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Multivariable Christoffel–Darboux Kernels and Characteristic Polynomials of Random Hermitian Matrices*

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Abstract. We study multivariable Christoffel–Darboux kernels, which may be viewed as reproducing kernels for antisymmetric orthogonal polynomials, and also as correlation functions for products of characteristic polynomials of random Hermitian matrices. Using their interpretation as reproducing kernels, we obtain simple proofs of Pfaffian and determinant formulas, as well as Schur polynomial expansions, for such kernels. In subsequent work, these results are applied in combinatorics (enumeration of marked shifted tableaux) and number theory (representation of integers as sums of squares).

Key words: Christoffel–Darboux kernel; multivariable orthogonal polynomial; Pfaffian; determinant; correlation function; random Hermitian matrix; orthogonal polynomial ensemble; Sundquist's identities

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Dedicated to the memory of Vadim Kuznetsov

1 Introduction

The Christoffel–Darboux kernel plays an important role in the theory of one-variable orthogonal polynomials. In the present work, we study a multivariable extension, which can be viewed as a reproducing kernel for *anti-symmetric* polynomials. As is explained below, our original motivation came from a very special case, having applications in number theory (sums of squares) and combinatorics (tableaux enumeration). More generally, this kind of kernels occur in random matrix theory as correlation functions for products of characteristic polynomials of random Hermitian matrices. The purpose of the present note is to highlight a number of useful identities for such kernels. Although, as we will make clear, the main results can be found in the literature, they are scattered in work belonging to different disciplines, so it seems useful to collect them in one place. Moreover, our proofs, with the interpretation as reproducing kernels, are new and conceptually very simple.

We first recall something of the one-variable theory. Let

$$f \mapsto \int f(x)\, d\mu(x)$$

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html be a linear functional defined on polynomials of one variable. We denote by V_n the space of polynomials of degree at most $n-1$. Assuming that the pairing

$$\langle f, g\rangle = \int f(x)g(x)\,d\mu(x)$$

is non-degenerate on each V_n (for instance, if it is positive definite), there exists a corresponding system $(p_k(x))_{k=0}^{\infty}$ of monic orthogonal polynomials. We may then introduce the Christoffel–Darboux kernel

$$K(x,y) = \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{\langle p_k, p_k\rangle},$$

which is the *reproducing kernel* of V_n, that is, the unique function such that $(y \mapsto K(x,y)) \in V_n$ and

$$f(x) = \int f(y)K(x,y)\,d\mu(y), \qquad f \in V_n.$$

The Christoffel–Darboux formula states that

$$K(x,y) = \frac{1}{\langle p_{n-1}, p_{n-1}\rangle} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}.$$

More generally, let V_n^m, $0 \leq m \leq n$, denote the mth exterior power of V_n. It will be identified with the space of antisymmetric polynomials $f(x) = f(x_1, \dots, x_m)$ that are of degree at most $n-1$ in each x_i. Writing

$$\int f(x)\,d\mu_m(x) = \frac{1}{m!}\int f(x_1,\dots,x_m)\,d\mu(x_1)\cdots d\mu(x_m),$$

we equip V_n^m with the pairing

$$\langle f, g\rangle_{V_n^m} = \int f(x)g(x)\,d\mu_m(x).$$

Equivalently, in terms of the spanning vectors $\det_{1\leq i,j\leq m}(f_j(x_i))$, $f_j \in V_n$,

$$\left\langle \det_{1\leq i,j\leq m}(f_j(x_i)), \det_{1\leq i,j\leq m}(g_j(x_i))\right\rangle_{V_n^m} = \det_{1\leq i,j\leq m}(\langle f_i, g_j\rangle_{V_n}). \tag{1}$$

Every element of V_n^m is divisible by the polynomial

$$\Delta(x) = \prod_{1\leq i<j\leq m}(x_j - x_i).$$

The map $f \mapsto f/\Delta$ is an isometry from V_n^m to the space W_n^m, consisting of *symmetric* polynomials in $x_1, \dots, x_m$ that are of degree at most $n-m$ in each x_i, equipped with the pairing

$$\langle f, g\rangle_{W_n^m} = \frac{1}{m!}\int f(x)g(x)\Delta(x)^2\,d\mu(x_1)\cdots d\mu(x_m).$$

We remark that, normalizing

$$\Delta(x)^2\,d\mu(x_1)\cdots d\mu(x_m) \tag{2}$$

to a probability distribution, it defines an *orthogonal polynomial ensemble* (the term is sometimes used also for the more general weights $|\Delta(x)|^\beta$). Such ensembles are important in a variety of contexts, including the theory of random Hermitian matrices [12].

An orthogonal basis of V_n^m is given by $(p_S(x))_{S\subseteq[n],\,|S|=m}$, where

$$p_S(x) = \det_{1\leq i\leq m,\, j\in S}(p_{j-1}(x_i)).$$

Here and throughout, we write $[n]=\{1,\dots,n\}$, and we assume that the columns are ordered in the natural way. Indeed, it follows from (1) that

$$\langle p_S, p_T\rangle = \delta_{ST}\prod_{i\in S}\langle p_{i-1}, p_{i-1}\rangle. \tag{3}$$

We denote by $\Delta(x)\Delta(y)K_m(x,y)$ the reproducing kernel of V_n^m, that is, the unique element of $V_n^m\otimes V_n^m$ such that

$$f(x) = \int f(y)\Delta(x)\Delta(y)K_m(x,y)\,d\mu_m(y), \qquad f\in V_n^m. \tag{4}$$

Equivalently, $K_m(x,y)$ is the reproducing kernel of W_n^m.

It is easy to see that

$$K_m(x,y) = \frac{1}{\Delta(x)\Delta(y)}\sum_{S\subseteq[n],\,|S|=m}\frac{p_S(x)p_S(y)}{\langle p_S,p_S\rangle} = \frac{\det_{1\leq i,j\leq m}(K(x_i,y_j))}{\Delta(x)\Delta(y)}. \tag{5}$$

Indeed, it follows from (1) and (3) that both sides satisfy (4). The equality of the two expressions can also be derived using the Cauchy–Binet formula.

The following elegant integral formula was recently obtained by Strahov and Fyodorov [29]. In Section 2 we will give a simple proof, using the interpretation of the left-hand side as a reproducing kernel.

Proposition 1 (Strahov and Fyodorov). *One has*

$$K_m(x,y) = \frac{1}{\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle}\int\prod_{\substack{1\leq j\leq 2m\\ 1\leq k\leq n-m}}(z_j-w_k)\,\Delta(w)^2\,d\mu_{n-m}(w), \tag{6}$$

where

$$(z_1,\dots,z_{2m}) = (x_1,\dots,x_m,y_1,\dots,y_m). \tag{7}$$

As a consequence, the obvious $S_m\times S_m\times\mathbb{Z}_2$ symmetry of $K_m(x,y)$ extends to a non-trivial S_{2m} symmetry.

Corollary 1. *The polynomial $K_m(z)=K_m(z_1,\dots,z_{2m})$ is symmetric in all its variables.*

The motivation for the work of Strahov and Fyodorov is an interpretation of the integral (6) as a correlation function for the product of characteristic polynomials of random Hermitian matrices. The case $x=y$ is of particular interest, since (6) may then be written as

$$\begin{aligned}&\Delta(x_1,\dots,x_m)^2K_m(x_1,\dots,x_m,x_1,\dots,x_m)\\ &\qquad=\frac{1}{(n-m)!\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle}\int\Delta(x_1,\dots,x_n)^2\,d\mu(x_{m+1})\cdots d\mu(x_n),\end{aligned} \tag{8}$$

which exhibits $\Delta(x)^2 K_m(x,x)$ as a correlation function for the measure (2). In that case, the determinant formula (5) is classical, see [5, Chapter 5]. In our opinion, our proof of Proposition 1 is more illuminating than the inductive proof usually given for the special case $x = y$. For applications of Proposition 1 and for further related results, see [1, 2, 4].

An alternative determinant formula for the integral (6) may be obtained by combining two classical results of Christoffel [9, Theorem 2.7.1] and Heine [9, Theorem 2.1.2], see also [4]. We will obtain it as a by-product of the proof of Proposition 1.

Proposition 2. *In the notation above,*

$$K_m(z) = \frac{1}{\prod_{i=1}^{m} \langle p_{n-i}, p_{n-i}\rangle} \frac{\det_{1\le i,j\le 2m} \left(p_{n-m+j-1}(z_i)\right)}{\Delta(z)}.$$

More generally, such a determinant formula holds for the integrals

$$\int \prod_{\substack{1\le j\le m\\ 1\le k\le n}} (z_j - w_k)\, \Delta(w)^2\, d\mu_n(w),$$

but we focus on the case when m is even.

The fact that the "two-point" determinant in (5) and the "one-point" determinant in Proposition 2 agree can also be derived from the work of Lascoux [15, Propositions 8.4.1 and 8.4.3], see [15, Exercise 8.33] for the integral formula in this context. Note that the reproducing property mentioned by Lascoux, and given explicitly in [17, Proposition 3], is of a different nature from (4), pertaining to integration against *one-variable* polynomials.

The special case of Proposition 2 obtained by subtracting the kth row from the $(m+k)$th, for $1 \le k \le m$, and then letting $z_{m+k} \to z_k$, gives the following formula for the correlation function (8).

Corollary 2. *In the notation above,*

$$K_m(x,x) = \frac{(-1)^{\frac{1}{2}m(m-1)}}{\prod_{i=1}^{m} \langle p_{n-i}, p_{n-i}\rangle \Delta(x)^4} \det_{1\le i,j\le 2m} \left(\begin{cases} p_{n-m+j-1}(x_i), & 1\le i\le m, \\ p'_{n-m+j-1}(x_i), & m+1 \le i\le 2m \end{cases} \right).$$

Next, we give Pfaffian formulas for K_m. As is explained below, they can be deduced from results of Ishikawa and Wakayama [8], Lascoux [16], and Okada [22]. Nevertheless, we will give an independent proof, using Corollary 1. Recall that the *Pfaffian* of a skew-symmetric even-dimensional matrix is given by

$$\operatorname*{pfaff}_{1\le i,j\le 2m} (a_{ij}) = \frac{1}{2^m m!} \sum_{\sigma\in S_{2m}} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1),\sigma(2i)}.$$

Proposition 3. *For any choice of square roots* $\sqrt{z_i}$,

$$K_m(z) = \frac{1}{\prod_{1\le i<j\le 2m} \left(\sqrt{z_j} - \sqrt{z_i}\right)} \operatorname*{pfaff}_{1\le i,j\le 2m} \left((\sqrt{z_j} - \sqrt{z_i}) K(z_i, z_j) \right).$$

Moreover, for any choice of ζ_i *such that*

$$\zeta_i + \zeta_i^{-1} = z_i + 2, \tag{9}$$

$$K_m(z) = \frac{\prod_{i=1}^{2m} \zeta_i^{m-1}}{\prod_{1\le i<j\le 2m} (\zeta_j - \zeta_i)} \operatorname*{pfaff}_{1\le i,j\le 2m} \left((\zeta_j - \zeta_i) K(z_i, z_j) \right).$$

Note that (9) implies

$$z_j - z_i = -\frac{1}{\zeta_i\zeta_j}(\zeta_j-\zeta_i)(1-\zeta_i\zeta_j). \tag{10}$$

In the special case $z = (x, x)$, choosing

$$(\sqrt{z_1}, \dots, \sqrt{z_{2m}}) = (-\sqrt{x_1}, \dots, -\sqrt{x_m}, \sqrt{x_1}, \dots, \sqrt{x_m}),$$
$$(\zeta_1, \dots, \zeta_{2m}) = (\xi_1^{-1}, \dots, \xi_m^{-1}, \xi_1, \dots, \xi_m),$$

Proposition 3 reduces to special cases of the identity

$$\operatorname{pfaff}\begin{pmatrix} A & B \\ -B & -A \end{pmatrix} = (-1)^{\frac{1}{2}m(m-1)} \det(B - A).$$

The general case seems to lie deeper.

Proposition 3 is actually equivalent to Proposition 4 below, which can be deduced from known results. Indeed, rewriting (11a) using [22, Theorem 4.2] and (11b) using the case $n = m$ of [22, Theorem 4.7], we can easily see the resulting expressions to agree. More explicitly, this identity appears in [16], with a simple proof. The Pfaffian (11c) can be treated similarly, or else shown to agree with (11b) by means of a result of Ishikawa and Wakayama [8, Theorem 5.1], see Remark 3 below.

We remark that the relevant results of [8], [16] and [22] are closely related to Sundquist's identities [30], see also [6] and [7]. Moreover, (11a) is related to the Izergin–Korepin determinant for the partition function of the six-vertex model [10], which, as well as the Pfaffians in (11), has applications to alternating sign matrices [13, 14, 23, 32].

Proposition 4. *Let a_i and z_i, $1 \le i \le 2m$, be free variables, and let ζ_i be as in (9). Moreover, let $S \subseteq [2m]$ be an arbitrary subset of cardinality m. Then,*

$$(-1)^{\frac{1}{2}m(m+1)+\sum_{s\in S} s} \prod_{i\in S,\, j\notin S}(z_j - z_i) \det_{i\in S,\, j\notin S}\left(\frac{a_j - a_i}{z_j - z_i}\right) \tag{11a}$$

$$= \prod_{1\le i<j\le 2m}(\sqrt{z_i}+\sqrt{z_j}) \operatorname*{pfaff}_{1\le i,j\le 2m}\left(\frac{a_j - a_i}{\sqrt{z_j}+\sqrt{z_i}}\right) \tag{11b}$$

$$= \prod_{i=1}^{2m}\zeta_i^{1-m} \prod_{1\le i<j\le 2m}(1-\zeta_i\zeta_j) \operatorname*{pfaff}_{1\le i,j\le 2m}\left(\frac{a_j - a_i}{1-\zeta_i\zeta_j}\right). \tag{11c}$$

Note that Proposition 3 is a special case of Proposition 4 when $a_i = p_n(z_i)/p_{n-1}(z_i)$, $\{x_1, \dots, x_m\} = \{z_i\}_{i\in S}$ and $\{y_1, \dots, y_m\} = \{z_i\}_{i\notin S}$. Conversely, it is not hard to deduce Proposition 4 from Proposition 3 using an interpolation argument.

Remark 1. Strahov and Fyodorov found one-point and two-point determinant formulas for more general correlation functions than those of Proposition 1. In the case of

$$\int \prod_{\substack{1\le j\le m\\ 1\le k\le n}} \frac{x_j - z_k}{y_j - z_k}\, \Delta(z)^2\, d\mu_n(z),$$

their two-point formula is of the type (11a), see [29, Proposition 4.2]. Thus, Proposition 4 implies Pfaffian formulas for this correlation function.

In [27], we need an elementary result on the expansion of the kernel $K_m(x,y)$ into Schur polynomials $s_\lambda(x)s_\mu(y)$. Since we have not found a suitable reference, we include it here. The proof is given in Section 4.

Proposition 5. *One has*

$$K_m(x,y) = \sum_{\substack{0\le\lambda_m\le\cdots\le\lambda_1\le n-m\\ 0\le\mu_m\le\cdots\le\mu_1\le n-m}} \frac{(-1)^{\sum_{i=1}^m(\lambda_i+\mu_i)}}{\prod_{i=1}^n\langle p_{i-1},p_{i-1}\rangle} \det_{i\in[n]\setminus S, j\in[n]\setminus T}(c_{i+j-2})\, s_\lambda(x)s_\mu(y),$$

where

$$S=\{\lambda_k+m+1-k;\ 1\le k\le m\},\qquad T=\{\mu_k+m+1-k;\ 1\le k\le m\},$$

and

$$c_k=\int x^k\,d\mu(x).$$

Finally, let us describe our original motivation, which comes from applications that seem completely unrelated to random matrix theory. Motivated by the theory of affine superalgebras, Kac and Wakimoto [11] conjectured certain new formulas for the number of representations of an integer as the sum of $4m^2$ or $4m(m+1)$ triangular numbers. These conjectures were first proved by Milne [19, 20, 21], and later by Zagier [31]. In [25], we re-derived and generalized the Kac–Wakimoto identities using elliptic Pfaffian evaluations. Extension of this analysis from triangles to squares leads to formulas involving *Schur Q-polynomials* [18] evaluated at the point $(1,\dots,1)$. (More precisely, these polynomials are normally labelled by positive integer partitions. Here, we need an extension to the case when some indices are *negative*.) Later, we realized that the resulting sums of squares formulas are equivalent to those of Milne [21]. Seeing this is far from obvious and requires an identification of the Schur Q-polynomials with kernels $K_m(z)$, where the underlying polynomials $p_k(x)$ are continuous dual Hahn polynomials. The key fact for obtaining this identification is the second part of Proposition 3. We refer to [26] for applications of the results above to Schur Q-polynomials and marked shifted tableaux, and to [27] for the relation to sums of squares.

2 Proof of Propositions 1 and 2

Lemma 1. *Let $\phi: V_n^m\to V_n^{n-m}$ be defined by*

$$(\phi f)(x)=\int f(y)\Delta(y,x)\,d\mu_m(y).$$

Then,

$$(\phi p_S)(x)=(-1)^{\frac12 m(m+1)+\sum_{s\in S}s}\prod_{i\in S}\langle p_{i-1},p_{i-1}\rangle\cdot p_{S^c}(x),$$

where $S^c=\{1,\dots,n\}\setminus S$.

A corresponding statement holds when V_n is a general n-dimensional vector space and Δ an element of the one-dimensional space $(V_n^*)^n$ [3, § 8.5], the map $\phi: V_n^m\to(V_n^*)^{n-m}$ often being called *Hodge star* or *Poincaré isomorphism*. For completeness, we provide a proof in the present setting.

Proof. The Vandermonde determinant evaluation

$$\Delta(x) = \det_{1\leq i,j\leq n}(p_{j-1}(x_i)) \tag{12}$$

gives

$$(\phi p_S)(x) = \sum_{\substack{\sigma\colon [m]\to S\\ \tau\colon [n]\to[n]}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\int \prod_{i=1}^{m} p_{\sigma(i)-1}(y_i)p_{\tau(i)-1}(y_i)\, d\mu_m(y) \prod_{i=m+1}^{n} p_{\tau(i)-1}(x_i),$$

where the sum is over bijections. By orthogonality, we may assume

$$(\tau(1),\dots,\tau(n)) = (\sigma(1),\dots,\sigma(m),\rho(1),\dots,\rho(n-m)),$$

with ρ a bijection $[n-m]\to S^c$. One easily checks that

$$\operatorname{sgn}(\tau) = (-1)^{\frac12 m(m+1)+\sum_{s\in S}s}\operatorname{sgn}(\sigma)\operatorname{sgn}(\rho),$$

which gives indeed

$$\begin{aligned}(\phi p_S)(x) &= (-1)^{\frac12 m(m+1)+\sum_{s\in S}s}\frac{1}{m!}\sum_{\sigma\colon[m]\to S}\prod_{i=1}^{m}\langle p_{\sigma(i)-1},p_{\sigma(i)-1}\rangle\\ &\quad\times\sum_{\rho\colon[n-m]\to S^c}\operatorname{sgn}(\rho)\prod_{i=1}^{n-m}p_{\rho(i)-1}(x_i)\\ &= (-1)^{\frac12 m(m+1)+\sum_{s\in S}s}\prod_{i\in S}\langle p_{i-1},p_{i-1}\rangle\, p_{S^c}(x).\end{aligned}$$

■

By iteration, it follows from Lemma 1 that

$$\phi\circ\phi = (-1)^{m(n-m)}\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle\cdot\mathrm{id}_{V_n^m}.$$

By means of $\Delta(w,x) = (-1)^{m(n-m)}\Delta(x,w)$, $x\in\mathbb{R}^m$, $w\in\mathbb{R}^{n-m}$, this fact can be expressed as

$$f(x) = \frac{1}{\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle}\int f(y)\Delta(x,w)\Delta(y,w)\, d\mu_m(y)d\mu_{n-m}(w), \qquad f\in V_n^m.$$

By the uniqueness of the reproducing kernel, it follows that

$$\Delta(x)\Delta(y)K_m(x,y) = \frac{1}{\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle}\int\Delta(x,w)\Delta(y,w)\, d\mu_{n-m}(w). \tag{13}$$

This is equivalent to Proposition 1.

Remark 2. The equation (13) can also be obtained as the special case $l=n$ of the contraction formula

$$\Delta(x)\Delta(y)K_m(x,y) = \frac{(n-l)!(l-m)!}{(n-m)!}\int\Delta(x,w)\Delta(y,w)K_l(x,w,y,w)\, d\mu_{l-m}(w), \tag{14}$$

$0\leq m\leq l\leq n$. Conversely, (14) follows easily from (13).

To prove Proposition 2, we note that

$$\Delta(x,y,w)=\frac{\Delta(x,y)\Delta(x,w)\Delta(y,w)}{\Delta(x)\Delta(y)\Delta(w)}.$$

Applying this to (13) gives

$$\begin{aligned}K_m(x,y)&=\frac{1}{\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle}\frac{1}{\Delta(x,y)}\int\Delta(x,y,w)\Delta(w)\,d\mu_{n-m}(w)\\&=\frac{1}{\prod_{i=1}^{n}\langle p_{i-1},p_{i-1}\rangle}\frac{(\phi\Delta)(z)}{\Delta(z)},\end{aligned}$$

where, as in (7), $z=(x,y)$. Next we observe that, by (12), $\Delta(w)=p_S(w)$, with $S=[n-m]$. Lemma 1 then gives indeed

$$K_m(x,y)=\frac{1}{\prod_{i=1}^{m}\langle p_{n-i},p_{n-i}\rangle}\frac{\det_{1\leq i,j\leq 2m}(p_{n-m+j-1}(z_i))}{\Delta(z)}.$$

3 Proof of Proposition 3

Our main tool is the following elementary property of Pfaffians, which we learned from an unpublished manuscript of Eric Rains [24].

Lemma 2 (Rains). *For arbitrary $(a_{ij})_{1\leq i,j\leq 2m}$,*

$$\operatorname*{pfaff}_{1\leq i,j\leq 2m}(a_{ij}-a_{ji})=\sum_{S\subseteq[2m],\,|S|=m}(-1)^{\chi(S)}\det_{i\in S,j\notin S}(a_{ij}),$$

where $\chi(S)$ denotes the number of even elements in S.

For completeness, we sketch Rains' proof.

Proof. The left-hand side is given by

$$\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\prod_{i=1}^{m}(a_{\sigma(2i-1),\sigma(2i)}-a_{\sigma(2i),\sigma(2i-1)})=\frac{1}{m!}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\prod_{i=1}^{m}a_{\sigma(2i-1),\sigma(2i)}.$$

Write

$$\prod_{i=1}^{m}a_{\sigma(2i-1),\sigma(2i)}=\prod_{i\in S}a_{i,\tau(i)},$$

with $S=\{\sigma(1),\sigma(3),\dots,\sigma(2m-1)\}$ and τ a bijection $S\to S^c$. Identifying τ as an element of S_m, using the natural orderings on S and S^c, it is easy to check that $\operatorname{sgn}(\sigma)=(-1)^{\chi(S)}\operatorname{sgn}(\tau)$. Since the map $\sigma\mapsto(S,\tau)$ is $m!$ to one, we obtain indeed

$$\sum_{S\subseteq[2m],\,|S|=m}(-1)^{\chi(S)}\sum_{\tau}\operatorname{sgn}(\tau)\prod_{i\in S}a_{i,\tau(i)}. \qquad\blacksquare$$

The following identity appeared as [8, Theorem A.1].

Corollary 3 (Ishikawa and Wakayama). *One has*

$$\begin{aligned}
&\operatorname*{pfaff}_{1\le i,j\le 2m}\left(\frac{z_j-z_i}{a+b(x_i+x_j)+cx_ix_j}\right)\\
&\quad=(b^2-ac)^{\frac12 m(m-1)}\prod_{1\le i<j\le 2m}\frac{1}{a+b(x_i+x_j)+cx_ix_j}\\
&\qquad\times\sum_{S\subseteq[2m],\,|S|=m}(-1)^{\chi(S)}\prod_{j\notin S}z_j\prod_{\substack{1\le i<j\le 2m\\ i,j\in S\text{ or }i,j\notin S}}(x_j-x_i)(a+b(x_i+x_j)+cx_ix_j).
\end{aligned}$$

This follows immediately from Lemma 2, by use of the Cauchy determinant

$$\det_{1\le i,j\le m}\left(\frac{1}{a+b(x_i+y_j)+cx_iy_j}\right)=(b^2-ac)^{\frac12 m(m-1)}\frac{\prod\limits_{1\le i<j\le m}(x_j-x_i)(y_j-y_i)}{\prod\limits_{i,j=1}^{m}(a+b(x_i+y_j)+cx_iy_j)},$$

which is reduced to its more well-known special case $a=1$, $b=0$, $c=-1$ through the elementary identity

$$a+b(x_i+x_j)+cx_ix_j=\frac1c\left(ac-b^2+(cx_i+b)(cx_j+b)\right). \tag{15}$$

The proof of Corollary 3 given in [8] is more complicated.

Remark 3. The equality of (11b) and (11c) can be deduced by applying Corollary 3 to both Pfaffians, using also (10).

We only need Corollary 3 in the case when $x_i=z_i$. Then, the Pfaffian is given by

$$(b^2-ac)^{m(m-1)}\prod_{1\le i<j\le 2m}\frac{x_j-x_i}{a+b(x_i+x_j)+cx_ix_j}.$$

Indeed, one may use (15) to reduce oneself to the case $a=1$, $b=0$, $c=-1$, which is the Pfaffian evaluation in [28, Proposition 2.3].

Corollary 4. *One has*

$$\begin{aligned}
&\sum_{S\subseteq[2m],\,|S|=m}(-1)^{\chi(S)}\prod_{j\notin S}x_j\prod_{\substack{1\le i<j\le 2m\\ i,j\in S\text{ or }i,j\notin S}}(x_j-x_i)(a+b(x_i+x_j)+cx_ix_j)\\
&\quad=(b^2-ac)^{\frac12 m(m-1)}\prod_{1\le i<j\le 2m}(x_j-x_i).
\end{aligned}$$

We are now ready to prove Proposition 3. By Lemma 2, we have in general

$$\operatorname*{pfaff}_{1\le i,j\le 2m}((a_j-a_i)K(z_i,z_j))=\sum_{S\subseteq[2m],\,|S|=m}(-1)^{\chi(S)}\prod_{j\notin S}a_j\det_{i\in S,j\notin S}(K(z_i,z_j)),$$

which, by Corollary 1, equals

$$K_m(z)\sum_{S\subseteq[2m],\,|S|=m}(-1)^{\chi(S)}\prod_{j\notin S}a_j\prod_{\substack{1\le i<j\le 2m\\ i,j\in S\text{ or }i,j\notin S}}(z_j-z_i). \tag{16}$$

Consider first the case $a_i = \sqrt{z}_i$. Then, by the case $a = c = 0$, $b = 1$, $x_i = \sqrt{z_i}$ of Corollary 4, the sum in (16) equals

$$\prod_{1\leq i<j\leq 2m} \left(\sqrt{z_j} - \sqrt{z_i}\right).$$

This yields the first part of Proposition 3. Similarly, letting $a_i = \zeta_i$ and using (10), we can compute the sum in (16) by the case $a = 1$, $b = 0$, $c = -1$, $x_i = \zeta_i$ of Corollary 4 as

$$\prod_{i=1}^{2m} \zeta_i^{1-m} \prod_{1\leq i<j\leq 2m} (\zeta_j - \zeta_i).$$

This completes the proof of Proposition 3.

4 Proof of Proposition 5

When $(e_k)_{k=1}^n$ is a basis of V_n, let $(e_S)_{S\subseteq[n],\,|S|=m}$ be the corresponding basis of V_n^m defined by

$$e_S(x) = \det_{1\leq i\leq m,\, j\in S}(e_j(x_i)).$$

We then have the following general expansion formula. If $e_i(x) = f_i(x) = p_{i-1}(x)$, this is (5).

Lemma 3. *Let $(e_k)_{k=1}^n$ and $(f_k)_{k=1}^n$ be arbitrary bases of V_n. Then,*

$$\Delta(x)\Delta(y)K_m(x,y) = \sum_{\substack{S,T\subseteq[n]\\|S|=|T|=m}} (-1)^{\sum\limits_{s\in S} s+\sum\limits_{t\in T} t} \frac{\det\limits_{i\in S^c,j\in T^c}(\langle e_i, f_j\rangle)}{\det\limits_{1\leq i,j\leq n}(\langle e_i, f_j\rangle)}\, e_S(x) f_T(y). \tag{17}$$

Proof. It is enough to show that

$$e_U(x) = \int e_U(y) R(x,y)\, d\mu_m(y), \qquad U\subseteq[n], \quad |U| = m,$$

where R denotes the right-hand side of (17). Equivalently, we need to show that

$$\sum_{T\subseteq[n],\,|T|=m} (-1)^{\sum\limits_{s\in S} s+\sum\limits_{t\in T} t} \frac{\det\limits_{i\in S^c,j\in T^c}(\langle e_i, f_j\rangle)\det\limits_{i\in U,j\in T}(\langle e_i, f_j\rangle)}{\det\limits_{1\leq i,j\leq n}(\langle e_i, f_j\rangle)} = \delta_{SU}.$$

To verify this, write $U = \{u_1 < \cdots < u_m\}$, $S^c = \{s_1' < \cdots < s_{n-m}'\}$, and let X be the ordered sequence $(u_1, \ldots, u_m, s_1', \ldots, s_{n-m}')$. Consider the determinant

$$D = \det_{i\in X, 1\leq j\leq n}(\langle e_i, f_j\rangle).$$

On the one hand, reordering the rows gives

$$D = (-1)^{\frac{1}{2}m(m+1)+\sum\limits_{s\in S} s} \delta_{SU} \det_{1\leq i,j\leq n}(\langle e_i, f_j\rangle).$$

On the other hand, applying Laplace expansion to the first m rows gives

$$D = \sum_{T\subseteq[n],\,|T|=m} (-1)^{\frac{1}{2}m(m+1)+\sum\limits_{t\in T} t} \det_{i\in U,j\in T}(\langle e_i, f_j\rangle) \det_{i\in S^c,j\in T^c}(\langle e_i, f_j\rangle).$$

This completes the proof. ∎

Consider the case of Lemma 3 when $e_i(x) = f_i(x) = x^{i-1}$. Then, $e_S(x) = \Delta(x)s_\lambda(x)$, where $S = \{s_1 < \cdots < s_m\}$ and $\lambda_i = s_{m+1-i} - i$. Thus, noting also that

$$\det_{1\le i,j\le n}(\langle e_i, f_j\rangle) = \det_{1\le i,j\le n}(\langle p_{i-1}, p_{j-1}\rangle) = \prod_{i=1}^{n}\langle p_{i-1}, p_{i-1}\rangle,$$

we obtain Proposition 5.

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