\sigma)$ is hereditary in $C_c(\mathcal{A}_{\sigma^e})$, by using the approximate units constructed in [12, VIII 16.4] we get that for all $k\in C_c(\mathcal{A}_\sigma)$ there exists a compact set $L\subset G$ and a net

$$\{k_i\}_{i\in I}\subset \operatorname{span}\{u*\langle\langle f,g\rangle\rangle_{\sigma^e}*v:\ f,g\in C_c(X^e),\ u,v\in C_c(\mathcal{A}_\sigma)\}$$

such that $\operatorname{supp}(k_i)\subset L$, for all $i\in I$, and $\|k_i-k\|_\infty\to 0$. But since $C_c(X^e)C_c(\mathcal{A}_\sigma)\subset E^0_\sigma$, the last approximate unit $\{k_i\}_{i\in I}$ is included in

$$I_\sigma := \operatorname{span}\{\langle\langle f,g\rangle\rangle_\sigma :\ f,g\in E^0_\sigma\}\subset C_c(\mathcal{A}_\sigma).$$

Now define

$$I_{\mathcal{B}} := \operatorname{span}\{\langle\langle x,y\rangle\rangle :\ x,y\in E^0_{\mathcal{B}}\}\subset C_c(\mathcal{B})$$

and let $\overline{I_{\mathcal{B}}}$ be the closure of $I_{\mathcal{B}}$ in the inductive limit topology of $C_c(\mathcal{B})$. For $\overline{I_{\mathcal{B}}}$ to be equal to $C_c(\mathcal{B})$ we just need to show, by [11, II 14.6],

(i) $C_c(G)\overline{I_{\mathcal{B}}}\subset\overline{I_{\mathcal{B}}}$.
(ii) For all $t\in G$, $\overline{I_{\mathcal{B}}}(t) := \{z(t):\ z\in\overline{I_{\mathcal{B}}}\}$ is dense in B_t.

Given $f\in C_c(G)$ and $k\in\overline{I_{\mathcal{B}}}$ take a net $\{k_i\}_{i\in I}\subset I_{\mathcal{B}}$ converging to k in the inductive limit topology (and hence uniformly over compact sets). Thus $\{fk_i\}_{i\in I}$ converges to fk in the inductive limit topology and to show $fk\in\overline{I_{\mathcal{B}}}$ it suffices to show that $fk_i\in\overline{I_{\mathcal{B}}}$, for all $i\in I$. In other words, we can assume from the beginning that $k\in I_{\mathcal{B}}$. Moreover, by linearity we may assume $k=\langle\langle g\delta_e\cdot x, h\delta_e\cdot y\rangle\rangle_{\mathcal{B}}$ with $g,h\in E^0_\sigma$ and $x,y\in B_e$. For all $t\in G$ we have

$$(fk)(t)=x^*(f(t)\langle\langle g,h\rangle\rangle_\sigma(t)\cdot y).$$

Now take a compact set L and a net $\{m_j\}_{j\in J}\subset I_\sigma\cap C_L(\mathcal{A}_\sigma)$ converging uniformly to the function $t\mapsto f(t)\langle\langle g,h\rangle\rangle_\sigma(t)$. Then the net $\{t\mapsto x^*(m_j(t)\cdot y)\}_{j\in J}$ is contained in $I_{\mathcal{B}}$ and converges to fk in the inductive limit topology. Thus $fk\in\overline{I_{\mathcal{B}}}$.

To prove (ii) take $t\in G$ and $b\in B_t$. By the Cohen-Hewitt factorization Theorem and the non degeneracy of the action of $\mathcal{A}_\sigma$ on $\mathcal{B}$ there exists $c,d\in B_e$ and $g\in C_0(X_t)$ such that $b=c^*(g\delta_t\cdot d)$. Since there exists an element of $C_c(\mathcal{A}_\sigma)$ taking the value $g\delta_t$ at t, there exists a net $\{m_j\}_{j\in J}\subset I_\sigma$ such that $\|m_j(t)-g\delta_t\|\to 0$. Then the net $\{s\mapsto c^*(m(s)\cdot d)\}_{j\in J}$ lies in $I_{\mathcal{B}}$ and, after evaluation at t, converges to b. Thus $b\in\overline{I_{\mathcal{B}}(t)}$.

The rest of the proof is straightforward because the inductive limit topology is stronger than any topology coming from a crossed product norm. □

3.3. The module $E^0_{\mathcal{B}}$ as a tensor product. For this section we need exactly the same setting we used in the first paragraph of Section 3.2.

In Section 3.1 we constructed the $C^*(\mathcal{A}_\sigma)$–module E_σ, notice there is only one crossed product norm to consider because $\mathcal{A}_\sigma$ is amenable. The action of $\mathcal{A}_\sigma$ on $\mathcal{B}$ passes to an action of $C^*(\mathcal{A}_\sigma)$ on $C^*_\mu(\mathcal{B})$, for any crossed product norm μ.

Proposition 3.18. *For every crossed product norm μ of $C_c(\mathcal{B})$ there exists a unique *-representation $S^\mu : \mathcal{A}_\sigma \to M(C^*_\mu(\mathcal{B}))$ such that:*

- *For all $a \in \mathcal{A}_\sigma$, $S^\mu_a C_c(\mathcal{B}) \subset C_c(\mathcal{B})$.*
- *For all $s, t \in G$, $a \in C_0(X_t)\delta_t$ and $f \in C_c(\mathcal{B})$, $S^\mu_a f(s) = a \cdot f(t^{-1}s)$.*

*Moreover, S^μ is nondegenerate and $\widetilde{S^\mu} : C^*(\mathcal{A}_\sigma) \to M(C^*_\mu(\mathcal{B}))$ is the unique *-homomorphism satisfying the following*

- *For all $f \in C_c(\mathcal{A}_\sigma)$, $\widetilde{S^\mu}_f C_c(\mathcal{B}) \subset C_c(\mathcal{B})$.*
- *For all $f \in C_c(\mathcal{A}_\sigma)$, $g \in C_c(\mathcal{B})$ and $t \in G$, $\widetilde{S^\mu}_f g(t) = \int_G f(s) \cdot g(s^{-1}t)\, dt$.*

Proof. Uniqueness claims are immediate, we will only prove the existence. For convenience we write A_t instead of $C_0(X_t)\delta_t$.

Let $T : \mathcal{B} \to \mathbb{B}(V)$ be a nondegenerate *-representation on a Hilbert space whose integrated form $\widetilde{T} : C^*(\mathcal{B}) \to \mathbb{B}(V)$ factors through a faithful representation of $C^*_\mu(\mathcal{B})$. Thus we can actually think of $\widetilde{T}$ as a nondegenerate and faithful *-representation of $C^*_\mu(\mathcal{B})$. We will denote D the image of $\widetilde{T}$. The canonical extension of $\widetilde{T}$ to $M(C^*_\mu(\mathcal{B}))$ will be denoted $\overline{T}$. This extension is injective and its image is

$$MD := \{R \in \mathbb{B}(V) : RD \cup DR \subset D\}.$$

Given $a \in A_t$ and $f \in C_c(\mathcal{B})$ we define the function $a \cdot f \in C_c(\mathcal{B})$ by $a \cdot f(s) := a \cdot f(t^{-1}s)$.

Let $\hat{T} : \mathcal{A}_\sigma \to \mathbb{B}(V)$ be the *-representation given by Lemma 3.8. Then for all $a \in A_t$, $f \in C_c(\mathcal{B})$ and $\xi \in V$:

$$\hat{T}_a \widetilde{T}_f \xi = \int_G T_{a\cdot f(t)} \xi\, dt = \int_G T_{a\cdot f(s^{-1}t)} \xi\, dt = \widetilde{T}_{a\cdot f} \xi.$$

This implies $\hat{T}_a \widetilde{T}(C_c(\mathcal{B})) \subset D$ and by continuity we get $\hat{T}_a D \subset D$. Now define $g \in C_c(\mathcal{B})$ by $g(t) := (a^* \cdot f(t)^*)^*$ and take a factorization $\xi = T_b \eta$, with $b \in B_e$ and $\eta \in V$. Then

$$\widetilde{T}_f \hat{T}_a \xi = \int_G T_{f(t)(a\cdot b)} \eta\, dt = \int_G T_{(a^*\cdot f(t)^*)^* b} \eta\, dt = \int_G T_{(a^*\cdot f(t)^*)^*} \xi\, dt = \widetilde{T}_g \xi.$$

This implies $D\hat{T}_a \subset D$ and we conclude that $\hat{T}_a \in MD$.

By thinking of $\overline{T}$ as in isomorphism between $M(C^*_\mu(\mathcal{B}))$ and MD one just needs to set $S^\mu := \overline{T}^{-1} \circ \hat{T}$. The computations above show S^μ satisfies the desired properties.

Under the isomorphism $M(C^*_\mu(\mathcal{B})) \approx MD$, S^μ is identified with $\hat{T}$ (that is the whole point of the proof). Then we can think of the integrated form of $\hat{T}$ as the integrated form of S^μ.

We now construct an action $C_c(\mathcal{A}_\sigma) \times C_c(\mathcal{B}) \to C_c(\mathcal{B})$, $(f, g) \mapsto f \cdot g$, such that $f \cdot g(t) = \int_G f(s) \cdot g(s^{-1}t)\, ds$. If $\mathcal{A}_\sigma$ where $\mathcal{B}$ and $\mathcal{A}_\sigma \times \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto a \cdot b$, were the multiplication of $\mathcal{B}$, the action we want to construct would be the convolution product of $C_c(\mathcal{B})$. So the best one can do is to consult [12, pp 803], where such product is constructed. After this, one realizes the arguments found there can be easily adapted (almost copied) to solve our problem. We leave this to the reader, who may find useful to use claim (4) of Proposition 2.8.

For all $f \in C_c(\mathcal{A}_\sigma)$, $g \in C_c(\mathcal{B})$ and $\xi \in V$ we have

$$\begin{aligned}\widetilde{T}_f \widetilde{T}_g \xi &= \int_G \widetilde{T}_f T_{g(t)} \xi \, dt = \int_G \int_G \hat{T}_{f(s)} T_{g(t)} \xi \, dsdt = \int_G \int_G T_{f(s) \cdot g(t)} \xi \, dsdt \\ &= \int_G \int_G T_{f(s) \cdot g(s^{-1}t)} \xi \, dtds = \widetilde{T}_{f \cdot g} \xi.\end{aligned}$$

Then we must have $S^\mu_f g = f \cdot g$, and this identity completes the proof. □

Theorem 3.19. *For every crossed product norm μ of $C_c(\mathcal{B})$, $E^\mu_{\mathcal{B}}$ is unitarily equivalent to $E_\sigma \otimes_{\widetilde{S^\mu}} C^*_\mu(\mathcal{B})$, where $\widetilde{S^\mu} \colon C^*(\mathcal{A}_\sigma) \to M(C^*_\mu(\mathcal{B}))$ is the integrated form given by Proposition* 3.18.

Proof. Let $E_\sigma \otimes C_c(\mathcal{B})$ be the subspace of $E_\sigma \otimes_{\widetilde{S^\mu}} C^*_\mu(\mathcal{B})$ spanned by the elementary tensor product $f \otimes g$ with $f \in E^0_\sigma$ and $g \in C_c(\mathcal{B})$.

Take $f_1, \ldots, f_n \in E^0_\sigma$ and $g_1, \ldots, g_n \in C_c(\mathcal{B})$. By considering the action of B_e on $C_0(\mathcal{B})$ by multiplication we can get factorizations $g_i = b_i h_i$ with $b_i \in B_e$ and $h_i \in C_c(\mathcal{B})$, for $i = 1, \ldots, n$. We claim that

$$\| \sum_{i=1}^n (f_i \cdot b_i) \diamond h_i \| = \| \sum_{i=1}^n f_i \otimes b_i h_i \|. \tag{3.9}$$

To show this it suffices to prove that

$$\langle\langle (f_i \cdot b_i) \diamond h_i, (f_j \cdot b_j) \diamond h_j \rangle\rangle_{\mathcal{B}} = (b_i h_i)^* (\widetilde{S^\mu}_{\langle\langle f_i, f_j \rangle\rangle_\sigma} (b_j h_j)), \ \forall\, i, j = 1, \ldots, n. \tag{3.10}$$

For any $i, j = 1, \ldots, n$ and $t \in G$ we have

$$\begin{aligned}
\langle\langle (f_i \cdot b_i) \diamond h_i, (f_j \cdot b_j) \diamond h_j \rangle\rangle_{\mathcal{B}}(t) &= h_i^* * \langle\langle f_i \cdot b_i, f_j \cdot b_j \rangle\rangle_{\mathcal{B}} * h_j(t) \\
&= \int_G \int_G h_i^*(r) \langle\langle f_i \cdot b_i, f_j \cdot b_j \rangle\rangle_{\mathcal{B}}(s) h_j](s^{-1}r^{-1}t)\, ds dr \\
&= \int_G \int_G h_i^*(r) b_i^* [\langle\langle f_i, f_j \rangle\rangle_\sigma(s) \cdot b_j] h_j(s^{-1}r^{-1}t)\, ds dr \\
&= \int_G \int_G (b_i h_i)^*(r) [\langle\langle f_i, f_j \rangle\rangle_\sigma(s) \cdot (b_j h_j(s^{-1}r^{-1}t))]\, ds dr \\
&= \int_G (b_i h_i)^*(r) [\widetilde{S^\mu}_{\langle\langle f_i, f_j \rangle\rangle_\sigma}(b_j h_j)(r^{-1}t)]\, dr \\
&= (b_i h_i)^* (\widetilde{S^\mu}_{\langle\langle f_i, f_j \rangle\rangle_\sigma}(b_j h_j))(t). \qquad (3.11)
\end{aligned}$$

Now that we know (3.9) holds we can construct a unique bounded linear operator $U : E_\sigma \otimes_{\widetilde{S^\mu}} C^*_\mu(\mathcal{B}) \to E^\mu_{\mathcal{B}}$ such that $U(f \otimes bh) = (f \cdot b) \diamond h$, for all $f \in E^0_\sigma$, $b \in B_e$ and $h \in C_c(\mathcal{B})$. Moreover, U is an isometry with dense range, thus it is an isometric isomorphism of Banach spaces. But now (3.10) says U preserves the inner products, thus it is a unitary operator. □

After the Theorem above Proposition 3.15 should be completely natural. We leave to the reader the verification of the fact that the unitary constructed in our last proof intertwines the action constructed in Proposition 3.15 with the natural action of $C_0(X/\sigma)$ on $E_\sigma \otimes_{\widetilde{S^\mu}} C^*_\mu(\mathcal{B})$.

Theorem 3.19 can also be used to give an alternative proof of Theorem 3.17. Indeed, in case σ is free then E_σ is full on the right, and since $\widetilde{S^\mu}$ is nondegenerate we conclude that $E^\mu_\sigma = E_\sigma \otimes_{\widetilde{S^\mu}} C^*_\mu(\mathcal{B})$ is full on the right and hence a $\mathbb{F}^\mu_{\mathcal{B}} - C^*_\mu(\mathcal{B})$−equivalence bimodule.

Our last Theorem also implies our fixed-point algebras (and even the modules used to construct them) are generalizations of those constructed in [6] for weakly proper actions (see the discussion preceding Definition 2.4 and Example 2.7).

3.4. Bra-ket operators and the fixed-point algebra. In [14, 15] Meyer defines square integrable actions, which are a generalization of proper actions on C*-algebras (or even of weakly proper actions). One can extend Meyer's definition to partial action on C*-algebras, but we will not pursue this goal here. We are more interested in the so called bra-ket operators in the context of Fell bundles.

Assume α is an action of G on the C*-algebra A and assume there exists a dense subset A_0 of A such that for all $a, b \in A_0$ the function $\langle\langle a, b \rangle\rangle : G \to A,\ t \mapsto \alpha_t(a)^* b$, has compact support. Then the element $a \in A_0$ is said to be square integrable if the bra-operator

$$\langle\langle a| : A_0 \to C_c(G, A),\ b \mapsto \langle\langle a, b \rangle\rangle,$$

is the restriction of some adjointable operator from A to $L^2(G,A)$. If such an extension exists, it is unique and it is denoted $\langle\langle a|$. The ket-operator is $|a\rangle\rangle := \langle\langle a|^*$ and it should satisfy

$$|a\rangle\rangle(f) = \int_G \alpha_t(a)f(t)\,dt,\ \forall\, f \in C_c(G,A).$$

If $a, b \in A_0$ are square integrable then $\langle\langle a|\circ|b\rangle\rangle \in \mathbb{B}(L^2(G,A))$ and in case α is weakly proper one gets that

$$\langle\langle a|\circ|b\rangle\rangle \in C_c(G,A) \subset A \rtimes_{\mathrm{r}\alpha} G \subset \mathbb{B}(L^2(G,A)).$$

In order to translate the previous construction to weakly proper Fell bundles one must first note that $L^2(G,A)$ is not equal to $L^2_e(\mathcal{B}_\alpha)$, but it is unitary equivalent. This explains the absence of the modular function in the formula $\langle\langle a,b\rangle\rangle(t) = \alpha_t(a)^*b$. The inclusion $A \rtimes_{\mathrm{r}\alpha} G \subset \mathbb{B}(L^2(G,A))$, as given in [14, Section 3], takes this equivalence into account. All we will do here will be compatible with that identification.

Take a Fell bundle $\mathcal{B} = \{B_t\}_{t\in G}$ which is weakly proper with respect to the proper LCH partial action σ of G on X. As usual we denote θ the partial action of G on $C_0(X)$ defined by σ. The action of $\mathcal{A}_\sigma$ on $\mathcal{B}$ will be denoted $\mathcal{A}_\sigma \times \mathcal{B} \to \mathcal{B},\ (a,b) \mapsto a\cdot b$. The space $E^0_{\mathcal{B}} \subset B_e$ is that of Section 3.2.

Theorem 3.20. *For every $x \in E^0_{\mathcal{B}}$ there exists a unique adjointable operator $\langle\langle x| : B_e \to L^2_e(\mathcal{B})$ such that $\langle\langle x|y = \langle\langle x,y\rangle\rangle_{\mathcal{B}}$ for all $y \in E^0_{\mathcal{B}}$. The adjoint $|x\rangle\rangle := \langle\langle x|^*$ is the unique linear operator from $L^2_e(\mathcal{B})$ to B_e such that $|x\rangle\rangle f = x \diamond f$, for all $f \in C_c(\mathcal{B})$. Moreover, if $\Lambda : C^*(\mathcal{B}) \to \mathbb{B}(L^2_e(\mathcal{B}))$ is the regular representation, then for all $x,y,z \in E^0_{\mathcal{B}}$ and $f \in C_c(\mathcal{B})$ we have*

$$\Lambda_{\langle\langle x,y\rangle\rangle_{\mathcal{B}}} = \langle\langle x|\circ|y\rangle\rangle;\quad |x\rangle\rangle\langle\langle y|z = x \diamond \langle\langle y,z\rangle\rangle_{\mathcal{B}};\quad |\,x\diamond f\rangle\rangle = |x\rangle\rangle\circ\Lambda_f.$$

Proof. Fix $x \in E^0_{\mathcal{B}}$ and define the functions

$$P : E^0_{\mathcal{B}} \to C_c(\mathcal{B}),\, y \mapsto \langle\langle x,y\rangle\rangle_{\mathcal{B}},$$
$$Q : C_c(\mathcal{B}) \to E^0_{\mathcal{B}},\, f \mapsto x \diamond f.$$

Note both P and Q are linear. If P and Q are to be extended to adjointable operators, then we must have, for all $y \in E^0_{\mathcal{B}}$ and $f \in C_c(\mathcal{B})$,

$$\langle f, \langle\langle x,y\rangle\rangle\,\rangle_{L^2_e(\mathcal{B})} = \langle f, Py\rangle_{L^2_e(\mathcal{B})} = \langle Qf, y\rangle_{B_e} = (Qf)^*y = (x\diamond f)^*y. \tag{3.12}$$

To prove the identities above it suffices to show the first term equals the last one.

Take $f \in C_c(\mathcal{B})$ and $y \in E^0_{\mathcal{B}}$ and consider factorizations $x = g\delta_e \cdot u$ and $y = h\delta_e\cdot v$ with $g, h \in E^0_\sigma$ and $u, v \in B_e$. Using an approximate unit $\{u_i\}_{i\in I}$ of

$C_0^\sigma(G,X)$ as the one in Lemma 3.11 we deduce that

$$\begin{aligned}(x\diamond f)^*y &= \int_G \lim_i \Delta(t)^{-1/2}(u_i(t^{-1})\delta_{t^{-1}}\cdot xf(t))^*y\,dt\\ &= \int_G \lim_i \Delta(t)^{-1/2}(u_i(t^{-1})\delta_{t^{-1}}\cdot g\delta_e\cdot uf(t))^*(h\cdot\delta_e v)\,dt\\ &= \int_G \lim_i \Delta(t)^{-1/2}(h^*\delta_e u_i(t^{-1})\delta_{t^{-1}}g\delta_e\cdot uf(t))^*v\,dt\\ &= \int_G \lim_i \Delta(t)^{-1/2}(u_i(t^{-1})h^*\theta^e_{t^{-1}}(g)\delta_{t^{-1}}\cdot uf(t))^*v\,dt\\ &= \int_G \Delta(t)^{-1/2}(h^*\theta^e_{t^{-1}}(g)\delta_{t^{-1}}\cdot uf(t))^*v\,dt\\ &= \int_G f(t)^*(u^*\Delta(t)^{-1/2}g^*\theta^e_t(h)\delta_t\cdot v)\,dt = \int_G f(t)^*\langle\langle x,y\rangle\rangle(t)\,dt\\ &= \langle f,\langle\langle x,y\rangle\rangle\rangle_{L^2_e(\mathcal{B})}\end{aligned}$$

This completes the proof of (3.12).

By taking $y=x\diamond f$ in (3.12) and recalling that $\Lambda_g h = g*h$ for all $g,h\in C_c(\mathcal{B})$ we get

$$\begin{aligned}(Qf)^*(Qf) &= (x\diamond f)^*(x\diamond f) = \langle f,\langle\langle x,x\diamond f\rangle\rangle_{\mathcal{B}}\rangle_{L^2_e(\mathcal{B})}\\ &= \langle f,\langle\langle x,x\rangle\rangle_{\mathcal{B}}*f\rangle_{L^2_e(\mathcal{B})} \le \|\langle\langle x,x\rangle\rangle_{\mathcal{B}}\|_{C^*_r(\mathcal{B})}\langle f,f\rangle_{L^2_e(\mathcal{B})}\end{aligned}\tag{3.13}$$

Then Q is bounded and $\|Q\|^2\le\|\langle\langle x,x\rangle\rangle_{\mathcal{B}}\|_{C^*_r(\mathcal{B})}$.

Using (3.12) and that Q is bounded we deduce that

$$\begin{aligned}\|Py\| &= \sup\{\|\langle f,Py\rangle_{L^2_e(\mathcal{B})}\|:\ f\in C_c(\mathcal{B}),\ \|f\|_{L^2_e(\mathcal{B})}\le 1\}\\ &= \sup\{\|\langle Qf,y\rangle_{L^2_e(\mathcal{B})}\|:\ f\in C_c(\mathcal{B}),\ \|f\|_{L^2_e(\mathcal{B})}\le 1\}\\ &\le \|Q\|\|y\|.\end{aligned}$$

Hence P is also bounded.

Let $\langle\langle x|: B_e\to L^2_e(\mathcal{B})$ and $|x\rangle\rangle: L^2_e(\mathcal{B})\to B_e$ be the unique continuous extensions of P and Q, respectively. By (3.12) $\langle\langle x|$ is adjointable with adjoint $|x\rangle\rangle$.

Now (3.13) can be used to deduce that $\Lambda_{\langle\langle x,x\rangle\rangle_{\mathcal{B}}} = \langle\langle x|\circ|x\rangle\rangle$ for all $x\in E^0_{\mathcal{B}}$. Then the polarization identity implies that $\Lambda_{\langle\langle x,y\rangle\rangle_{\mathcal{B}}} = \langle\langle x|\circ|y\rangle\rangle$ for all $x,y\in E^0_{\mathcal{B}}$. Finally, for all $x,y,z\in E^0_{\mathcal{B}}$ and $f,g\in C_c(\mathcal{B})$ one has $|x\rangle\rangle\langle\langle y|z = x\diamond(\langle\langle y|z) = x\diamond\langle\langle y,z\rangle\rangle_{\mathcal{B}}$ and

$$|x\diamond f\rangle\rangle g = (x\diamond f)\diamond g = x\diamond(f*g) = x\diamond(\Lambda_f g) = \big(|x\rangle\rangle\circ\Lambda_f\big)\,g.$$

Then the last identity holds for all $g\in L^2_e(\mathcal{B})$ and the proof is complete. □

Consider the subspace

$$\mathbb{F}^0_{\mathcal{B}} := \operatorname{span}\{|x\rangle\rangle\langle\langle y|:\ x,y\in E^0_{\mathcal{B}}\}\subset M(B_e),$$

which is in fact a *-subalgebra of $M(B_e)$ because

$$|x\rangle\rangle\langle\langle y||z\rangle\rangle\langle\langle w| = |x\rangle\rangle\Lambda_{\langle\langle y,z\rangle\rangle_{\mathcal{B}}}\langle\langle w| = |\, x \diamond \langle\langle y, z\rangle\rangle_{\mathcal{B}}\rangle\rangle\langle\langle w| \in \mathbb{F}^0_{\mathcal{B}}.$$

Given a C*-seminorm μ of $C_c(\mathcal{B})$, the generalized compact operator

$$|x\rangle\rangle\circ\langle\langle y| \in \mathbb{F}^\mu_{\mathcal{B}} = \mathbb{B}(E^\mu_{\mathcal{B}})$$

corresponding to the elements $x, y \in E^0_{\mathcal{B}}$ is given by

$$|x\rangle\rangle\circ\langle\langle y|(z) = x \diamond \langle\langle y, z\rangle\rangle_{\mathcal{B}}$$

for all $z \in E^0_{\mathcal{B}}$. Thus one gets a unique morphism of *-algebras

$$\pi_\mu : \mathbb{F}^0_{\mathcal{B}} \to \mathbb{F}^\mu_{\mathcal{B}} \subset \mathbb{B}(E^\mu_{\mathcal{B}}), \text{ such that } \pi_\mu(T)x = Tx \;\forall\, T \in \mathbb{F}^0_{\mathcal{B}},\; x \in E^0_{\mathcal{B}}.$$

In fact π_μ is injective and has dense range. Then the fixed-point algebra $\mathbb{F}^\mu_{\mathcal{B}}$ is a C*-completion of $\mathbb{F}^0_{\mathcal{B}} \subset M(B_e)$.

We need a Lemma to determine the fixed-point algebra corresponding to the closure (completion) of $\mathbb{F}^0_{\mathcal{B}}$ in $M(B_e)$.

Lemma 3.21. *For all $x \in E^0_{\mathcal{B}}$, x belongs to the norm closure of $|x\rangle\rangle(L^2_e(\mathcal{B})_1)$, $L^2_e(\mathcal{B})_1$ being the closed unit ball of $L^2_e(\mathcal{B})$. In particular $\|x\|_{B_e} \le \||x\rangle\rangle\|$.*

Proof. The thesis follows immediately if $x = 0$, otherwise we may assume $\|x\|_{B_e} = 1$ without loss of generality.

Given $\varepsilon > 0$ take $b \in B_e$ such that $\|x - xb\|_{B_e} < \varepsilon$ and $\|b\| < 1$. Now take $f \in C_c(\mathcal{B})$ such that $f(e) = b$ and set $g := x \lhd f \in C_c(G, B_e)$ as in Lemma 3.11. By construction $g(e) = xf(e) = xb$, thus there exists a compact neighborhood V of $e \in G$ such that: (a) its measure $\mu(V)$ is less than 1; (b) $\|x - \Delta(r)^{-1/2}g(r)\|^2 < \varepsilon$ and $\|f(r)\| < 1$, for all $r \in V$.

Take $a \in C_c(G)^+$ with support contained in V and such that $\int_G a(r)^2\, dr = 1$. Then

$$\|af\|_{L^2_e(\mathcal{B})} = \|\int_G a(r)^2 f(r)^* f(r)\, dr\| \le \int_G a(r)^2 \|f(r)\|^2, dr \le 1$$

and $x \lhd (af) = a(x \lhd f) = ag$. Thus

$$\begin{aligned}
\|x - |x\rangle\rangle(af)\|_{B_e} &= \|x - \int_G \Delta(r)^{-1/2} a(r) g(r)\, dr\| \\
&\le \int_V a(r)\|x - \Delta(r)^{-1/2} g(r)\|\, dr \\
&\le \left(\int_V a(r)^2\, dr\right)^{1/2} \left(\int_V \|x - \Delta(r)^{-1/2} g(r)\|^2\, dr\right)^{1/2} \\
&\le \varepsilon\mu(V)^{1/2} < \varepsilon.
\end{aligned}$$

The proof is complete because we have been able to find, for every $\varepsilon > 0$, a function $h = ag \in L^2_e(\mathcal{B})$ such that $\|h\|_{L^2_e(\mathcal{B})} \le 1$ and $\|x - |x\rangle\rangle h\|_{B_e} < \varepsilon$. □

Proposition 3.22. *For every $T \in \mathbb{F}^0_{\mathcal{B}}$ and $x \in E^0_{\mathcal{B}}$ one has $Tx \in E^0_{\mathcal{B}}$ and $|Tx\rangle\rangle = T|x\rangle\rangle$. Besides, the completion of $\mathbb{F}^0_{\mathcal{B}}$ in $M(B_e)$ is the fixed-point algebra corresponding to the reduced cross-sectional C*-algebra norm on $C_c(\mathcal{B})$.*

Proof. If $T = \sum_{i=1}^n |y_i\rangle\rangle\langle\langle z_i|$, then $Tx = \sum_{i=1}^n y_i \diamond \langle\langle z_i, x\rangle\rangle_{\mathcal{B}} \in E^0_{\mathcal{B}}$. Besides, for all $f \in C_c(\mathcal{B})$

$$\begin{aligned} |Tx\rangle\rangle f = (Tx)\diamond f &= \sum_{i=1}^n (y_i \diamond \langle\langle z_i, x\rangle\rangle_{\mathcal{B}}) \diamond f = \sum_{i=1}^n y_i \diamond \langle\langle z_i, x\diamond f\rangle\rangle_{\mathcal{B}} \\ &= \sum_{i=1}^n |y_i\rangle\rangle\langle\langle z_i|(x\diamond f\rangle\rangle_{\mathcal{B}}) = (T|x\rangle\rangle)f. \end{aligned}$$

Hence $|Tx\rangle\rangle = T|x\rangle\rangle$.

The norm of T in the reduced fixed-point algebra $\mathbb{F}^{\mathrm{r}}_{\mathcal{B}}$ satisfies

$$\begin{aligned} \|T\|^2_{\mathbb{F}^{\mathrm{r}}_{\mathcal{B}}} &= \sup\{\|\langle\langle Tx, Tx\rangle\rangle_{\mathcal{B}}\|_{\mathrm{r}} : x \in E^0_{\mathcal{B}},\ \|\langle\langle x, x\rangle\rangle_{\mathcal{B}}\|_{\mathrm{r}} \le 1\} \\ &= \sup\{\||Tx\rangle\rangle\|^2 : x \in E^0_{\mathcal{B}},\ \|\langle\langle x, x\rangle\rangle_{\mathcal{B}}\|_{\mathrm{r}} \le 1\} \\ &= \sup\{\|T|x\rangle\rangle\|^2 : x \in E^0_{\mathcal{B}},\ \|\langle\langle x, x\rangle\rangle_{\mathcal{B}}\|_{\mathrm{r}} \le 1\} \\ &\le \sup\{\|T\|^2_{M(B_e)}\||x\rangle\rangle\|^2 : x \in E^0_{\mathcal{B}},\ \|\langle\langle x, x\rangle\rangle_{\mathcal{B}}\|_{\mathrm{r}} \le 1\} \\ &\le \|T\|^2_{M(B_e)}. \end{aligned}$$

Hence $\|T\|_{\mathbb{F}^{\mathrm{r}}_{\mathcal{B}}} \le \|T\|_{M(B_e)}$.

Let D be the completion of $\mathbb{F}^0_{\mathcal{B}}$ in $M(B_e)$. Then the conclusion of the last paragraph implies the existence of a unique surjective *-homomorphism $\pi : D \to \mathbb{F}^{\mathrm{r}}_{\mathcal{B}}$ extending the identity operator of $\mathbb{F}^0_{\mathcal{B}}$. The proof will be completed if we can show π is injective, because in that case it is isometric.

Suppose $T \in D$ satisfies $\mu(T) = 0$ and take a sequence $\{T_n\}_{n\ge 0} \in \mathbb{F}^0_{\mathcal{B}} \subset D$ approximating T. Then by Theorem 3.20 and Lemma 3.21, for all $x \in E^0_{\mathcal{B}}$ we have

$$\begin{aligned} \|Tx\|_{B_e} = \lim_n \|T_n x\|_{B_e} &\le \limsup_n \||T_n x\rangle\rangle\| = \limsup_n \|\langle\langle T_n x, T_n x\rangle\rangle_{\mathcal{B}}\|^{1/2}_{\mathrm{r}} \\ &\le \limsup_n \|T_n x\|_{E^{\mathrm{r}}_{\mathcal{B}}} = \limsup_n \|\pi(T_n)x\|_{E^{\mathrm{r}}_{\mathcal{B}}} = \|\pi(T)x\|_{E^{\mathrm{r}}_{\mathcal{B}}} = 0. \end{aligned}$$

This shows $T \in M(B_e)$ vanishes in the dense set $E^0_{\mathcal{B}} \subset B_e$, thus $T = 0$ and π is injective. □

The results presented above for bra-ket operators are generalizations, to Fell bundles, of those presented in [14, 15]. In a forthcoming article we will prove an imprimitivity theorem for (exotic) cross-sectional C*-algebras of Fell bundles using the (exotic) fixed-point algebras constructed here.

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