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Multipliers of the Hilbert spaces of Dirichlet series

Chaman Kumar Sahu

ABSTRACT. For a sequence $\mathbf{w} = \{w_j\}_{j=2}^{\infty}$ of positive real numbers, consider the positive semi-definite kernel $\kappa_{\mathbf{w}}(s,u) = \sum_{j=2}^{\infty} w_j j^{-s-\overline{u}}$ defined on some right-half plane $\mathbb{H}_\rho$ for a real number ρ. Let $\mathscr{H}_{\mathbf{w}}$ denote the reproducing kernel Hilbert space associated with $\kappa_{\mathbf{w}}$. Let

$$\delta_{\mathbf{w}} = \inf\left\{\Re(s) : \sum_{\substack{j\geqslant 2 \\ \mathbf{gpf}(j)\leqslant p_n}} w_j j^{-s} < \infty \text{ for all } n \in \mathbb{Z}_+\right\},$$

where $\{p_j\}_{j\geqslant 1}$ is an increasing enumeration of prime numbers and $\mathbf{gpf}(n)$ denotes the greatest prime factor of an integer $n \geqslant 2$. If $\mathbf{w}$ satisfies

$$\sum_{\substack{j\geqslant 2 \\ j|n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big) \geqslant 0, \quad n \geqslant 2,$$

where μ is the Möbius function, then the multiplier algebra $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ of $\mathscr{H}_{\mathbf{w}}$ is isometrically isomorphic to the space of all bounded and holomorphic functions on $\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}$ that are representable by a convergent Dirichlet series in some right half plane. As a consequence, we describe the multiplier algebra $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ when $\mathbf{w}$ is an additive function satisfying $\delta_{\mathbf{w}} \leqslant 0$ and

$$\frac{w_{p^{j-1}}}{w_{p^j}} \leqslant p^{-\delta_{\mathbf{w}}} \text{ for all integers } j \geqslant 2 \text{ and all prime numbers } p.$$

Moreover, we recover a result of Stetler that describes the multipliers of $\mathscr{H}_{\mathbf{w}}$ when $\mathbf{w}$ is multiplicative. The proof of the main result is a refinement of the techniques of Stetler.

CONTENTS

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1. Preliminaries

Let $\mathbb{Z}_+$ and $\mathbb{N}$ denote the sets of positive and non-negative integers, respectively. For $k \in \mathbb{N}$, let $\mathbb{N}_k = \{m \in \mathbb{N} : m \leqslant k\}$. Denote by $\mathbb{R}$ and $\mathbb{C}$, the sets of real and complex numbers, respectively. For $s \in \mathbb{C}$, let $\Re(s)$, $|s|$, $\overline{s}$ and $\arg(s)$ denote the real part, the modulus, the complex conjugate and the argument of s, respectively. The open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathbb{D}$. For $\rho \in \mathbb{R}$, let $\mathbb{H}_\rho$ denote the right half-plane $\{s \in \mathbb{C} : \Re(s) > \rho\}$.

We invoke here some known arithmetic functions on $\mathbb{Z}_+$ used in the sequel. The *prime omega function* ω that counts the total number of all distinct prime factors of a positive integer. The *divisor function* $\mathbf{d}$ counts all divisors of a positive integer. The *Möbius function* $\mu : \mathbb{Z}_+ \to \mathbb{C}$ is given by

$$\mu(n) = \begin{cases} 1 & \text{if n} = 1, \\ (-1)^j & \text{if } n \text{ is a product of } j \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

A *Dirichlet series* is a series of the form

$$f(s) = \sum_{j=1}^{\infty} a_j j^{-s},$$

where $a_j \in \mathbb{C}$. If $a_j = 1$ for all $j \geqslant 1$, then we have the Riemann zeta function, denoted by ζ. If f is convergent at $s = s_0$, then it converges uniformly throughout the angular region $\{s \in \mathbb{C} : |\arg(s - s_0)| < \frac{\pi}{2} - \delta\}$ for every positive real number $\delta < \frac{\pi}{2}$. Consequently, f defines a holomorphic function on $\mathbb{H}_{\Re(s_0)}$ (refer to [13, Chapter IX] for the basic theory of Dirichlet series). Let $\mathcal{D}$ denotes the set of all functions which are representable by a convergent Dirichlet series in some right half plane.

The following proposition describes the product of two Dirichlet series (see [10, Theorem 4.3.1 and discussion prior to Theorem 4.3.4]).

Proposition 1.1. *Let $f(s) = \sum_{j=1}^{\infty} a_j j^{-s}$ and $g(s) = \sum_{j=1}^{\infty} b_j j^{-s}$ be two convergent Dirichlet series on $\mathbb{H}_\rho$. If g converges absolutely on $\mathbb{H}_\rho$, then*

$$f(s)g(s) = \sum_{j=1}^{\infty} \Big(\sum_{m|j} a_m b_{\frac{j}{m}} \Big) j^{-s},$$

which converges on $\mathbb{H}_\rho$.

For an integer $n \geqslant 2$, let $\mathbf{gpf}(n)$ denote the greatest prime factor of n. For a sequence $\mathbf{x} = \{x_j\}_{j\geqslant 2}$ of non-negative real numbers, let

$$\sigma_{\mathbf{x}} := \inf \Big\{ \Re(s) : \sum_{j=2}^{\infty} x_j j^{-s} < \infty \Big\},$$

$$\delta_{\mathbf{x}} := \inf\left\{\Re(s) : \sum_{\substack{j\geqslant 2\\ \mathbf{gpf}(j)\leqslant p_n}} x_j j^{-s} < \infty \text{ for all } n \in \mathbb{Z}_+\right\}, \tag{2}$$

where $\{p_j\}_{j=1}^{\infty}$ is the increasing enumeration of prime numbers. Note that $\delta_{\mathbf{x}} \leqslant \sigma_{\mathbf{x}}$.

1.1. Multipliers of a reproducing kernel Hilbert space. Let X be a non-empty open subset of $\mathbb{C}$ and let $H^\infty(X)$ denote the Banach space of bounded holomorphic functions on X endowed with supremum norm. Let $\mathcal{H}$ be a reproducing kernel Hilbert space of complex-valued holomorphic functions on X. A function $\varphi : X \to \mathbb{C}$ is called *multiplier of* $\mathcal{H}$ if $\varphi \cdot f \in \mathcal{H}$ for every $f \in \mathcal{H}$. Clearly, $\mathcal{M}(\mathcal{H})$ is an algebra. Denote by $\mathcal{M}(\mathcal{H})$, the set of all multipliers of $\mathcal{H}$. Note that if the constant function equal to 1 belongs to $\mathcal{H}$, then by [9, Corollary 5.22], $\mathcal{M}(\mathcal{H})$ is contractively contained in $H^\infty(X)$.

2. The multiplier algerba of $\mathscr{H}_{\mathbf{w}}$

For a sequence $\mathbf{w} = \{w_j\}_{j=2}^{\infty}$ of positive real numbers, consider the weighted Hilbert space of the formal Dirichlet series

$$\mathscr{H}_{\mathbf{w}} := \left\{f(s) = \sum_{j=2}^{\infty} \hat{f}(j) j^{-s} : \|f\|_{\mathbf{w}}^2 := \sum_{j=2}^{\infty} \frac{|\hat{f}(j)|^2}{w_j} < \infty\right\}$$

endowed with the norm $\|\cdot\|_{\mathbf{w}}$. If $\sigma_{\mathbf{w}} < +\infty$, then $\mathscr{H}_{\mathbf{w}}$ is a reproducing kernel Hilbert space associated with $\kappa_{\mathbf{w}}$ given by

$$\kappa_{\mathbf{w}}(s,u) = \sum_{j=2}^{\infty} w_j j^{-s-\overline{u}}, \quad s, u \in \mathbb{H}_{\frac{\sigma_{\mathbf{w}}}{2}}. \tag{3}$$

For every $s \in \mathbb{H}_{\frac{\sigma_{\mathbf{w}}}{2}}$, $\kappa_{\mathbf{w}}(\cdot, s) \in \mathscr{H}_{\mathbf{w}}$ and the following holds:

$$\langle f, \kappa_{\mathbf{w}}(\cdot, s)\rangle = f(s), \quad f \in \mathscr{H}_{\mathbf{w}}. \tag{4}$$

Note that $\kappa_{\mathbf{w}}$ converges absolutely on $\mathbb{H}_{\frac{\sigma_{\mathbf{w}}}{2}} \times \mathbb{H}_{\frac{\sigma_{\mathbf{w}}}{2}}$.

The following is immediate from [12, Theorem 3.1] and the discussion prior to [12, Theorem 2.7].

Theorem 2.1. *$\mathcal{M}(\mathscr{H}_{\mathbf{w}}) \subseteq H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$ and*

$$\|\varphi\|_{\infty, \mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}} \leqslant \|M_{\varphi,\mathbf{w}}\|_{\mathscr{H}_{\mathbf{w}}}, \quad \varphi \in \mathcal{M}(\mathscr{H}_{\mathbf{w}}).$$

This raises the following question.

Question 2.2. For which sequences $\mathbf{w}$, the multiplier algebra $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ of $\mathscr{H}_{\mathbf{w}}$ is isometrically isomorphic to $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$?

In the following cases, Question 2.2 is answered affirmatively.

(i) If $\mathbf{1}$ denotes the constant sequence with value 1, then $\mathcal{M}(\mathscr{H}_{\mathbf{1}})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$ (see [5, Theorem 3.1]).

(ii) If, for some positive Radon measure η on $[0, \infty)$ with $0 \in \text{supp}(\eta)$ and $n_0 \in \mathbb{Z}_+$,

$$\frac{1}{w_j} = \int_0^\infty j^{-2\sigma} d\eta(\sigma), \quad j \geqslant n_0, \tag{5}$$

then $\mathcal{M}(\mathscr{H}_{\mathbf{w}_\eta})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$ (see [6, Theorem 1.11]).

(iii) If $\mathbf{w}$ is either $\{\frac{1}{d_\beta(n)}\}_{n=1}^\infty$, where $\beta > 0$ and $d_\beta(n)$ denotes n-th coefficient of the Dirichlet series ζ^β, or $\{(\mathbf{d}(n))^\alpha\}_{n=1}^\infty$ for $\alpha \in \mathbb{R}$, then $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$ (see [8, Section 8]).

(iv) If $\mathbf{w} = \{w_n\}_{n=1}^\infty$ is a multiplicative function (i.e., $w_{mn} = w_m w_n$ for all integers $m, n \geqslant 1$ such that $\gcd(m, n) = 1$) satisfying

$$\frac{w_{p^{j-1}}}{w_{p^j}} \leqslant p^{-\delta_{\mathbf{w}}} \text{ for all integers } j \geqslant 1 \text{ and prime numbers } p, \tag{6}$$

then $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$ (see [12, Corollary 4.2]).

It follows from the cases above that the multipliers of $\mathscr{H}_{\mathbf{w}}$ may be extended beyond the common domain of $\mathscr{H}_{\mathbf{w}}$. However, this is not always true (see [6, Theorem 3.6]).

The following is the main theorem of this paper, which generalizes the aforementioned results (i) and (iv).

Theorem 2.3. *Suppose that*

$$\sum_{\substack{j \geqslant 2 \\ j | n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big) \geqslant 0, \quad n \geqslant 2, \tag{7}$$

where μ and $\delta_{\mathbf{w}}$ are given by (1) *and* (2)*, respectively. Then $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$.*

Remark 2.4. Here we make some observations:

(a) If $\delta_{\mathbf{w}} = -\infty$, then there are no non-constant multipliers of $\mathscr{H}_{\mathbf{w}}$. Indeed, if $\varphi \in \mathcal{M}(\mathscr{H}_{\mathbf{w}})$, then by Theorem 2.1, φ is entire and bounded. Hence, by the Liouville's theorem (see [11, Corollary 4.5]), φ is constant.

(b) If $\mathbf{w} = \{w_j\}_{j=2}^\infty$ in the above theorem is replaced by $\{w_j\}_{j=k}^\infty \subseteq (0, \infty)$ for an integer $k \in \mathbb{Z}_+$ and

$$\sum_{\substack{j \geqslant k \\ j | n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big) \geqslant 0, \quad n \geqslant k, \tag{8}$$

then the conclusion of Theorem 2.3 will be same.

The following consequence of Theorem 2.3 deals with additive functions.

Corollary 2.5. *Let* $\mathbf{w}$ *be an additive function (i.e.,* $w_{mn} = w_m + w_n$ *for all integers* $m, n \geqslant 2$ *such that* $\gcd(m,n) = 1$*) satisfying*

$$\frac{w_{p^{j-1}}}{w_{p^j}} \leqslant p^{-\delta_{\mathbf{w}}} \text{ for all integers } j \geqslant 2 \text{ and all prime numbers } p. \tag{9}$$

If $\delta_{\mathbf{w}} \leqslant 0$*, then* $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ *is isometrically isomorphic to* $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$.

The proofs of Theorem 2.3 and Corollary 2.5 will be presented in Section 3. In Section 4, we discuss some consequences of Theorem 2.3 (see Theorem 4.1 and Proposition 4.2). In particular, we recover [12, Corollary 4.2]. We also present some examples of the Hilbert spaces $\mathscr{H}_{\mathbf{w}}$ illustrating the results of this paper.

3. Proof of Theorem 2.3

In view of Remark 2.4 (a), we assume that $\delta_{\mathbf{w}} \in \mathbb{R}$. To prove Theorem 2.3, we require several lemmas.

The first one relates the condition (7) with the positive semi-definiteness of a kernel.

Lemma 3.1. *Let* $\tilde{\mathbf{w}} = \{n^{-\delta_{\mathbf{w}}} w_n\}_{n=2}^\infty$ *and* $\beta = \frac{1}{2}\max\{\sigma_{\mathbf{w}} - \delta_{\mathbf{w}}, 1\}$*. Then the kernel* $\eta : \mathbb{H}_\beta \times \mathbb{H}_\beta \to \mathbb{C}$ *defined by*

$$\eta(s,u) = \frac{\kappa_{\tilde{\mathbf{w}}}(s,u)}{\kappa_{\mathbf{1}}(s,u)}, \quad s, u \in \mathbb{H}_\beta \tag{10}$$

is positive semi-definite if and only if (7) *holds.*

Proof. By [2, Example 11.4.1],

$$\frac{1}{\zeta(s)} = \sum_{j=1}^\infty \mu(j) j^{-s}, \quad s \in \mathbb{H}_1.$$

This, combined with Proposition 1.1 and (3), yields

$$\begin{aligned}\eta(s,u) &= \Big(\sum_{j=2}^\infty j^{-\delta_{\mathbf{w}}} w_j j^{-s-\overline{u}}\Big)\Big(\sum_{j=1}^\infty \mu(j) j^{-s-\overline{u}}\Big) \\ &= \sum_{n=2}^\infty \Big(\sum_{\substack{j\geqslant 2\\ j|n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big)\Big) n^{-s-\overline{u}}, \quad s,u \in \mathbb{H}_\beta.\end{aligned}$$

The desired equivalence is now immediate from [7, Lemma 20]. □

The following lemma is needed in the proof of Theorem 2.3.

Lemma 3.2. *For* $t \in \mathbb{R}$*, let* $\tilde{\mathbf{w}} = \{n^{-t} w_n\}_{n=2}^\infty$*. If* $\varphi \in \mathcal{M}(\mathscr{H}_{\mathbf{w}})$*, then* $\tilde{\varphi} : \mathbb{H}_{\frac{\sigma_{\mathbf{w}}-t}{2}} \to \mathbb{C}$ *defined by*

$$\tilde{\varphi}(s) = \varphi\Big(s + \frac{t}{2}\Big), \quad s \in \mathbb{H}_{\frac{\sigma_{\mathbf{w}}-t}{2}}$$

is a multiplier of $\mathscr{H}_{\tilde{\mathbf{w}}}$ *(see* (3)*).*

Proof. Let $j : \mathscr{H}_{\mathbf{w}} \to \mathscr{H}_{\tilde{\mathbf{w}}}$ be a map defined by

$$j(f)(s) = f\Big(s + \frac{t}{2}\Big), \quad s \in \mathbb{H}_{\frac{\sigma_{\mathbf{w}}-t}{2}}.$$

Since $\varphi \in \mathcal{M}(\mathscr{H}_{\mathbf{w}})$, for any $f \in \mathscr{H}_{\mathbf{w}}$,

$$\begin{aligned} \tilde{\varphi}(s) j(f)(s) &= \varphi\Big(s + \frac{t}{2}\Big) f\Big(s + \frac{t}{2}\Big) \\ &= (\varphi \cdot f)\Big(s + \frac{t}{2}\Big) = j(\varphi \cdot f)(s), \ s \in \mathbb{H}_{\frac{\sigma_{\mathbf{w}}-t}{2}}. \end{aligned}$$

As j is surjective, $\tilde{\varphi} \cdot g \in \mathscr{H}_{\tilde{\mathbf{w}}}$ for every $g \in \mathscr{H}_{\tilde{\mathbf{w}}}$. This concludes the proof. □

The following fact shows that the inclusion map $\mathcal{M}(\mathscr{H}_{\mathbf{w}}) \hookrightarrow H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$ is surjective (cf. [12, Theorem 4.5]).

Lemma 3.3. *If* (7) *holds, then* $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D} = \mathcal{M}(\mathscr{H}_{\mathbf{w}})$ *(as a set).*

Proof. Assume that (7) holds. In view of Theorem 2.1, it suffices to check that $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D} \subseteq \mathcal{M}(\mathscr{H}_{\mathbf{w}})$. To see that, let $\varphi \in H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$. Then $\tilde{\varphi} : \mathbb{H}_0 \to \mathbb{C}$ defined by $\tilde{\varphi}(s) = \varphi(s + \frac{\delta_{\mathbf{w}}}{2})$, $s \in \mathbb{H}_0$, belongs to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$. Hence, by [5, Theorem 3.1], $\tilde{\varphi}$ is a multiplier of $\mathscr{H}_{\mathbf{1}}$. Therefore, by [9, Theorem 5.21], there exists $c \geqslant 0$ such that

$$(c^2 - \tilde{\varphi}(s)\overline{\tilde{\varphi}(u)})\kappa_{\mathbf{1}}(s, u) \geqslant 0.$$

Let $\tilde{\mathbf{w}} = \{n^{-\delta_{\mathbf{w}}} w_n\}_{n=2}^\infty$. Then, by Lemma 3.1 and [9, Theorem 4.8],

$$(c^2 - \tilde{\varphi}(s)\overline{\tilde{\varphi}(u)})\kappa_{\tilde{\mathbf{w}}}(s, u) \geqslant 0.$$

Thus, by [9, Theorem 5.21], $\tilde{\varphi} \in \mathcal{M}(\mathscr{H}_{\tilde{\mathbf{w}}})$. An application of Lemma 3.2 now completes the proof. □

Proof of Theorem 2.3. By Lemma 3.3,

$$H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D} = \mathcal{M}(\mathscr{H}_{\mathbf{w}}) \text{ (as a set)}. \tag{11}$$

Hence, in view of Theorem 2.1, it is sufficient to check that

$$\|M_{\varphi,\mathbf{w}}\|_{\mathscr{H}_{\mathbf{w}}} \leqslant \|\varphi\|_{\infty, \mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}}, \quad \varphi \in H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}.$$

To see the above estimate, let $\varphi \in H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$. Note that $\tilde{\varphi} : \mathbb{H}_0 \to \mathbb{C}$, defined by $\tilde{\varphi}(s) = \varphi(s + \frac{\delta_{\mathbf{w}}}{2})$, $s \in \mathbb{H}_0$, belongs to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$. Therefore, by an application of [5, Theorem 3.1],

the multiplication operator $M_{\tilde{\varphi},\mathbf{1}}$ is bounded on $\mathscr{H}_{\mathbf{1}}$.

In particular, the Hilbert space adjoint $M^*_{\tilde{\varphi},\mathbf{1}}$ of $M_{\tilde{\varphi},\mathbf{1}}$ is well-defined as a bounded linear operator.

Note that Lemma 3.1 combined with (7) yields that the kernel η defined by (10), is positive semi-definite on $\mathbb{H}_\beta \times \mathbb{H}_\beta$, where $\beta = \frac{1}{2}\max\{\sigma_{\mathbf{w}} - \delta_{\mathbf{w}}, 1\}$. Hence, by [9, Theorem 2.14], there exists a reproducing kernel Hilbert space $\mathcal{H}(\eta)$ associated with η. Thus, by (4), for any $s, u \in \mathbb{H}_\beta$,

$$\langle \kappa_{\tilde{\mathbf{w}}}(\cdot, u), \kappa_{\tilde{\mathbf{w}}}(\cdot, s)\rangle_{\mathscr{H}_{\tilde{\mathbf{w}}}} = \langle \kappa_{\mathbf{1}}(\cdot, u) \otimes \eta(\cdot, u), \kappa_{\mathbf{1}}(\cdot, s) \otimes \eta(\cdot, s)\rangle_{\mathscr{H}_{\mathbf{1}} \otimes \mathcal{H}(\eta)}.$$

An application of the Lurking isometry Lemma (see [1, Lemma 2.18]) yields a linear isometry $V : \mathscr{H}_{\tilde{\mathbf{w}}} \to \mathscr{H}_{\mathbf{1}} \otimes \mathcal{H}(\eta)$ such that

$$V(\kappa_{\tilde{\mathbf{w}}}(\cdot, u)) = \kappa_{\mathbf{1}}(\cdot, u) \otimes \eta(\cdot, u), \quad u \in \mathbb{H}_\beta. \tag{12}$$

Further, since $\varphi \in \mathcal{M}(\mathscr{H}_{\mathbf{w}})$ (see (11)), by Lemma 3.2, $\tilde{\varphi} \in \mathcal{M}(\mathscr{H}_{\tilde{\mathbf{w}}})$. This implies that the multiplication operator $M_{\tilde{\varphi}, \tilde{\mathbf{w}}}$ is bounded on $\mathscr{H}_{\tilde{\mathbf{w}}}$, and hence, by [9, Corollary 5.22],

$$\begin{aligned} V^*(M^*_{\tilde{\varphi},\mathbf{1}} \otimes I)V(\kappa_{\tilde{\mathbf{w}}}(\cdot, u)) &\overset{(12)}{=} V^*(M^*_{\tilde{\varphi},\mathbf{1}} \otimes I)(\kappa_{\mathbf{1}}(\cdot, u) \otimes \eta(\cdot, u)) \\ &= \overline{\tilde{\varphi}(u)} V^*(\kappa_{\mathbf{1}}(\cdot, u) \otimes \eta(\cdot, u)) \\ &= M^*_{\tilde{\varphi},\tilde{\mathbf{w}}}(\kappa_{\tilde{\mathbf{w}}}(\cdot, u)), \quad u \in \mathbb{H}_\beta. \end{aligned}$$

Since $V^*(M^*_{\tilde{\varphi},\mathbf{1}} \otimes I)V$ and $M^*_{\tilde{\varphi},\tilde{\mathbf{w}}}$ are bounded linear operators on $\mathscr{H}_{\tilde{\mathbf{w}}}$,

$$V^*(M^*_{\tilde{\varphi},\mathbf{1}} \otimes I)V = M^*_{\tilde{\varphi},\tilde{\mathbf{w}}}.$$

Because V is an isometry, we obtain

$$\|M_{\tilde{\varphi},\tilde{\mathbf{w}}}\|_{\mathscr{H}_{\tilde{\mathbf{w}}}} \leqslant \|V^*\|\|M^*_{\tilde{\varphi},\mathbf{1}}\|\|V\| \leqslant \|\tilde{\varphi}\|_{\infty, \mathbb{H}_0} = \|\varphi\|_{\infty, \mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}},$$

which completes the proof. □

Remark 3.4. Let κ be given by

$$\kappa(z, w) = \sum_{j=0}^{\infty} a_j z^j \overline{w}^j.$$

Suppose that κ converges on $\mathbb{D} \times \mathbb{D}$. If $\{a_j\}_{j=0}^{\infty}$ is an increasing sequence of non-negative real numbers, then the multiplier algebra $\mathcal{M}(\mathscr{H}(\kappa))$ of $\mathscr{H}(\kappa)$ is isometrically isomorphic to $H^\infty(\mathbb{D})$. This can be shown using the arguments in the proof of Theorem 2.3 with the essential change that $\kappa_{\mathbf{1}}$ is replaced by the Szegö kernel $S : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$S(z, w) = \frac{1}{1 - z\overline{w}}, \quad z, w \in \mathbb{D}.$$

Proof of Corollary 2.5. Fix an integer $n \geqslant 2$, let $\prod_{m=1}^{\omega(n)} p_{i_m}^{r_m}$, $r_m \in \mathbb{Z}_+$, be the prime factorization of n. If $j \in \mathbb{Z}_+$ such that $j | n$, then

$$j = \prod_{m=1}^{\omega(n)} p_{i_m}^{s_{m,j}}, \quad (s_{m,j})_{m=1}^{\omega(n)} \in \prod_{m=1}^{\omega(n)} \mathbb{N}_{r_m}. \tag{13}$$

Hence, there exists a bijective map $\psi : \{j \in \mathbb{Z}_+ : j|n,\ j \geqslant 2\} \to \prod_{m=1}^{\omega(n)} \mathbb{N}_{r_m} - \{\mathbf{0}\}$ defined by $\psi(j) = (s_{m,j})_{m=1}^{\omega(n)}$. Therefore, by (13),

$$\sum_{\substack{j\geqslant 2\\ j|n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big) = \sum_{(s_m)_{m=1}^{\omega(n)} \in \operatorname{Im}(\psi)} \Big(\prod_{m=1}^{\omega(n)} p_{i_m}^{-\delta_{\mathbf{w}} s_m}\Big)\big(w_{\prod_{m=1}^{\omega(n)} p_{i_m}^{s_m}}\big)\mu\Big(\prod_{m=1}^{\omega(n)} p_{i_m}^{r_m - s_m}\Big). \tag{14}$$

Let $\tilde{\mathbf{w}}$ be the extension of $\mathbf{w}$ to $\mathbb{Z}_+$ by letting $w_1 = 0$. Since $\mathbf{w}$ is additive, $\tilde{\mathbf{w}}$ is additive. This together with (14) and the multiplicativity of μ gives

$$\begin{aligned}
\sum_{\substack{j\geqslant 2\\ j|n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big) &= \sum_{(s_m)_{m=1}^{\omega(n)} \in \operatorname{Im}(\psi)} \sum_{t=1}^{\omega(n)} \Bigg(w_{p_{i_t}^{s_t}}\Big(\prod_{m=1}^{\omega(n)} p_{i_m}^{-\delta_{\mathbf{w}} s_m}\Big)\mu\Big(\prod_{m=1}^{\omega(n)} p_{i_m}^{r_m - s_m}\Big)\Bigg) \\
&= \sum_{t=1}^{\omega(n)} \sum_{(s_m)_{m=1}^{\omega(n)} \in \operatorname{Im}(\psi)} \Bigg(w_{p_{i_t}^{s_t}}\Big(\prod_{m=1}^{\omega(n)} p_{i_m}^{-\delta_{\mathbf{w}} s_m}\Big)\Big(\prod_{m=1}^{\omega(n)} \mu(p_{i_m}^{r_m - s_m})\Big)\Bigg) \\
&\overset{(1)}{=} \sum_{t=1}^{\omega(n)} \sum_{(s_m)_{m=1}^{\omega(n)} \in \mathcal{U}_n} \Bigg(w_{p_{i_t}^{s_t}} \prod_{m=1}^{\omega(n)} \Big(p_{i_m}^{-\delta_{\mathbf{w}} s_m} \mu(p_{i_m}^{r_m - s_m})\Big)\Bigg),
\end{aligned}$$

where $\mathcal{U}_n = \prod_{m=1}^{\omega(n)} \{r_m - 1, r_m\}$. For any positive integer $t \leqslant \omega(n)$, let

$$T_t = \sum_{(s_m)_{m=1}^{\omega(n)} \in \mathcal{U}_n} \Bigg(w_{p_{i_t}^{s_t}} \prod_{m=1}^{\omega(n)} \Big(p_{i_m}^{-\delta_{\mathbf{w}} s_m} \mu(p_{i_m}^{r_m - s_m})\Big)\Bigg).$$

Then

$$\begin{aligned}
T_t &= \sum_{s_1 = r_1 - 1}^{r_1} \sum_{s_2 = r_2 - 1}^{r_2} \cdots \sum_{s_{\omega(n)} = r_{\omega(n)} - 1}^{r_{\omega(n)}} \Bigg(w_{p_{i_t}^{s_t}} \prod_{m=1}^{\omega(n)} \Big(p_{i_m}^{-\delta_{\mathbf{w}} s_m} \mu(p_{i_m}^{r_m - s_m})\Big)\Bigg) \\
&= \Big(\sum_{s_t = r_t - 1}^{r_t} p_{i_t}^{-\delta_{\mathbf{w}} s_t} w_{p_{i_t}^{s_t}} \mu(p_{i_t}^{r_t - s_t})\Big) \prod_{\substack{m=1\\ m\neq t}}^{\omega(n)} \Big(\sum_{s_m = r_m - 1}^{r_m} p_{i_m}^{-\delta_{\mathbf{w}} s_m} \mu(p_{i_m}^{r_m - s_m})\Big) \\
&= \big(p_{i_t}^{-\delta_{\mathbf{w}}(r_t - 1)} w_{p_{i_t}^{r_t - 1}} \mu(p_{i_t}) + p_{i_t}^{-\delta_{\mathbf{w}} r_t} w_{p_{i_t}^{r_t}} \mu(1)\big) \\
&\qquad \prod_{\substack{m=1\\ m\neq t}}^{\omega(n)} \big(p_{i_m}^{-\delta_{\mathbf{w}}(r_m - 1)} \mu(p_{i_m}) + p_{i_m}^{-\delta_{\mathbf{w}} r_m} \mu(1)\big)
\end{aligned}$$

$$\overset{(1)}{=} \quad p_{i_t}^{-\delta_{\mathbf{w}}(r_t-1)}(p_{i_t}^{-\delta_{\mathbf{w}}} w_{p_{i_t}^{r_t}} - w_{p_{i_t}^{r_t-1}}) \prod_{\substack{m=1 \\ m\neq t}}^{\omega(n)} (p_{i_m}^{-\delta_{\mathbf{w}}(r_m-1)}(p_{i_m}^{-\delta_{\mathbf{w}}} - 1)).$$

This combined with the assumptions $\delta_{\mathbf{w}} \leqslant 0$ and (9) (which imply that $p_{i_m}^{-\delta_{\mathbf{w}}} \geqslant 1$ and $p_{i_t}^{-\delta_{\mathbf{w}}} w_{p_{i_t}^{r_t}} \geqslant w_{p_{i_t}^{r_t-1}}$) shows that $T_t \geqslant 0$ for every positive integer $t \leqslant \omega(n)$. Since n is arbitrary, (7) is valid for all integers $n \geqslant 2$. An application of Theorem 2.3 now completes the proof. □

We provide an example illustrating Corollary 2.5.

Example 3.5. Let $\mathbf{w} = \omega$. Note that ω is additive and satisfies

$$\omega(p^j) = 1 \text{ for every prime } p \text{ and integer } j \geqslant 1. \tag{15}$$

By [13, Chap. I, Equations (1.6.1) and (1.6.2)],

$$\sum_{j=2}^{\infty} \omega(j) j^{-s} = \zeta(s) \sum_{j=1}^{\infty} p_j^{-s}, \quad s \in \mathbb{H}_1. \tag{16}$$

It follows from (15), (16) and [2, Theorem 1.13] that $\sigma_\omega = 1$. To compute δ_ω, consider any $\epsilon > 0$ and an integer $n \geqslant 1$,

$$\begin{aligned} \sum_{\mathbf{gpf}(j)\leqslant p_n} \omega(j) j^{-\epsilon} &= \sum_{j=1}^{n} \sum_{\substack{i_1,i_2,\dots,i_j=1 \\ i_s\neq i_t,\ s\neq t}}^{n} \left(\sum_{m_{i_1},\dots,m_{i_j}=1}^{\infty} j(p_{i_1}^{m_{i_1}} p_{i_2}^{m_{i_2}} \dots p_{i_j}^{m_{i_j}})^{-\epsilon} \right) \\ &= \sum_{j=1}^{n} \left(\sum_{\substack{i_1,i_2,\dots,i_j=1 \\ i_s\neq i_t,\ s\neq t}}^{n} \left(j \prod_{r=1}^{j} \sum_{m_{i_r}=1}^{\infty} p_{i_r}^{-\epsilon m_{i_r}} \right) \right) < \infty. \end{aligned} \tag{17}$$

In view of (15), $\sum_{\mathbf{gpf}(j)\leqslant p_n} \omega(j)$ diverges for all $n \geqslant 1$. This combined with (17) yields $\delta_\omega = 0$, and hence, (9) holds. Therefore, by Corollary 2.5, $\mathcal{M}(\mathscr{H}_\omega)$ is isometrically isomorphic to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$. Moreover,

$$\omega(pq) = 2 \overset{(15)}{\neq} \omega(p)\omega(q) \text{ for distinct primes } p \text{ and } q.$$

So, ω is not multiplicative. Also, because ω is not monotone, (5) does not hold.

4. Applications

In this section, we discuss some applications of Theorem 2.3. The first one recovers [12, Corollary 4.2].

Theorem 4.1. *Let $\mathbf{w} = \{w_j\}_{j=1}^{\infty}$ be an multiplicative function satisfying* (6)*. Then $\mathcal{M}(\mathscr{H}_{\mathbf{w}})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_{\frac{\delta_{\mathbf{w}}}{2}}) \cap \mathcal{D}$.*

Proof. Fix an integer $n \geqslant 2$, let $\prod_{m=1}^{\omega(n)} p_{i_m}^{r_m}$, $r_m \in \mathbb{Z}_+$, be the prime factorization of n. Since any divisor of n is given by $\prod_{m=1}^{\omega(n)} p_{i_m}^{s_m}$, $s_m \in \mathbb{N}_{r_m}$,

$$\begin{aligned}\sum_{\substack{j\geqslant 1\\ j|n}} j^{-\delta_{\mathbf{w}}} w_j \mu\Big(\frac{n}{j}\Big) &= \prod_{m=1}^{\omega(n)} \sum_{s_m=0}^{r_m} p_{i_m}^{-\delta_{\mathbf{w}} s_m} w_{p_{i_m}^{s_m}} \mu(p_{i_m}^{r_m - s_m}) \\ &\overset{(1)}{=} \prod_{m=1}^{\omega(n)} (p_{i_m}^{-\delta_{\mathbf{w}}(r_m-1)})(p_{i_m}^{-\delta_{\mathbf{w}}} w_{p_{i_m}^{r_m}} - w_{p_{i_m}^{r_m-1}}) \overset{(6)}{\geqslant} 0.\end{aligned}$$

Thus, (8) is valid for all integers $n \geqslant 2$ and is trivially true for $n = 1$. Hence by Theorem 2.3 and Remark 2.4(b), the conclusion now follows. □

As a consequence of the previous theorem, we recover [3, Theorem 3]. Indeed, for any $\alpha > 0$, consider $\mathbf{w}_\alpha = \{(\mathbf{d}(j))^\alpha\}_{j=1}^\infty$. Since $\mathbf{d}(n) = o(n^\epsilon)$ for every $\epsilon > 0$ (see [2, Equation (31)]), $\sigma_{\mathbf{w}_\alpha} = 1$ and $\delta_{\mathbf{w}_\alpha} = 0$. For any prime p and integer $j \in \mathbb{Z}_+$, $\mathbf{d}(p^j) = j+1$. Thus, (9) holds, and hence by Theorem 4.1, $\mathcal{M}(\mathscr{H}_{\mathbf{w}_\alpha})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$.

The following proposition yields a family of Hilbert spaces $\mathscr{H}_{\mathbf{w}}$ illustrating the main result.

Proposition 4.2. *Let $\mathbf{w} = \{w_j\}_{j=1}^\infty$ be a multiplicative function satisfying*

(*i*) $w_j = \mathcal{O}(j^\delta)$ *for every* $\delta > 0$,
(*ii*) *for each prime* p, $\{w_{p^j}\}_{j=0}^\infty$ *is increasing.*

If $\tilde{\mathbf{w}} = \{1 + w_j\}_{j=1}^\infty$, *then* $\mathcal{M}(\mathscr{H}_{\tilde{\mathbf{w}}})$ *is isometrically isomorphic to* $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$.

Proof. For a real number $r > 1$, let δ be a positive number such that $r > \delta + 1$. Then by (i),

$$\sum_{j=2}^\infty (1 + w_j) j^{-r} \leqslant \sum_{j=2}^\infty j^{-r} + C_\delta \sum_{j=2}^\infty j^{-(r-\delta)} < \infty, \tag{18}$$

which yields $\sigma_{\tilde{\mathbf{w}}} \leqslant 1$. Moreover, since $\sum_{j=2}^\infty j^{-1}$ diverges, $\sigma_{\tilde{\mathbf{w}}} = 1$. For any $\sigma > 0$, let δ be such that $0 < \delta < \sigma$. Then, [12, Lemma 3.2] combined with (i) yields

$$\sum_{\substack{j\geqslant 2\\ \mathbf{gpf}(j)\leqslant p_n}} w_j j^{-\sigma} \leqslant \sum_{\substack{j\geqslant 2\\ \mathbf{gpf}(j)\leqslant p_n}} j^{-(\sigma-\delta)} < \infty, \quad n \geqslant 1. \tag{19}$$

This implies $\delta_{\mathbf{w}} \leq 0$ and $\delta_{\tilde{\mathbf{w}}} \leq 0$. Moreover, since $\sum_{j=0}^\infty w_{2^j}$ diverges, $\delta_{\mathbf{w}} = \delta_{\tilde{\mathbf{w}}} = 0$. This together with (19) yields $\delta_{\mathbf{w}} = 0$. Hence, by (ii), (9) holds. Thus, by using the steps in the proof of Theorem 4.1, we obtain

$$\sum_{\substack{j\geqslant 1\\ j|n}} w_j \mu\Big(\frac{n}{j}\Big) \geqslant 0, \quad n \geqslant 1.$$

Combining this with [2, Theorem 2.1] gives

$$\sum_{\substack{j \geqslant 1 \\ j|n}} (1 + w_j)\mu\Big(\frac{n}{j}\Big) \geqslant 0, \quad n \geqslant 1.$$

The desired conclusion is now immediate from Theorem 2.3 together with Remark 2.4(b). □

Note that $\tilde{\mathbf{w}}$ defined above is never multiplicative. Moreover, Proposition 4.2 applies to $\mathbf{w} = \{\mathbf{d}(j)\}_{j=1}^{\infty}$. We conclude this paper by revealing that Theorem 2.3 also recovers [6, Theorem 1.11] for some particular sequences.

Example 4.3. Let $\mathbf{w}$ be given by (5). Then by [6, Equation (1.4)], $\sigma_{\mathbf{w}} \leqslant 1$. Since $\mathbf{w}$ is an increasing sequence, $\sigma_{\mathbf{w}} = 1$. So, $\kappa_{\mathbf{w}}$ converges absolutely on $\mathbb{H}_{1/2} \times \mathbb{H}_{1/2}$. For an integer $\alpha > 0$, consider the sequence $\mathbf{w}_\alpha = \{(\log(j))^\alpha\}_{j=2}^{\infty}$. By [6, Section 1],

$$(\log(j))^{-\alpha} = \int_{[0,\infty)} j^{-2\sigma} \frac{2^\alpha}{\Gamma(\alpha)} \sigma^{-1+\alpha} d\sigma, \quad j \geq 2,$$

where Γ denotes the Gamma function, and hence $\sigma_{\mathbf{w}_\alpha} = 1$. Also, combining [6, Equation (1.4)] with [12, Lemma 2] yields $\delta_{\mathbf{w}_\alpha} = 0$. In addition, by [4, Equation (3.16)], we obtain the non-negativity of the *Generalized von Mangoldt function*:

$$\sum_{\substack{j \geqslant 2 \\ j|n}} (\log(j))^\alpha \mu\Big(\frac{n}{j}\Big) \geqslant 0. \tag{20}$$

An application of Theorem 2.3 now shows that $\mathcal{M}(\mathscr{H}_{\mathbf{w}_\alpha})$ is isometrically isomorphic to $H^\infty(\mathbb{H}_0) \cap \mathcal{D}$.

The previous example yields that for any integer $\alpha > 0$, $\mathbf{w}_\alpha$ satisfies (20). In general, we don't know whether $\mathbf{w}$ satisfying (5) guarantees that

$$\sum_{\substack{j \geqslant n_0 \\ j|n}} w_j \mu\Big(\frac{n}{j}\Big) \geqslant 0, \quad n \geqslant n_0.$$

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References

[1] Agler, Jim; McCarthy, John E.; Young, Nicholas. Operator analysis — Hilbert space methods in complex analysis. Cambridge Tracts in Mathematics, 219. *Cambridge University Press, Cambridge,* 2020. xv+375 pp. ISBN: 978-1-108-48544-9. MR4411370, Zbl 1498.47001, doi: 10.1017/9781108751292. 329

[2] Apostol, Tom M. Introduction to analytic number theory. Undergraduate Texts in Mathematics. *Springer-Verlag, New York-Heidelberg,* 1976. xii+338 pp. MR0434929, Zbl 0335.10001. doi: 10.1007/978-1-4757-5579-4. 327, 331, 332, 333

[3] Bailleul, Maxime; Brevig, Ole F. Composition operators on Bohr–Bergman spaces of Dirichlet series. *Ann. Acad. Sci. Fenn. Math.* **41** (2016), no. 1, 129–142. MR3467701, Zbl 1359.47017, arXiv:1409.3017, doi: 10.5186/aasfm.2016.4104. 332

[4] Friedlander, John; Iwaniec, Henryk. Opera de cribro. American Mathematical Society Colloquium Publications, 57. American Mathematical Society, Providence, RI, 2010. xx+527 pp. MR2647984, Zbl 1226.11099, doi: http://dx.doi.org/10.1090/coll/057. 333

[5] Hedenmalm, Håkan; Lindqvist, Peter; Seip, Kristian. A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$. *Duke Math. J.* **86** (1997), no. 1, 1–37. MR1427844, Zbl 0887.46008, arXiv:math/9512211, doi: 10.1215/S0012-7094-97-08601-4. 326, 328

[6] McCarthy, John E. Hilbert spaces of Dirichlet series and their multipliers. *Trans. Amer. Math. Soc.* **356** (2004), no. 3, 881–893. MR1984460, Zbl 1039.30001, doi: 10.1090/S0002-9947-03-03452-4. 326, 333

[7] McCarthy John E.; Shalit, Orr M. Spaces of Dirichlet series with the complete Pick property *Israel J. Math.* **220** (2017), no. 2, 509–530. MR3666434, Zbl 1482.46030, arXiv:1507.04162, doi: 10.1007/s11856-017-1527-6. 327

[8] Olsen, Jan-Fredrik. Local properties of Hilbert spaces of Dirichlet series *J. Funct. Anal.* **261** (2011), no. 9, 2669-2696. MR2826411, Zbl 1236.46022, arXiv:1011.3370, doi: 10.1016/j.jfa.2011.07.007. 326

[9] Paulsen, Vern I.; Raghupathi, Mrinal. An introduction to the theory of reproducing kernel Hilbert spaces. Cambridge Studies in Advanced Mathematics, 152. *Cambridge University Press, Cambridge,* 2016. x+182 pp. ISBN: 978-1-107-10409-9. MR3526117, Zbl 1364.46004, doi: 10.1017/CBO9781316219232. 325, 328, 329

[10] Queffelec, Hervé; Queffelec, Martine. Diophantine approximation and Dirichlet series. Second edition. Texts and Readings in Mathematics, 80. *Hindustan Book Agency, New Delhi; Springer, Singapore*, 2020. xvii+287 pp. ISBN: 978-981-15-9350-5; 978-981-15-9351-2; 978-93-86279-82-8. MR4241378, Zbl 1478.11002, doi: 10.1007/978-981-15-9351-2. 324

[11] Stein, Elias M.; Shakarchi Rami. Complex analysis. Princeton Lectures in Analysis, 2. *Princeton University Press, Princeton, NJ*, 2003. xviii+379 pp. ISBN: 0-691-11385-8. MR1976398, Zbl 1020.30001. 326

[12] Stetler, Eric B. Multiplication operators on Hilbert spaces of Dirichlet series. Thesis (Ph.D.) - University of Florida, 2014. *ProQuest LLC, Ann Arbor, MI*, 102 pp. ISBN: 978-1321-42187-3. MR3321769, https://www.proquest.com/docview/1645427826. 325, 326, 327, 328, 331, 332, 333

[13] Titchmarsh, Edward C. The theory of the Riemann zeta-function, Second edition. *The Clarendon Press, Oxford University Press, New York,* 1986. x+412 pp. ISBN: 0-19-853369-1. MR0882550, Zbl 0601.10026. 324, 331

(Chaman Kumar Sahu) Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India
chamanks@iitk.ac.in

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