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A REPRESENTATION OF INFINITELY DIVISIBLE SIGNED RANDOM MEASURES

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Abstract: The study of the measurable space of signed Radon measures on a metric space is carried, establishing results of characterization of distributions of signed random measures which generalize similar results about the nonnegative case. These results enable the study of a Lévy–Khintchine type characterization for infinite divisible signed random measures.

1 – Introduction

There has been a wide interest in the study of random measures in the sense of random variables taking values in the space of non negative Radon measures defined on some separable, complete and locally compact metric space, with it's natural σ-field induced by the vague topology. This space of measures with the vague topology has very nice properties in what concerns the study of convergence and compactness characterizations. In fact, it is well known that this space is metrizable and separable, so providing a convenient setting to the treatment of random measures. A basic reference is the book by Kallenberg [4] where it is possible to find an exposition of the basic theory about random measures as well as a more complete list of references. However, if we turn our attention to the space of signed Radon measures the references are rare. It is well known that this space is, in general, not metrizable nor separable, what is obviously a source of problems that did not appear in the non negative setting. An important reference in the treatment of the vague topology of the signed Radon measures space is Varadarajan [5]. The aim of this paper is the study of some basic results concerning random measures with values in the signed Radon measures space, namely the identification of classes of functions inducing the Borel

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σ-algebra, which then enables the proof of uniqueness characterizations parallel to the non negative case. Once these questions resolved we turn our attention to another classical problem: the representation of infinitely divisible random measures, giving a Kolmogorov type representation of the characteristic function.

In what follows $\mathbf{S}$ represents a separable, complete and locally compact metric space, $\mathcal{B}$ the ring of bounded Borel subsets of $\mathbf{S}$ and $C_c(\mathbf{S})$ the space of real valued continuous functions defined on $\mathbf{S}$ with a compact support. Furthermore, $\mathcal{M}$ stands for the space of signed Radon measures on $\mathbf{S}$, that is, $\mu \in \mathcal{M}$ if and only if $|\mu|$, the total variation of μ, is a non negative Radon measure on $\mathbf{S}$. On $\mathcal{M}$ we consider the vague topology, that is, the topology induced by the functions $\pi_f(\mu) = \int f d\mu = \mu f$, with $f \in C_c(\mathbf{S})$, we will represent by $\mathcal{T}$ the Borel σ-algebra on $\mathcal{M}$ induced by the vague topology. Finally, $\mathcal{M}^+$ will denote the cone of non negative Radon measures on $\mathbf{S}$ and $\mathcal{T}^+$ the trace σ-algebra of $\mathcal{T}$ on $\mathcal{M}^+$. Evidently $\mathcal{T}^+$ is the Borel σ-algebra induced by the vague topology on $\mathcal{M}^+$.

2 – The measurable space of signed measures

Most of characterizations of $\mathcal{M}^+$ valued random measures depend on the fact that the σ-algebra $\mathcal{T}^+$ is induced by the family of functions π_f, $f \in C_c(\mathbf{S})$ or by the family $\pi_B(\mu) = \mu(B)$, $B \in \mathcal{B}$. This fact is evident as $\mathcal{M}^+$ with the vague topology is separable, so it's Baire and Borel σ-algebras coincide. In $\mathcal{M}$, as we no longer have separability this is not, at least, immediate. It is a simple matter that $\sigma\{\pi_f,\ f \in C_c(\mathbf{S})\} \subset \sigma\{\pi_B,\ B \in \mathcal{B}\}$. The other inclusion is not obvious, but still holds. In fact, let C be any compact set of $\mathbf{S}$ it is possible to find a sequence of functions f_n in $C_c(\mathbf{S})$ such that $f_n \downarrow 1\!\!1_C$. Then, the dominated convergence theorem, applied to the positive and negative variations of any measure $\mu \in \mathcal{M}$, gives $\int f_n \longrightarrow \mu(C)$ so it follows that π_C is measurable with respect to $\sigma\{\pi_f,\ f \in C_c(\mathbf{S})\}$ for any compact set C. From here the $\sigma\{\pi_f,\ f \in C_c(\mathbf{S})\}$-measurability of π_B for every $B \in \mathcal{B}$ follows using the $\pi - \lambda$ device.

Denote $\mathcal{T}_1 = \sigma\{\pi_f,\ f \in C_c(\mathbf{S})\} = \sigma\{\pi_B,\ B \in \mathcal{B}\}$. As we do not have separability all we can state for the moment is $\mathcal{T}_1 \subset \mathcal{T}$. It is interesting to note that the traces of these σ-algebras coincide not only on $\mathcal{M}^+$, which gives the characterizations proved in [4], but also in some other subspaces of $\mathcal{M}$. Define, for each $k > 0$, $\mathcal{M}^k = \{\mu \in \mathcal{M} \colon |\mu|(\mathbf{S}) \le k\}$. Then it is possible to prove that $\mathcal{M}^k$ is metrizable, separable and even compact with respect to the vague topology. Using this fact it follows easily that the traces of $\mathcal{T}_1$ and $\mathcal{T}$ coincide on each $\mathcal{M}^k$.

In order to prove that $\mathcal{T}_1$ and $\mathcal{T}$ are in fact the same σ-algebra we will need to prove the measurability of some functions. We will denote by μ^+ and μ^- the positive and negative variations of a given measure $\mu \in \mathcal{M}$, respectively.

Lemma 1. *Let $B \in \mathcal{B}$ be fixed. The functions $\pi_B^-(\mu) = \mu^-(B)$, $\pi_B^+(\mu) = \mu(B)$ and $|\pi_B|(\mu) = |\mu|(B)$ are $\mathcal{T}$-measurable.*

Proof: Given that π_B is a $\mathcal{T}$-measurable function it is enough to prove the $\mathcal{T}$-measurability of $|\pi_B|$, and for this we will verify that $|\pi_B|$ is lower semi-continuous. Moreover, it suffices to prove the measurability when B is an open set. Using the same procedure as in the proof of theorem II [2] we may, for each $\varepsilon > 0$ and $\mu \in \mathcal{M}$ construct a function $f \in C_c(\mathbf{S})$ depending only on μ with values in $[-1,+1]$, and being zero outside B, such that

$$\Big|\int f\,d\mu\Big| > |\mu|(B) - \frac{2\varepsilon}{3}\ .$$

Consider the set $\{\nu \in \mathcal{M}\colon\ |\int f\,d\mu - \int f\,d\nu| < \frac{\varepsilon}{3}\}$, this is a vague neighbourhood of μ, such that for each ν in this neighbourhood we have

$$|\nu|(B) \geq \Big|\int f\,d\nu\Big| > \Big|\int f\,d\mu\Big| - \frac{\varepsilon}{3} > |\mu|(B) - \varepsilon$$

that is, $|\pi_B|(\nu) \geq |\pi_B|(\mu) - \varepsilon$, so $|\pi_B|$ is lower semi-continuous. ∎

Lemma 2. *Let $f \in C_c(\mathbf{S})$ be fixed. The functions $\pi_f^-(\mu) = \mu^- f$, $\pi_f^+(\mu) = \mu^+ f$ and $|\pi_f|(\mu) = |\mu| f$ are $\mathcal{T}_1$-measurable (so, also $\mathcal{T}$-measurable).*

Proof: Again it suffices to prove the $\mathcal{T}_1$-measurability of $|\pi_f|$ and it is enough to check for f positive. For this notice that we may put

$$|\pi_f|(\mu) = \sup\bigg\{\Big|\int h\,d\mu\Big|,\ \ 0 \leq |h| \leq f,\ \ h \in C_c(\mathbf{S})\bigg\}\ .$$

Putting $C_f = \{h \in C_c(\mathbf{S})\colon\ 0 \leq |h| \leq f\}$ we define a separable subspace of $C_c(\mathbf{S})$ and, using the separability of $C_c(\mathbf{S})$, we may find a sequence h_n in $C_c(\mathbf{S})$, independent of the function f such that

$$|\pi_f|(\mu) = \sup_{n\in\mathbb{N}} \Big|\int h_n\,d\mu\Big|\ ,$$

so the $\mathcal{T}_1$-measurability follows. ∎

These two lemmas enable the proof of the measurability of the Hahn-Jordan decomposition.

Theorem 3. *The functions $H^-(\mu) = \mu^-$, $H^+(\mu) = \mu^+$ and $H^{||}(\mu) = |\mu|$ defined on $\mathcal{M}$ with values on $\mathcal{M}^+$ are $\mathcal{T}_1$-measurable and $\mathcal{T}$-measurable.*

Proof: To check the $\mathcal{T}$-measurability of H^+, for example, decompose π_B^+ in the following way

$$
\begin{tikzcd}
(\mathcal{M},\mathcal{T}) \arrow[rr, "\mathrm{H}^+"] & & (\mathcal{M}^+,\mathcal{T}^+) \arrow[dd, "\pi^+_{B|_{\mathcal{M}^+}}"] \\
\mu \arrow[r] \arrow[dr, "\pi^+_B"'] & \mu^+ \arrow[d] & \\
& \mu^+(\mathrm{B}) & (\mathbb{R},\mathcal{B}(\mathbb{R}))
\end{tikzcd}
$$

As π_B^+ defined on $(\mathcal{M}^+, \mathcal{T}^+)$ is measurable and the diagram commutes, it follows the $\mathcal{T}$-measurability of H^+. For the $\mathcal{T}_1$-measurability it is enough to remark that the trace of $\mathcal{T}_1$ on $\mathcal{M}^+$ is also $\mathcal{T}^+$. ∎

At this point we will define two families of random measures according to the σ-algebra we use.

Definition 4. A $\mathcal{T}$-random measure is a measurable function defined on some probability space with values in $(\mathcal{M}, \mathcal{T})$.
A $\mathcal{T}_1$-random measure is a measurable function defined on some probability space with values in $(\mathcal{M}, \mathcal{T}_1)$.

It is evident that every $\mathcal{T}$-random measure is a $\mathcal{T}_1$-random measure as $\mathcal{T}_1 \subset \mathcal{T}$. Remark that if we are interested on non negative random measures we do not have to distinguish between these two types of random measures as $\mathcal{T}^+ = \mathcal{T}_1^+$.

Theorem 5. *Every $\mathcal{T}$-random measure ξ may be written as the difference of two non negative random measures.*
The same result holds for $\mathcal{T}_1$-random measures.

Proof: From theorem 3 it follows that $H^+ \circ \xi$ is measurable, so ξ^+ is a non negative random measure. The same holds for $H^- \circ \xi = \xi^-$. Evidently $\xi = \xi^+ - \xi^-$. The proof for $\mathcal{T}_1$-random measures is done analogously. ∎

Theorem 6. *The difference of two non negative random measures is both a $\mathcal{T}_1$-random measure and a $\mathcal{T}$-random measure.*

Proof: If ξ and η are non negative random measures then (ξ, η) is a random variable with values on $(\mathcal{M}^+ \otimes \mathcal{M}^+, \mathcal{T}^+ \otimes \mathcal{T}^+)$. As $d(\mu_1, \mu_2) = \mu_1 - \mu_2$ is a continuous function the $\mathcal{T}$-measurability of $d(\xi, \eta) = \xi - \eta$ follows. As every $\mathcal{T}$-random measure is a $\mathcal{T}_1$-random measure the theorem is proved. ∎

These two last theorems togethter give the following corollary.

Corollary 7. *Every $\mathcal{T}_1$-random measure is a $\mathcal{T}$-random measure.*

And finally we may prove the equality of the σ-algebras defined on $\mathcal{M}$.

Theorem 8. $\mathcal{T}_1$=$\mathcal{T}$.

Proof: We have to prove that $\mathcal{T} \subset \mathcal{T}_1$. Define the $\mathcal{T}_1$-random measure on the probability space $(\mathcal{M}, \mathcal{T}_1, \mathbf{P})$ as the identity function, where $\mathbf{P}$ is any probability measure. According to the preceeding corollary this is also a $\mathcal{T}$-random measure, from what follows that $\mathcal{T} \subset \mathcal{T}_1$. ∎

3 – Identification of random measures

From what was proved before it follows that in fact we have only one family of random measures and, moreover, that $\mathcal{T} = \sigma\{\pi_f,\ f \in C_c(\mathbf{S})\} = \sigma\{\pi_B,\ B \in \mathcal{B}\}$. The identification of these families that induce the Borel σ-algebra $\mathcal{T}$ permits the following statement characterizing the distribution of a random measure. Before stating the result define the characteristic function of a random measure by

$$F_\xi(f) = \mathbf{E}\left(e^{i\xi f}\right) \ ,$$

for $f \in C_c(\mathbf{S})$.

Theorem 9. *Suppose that ξ and η are random measures. The following conditions are equivalent.*

1. *ξ and η have the same distribution.*

2. *For each $f \in C_c(\mathbf{S})$, ξf and ηf have the same distribution.*

3. *$F_\xi(f) = F_\eta(f)$, for every $f \in C_c(\mathbf{S})$.*

4. *$(\xi(B_1), \ldots, \xi(B_k))$ and $(\eta(B_1), \ldots, \eta(B_k))$ have the same distribution for each choice $B_1, \ldots, B_k \in \mathcal{B}$ and $k \in \mathbb{N}$.*

Another problem is suggested by condition 4. Given a family of probability distributions $\{\mathbf{P}_{B_1,\ldots,B_k}\}$, where $\mathbf{P}_{B_1,\ldots,B_k}$ is a distribution on $\mathbb{R}^k$, what conditions should we impose in order to assure that there exists a random measure ξ such that $\mathbf{P}_{B_1,\ldots,B_k}$ is the distribution of $(\xi(B_1), \ldots, \xi(B_k))$? Besides the compatibility in the sense of Kolmogorov's theorem of the family of probability distributions we must assure that the stochastic process defined is σ-additive in order to define a random measure.

Let $\mathcal{D}$ be a countable basis of the topology of $\mathbf{S}$ that we will suppose closed under finite unions and intersections, and $\mathcal{D} \subset \mathcal{B}$. Denote by $\mathcal{A}$ the ring induced by $\mathcal{D}$. This ring $\mathcal{A}$ is still countable and $\mathcal{A} \subset \mathcal{B}$. Given a family of probability distributions $\{\mathbf{P}_{B_1,\ldots,B_k}\}$ for $B_1,\ldots,B_k \in \mathcal{A}$ and $k \in \mathbb{N}$, compatible in the sense of Kolmogorov's theorem in order to have the existence of the stochastic process indexed by $\mathcal{A}$, consider the conditions

(**M1**) $\forall\, A, B \in \mathcal{A},\ A \cap B = \emptyset, \quad \mathbf{P}_{A,B,A\cup B}\,\{(x,y,z)\colon\ x+y=z\} = 1,$

(**M2**) $\forall \{A_n\} \subset \mathcal{A},\ A_n \downarrow \emptyset,\ \forall\, t > 0, \quad \lim\limits_{n\to+\infty} \mathbf{P}\Big\{ \sup\limits_{A\subset A_n} |\xi(A)| \in [0,t] \Big\} = 1.$

These conditions are obviously necessary if ξ is to define a random measure. Now following the proof of proposition 1.3 of Jagers [3] we may prove the result.

Theorem 10. *Suppose* $\mathbf{S}$ *is compact and* $\mathbf{S} \in \mathcal{D}$. *If the given family of probability distributions* $\{\mathbf{P}_{B_1,\ldots,B_k}\}$ *is compatible in the sense of Kolmogorov's theorem and verifies* $(M1)$ *and* $(M2)$ *then there exists a random measure* ξ *such that the distribution of* $(\xi(B_1),\ldots\xi(B_k))$ *is* $\mathbf{P}_{B_1,\ldots,B_k}$ *for every* $B_1,\ldots,B_k \in \mathcal{A}$ *and* $k \in \mathbb{N}$.

Finally, to solve the general case, we recall that there exists an increasing sequence of compact sets K_n such that $S = \bigcup_{n=1}^{+\infty} K_n$. Applying the preceeding theorem to each K_n we will define a random measure on a compact that gets as close to $\mathbf{S}$ as we need with the required properties.

Corollary 11. *The preceeding theorem holds for any complete, separable and locally compact metric space* $\mathbf{S}$ *provided that* $\mathcal{D}$ *contains the increasing sequence of compacts* $\{K_n\}$.

Note that if only $(M1)$ is verified we can construct a finitely additive random measure.

4 – Infinite divisibility

We turn now our attention to the characterization of infinite divisible random measures, proving a general expression for the characteristic function of random measures ξ such that $\xi \otimes \xi$ is integrable on $\mathbf{S} \times \mathbf{S}$. Let us denote $\pi_{B_1,\ldots,B_k}(\mu) = (\mu(B_1),\ldots,\mu(B_k))$ for every $B_1,\ldots,B_k \in \mathcal{B}$ and $k \in \mathbb{N}$.

Theorem 12. *Let* ξ *be an infinitely divisible random measure such that* $\mathbf{E}(\xi \otimes \xi)$ *is a Radon measure on* $\mathbf{S} \times \mathbf{S}$. *Then there exists a finitely additive* *measure* λ *on* $(\mathcal{M},\mathcal{T})$ *verifying*

$$\int_{\mathbb{R}^k} \|x\|^2\, \lambda\, \pi^{-1}_{B_1,\ldots,B_k}(dx) < \infty\,, \qquad B_1,\ldots,B_k \in \mathcal{B},\;\; k\in\mathbb{N}\;,$$

and $\alpha\in\mathcal{M}$ *such that*

$$\mathbf{E}\left(e^{i\xi f}\right) = \exp\Big[i\,\alpha f + \int_{\mathcal{M}} e^{i\mu f} - 1 - i\mu f\,\lambda(d\mu)\Big]\;, \tag{1}$$

for each $f\in C_c(\mathbf{S})$.

Proof: We shall suppose that $\mathbf{E}(\xi)=0$ in order to simplify the notations. As $\mathbf{E}(\xi\otimes\xi)$ is a Radon measure on $\mathbf{S}\times\mathbf{S}$ it follows that, for each $B_1,\ldots,B_k\in\mathcal{B}$, and $k\in\mathbb{N}$, $\mathbf{E}\|(\xi(B_1),\ldots,\xi(B_k))\|^2<\infty$, where $\|\cdot\|$ stands for the euclidean norm of $\mathbb{R}^k$. So, Kolmogorov's representation of the characteristic function of $(\xi(B_1),\ldots,\xi(B_k))$ states the existence of a finite non negative measure on $\mathbb{R}^k$, $\gamma_{B_1,\ldots,B_k}$ with zero mass at origin, such that

$$\mathbf{E}\left[e^{it\cdot(\xi(B_1),\ldots,\xi(B_k))}\right] = \exp\Big[\int_{\mathbb{R}^k} \frac{e^{it\cdot x}-1-it\cdot x}{\|x\|^2}\,\gamma_{B_1,\ldots,B_k}(dx)\Big]\,, \qquad t\in\mathbb{R}^k\;.$$

If we define $\lambda_{B_1,\ldots,B_k}(dx) = \frac{1}{\|x\|^2}\,\gamma_{B_1,\ldots,B_k}(dx)$, then obviously

$$\mathbf{E}\left[e^{it\cdot(\xi(B_1),\ldots,\xi(B_k))}\right] = \exp\Big[\int_{\mathbb{R}^k} e^{it\cdot x} - 1 - it\cdot x\,\lambda_{B_1,\ldots,B_k}(dx)\Big] \tag{2}$$

and

$$\int_{\mathbb{R}^k} \|x\|^2\,\lambda_{B_1,\ldots,B_k}(dx) < \infty\;.$$

If in (2) we choose all t_j equal to 1 but one that is equal to 0 we deduce the compatibility of the family of probability measures $\{\lambda_{B_1,\ldots,B_k}\}$ in the sense of Kolmogorov's theorem. In fact, suppose $t_j=1$, $j=1,\ldots,k-1$, and $t_k=0$. Then we get

$$\int_{\mathbb{R}^k} e^{i\sum_{j=1}^{k-1} t_j x_j} - 1 - i\sum_{j=1}^{k-1} t_j x_j\;\lambda_{B_1,\ldots,B_k}(dx_1\cdots dx_k) =$$

$$= \int_{\mathbb{R}^{k-1}} e^{i\sum_{j=1}^{k-1} t_j x_j} - 1 - i\sum_{j=1}^{k-1} t_j x_j\;\lambda_{B_1,\ldots,B_{k-1}}(dx_1\cdots dx_{k-1})\;,$$

the exponentials of the integrals being the characteristic functions of the probability measures $s-Pois(\lambda_{B_1,\ldots,B_k}\,\pi^{-1}_{B_1,\ldots,B_{k-1}})$ and $s-Pois(\lambda_{B_1,\ldots,B_{k-1}})$, respectively (see, Araujo, Gine [1]) from what follows $\lambda_{B_1,\ldots,B_{k-1}} = \lambda_{B_1,\ldots,B_k}\,\pi^{-1}_{B_1,\ldots,B_{k-1}}$, that is, the compatibility of the family.

Now take $A, B \in \mathcal{B}$ such that $A \cap B = \emptyset$. Then, as ξ is a random measure, so additive we have the equality of the real parts of the exponents of the characteristic functions of $(\xi(A), \xi(B), \xi(A \cup B))$ and $(\xi(A), \xi(B))$ computed at convenient points

$$\int_{\mathbb{R}^3} \cos(s\,x_1 + t\,x_2 + u\,x_3) - 1\;\lambda_{A,B,A\cup B}(dx_1\,dx_2\,dx_3) =$$
$$= \int_{\mathbb{R}^2} \cos\Big[(s+u)\,x_1 + (t+u)\,x_2\Big] - 1\;\lambda_{A,B}(dx_1 dx_2)\ .$$

The compatibility of $\{\lambda_{B_1,\ldots,B_k}\}$ then implies

$$\int_{\{x_3 \neq x_1 + x_2\}} \cos(s\,x_1 + t\,x_2 + u\,x_3) - 1\;\lambda_{A,B,A\cup B}(dx_1\,dx_2\,dx_3) = 0\ ,$$

so, $\lambda_{A,B,A\cup B}\,(\{x_3 \neq x_1 + x_2\} \cap \{s x_1 + t x_2 + u x_3 \neq 2k\pi,\ k \in \mathbb{Z}\}) = 0$ for every fixed $s, t, u \in \mathbb{R}$, as the integrand on this set is a strictly negative function. Choosing four triplets (s_i, t_i, u_i), $i = 1,2,3,4$ such that

$$\bigcup_{i=1}^{4} \Big\{ s_i\,x_1 + t_i\,x_2 + u_i\,x_3 \neq 2k\pi,\ k \in \mathbb{Z} \Big\} = \mathbb{R}^3 \backslash \{0\}$$

it follows that $\lambda_{A,B,A\cup B}\,(\{x_3 \neq x_1 + x_2\}) = 0$ as $\lambda_{A,B,A\cup B}$ has mass zero at the origin, that is, the family $\{\lambda_{B_1,\ldots,B_k}\}$ verifies $(M1)$. The choice of the triplets (s_i, t_i, u_i), $i = 1,2,3,4$ should be made such that the linear system

$$s_i\,x_1 + t_i\,x_2 + u_i\,x_3 = 2k_i\pi\,, \qquad i = 1,2,3,4\ ,$$

has no solution for any choice of $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. This can be obtained choosing real numbers a, b, c, d linearly independent over $\mathbb{N}$, and setting

$$\begin{bmatrix} s_1 & t_1 & u_1 \\ s_2 & t_2 & u_2 \\ s_3 & t_3 & u_3 \\ s_4 & t_4 & u_4 \end{bmatrix} = \begin{bmatrix} b & 0 & 0 \\ -a & -\frac{c}{d} & 0 \\ 0 & 1 & -\frac{d}{c} \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, in order to use theorem 10 and corollary 11 we need a family of finite measures. Recall the above mentioned sequence of compacts of $\mathbf{S}$, (K_n), such that $\mathbf{S} = \bigcup_n K_n$, and for each $n \in \mathbb{N}$, define

$$\lambda'_{n,B_1,\ldots,B_k}(A) = \int_{\mathbb{R}\times A} x^2\,\lambda_{K_n,B_1,\ldots,B_k}(dx\,dx_1 \cdots dx_k)\,, \qquad A \in \mathcal{B}(\mathbb{R}^k)\ ,$$

so $\lambda'_{n,B_1,\ldots,B_k}(\mathbb{R}^k) < +\infty$ and, for each $n \in \mathbb{N}$ fixed, the family $\left\{\lambda'_{n,B_1,\ldots,B_k}\right\}$ has the same properties as $\{\lambda_{B_1,\ldots,B_k}\}$. So we may construct a finitely additive measure λ'_n on the set of Radon measures on K_n. Put

$$\lambda_n(d\mu) = \frac{1}{\mu^2(K_n)}\,\lambda'_n(d\mu) \;.$$

This way we define a finitely additive measure with marginal distributions $\lambda_{B_1,\ldots,B_k}$:

$$\begin{aligned}\lambda_n\,\pi^{-1}_{B_1,\ldots,B_k}(A) &= \int_{\{(\mu(B_1),\ldots,\mu(B_k)\in A\}} \frac{1}{\mu^2(K_n)}\,\lambda'_n(d\mu)\\ &= \int_{\mathbb{R}\times\mathbb{R}\times A} y^2\,\frac{1}{x^2}\,\lambda_{K_n,K_n,B_1,\ldots,B_k}(dy\,dx\,dx_1\cdots dx_k) = \lambda_{B_1,\ldots,B_k}(A)\end{aligned}$$

from the fact that $\lambda_{K_n,K_n,B_1,\ldots,B_k}$ is concentrated on the set of vectors with first coordinate equal to the second and using the compatibility. As K_n is increasing to $\mathbf{S}$ from (λ_n) we find a finitely additive measure λ on $\mathcal{M}$.

Remark now that the term $i\alpha f$ in (1) appears taking account of the simplification we made during the proof, that is, $\mathbf{E}(\xi) = 0$. ∎

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