PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 1 – 1999

CONTINUOUS NORMS ON LOCALLY CONVEX SPACES

Vítor Neves *

Presented by J.P. Dias

Abstract: Given a locally convex space E with nonstandard extension ${}^*E$ in a polysaturated model of Analysis, we distinguish *very large infinite* and *very small infinitesimal* elements of ${}^*E$, show that *E is normable if and only if the former do not exist* (Theorem 3.1) and show that the *existence of continuous norms on E is a necessary condition for validity of Inverse Function Theorems* (Theorem 2.2). We use a stronger version of the embedding of standard sets in hyperfinite sets (Lemma 4.1).

1 – Introduction

All we need from the basic Theory of Infinitesimals and non-standard extensions ${}^*(\cdot)$ is contained in [4] and a good introduction to this subject can be found in [3]; nevertheless we describe some notation and terminology for the reader's convenience. As for the general Theory of Locally Convex Spaces we use [1].

The set of standard elements of an extension ${}^*X$ is denoted ${}^\sigma X$; we assume that our non-standard models are *polysaturated* ([4, 7.6]); $\mathbb{K}$ denotes any of the scalar fields $\mathbb{R}$ — of real numbers — or $\mathbb{C}$ — of complex numbers. A scalar is *finite* if its absolute value is bounded above by *some* standard hyperreal number, and is *infinitesimal* if its absolute value is bounded above by *all* standard positive hyperreal numbers. Nonfinite scalars are called *infinite*.

Let E denote a vector space over $\mathbb{K}$ and Γ be a right directed family of seminorms making E a locally convex space. We say that a vector $v \in {}^*E$ is *finite* if $\gamma(v)$ is finite in ${}^*\mathbb{R}$ for every standard seminorm $\gamma \in {}^\sigma\Gamma$ and write $fin({}^*E)$ for the set of finite vectors; $\mathcal{O}$ denotes the set of finite scalars. The vector $v \in {}^*E$

Received: March 7, 1997; *Revised*: October 23, 1997.
AMS Subject Classification: 46A03, 46B99, O3H05.
Keywords: locally convex space, polysaturated model, finite, infinitesimal, perturbation.
* This work was partially supported by scholarship 10/c/92/PO of JNICT-INVOTAN and Projecto PRAXIS XXI de Física Matemática, 1996.

is *infinitesimal* if $\gamma(v)$ is infinitesimal in ${}^*\mathbb{R}$, for all $\gamma \in {}^\sigma\Gamma$. We write $v \approx w$ when $v - w$ is infinitesimal and say that *v and w are infinitely near*. The set of infinitesimal vectors is denoted $\mu({}^*E)$.

If $\gamma \in \Gamma$, γ is a norm and v is a non zero vector, then ${}^*\mathbb{R}\gamma(v) = {}^*\mathbb{R}$, but, as shown in the following example, one cannot guarantee that ${}^*\mathbb{C}v$ contains finite non infinitesimal vectors: it may happen that scalar multiples of an infinite vector are either infinite or infinitesimal; one might say that such a vector is "abnormaly" large.

Example 1.1. Say E is the space of entire functions, with the usual gauge of norms γ_n $(n \in \mathbb{N})$ given by

$$\gamma_n(f) = \max\Big\{|f(z)|\colon\ |z| \le n\Big\}$$

and define

$$v(z) = e^{\Omega z}\ (z \in \mathbb{C}) \qquad \text{where}\ \ 0 < \Omega \in {}^*\mathbb{R}\backslash\mathcal{O}\ .$$

The function v is infinite, for $\gamma_1(v) = e^{\Omega}$. Moreover, if $0 \ne \lambda = e^{a+ib} \in {}^*\mathbb{C}$ we have

$$\gamma_n(\lambda v) = e^{a+\Omega n} \qquad (n \in \mathbb{N})\ .$$

We show that for any $\lambda \in {}^*\mathbb{C}$,

$$\lambda v \in fin({}^*E)\ \Rightarrow\ \lambda v \in \mu({}^*E)\ .$$

Suppose λv is finite. If, for some $N \in {}^\sigma\mathbb{N}$ there was $r \in {}^\sigma\mathbb{R}$ such that $a+\Omega N > r$, then $a+\Omega(N+1) > (r+\Omega) \in {}^*\mathbb{R}^+\backslash\mathcal{O}$ so λv would be infinite; hence $a+\Omega n \le r$, for all $r \in {}^\sigma\mathbb{R}$ and all $n \in {}^\sigma\mathbb{N}$, in particular $e^{a+\Omega n} \approx 0$ for all standard n, i.e., λv is infinitesimal.

On the other direction: an infinitesimal vector v may be so small that its scalar multiples λv $(\lambda \in {}^*\mathbb{K})$ are always infinitesimal.

Example 1.2 Let $\mathcal{C}(\mathbb{R})_c$ denote the space of continuous real functions of one real variable with the topology of compact convergence. The scalar multiples of a function $f \in {}^*\mathcal{C}(\mathbb{R})_c$ such that $f^{-1}(0) \supseteq \mathcal{O}$ are all infinitesimal, since they all have standard seminorms zero.

The existence of very large or very small vectors does point to particular standard properties of the locally convex space, as we intend to show from section 2 on.

With the exception of Lemma 4.1 on polysaturation and the **Main Theorem** 3.1, we postpone proofs of lemmas, corollaries and other theorems to section 4.

2 – Very small infinitesimal vectors

From now on E denotes a non trivial locally convex space with gauge of continuous seminorms Γ.

A vector $v \in {}^*E$ is a *very small infinitesimal* if all its scalar multiples λv ($\lambda \in {}^*\mathbb{K}$) are infinitesimal.

The following theorem shows that Example 1.2 actually describes very small infinitesimals:

Theorem 2.1. *Given a locally convex space E, the very small infinitesimal vectors of* ${}^*E$ *are the elements of* $\bigcap\{\gamma^{-1}(0)\colon\ \gamma \in {}^\sigma\Gamma\}$; *therefore* ${}^*E$ *has non zero such vectors if and only if its gauge of continuous seminorms* Γ *does not contain norms.*

Proof: Just observe that zero is the only hyperreal number whose hyperreal multiples are all infinitesimal. ∎

These non zero very small elements also have to do with Inverse Function Theorems: if they exist one cannot expect the set of linear homeomorphisms between a pair of locally convex spaces to be open, even in the weak sense of convergence structures ([2]), as we shall now explain.

Given internally linear maps $\phi, \psi \in {}^*L(E,E)$, we say that ϕ is an *infinitesimal perturbation of* ψ if they differ infinitesimally on finite elements, i.e., $\phi(v) \approx \psi(v)$ whenever $v \in fin({}^*E)$.

Theorem 2.2. *If a locally convex space has very small infinitesimal vectors, i.e., it does not admit continuous norms, then there exists an infinitesimal perturbation of the identity function which is not injective.*

Proof. Given a very small infinitesimal vector $v \in {}^*E$, Transfer of [1, 7.2.2c] implies that the internally finite dimensional space ${}^*\mathbb{K}v$ has an internal topological supplement F, i.e., ${}^*E$ is the internal topological direct sum ${}^*\mathbb{K}v \oplus F$. The projection $\pi_F\colon {}^*E \to F$ is an internally continuous infinitesimal perturbation of the identity. ∎

In other words, the existence of continuous norms is a necessary condition for the preservation of homeomorphy with nearness.

Large vectors have a stronger relation to norms.

3 – Ultra infinite vectors

Definition 3.1. A vector $v \in {}^*E$ is *ultra infinite* if v is infinite and ${}^*\mathbb{K}v \cap fin({}^*E) \subseteq \mu({}^*E)$.

The following examples essentially describe ultra infinite vectors.

Example 3.1. $E = \mathbb{R}^{\mathbb{N}}$ with pointwise convergence. Take an internal sequence of infinite Ω_n such that

$$0 < \frac{\Omega_{n+1}}{\Omega_n} \in {}^*\mathbb{N} \backslash {}^\sigma\mathbb{N} \qquad (n \in \mathbb{N})$$

— e.g. (Ω^n) with infinite $\Omega \in {}^*\mathbb{N}$ — This sequence is ultra infinite.

Going a little further:

Example 3.2. If E is the countable product of normed spaces $(E_n, \|\cdot\|_n)$, $E = \prod_{n=1}^{\infty} E_n$, then for all $r > 0$, there exists a sequence (v_n), such that

$$\forall\, n \in \mathbb{N} \quad \Big[0 \neq v_n \in E_n \ \wedge\ \|v_{n+1}\|_{n+1} \geq r\|v_n\|_n \geq r\Big] .$$

Therefore, by Transfer ([4]), there exists an internal sequence $v = (v_n) \in {}^*E$ such that

$$\|v_{n+1}\|_{n+1} \geq \Omega\|v_n\|_n \geq \Omega \qquad (n \in {}^*\mathbb{N})$$

with Ω a fixed infinite positive hyperreal. The vector v is ultra infinite.

And we shall generalize these ideas in order to prove our **Main Theorem**:

Theorem 3.1. *A locally convex space E is normable if and only if ${}^*E$ does not contain ultra infinite vectors.*

By a *normable space* we mean a locally convex space whose topology is definable by a single norm.

As we observed above, if E admits a continuous norm $\|\cdot\|$ say, then $\{\|\lambda v\| : \lambda \in {}^*\mathbb{K}\} = {}^*[0, +\infty[$ whenever $v \neq 0$, thus ${}^*E$ contains no ultra infinite vectors. This shows that the "only if" statement holds.

For the proof of the "if" part, recall the following from [1, 6.8.5].

Theorem 3.2. *Every locally convex space is a dense subspace of a reduced projective limit of Banach spaces.*

Then consider a refinement of Example 3.2:

Theorem 3.3. *Let A be an infinite set and E_α a non trivial semi normed space, for each $\alpha \in A$. The product ${}^*E = {}^*\prod_{\alpha\in A} E_\alpha$ has ultra infinite elements.*

Now observe that:

Lemma 3.1. *If v is a ultra infinite vector of ${}^*E$ and $w \approx v$, then w is also ultra infinite.*

And conclude:

Corollary 3.1. *If E is the reduced projective limit of a system $((E_i)_{i\in J}; (T_{j,k})_{j\le k})$ of (semi) normed spaces and E is not (semi) normable, then E has ultra infinite vectors.*

This together with Lemma 3.1 proves the **Main Theorem** 3.1 for, if E is not normable the corresponding reduced limit is of course not normable.

4 – Proofs

We need the following consequence of polysaturation

Lemma 4.1. *If A is an infinite set, there exists a hyperfinite set $\mathcal{C} = \{\alpha_1, \alpha_2, ..., \alpha_\omega\}$ such that the following two conditions hold.*

1) ${}^\sigma A \subseteq \mathcal{C} = \{\alpha_1, \alpha_2, ..., \alpha_\omega\} \subseteq {}^*A$, *for some* $\omega \in {}^*\mathbb{N}$.

2) *For all standard α_i, there exists a standard α_j such that $i < j$.*

Observe that the indices in assertion 2 above need not be standard.

Proof of Lemma 4.1: Take A well ordered without last element. A set $\mathcal{C}$ verifying 1 exists because our model is polysaturated ([4, 7.6.2]). If necessary, *internally* reorder $\mathcal{C}$ so that the hyperfinite order given by the indices in ${}^*\mathbb{N}$ is induced by the extended order on ${}^*A$; in so doing the order on $\mathcal{C}$ induces the original order in ${}^\sigma A$ too. Now, if $\alpha_i \in {}^\sigma A$, there exists $b \in {}^\sigma A$, such that $\alpha_i < b$; by 1, b must be some α_j and, as the order in ${}^\sigma A$ and in (the indices of) $\mathcal{C}$ agree, one must have $i < j$. ∎

We have all we need from Model Theory.

Proof of Theorem 3.3: Since each E_α is non trivially semi normed, the semi norms γ_α on E_α $(\alpha \in A)$ have the following property:

(**P**) Given $n\in\mathbb{N}$, $\{\alpha_1, ..., \alpha_n\} \subseteq A$ and $r \in\,]0, +\infty[$, there exists $(v_{\alpha_1}, ..., v_{\alpha_n}) \in \prod_{i=1}^n E_{\alpha_i}$, such that

$$\gamma_{\alpha_{i+1}}(v_{\alpha_{i+1}}) \geq r\, \gamma_{\alpha_i}(v_{\alpha_i}) > r \qquad (1 \leq i \leq n-1)\ .$$

By Lemma 4.1, there exists a strictly hyperfinite set such that

$${}^\sigma\! A \subseteq \mathcal{C} = \{\alpha_1, ..., \alpha_n, ..., \alpha_\omega\} \subseteq {}^*\! A\ .$$

If $\alpha \in {}^\sigma\! A$ and $\alpha = \alpha_i$, then there exists $j > i$ such that $\alpha_j \in {}^\sigma\! A$. Pick an infinite positive hyperreal number Ω. By Transfer of the above property (**P**), there exists $(v_{\alpha_1}, ..., v_{\alpha_\omega}) \in \prod_{i=1}^\omega E_{\alpha_i}$, such that

$$\gamma_{\alpha_{i+1}}(v_{\alpha_{i+1}}) \geq \Omega\, \gamma_{\alpha_i}(v_{\alpha_i}) > \Omega \qquad (1 \leq i \leq \omega-1)\ .$$

${}^*\! A\backslash\mathcal{C}$ is internal, therefore the vector $v = (v_\alpha; \alpha \in {}^*\! A)$, given below is in ${}^*\! E$.

$$v_\alpha = \begin{cases} v_\alpha & \text{if } \alpha \in \mathcal{C}, \\ 0 & \text{if } \alpha \in {}^*\! A\backslash\mathcal{C}\ . \end{cases}$$

Now, as $\mathcal{C} \supseteq {}^\sigma\! A$, v is infinite. Suppose λv is finite for some scalar λ. Then for each standard α, there exists a finite positive hyperreal number k_α, such that $\gamma_\alpha(\lambda v_\alpha) \leq k_\alpha$, so that if $\alpha = \alpha_i$, then $i < \omega$ and there also exists j such that $i < j < \omega$, α_j is standard and

$$\Omega^{j-i}\, \gamma_{\alpha_i}(\lambda v_{\alpha_i}) \leq \gamma_{\alpha_j}(\lambda v_{\alpha_j}) \leq k_{\alpha_j} \in \mathcal{O}\ ;$$

it follows that

$$\gamma_\alpha(\lambda v_\alpha) \approx 0 \qquad (\alpha \in {}^\sigma\! A)$$

i.e. λv is infinitesimal. As λ was arbitrary, v is ultra infinite. ∎

Proof of Lemma 3.1: Say v is ultra infinite and $v \approx w$, so that

$$\gamma(v) \approx \gamma(w) \qquad (\gamma \in {}^\sigma\Gamma)\ . \tag{1}$$

If γ_0 is a standard semi norm such that $\gamma_0(v)$ is infinite, then $\gamma_0(w)$ is infinite too, by (1) and w itself is infinite. If $\lambda \in {}^*\mathbb{K}$ and $\gamma(\lambda w)$ is finite, for *every* standard seminorm γ, then $|\lambda|$ *must be infinitesimal* — otherwise $\gamma_0(\lambda w)$ would be infinite — and, by (1), $\gamma(\lambda v)$ is also finite for *all* standard seminorms γ, forcing $\gamma(\lambda w)$ to be infinitesimal too, again by (1), because v is ultra infinite. ∎

Proof of Corollary 3.1: Let π_j denote the projection into the j-th factor of $E = \prod_{i\in J} E_i$. Since E is not (semi) normable we may assume

$$\pi_j(E) \neq \{0\} \quad (j \in J) \ . \tag{2}$$

In particular $E_j \neq \{0\}$ $(j \in J)$ so that, by Theorem 3.3, ${}^*F$ has a ultra infinite vector say y. Let $\mathcal{J}$ be a hyperfinite set such that ${}^\sigma J \subseteq \mathcal{J} \subseteq {}^*J$. Recall that J is directed and pick $k \in {}^*J$ such that $j \leq k$ for all $j \in \mathcal{J}$.

As the $T_{jk}\colon E_k \to E_j$ are internally linear continuous for all $j \in {}^*J$, we have

$$\gamma_j(x_j) \leq M_{jk}\, \gamma_k(x_k) \quad (x = (x_j) \in {}^*E;\ j \in \mathcal{J})$$

and, by (2), we may assume all the $M_{jk} > 0$.

Choose a positive infinitesimal number ε such that $M_{jk}\, \varepsilon \approx 0$ $(j \in \mathcal{J})$. Since E is reduced, we may pick $x \in {}^*E$ such that $\gamma_k(x_k - y_k) < \varepsilon$. For each $j \in {}^\sigma J$ we have

$$\gamma_j(x_j - y_j) \leq M_{jk}\, \gamma_k(x_k - y_k) < \varepsilon\, M_{jk} \approx 0$$

because ${}^\sigma J \subseteq \mathcal{J}$. Therefore $x \approx y$ and, by Lemma 3.1, x is also ultra infinite. ∎

ACKNOWLEDGEMENTS – The author thanks Keith Stroyan, Peter Loeb and Ward Henson for helpful remarks and encouragment.

REFERENCES

[1] Jarchow, H. – *Locally Convex Spaces*, Teubner, Stuttgart 1981.

[2] Neves, V. – Infinitesimal Calculus in HM-spaces*, Bulletin de la Societé Mathématique de Belgique,* 40(2), Ser. B (1988).

[3] Stroyan, K.D. – *The infinitesimal rule of three*, in "Developments in Nonstandard Mathematics" (Nigel J. Cutland, Vítor Neves, Franco Oliveira and José Sousa Pinto, Eds.), Pitman Research Notes in Math. 336, 1995.

[4] Stroyan, K.D. and Luxemburg, W.A.J. – *Introduction to the Theory of Infinitesimals*, Acad. Press, Mathematics 72, 1976.

Vítor Neves,
Dep. de Matemática, Univ. de Aveiro,
3810 Aveiro – PORTUGAL
E-mail: v_neves@mat.ua.pt