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POSITIVE SOLUTIONS FOR SEMIPOSITONE (n, p) BOUNDARY VALUE PROBLEMS *

Xiaoming He and Weigao Ge

Abstract: This paper is concerned with the existence of positive solutions to the (n, p) boundary value problem

$$\begin{aligned} &u^{(n)} + \lambda f(t,u) = 0\,, && 0 < t < 1\ ,\\ &u^{(i)}(0) = 0\,, && 0 \le i \le n-2\ ,\\ &u^{(p)}(1) = 0\,, && 1 \le p \le n-1\ , \end{aligned}$$

where p is fixed and $\lambda > 0$. We shall use a fixed point theorem in a cone to obtain positive solutions of the above problem for λ on a suitable interval.

1 – Introduction

Let $n \ge 2$ and $1 \le p \le n-1$ be fixed, in this paper we study the existence of positive solutions to the (n, p) boundary value problem

$$\begin{aligned} &u^{(n)} + \lambda f(t,u) = 0\,, && 0 < t < 1\ ,\\ &u^{(i)}(0) = 0\,, && 0 \le i \le n-2\ ,\\ &u^{(p)}(1) = 0\ , \end{aligned} \tag{1}$$

where $f\colon\ [0,1]{\times}[0,\infty) \to R$ is continuous and satisfies:

(H1) there exists $M > 0$ such that

$$f(t,u) \ge -M, \qquad \text{for}\ \ (t,u) \in [0,1]{\times}[0,\infty)\ ,$$

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(**H2**) let

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = \infty$$

uniformly on a compact subinterval $[\alpha,\beta]$ of $[0,1]$.

Recently, the (n,p) boundary value problems have been given extensive attention to the existence of positive solutions, for some excellent results, we refer to R.P. Agarwal, D. O'Regan and V. Lakshmikantham [1, 2], R.P. Agarwal and D. O'Regan [3], P.J.Y. Wong and R.P. Agarwal [4], etc. The key condition they employed was that the nonlinearity f is nonnegative. In the case $n = 2$, if the nonlinearity f is nonnegative, then the positive solution u is concave down. If the nonlinearity f is negative somewhere, then the concavity is no longer kept. The purpose of this paper is to establish sufficient conditions which ensure the existence of positive solutions of (1) for λ on a suitable interval, the nonlinearity f is allowed to be negative somewhere.

2 – The preliminary lemmas

In order to prove our main results, we first present several useful lemmas, which are fundamental in our arguments. The first three lemmas are derived from the recent literature [1–3]. The fourth one is due to Krasnosel'skii [10, 11].

Let $G(t,s)$ be the Green's function for

$$\begin{aligned} &-u^{(n)} = 0\,, && 0 < t < 1\,,\\ &u^{(i)}(0) = 0\,, && 0 \le i \le n-2\,,\\ &u^{(p)}(1) = 0\,. && \end{aligned} \tag{2}$$

Recalling [2] we see that $G(t,s)$ can be expressed explicitly as

$$G(t,s) = \begin{cases} \dfrac{1}{(n-1)!}\Big[t^{n-1}(1-s)^{n-p-1} - (t-s)^{n-1}\Big]\,, & 0 \le s \le t\,,\\ \dfrac{1}{(n-1)!}\Big[t^{n-1}(1-s)^{n-p-1}\Big]\,, & t \le s \le 1\,. \end{cases}$$

Lemma 1 (see [2]). *For $(t,s) \in [0,1]\times[0,1]$, then*

$$G(t,s) \le G(1,s) = \frac{1}{(n-1)!}\Big[(1-s)^{n-p-1} - (1-s)^{n-1}\Big]\,.$$

Lemma 2 (see [1, 3]). *Suppose $u \in C^{n-1}[0,1] \cap C^n(0,1)$ satisfies*

$$\begin{aligned}
&u^{(n)}(t) \le 0\,, && t \in [0,1]\ ,\\
&u(0) = a \ge 0\ ,\\
&u^{(i)}(0) = 0\,, && 1 \le i \le n-2\ ,\\
&u^{(p)}(1) = 0\ .
\end{aligned}$$

Then

$$u(t) \ge q(t)\,\|u\|, \qquad \text{for} \;\; t \in [0,1]\ ;$$

where $\|u\| = \sup_{t\in[0,1]} |u(t)|$, $q(t) = t^{n-1}$.

Lemma 3. *Let $\overline{w}(t)$ be the solution of the boundary value problem*

$$\begin{aligned}
&u^{(n)}(t) = -1\,, && 0 < t < 1\ ,\\
&u^{(i)}(0) = 0\,, && 0 \le i \le n-2\ ,\\
&u^{(p)}(1) = 0\ .
\end{aligned} \tag{3}$$

Then, there exists a positive constant Γ such that $\overline{w}(t) \le \Gamma q(t)$ for every $t \in [0,1]$, where $\Gamma = \frac{1}{(n-1)!\,(n-p)}$, $q(t)$ is defined in Lemma 2.

Proof: It is easy to see that if $\overline{w}(t)$ is a solution of (3), then

$$\begin{aligned}
\overline{w}(t) &= \int_0^t G(t,s)\,ds + \int_t^1 G(t,s)\,ds\\
&= \int_0^t \frac{1}{(n-1)!}\Big[t^{n-1}(1-s)^{n-p-1} - (t-s)^{n-1}\Big]\,ds\\
&\quad + \int_t^1 \frac{1}{(n-1)!}\Big[t^{n-1}(1-s)^{n-p-1}\Big]\,ds\\
&\le \int_0^t \frac{1}{(n-1)!}\,t^{n-1}(1-s)^{n-p-1}\,ds + \int_t^1 \frac{1}{(n-1)!}\Big[t^{n-1}(1-s)^{n-p-1}\Big]\,ds\\
&= \frac{t^{n-1}}{(n-1)!}\int_0^1 (1-s)^{n-p-1}\,ds\\
&= \frac{1}{(n-1)!\,(n-p)}\,t^{n-1}\\
&= \Gamma\,q(t)\ .
\end{aligned}$$

Lemma 4 (see [10, 11]). *Let E be a Banach space and $P \subset E$ be a cone in E. Suppose Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon\; P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

be a completely continuous operator such that either

(**a**) $\|Tu\| \le \|u\|,\ u \in P \cap \partial\Omega_1,$ *and* $\|Tu\| \ge \|u\|,\ u \in P \cap \partial\Omega_2;$ *or*

(**b**) $\|Tu\| \ge \|u\|,\ u \in P \cap \partial\Omega_1,$ *and* $\|Tu\| \le \|u\|,\ u \in P \cap \partial\Omega_2.$

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 – Main Results

We now turn our attention to the problem (1) with f possibly negative. We have the following result:

Theorem 1. *Assume (H1), (H2) hold. Then problem (1) has at least one positive solution if $\lambda > 0$ is small enough.*

Proof: Let $w(t) = \lambda\, M\, \overline{w}(t)$, where $\overline{w}(t)$ is defined in Lemma 3. In view of $q(t) = t^{n-1} \le 1$ on $[0,1]$, it follows from Lemma 3 that

$$w(t) \le \lambda\, M\, \Gamma\ , \tag{4}$$

for all $t \in [0,1]$. Thus $u_1(t)$ is a positive solution of (1) if and only if $\widetilde{u}(t) = u_1(t) + w(t)$ is a solution of the boundary value problem

$$\begin{aligned} &u^{(n)}(t) = -\lambda\, g(t, u(t) - w(t))\,, && 0 < t < 1\ , \\ &u^{(i)}(0) = 0\,, && 0 \le i \le n-2\ , \\ &u^{(p)}(1) = 0\ , \end{aligned} \tag{5}$$

with $\widetilde{u}(t) > w(t)$ on $(0,1)$, where

$$g(t,u) = \begin{cases} f(t,u) + M\,, & (t,u) \in [0,1] \times [0,\infty)\ , \\ f(t,0) + M\,, & (t,u) \in [0,1] \times (-\infty, 0)\ . \end{cases}$$

Then $g(t,u)$ is a nonnegative continuous function on $[0,1] \times R$.

Let $P = \{u \mid u \in C[0,1],\ u(t) \ge q(t)\|u\|,\ t \in [0,1]\}$, where $q(t)$ is defined in Lemma 2. Clearly, P is a cone. If $u(t)$ is a solution of problem (5), then $u(t)$ satisfies the integral equation

$$u(t) \;=\; \lambda \int_0^1 G(t,s)\, g(s, u(s) - w(s))\, ds\ .$$

Now define the operator T on P by

$$Tu(t) \;=\; \lambda \int_0^1 G(t,s)\, g(s, u(s) - w(s))\, ds\ .$$

From Lemma 2 and Ascoli's Lemma, it is easy to verify that $T\colon P \to P$ is completely continuous.

Let

$$\lambda \in (0, k)$$

be fixed, where

$$k \;=\; \min\left\{\frac{1}{\Gamma M}, \frac{1}{M_1 \int_0^1 G(1,s)\, ds}\right\} . \tag{6}$$

and $M_1 = \max\{g(t,u) \mid 0 \le t \le 1,\; 0 \le u \le 1\}$.

Take $\Omega_1 = \{u \in C[0,1]\colon \|u\| < 1\}$. Then for $u \in P \cap \partial\Omega_1$, by (6) and Lemma 1 we have

$$\begin{aligned} Tu(t) \;&=\; \lambda \int_0^1 G(t,s)\, g(s, u(s) - w(s))\, ds \\ &\le\; \lambda\, M_1 \int_0^1 G(t,s)\, ds \\ &\le\; \lambda\, M_1 \int_0^1 G(1,s)\, ds \;\le\; 1 . \end{aligned}$$

Hence,

$$\|Tu\| \le \|u\| , \qquad u \in P \cap \partial\Omega_1 . \tag{7}$$

Now denote

$$\overline{q} \;=\; \min_{\alpha \le t \le \beta} q(t) \;=\; \alpha^{n-1} . \tag{8}$$

We choose real number $N > 0$ such that

$$\frac{\lambda\, N\, \overline{q}}{2} \int_\alpha^\beta G\left(\frac{\alpha+\beta}{2}, s\right) ds \;\ge\; 1 . \tag{9}$$

Taking $R > 1$ large enough, then by (H2) we have

$$\frac{g(t,h)}{h} \ge N, \qquad \text{for} \;\; t \in [\alpha, \beta], \quad h \ge \frac{R\,\overline{q}}{2} \tag{10}$$

and

$$1 - \frac{\lambda M \Gamma}{R} \;\ge\; \frac{1}{2} . \tag{11}$$

Let $\Omega_2 = \{u \in C[0,1] : \|u\| < R\}$. Then for $u \in P \cap \partial\Omega_2$, it follows from Lemma 2 and Lemma 3 that

$$w(t) = \lambda M \overline{w}(t) \le \lambda M \Gamma q(t) \le \lambda M \Gamma \frac{u(t)}{\|u\|} = \frac{\lambda M \Gamma}{R} u(t) . \tag{12}$$

Thus,

$$u(t) - w(t) \ge \left(1 - \frac{\lambda M \Gamma}{R}\right) u(t) , \quad t \in [0,1] . \tag{13}$$

It follows from (11), (13) and Lemma 2 that

$$u(t) - w(t) \ge \frac{1}{2} u(t) \ge \frac{1}{2} \|u\| q(t) \ge \frac{1}{2} R \overline{q} , \quad t \in [\alpha, \beta] . \tag{14}$$

This together with (10) yields

$$g(t, u(t) - w(t)) \ge N\Big(u(t) - w(t)\Big) \ge \frac{N R \overline{q}}{2} , \quad t \in [\alpha, \beta] . \tag{15}$$

Therefore, from (9) we have

$$\begin{aligned} Tu\left(\frac{\alpha+\beta}{2}\right) &= \lambda \int_0^1 G\left(\frac{\alpha+\beta}{2}, s\right) g(s, u(s) - w(s)) \, ds \\ &\ge \lambda \int_\alpha^\beta G\left(\frac{\alpha+\beta}{2}, s\right) \frac{N R \overline{q}}{2} \, ds \\ &\ge R = \|u\| , \end{aligned}$$

for $u \in P \cap \partial\Omega_2$. Hence,

$$\|Tu\| \ge \|u\| , \quad u \in P \cap \partial\Omega_2 . \tag{16}$$

It follows from (7), (16) and Lemma 4 that there exists $\widetilde{u} \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $T\widetilde{u}(t) = \widetilde{u}(t)$ and $\|\widetilde{u}\|$ is between 1 and R. Moreover, in view of (6), Lemma 2 and Lemma 3, we know that

$$R \ge \widetilde{u}(t) > \|\widetilde{u}\| q(t) \ge \lambda M \Gamma q(t) \ge \lambda M \overline{w}(t) = w(t) , \tag{17}$$

for $t \in (0,1)$.

Hence $u_1(t) = \widetilde{u}(t) - w(t)$ is a positive solution of (1) for $\lambda \in (0, k)$. This completes the proof.

Theorem 2. *Let (H1) hold. Assume that the following conditions are satisfied:*

(H3) *$f(t,u)+M \le F(u)$ on $[0,1]\times[0,\infty)$ with $F>0$ continuous and non-decreasing on $[0,\infty)$.*

(H4) *there exist positive constants r,k such that*

$$\frac{x}{k\,F(x)\displaystyle\int_0^1 G(1,s)\,ds} > 1, \qquad \textit{for } \; x \ge r \;.$$

(H5) *there exists a continuous function $\psi\colon\, [0,1]\to[0,\infty)$ such that*

$$f(t,u)+M \ge \psi(t), \qquad \textit{on } \;\; [1/4,3/4]\times[0,k\Gamma(M{+}1)] \;,$$

and

$$\int_{1/4}^{3/4} G\Big(\frac{1}{2},s\Big)\,\psi(s)\,ds \;\ge\; \Gamma(M+1) \;.$$

Then (1) has a positive solution for $\lambda\in(0,k]$.

Proof: Let $w(t)$, $g(t,u)$, P and T be as in the proof of Theorem 1. Thus (12), (13) hold. Moreover, $u_1(t)$ is a positive solution of (1) if and only if $\widetilde{u}(t)=u_1(t)+w(t)$ is a solution of the boundary value problem (5). So, the task for us to do is prove that T has a fixed point $\widetilde{u}(t)\in P$ with $\widetilde{u}(t)>w(t)$ on $[0,1]$, then $u(t)=\widetilde{u}(t)-w(t)$ is a positive solution of (1).

Let $\lambda\in(0,k]$ and choose $\eta>\max\{\lambda\,\Gamma(M+1),r\}$. Furthermore, set

$$\Omega_3 = \Big\{u\in C[0,1]\colon\; \|u\|<\eta\Big\}$$

and

$$\Omega_4 = \Big\{u\in C[0,1]\colon\; \|u\|<\lambda\,\Gamma(M{+}1)\Big\} \;.$$

Then, for $u\in P\cap\partial\Omega_3$, we have from (13) and (H3) that

$$\begin{aligned} Tu(t) \;&=\; \lambda\int_0^1 G(t,s)\,g(s,u(s){-}w(s))\,ds \\ &\le\; \lambda\int_0^1 G(t,s)\,F(u(s)-w(s))\,ds \;. \end{aligned} \tag{18}$$

In view of $0<u(s)-w(s)<\eta$ for $s\in[0,1]$, we have, using (H3), (H4), that

$$F(u(s)-w(s)) \;\le\; F(\eta) \;<\; \frac{\eta}{k\displaystyle\int_0^1 G(1,s)\,ds} \;. \tag{19}$$

Combining (18) and (19) implies

$$Tu(t) \le \eta = \|u\|, \quad \text{for all} \quad t \in [0,1] \ .$$

Hence,

$$\|Tu\| \le \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_3 \ . \tag{20}$$

On the other hand, by Lemma 2 we have that

$$\begin{aligned} \Gamma\, k(M+1) \ &\ge\ \lambda\,\Gamma(M+1) \ \ge\ u(t) \ \ge\ u(t) - w(t) \\ &\ge\ \|u\|\, q(t) - \lambda\, M\, \overline{w}(t) \ \ge\ \lambda\,\Gamma\, q(t) \ \ge\ 0 \ , \end{aligned} \tag{21}$$

for $u \in \partial\Omega_4$. Combining (21) and (H5) gives

$$\begin{aligned} Tu\Big(\frac{1}{2}\Big) \ &=\ \lambda \int_0^1 G\Big(\frac{1}{2}, s\Big)\, g(s, u(s){-}w(s))\, ds \\ &\ge\ \lambda \int_{1/4}^{3/4} G\Big(\frac{1}{2}, s\Big)\, g(s, u(s){-}w(s))\, ds \\ &\ge\ \lambda \int_{1/4}^{3/4} G\Big(\frac{1}{2}, s\Big)\, \psi(s) \ \ge\ \lambda\,\Gamma(M{+}1) \ . \end{aligned}$$

Thus,

$$\|Tu\| \ge \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_4 \ . \tag{22}$$

By (20), (22) and the second part of Lemma 4, there exists $\widetilde{u} \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $T\widetilde{u}(t) = \widetilde{u}(t)$ and $\|\widetilde{u}\|$ is between $\lambda\,\Gamma(M{+}1)$ and η. By (4) and (21), we know that $\widetilde{u}(t) > w(t)$ on $[0,1]$, and so $u(t) = \widetilde{u}(t) - w(t)$ is a positive solution of (1) for $\lambda \in (0,k]$. This completes the proof. ∎

Finally, we present two examples to explain our main results. In what follows we will see that Theorem 1 is suitable to Example 1, but invalid to Example 2.

Example 1. Consider the boundary value problem (1) with $n = 2,\ p = 1$, i.e.,

$$\begin{aligned} &u'' + \lambda\, f(t,u) = 0\,, \qquad 0 < t < 1 \ , \\ &u(0) = u'(1) = 0\,, \end{aligned} \tag{23}$$

where $f(t,u) = t^{10} u^2 - 10\, t^2 \sin u \ge -10 = -M$ for $t \in [0,1]$ and $u \ge 0$.

It is clear that the Green's function of (23) is

$$G(t,s) = \begin{cases} s\,, & 0 \le s \le t, \\ t\,, & t \le s \le 1\ . \end{cases}$$

After some simple calculation, we have $\overline{w}(t) = t - t^2/2$, $\Gamma = 1$, $\int_0^1 G(1,s)\,ds = 1/2$. Moreover, $M_1 \le 11$, $f(t,u)$ satisfies

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = \infty \quad \text{uniformly on each compact subset of (0,1)}\ .$$

Hence, by Theorem 1, we see that (23) has at least one positive solution for

$$0 < \lambda < \min\left\{ \frac{1}{M_1 \int_0^1 G(1,s)\,ds}, \frac{1}{\Gamma M} \right\} \le \frac{1}{\Gamma M} = \frac{1}{10}\ . \ \square$$

Example 2. Consider the following boundary value problem

$$\begin{aligned} &u'' + \lambda\, f(t,u) = 0\,, \qquad 0 < t < 1\ , \\ &u(0) = u'(1) = 0\ , \end{aligned} \tag{24}$$

where $f(t,u) = 100\,t\,\sqrt{u+1} - 9\,t\cos u \ge -9 = -M$, for $t \in [0,1]$ and $u \ge 0$.

It is easy to see that $\overline{w}(t) = t - t^2/2$, $\Gamma = 1$, and $f(t,u)$ satisfies

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = 0 \quad \text{uniformly on each compact subset of (0,1)}\ .$$

Thus, Theorem 1 is invalid to this example. However, if we take $F(u) = 100\sqrt{u} + 118$, $\psi(t) = 100\,t$, $k = 100$, then $\Gamma\,k(M+1) = 1000$ and

$$f(t,u) + M \le 100\sqrt{u} + 118 = F(u), \qquad \text{for} \ \ t \in [0,1] \ \text{ and } \ u \ge 0\ ,$$

and

$$f(t,u) + M \ge 100\,t\sqrt{u+1} \ge 100\,t = \psi(t), \qquad \text{on} \ \ [0,1]\times[0,\infty)\ .$$

Since the Green's function of (24) is the same as in Example 1, it is easy to see that

$$\int_{1/4}^{3/4} G\left(\frac{1}{2}, s\right) \psi(s)\,ds = 100\int_{1/4}^{1/2} s^2\,ds + 50\int_{1/2}^{3/4} s\,ds = 275/24 > 10 = \Gamma(M+1)\,.$$

Therefore, (H3), (H5) are satisfied. Furthermore, if we choose $r = (2500 + 10\sqrt{62559})^2$ then condition (H4) holds. Thus, by Theorem 2 we, claim that (24) has at least one positive solution for $\lambda \in (0, 100]$. $\square$

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Xiaoming He and Weigao Ge,
Department of Applied Mathematics, Beijing Institute of Technology,
Beijing 100081 – PEOPLE'S REPUBLIC OF CHINA
E-mail: mingxiaohe@263.net