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On the Relationship between Two Notions of Compatibility for Bi-Hamiltonian Systems*

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Abstract. Bi-Hamiltonian structures are of great importance in the theory of integrable Hamiltonian systems. The notion of compatibility of symplectic structures is a key aspect of bi-Hamiltonian systems. Because of this, a few different notions of compatibility have been introduced. In this paper we show that, under some additional assumptions, compatibility in the sense of Magri implies a notion of compatibility due to Fassò and Ratiu, that we dub bi-affine compatibility. We present two proofs of this fact. The first one uses the uniqueness of the connection parallelizing all the Hamiltonian vector fields tangent to the leaves of a Lagrangian foliation. The second proof uses Darboux–Nijenhuis coordinates and symplectic connections.

Key words: bi-Hamiltonian systems; Lagrangian foliation; bott connection; symplectic connections

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1 Introduction

Let M be a smooth manifold of dimension $2n$, and let X be a vector field on M. Suppose that X is Hamiltonian with respect to two different symplectic structures ω_1 and ω_2, that is,

$$\mathbf{i}_X\omega_1 = \mathbf{d}H \qquad \text{and} \qquad \mathbf{i}_X\omega_2 = \mathbf{d}K,$$

where H and K are two, possibly distinct, Hamiltonian functions. Let us introduce the so called *recursion operator* $N = \omega_2^\sharp\omega_1^\flat\colon TM \to TM$, where $\omega^\flat\colon TM \to T^*M$ denotes the "musical" isomorphism induced by the symplectic form ω, and $\omega^\sharp$ is its inverse. Then, as a consequence of the fact that X is Hamiltonian with respect to two symplectic forms, the flow associated to X preserves the eigenvalues of the recursion operator. Hence, if N has n functionally independent eigenvalues in involution, then, it is completely integrable via the Liouville–Arnold theorem.

A natural approach to integrability is to try to find sufficient conditions for the eigenvalues of the recursion operator to be in involution. Several sufficient conditions of this type have been found.

One such condition is based on the pioneering work of Magri [10] in the infinite-dimensional case. Magri and Morosi [12] showed that, if the sum of the Poisson tensor associated to ω_1 and the one associated to ω_2 is still a Poisson tensor, then the eigenvalues of the recursion operator are in involution. A similar claim is also present in the work of Gel'fand and Dorfman [8]. In this case we say that ω_1 and ω_2 are *Magri-compatible*, the triple (M,ω_1,ω_2) is a *bi-Hamiltonian manifold*, and the quadruple (M,ω_1,ω_2,X) is a *bi-Hamiltonian system* in Magri's sense if there exist functions H_1 and H_2 such that $X = \omega_1^\sharp \cdot \mathbf{d}H_1 = \omega_2^\sharp \cdot \mathbf{d}H_2$. However, it is known that not all completely integrable Hamiltonian system are bi-Hamiltonian in Magri's sense. In fact

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there are results by Brouzet [3] and Fernandes [6] indicating that there are completely integrable Hamiltonian system that are not bi-Hamiltonian in Magri's sense. This limitation of Magri's definition stimulated the search for different notions of compatibility.

Bogoyavlenskij has given a sufficient condition that he calls *strong dynamical compatibility* [2]. This condition requires the existence of a vector field X that is Hamiltonian with respect to two symplectic structures ω_1 and ω_2, is completely integrable with respect to ω_1, and it is *non-degenerate* in the following sense. The orbits of X lie on Lagrangian tori and, in any local system of ω_1-action-angle coordinates (a, α), the Hamiltonian function H_1 of X associated to ω_1 satisfies the following equation

$$\det\left(\frac{\partial^2 H_1}{\partial a_i \partial a_j}\right)(a) \neq 0.$$

A third notion of compatibility, was introduced by Fassò and Ratiu (see [5]) in order to study superintegrable systems, that is, systems with more than n independent integrals of motion, and with motions on isotropic tori of dimension less than n, rather than on Lagrangian tori of dimension n. Let ω_1 and ω_2 be two symplectic forms on M. We say that a fibration (foliation) is bi-Lagrangian if the fibers (leaves) are Lagrangian with respect to both symplectic forms. Suppose there exist a bi-Lagrangian fibration of M. We say that ω_1 and ω_2 are *bi-affinely compatible* if the Bott connection (see Section 2 for a definition) associated to ω_1 and the one associated to ω_2 coincide.

Later we will explain the notion of Magri compatibility and bi-affine compatibility in more detail. In [5] Fassò and Ratiu wrote:

"It is not known to us whether our definition (as well as Bogoyavlenskij's) is more general than Magri's. If ω_1 and ω_2 are Magri compatible, then the eigenvalues of the recursion operator (if independent) define a bi-Lagrangian foliation. However, even when this foliation is a fibration and has compact and connected fibers, it is not clear whether, using the terminology of Definition 2, it is bi-affine."

This paper will be devoted to showing that Magri compatibility implies bi-affine compatibility in the simple case where the recursion operator has the maximal number of distinct eigenvalues. Note, however, that the converse is not true, since, as shown in [2] and [5], there exist bi-affinely compatible structures that are not compatible in Magri's sense. We believe it should be possible to tackle the general case by using Turiel's classification of Magri-compatible bi-Hamiltonian structures (see [13, 15]).

We are aware of three different ways of proving that Magri compatibility implies bi-affine compatibility. The first proof uses the property that the Bott connection is the unique connection parallelizing all the Hamiltonian vector fields tangent to the leaves of a Lagrangian foliation. The second proof employs Darboux–Nijenhuis coordinates and the fact (proved in Proposition 5) that the restriction of a torsion-free symplectic connection to an involutive Lagrangian distribution coincides with the Bott connection in L. A third proof can be obtained directly by using the definition of Bott's connection given in equation (1). In this paper we will present only the first two proofs.

The first proof is more direct and has the advantage of avoiding the introduction of Darboux–Nijenhuis coordinates and symplectic connections. The second proof, on the other hand, shows that Magri's compatibility condition allows to construct explicitly, in Darboux–Nijenhuis coordinates, two symplectic connections that have a special form.

Recall that, while Magri compatibility implies bi-affine compatibility, the converse, as mentioned above, does not hold. Since the restriction of a symplectic connection to an integrable Lagrangian distribution L is the Bott connection in L (as shown in Proposition 5) and bi-affine compatibility, by definition, concerns itself only with the Bott connection, it seems clear that

that the difference between the two types of compatibility lies in the restrictions Magri's condition imposes on the allowable symplectic connections. Therefore, in our opinion, the link between symplectic connections and the notion of compatibility for bi-Hamiltonain systems deserves further investigation.

2 Partial and symplectic connections

In this section we briefly review some facts about partial connections and symplectic connections. We refer the reader to [7] for further details.

2.1 Bott connection

We first recall the definition of a partial connection.

Definition 1. Let M be a manifold, V a vector bundle over M, L a distribution on M. Let $\Gamma(V)$ denote the space of sections of V, and $\Gamma(L)$ the space of vector fields tangent to L. A *partial linear connection* in V along L is a bilinear map

$$\nabla\colon\ \Gamma(L)\times\Gamma(V)\to\Gamma(V)\colon\ (X,Y)\to\nabla_X Y,$$

such that

1) $\nabla_{fX}Y = f\nabla_X Y$ (i.e., ∇ is linear in X),
2) $\nabla_X(fY) = f\nabla_X Y + (X[f])Y$ (i.e., ∇ is a derivation),

for $X\in\Gamma(L)$, $f\in C^\infty(M)$, and $Y\in\Gamma(V)$.

Suppose L is an involutive distribution on M, and let $L^\perp$ be the *annihilator* of L, that is, the vector subbundle of T^*M consisting of 1-forms that vanish on L. The partial linear connection $\tilde\nabla^B$ in $L^\perp$ along L defined by

$$\tilde\nabla^B_X\alpha = \mathcal{L}_X\alpha, \qquad \text{for} \quad X\in\Gamma(L), \quad \alpha\in\Gamma\big(L^\perp\big)$$

is the *Bott connection* associated with the distribution L.

Now assume that (M,ω) is an almost-symplectic manifold (that is, ω is a non-degenerate 2-form), and L is an involutive Lagrangian distribution with respect to ω. By Frobenius theorem L is also integrable, that is, each point of M is contained in an integral manifold of L. Moreover, the collection of all maximal connected integral manifolds of L forms a foliation of M (see [9] for more details). Since the "musical" isomorphism $\omega^\flat\colon TM\to T^*M$ and its inverse $\omega^\sharp\colon T^*M\to TM$ restrict to $\omega^\flat\colon\Gamma(L)\to\Gamma(L^\perp)$ and $\omega^\sharp\colon\Gamma(L^\perp)\to\Gamma(L)$ we can use them to define a partial linear connection in L along L.

Definition 2. Let L be an involutive Lagrangian distribution on the almost symplectic manifold (M,ω). The partial connection

$$\nabla^B\colon\ \Gamma(L)\times\Gamma(L)\to\Gamma(L)\colon\ (X,Y)\to\nabla_X Y$$

defined by

$$\nabla^B_X Y = \omega^\sharp\mathcal{L}_X(\omega^\flat Y) \tag{1}$$

is called, by an abuse of terminology, the *Bott connection* in L.

The Bott connection can also be defined with the following formula:

$$\omega\big(\nabla^B_X Y, Z\big) = X[\omega(Y,Z)] - \omega(Y,[X,Z]) \tag{2}$$

for $X, Y \in \Gamma(L)$, $Z \in \Gamma(TM)$.

It can be shown that ∇^B defines a flat partial connection. See [7] for more details. However, in general, the Bott connection is not torsion-free, in fact we have the following proposition.

Proposition 1. *Let (M,ω) be an almost symplectic manifold, and let L be an involutive Lagrangian distribution on M. Then, if ω is closed, the Bott connection ∇^B in L has zero torsion. More generally, the torsion tensor T^B of ∇^B, defined by*

$$T^B(X,Y) = \nabla^B_X Y - \nabla^B_Y X - [X,Y]$$

is related to the exterior derivative of ω by the formula

$$\mathbf{d}\omega(X,Y,Z) = \omega\big(T^B(X,Y),Z\big) \qquad \textit{for} \quad X,Y \in \Gamma(L), \quad Z \in \Gamma(TM).$$

A proof of this statement can be found in [7].

Another important property of the Bott connection is that it is the unique connection parallelizing all the Hamiltonian vector fields tangent to the leaves of a Lagrangian foliation. More precisely we have the following theorem.

Theorem 1. *Suppose (M,ω) is a $2n$-dimensional symplectic manifold, and U an open subset of M. Let $f_1,\ldots,f_n$ be a set of smooth functions defined on U such that*

1) *the f_i are pairwise in involution, that is, $\{f_i,f_j\}=0$,*
2) *the differentials are linearly independent, that is, $\mathbf{d}f_i \wedge \cdots \wedge \mathbf{d}f_n \neq 0$.*

Then the f_i define a Lagrangian foliation of U, with leaves of the form

$$N^c = \{x \,|\, f_1(x) = c_1, \ldots, f_n(x) = c_n\},$$

and the Bott connection

$$\nabla^B_X Y = \omega^\sharp \mathcal{L}_X(\omega^\flat Y)$$

is the unique connection parallelizing all the Hamiltonian vector fields tangent to the leaves.

Proof. The fact that the f_i define a Lagrangian foliation is well known. Let $Y = \omega^\sharp \cdot \mathbf{d}g$ be a Hamiltonian vector field tangent to the leaves of the foliation. We prove that the Bott connection parallelizes the Hamiltonian vector fields tangent to the leaves. Clearly, we have

$$\begin{aligned}\nabla^B_{X_i} Y &= \omega^\sharp \mathcal{L}_{X_i}(\omega^\flat Y) = \omega^\sharp \mathcal{L}_{X_i}(\mathbf{d}g) \overset{\text{(by Cartan's formula)}}{=} \omega^\sharp\big(\mathbf{i}_{X_i}\mathbf{d}^2 g + \mathbf{d}(\mathbf{i}_{X_i}\mathbf{d}g)\big)\\ &= \omega^\sharp(\mathbf{d}(\mathbf{i}_{X_i}\mathbf{d}g)) = \omega^\sharp(\mathbf{d}\langle \mathbf{d}g, X_i\rangle) = \omega^\sharp(\mathbf{d}\langle \omega^\flat Y, X_i\rangle) = \omega^\sharp(\mathbf{d}(\omega(Y,X_i))) = 0,\end{aligned}$$

since $\omega(Y,X_i) = 0$. This is because Y is tangent to the leaves and the leaves are Lagrangian submanifolds. Now, suppose $X = \sum_{i=1}^n a_i X_i$, then

$$\nabla_X Y = \sum a_i \nabla^B_{X_i} Y = 0.$$

Hence, ∇^B parallelizes all the Hamiltonian vector fields $Y = \omega^\sharp \cdot \mathbf{d}g$ tangent to the leaves.

To show uniqueness suppose there is another connection $(\nabla^B)'$ with the same property. Then, there is a tensor $S(X,Y)$ such that $(\nabla^B)'_X Y = \nabla^B_X Y + S(X,Y)$. Hence, we have

$$0 = \left(\nabla^B\right)'_{X_i} X_j = \nabla^B_{X_i} X_j + S(X_i, X_j) = S(X_i, X_j).$$

Consequently, $S(X,Y) = 0$ for all X, Y. In fact, let $X = \sum\limits_{i=1}^{n} a_n X_i$ and $Y = \sum\limits_{i=1}^{n} b_i X_i$, then

$$S(X,Y) = \sum_{i,j} a_i b_j S(X_i, X_j) = 0$$

by linearity, since $S(X_i, X_j) = 0$. ∎

Remark 1. Suppose the hypotheses of the theorem above are verified and let $Y_1, \dots, Y_n$ be Hamiltonian vector fields that, at each point, span the tangent plane to the leaves of the foliation. Then, the Bott connection is the unique connection parallelizing all the vector fields Y_i. The proof of this is given by a computation similar to the one in the proof of the theorem above.

2.2 Symplectic connections

Definition 3. Let (M,ω) be an almost symplectic manifold. A *symplectic connection* on (M,ω) is a bilinear map

$$\nabla\colon\ \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)\colon\ (X,Y) \to \nabla_X Y$$

such that ∇ is a linear connection, that is,

1) $\nabla_{fX} = f\nabla_X Y$,
2) $\nabla_X(fY) = f\nabla_X Y + (X[f])Y$,

and parallelizes ω, that is,

3) $(\nabla_X \omega)(Y,Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$, where $\nabla_X\omega$ denotes the covariant derivative of ω, given by the formula

$$(\nabla_X\omega)(Y,Z) = X[\omega(Y,Z)] - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z).$$

Note that here we adhere to the terminology of [7]. Other authors incorporate the requirement of being torsion-free, namely

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$$

for all $X, Y \in \Gamma(TM)$, in the definition of a symplectic connection.

The existence of torsion-free symplectic connections is ensured, in the case of symplectic manifolds, by the following proposition.

Proposition 2. *Let (M,ω) be a symplectic manifold. Then, there is a torsion-free symplectic connection on M.*

See [1] for a proof. On the other hand, if the manifold is not symplectic (i.e., ω is not closed) a torsion-free symplectic connection does not exist, in fact we have the following proposition.

Proposition 3. *Let (M,ω) be an almost symplectic manifold, and let ∇ be a symplectic connection on M. Then, the torsion tensor T of ∇ is related to the exterior derivative of ω by the formula*

$$\mathbf{d}\omega(X,Y,Z) = \omega(T(X,Y),Z) + \omega(T(Z,Y),X) + \omega(T(Z,X),Y).$$

Consequently, if there is a torsion-free symplectic connection ∇ on M, ω must be closed.

Suppose L is a distribution on M. Let $\Gamma(L)$ denote the set of all vector fields tangent to L, and let $\Gamma(TM)$ denote the set of all vector fields on M. Recall that a connection ∇ is said to *parallelize (or preserve) a distribution L* if $\nabla_X Y \in \Gamma(L)$ for all vector fields $X \in \Gamma(TM)$ and all vector fields in $Y \in \Gamma(L)$. The following result, proved in [7], links symplectic connections and Lagrangian distributions.

Proposition 4. *Let (M,ω) be an almost symplectic manifold and let L be a Lagrangian distribution on M. If there exists a torsion-free symplectic connection ∇ on M that preserves L, then ω must be closed (that is, symplectic), and L must be involutive.*

Finally, we state and prove a proposition, also proved in [7], that will be essential for the main result of this paper.

Proposition 5. *Let (M,ω) be a symplectic manifold and let L be an involutive Lagrangian distribution, then the restriction of any symplectic torsion-free connection ∇ preserving L to L coincides with the Bott connection in L.*

Proof. From the definition of symplectic connection:

$$\omega(\nabla_X Y, Z) = X[\omega(Y,Z)] - \omega(Y, \nabla_X Z) = X[\omega(Y,Z)] - \omega(Y, \nabla_Z X) - \omega(Y, [X,Z]),$$

where the second equality holds because, by hypothesis, the connection is torsion-free. Since ∇ preserves L we have that, if $X \in \Gamma(L)$, then $\nabla_Z X \in \Gamma(L)$. Furthermore, if $X, Y \in \Gamma(L)$, then, since the distribution is Lagrangian, we have that $\omega(Y, \nabla_Z X) = 0$, and

$$\omega(\nabla_X Y, Z) = X[\omega(Y,Z)] - \omega(Y, [X,Z]).$$

This last equation agrees with equation (2) for the Bott connection. ∎

3 Darboux–Nijenhuis coordinates

Suppose ω_1 and ω_2 are Magri-compatible symplectic forms on a smooth $2n$-dimensional manifold M, and let $N = \omega_2^\sharp \omega_1^\flat \colon TM \to TM$ be the recursion operator.

Magri's notion of compatibility can be equivalently expressed by saying that the *Nijenhuis torsion* of the recursion operator vanishes for all vector fields X, Y, that is,

$$T_N(X,Y) = [NX, NY] - N[NX, Y] - N[X, NY] + N^2[X,Y] = 0.$$

A proof of this fact can be found, for instance, in [11]. A tensor field with vanishing torsion is called a *Nijenhuis* tensor field. A Nijenhuis tensor field is *compatible* with a symplectic form ω if

1) $\omega^\flat N = N^* \omega^\flat$,
2) $\mathbf{d}\omega(NX, Y, \cdot) - \mathbf{d}\omega(NY, X, \cdot) + \mathbf{d}\Omega(Y, X, \cdot) = 0$ for all X, Y,

where N^* is the adjoint tensor of N, and Ω is defined by the following expression

$$\Omega(X,Y) = \langle \Omega^\flat X, Y\rangle = \langle (\omega^\flat N)X, Y\rangle.$$

The first condition ensures Ω is skew symmetric, while the second ensures that Ω is closed.

A triple (M,ω,N), where ω is a symplectic form and N is a compatible Nijenhuis tensor field, is called an *ωN manifold*. The definition of ωN manifold first appeared in the work of Magri and Morosi [12]. A manifold M with two Magri-compatible symplectic forms is an important example of an ωN manifold.

If (M,ω_1,ω_2,X) is a bi-Hamiltonian system (in Magri's sense) then it is possible to use the recursion operator to create a sequence of functions in involution that commute with the Hamiltonians H_1 and H_2. Then, if a sufficient number of integrals are functionally independent, the system is completely integrable. Such sequence of functions can be constructed by using the trace of powers of the recursion operator as follows

$$I_k = \frac{1}{k}\operatorname{Tr}\left(N^k\right).$$

Since $T_N(X,Y) = 0$ it can be shown that the differentials $\mathbf{d}I_k$ satisfy the Lenard recursion relation $N^*\mathbf{d}I_k = \mathbf{d}I_{k+1}$, or equivalently, since $N^* = \omega_1^\flat\omega_2^\sharp$, they satisfy $\omega_2^\sharp\cdot\mathbf{d}I_k = \omega_1^\sharp\cdot\mathbf{d}I_{k+1}$. Since the $\mathbf{d}I_k$'s fulfill the recursion relation it is easy to show that $\{I_i,I_j\}_1 = \{I_i,I_j\}_2 = 0$, where $\{\,,\,\}_1$ and $\{\,,\,\}_2$ are the Poisson brackets associated with ω_1 and ω_2, respectively. Thus, the I_j are in involution. For a more detailed account of this construction we refer the reader to [11, 12]. It is convenient to give the following definition (see for example [14]):

Definition 4. A point p of an ωN manifold is a *regular point* if N has the maximal number $n = \frac{1}{2}\dim(M)$ of distinct, functionally independent, eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ (i.e., the differentials $\mathbf{d}\lambda_1,\ldots,\mathbf{d}\lambda_n$ are linearly independent at p). An open set $U\subset M$ is called *regular* if each point of U is a regular point.

We remark that, in the definition above, it is enough to require that the eigenvalues are distinct and non-constant, since the latter automatically implies their independence.

Furthermore, note that, in a regular open set U, since the λ_i are functionally independent, it follows that the I_i's are also functionally independent (see, for instance, [4] for a proof), and so the I_i define a bi-Lagrangian foliation, namely, a foliation Lagrangian with respect to ω_1 and ω_2. In this setting, it can also be shown that the eigenvalues are in involution $\{\lambda_i,\lambda_j\}_1 = \{\lambda_i,\lambda_j\}_2 = 0$ (see [4, 11]), and the λ_i also define the same bi-Lagrangian foliation. With these definitions we have the following proposition.

Proposition 6. *Let (M,ω_1,N) be an ωN manifold. Each regular point has an open neighborhood where there exist coordinates $(\boldsymbol{\lambda},\boldsymbol{\mu}) = (\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n)$ (where the λ_i's are the eigenvalues of N), called Darboux–Nijenhuis coordinates, satisfying the following two properties:*

1) $\omega_1 = \sum_i \mathbf{d}\lambda_i\wedge\mathbf{d}\mu_i$, *that is, they are Darboux-coordinates for* ω_1,
2) $N^*\mathbf{d}\lambda_i = \lambda_i\mathbf{d}\lambda_i$ *and* $N^*\mathbf{d}\mu_i = \lambda_i\mathbf{d}\mu_i$.

See [11] for a sketch of the proof of this statement.

In these coordinates the tensor N takes the diagonal form

$$N = \begin{bmatrix}\boldsymbol{\Lambda}_n & \mathbf{0}_n\\ \mathbf{0}_n & \boldsymbol{\Lambda}_n\end{bmatrix},$$

where $\mathbf{\Lambda}_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, and $\mathbf{0}_n$ is the $n \times n$ matrix with zero entries. Moreover $\omega_2^\sharp = N(\omega_1^\flat)^{-1}$ takes the form

$$\omega_2^\sharp = \begin{bmatrix} \mathbf{0}_n & \mathbf{\Lambda}_n \\ -\mathbf{\Lambda}_n & \mathbf{0}_n \end{bmatrix},$$

and since the matrix of $\omega_2^\sharp$ is the inverse of that of $\omega_2^\flat$ and the matrix of $\omega_2^\flat$ is the negative of that of ω_2 we have

$$\omega_2 = \begin{bmatrix} \mathbf{0}_n & \mathbf{\Lambda}_n^{-1} \\ -\mathbf{\Lambda}_n^{-1} & \mathbf{0}_n \end{bmatrix}.$$

Remark 2. The definition of Darboux–Nijenhuis coordinates can be generalized to each set of Darboux coordinates in which N takes the diagonal form. With such more general definition, one can manage the cases in which the eigenvalues are not independent.

4 Bi-affine compatibility

We now recall the definition of compatibility due to Fassò and Ratiu (see [5]) in the special case the fibration is not only bi-isotropic but also bi-Lagrangian, since this is the only relevant case for our purposes. We refer to this type of compatibility as bi-affine compatibility.

Definition 5. Let M be a smooth manifold of dimension $2n$ and let ω_1 and ω_2 be two symplectic structures on M. Assume there exists a bi-Lagrangian fibration π of M with compact connected fibers which is bi-affine (i.e., the restriction of the Bott connections ∇_1^B and ∇_2^B associated with ω_1 and ω_2 to the fibers, coincide). Then we say that ω_1 and ω_2 are *bi-affinely compatible* or *π-compatible.*

5 Relationship between Magri's notion of compatibility and bi-affine compatibility

Suppose we have a manifold M with two Magri-compatible symplectic forms ω_1 and ω_2. If every point is regular, then, as we mentioned in Section 3, the I_k's (with $I_k = \frac{1}{k}\operatorname{Tr}(N^k)$) define a bi-Lagrangian foliation with associated distribution L. The link between Magri's notion of compatibility and bi-affine compatibility is given in the following theorem.

Theorem 2. *Under the hypothesis above the Bott connections*

$$\left(\nabla_B^1\right)_X Y = \omega_1^\sharp \mathcal{L}_X(\omega_1^\flat Y) \qquad \text{and} \qquad \left(\nabla_B^2\right)_X Y = \omega_2^\sharp \mathcal{L}_X(\omega_2^\flat Y)$$

in L coincide. If, in particular, the Lagrangian foliation is a Lagrangian fibration with compact connected fibers then Magri's compatibility implies bi-affine compatibility.

Proof. Let $X_k = \omega_1^\sharp \cdot \mathbf{d}I_{k+1}$ for $k = 1, \ldots, n$. Since ω_1 and ω_2 are Magri compatible, as we mentioned in Section 3, it follows that

$$X_k = \omega_1^\sharp \cdot \mathbf{d}I_{k+1} = \omega_2^\sharp \cdot \mathbf{d}I_k, \qquad k = 1, \ldots, n.$$

Hence, the X_k's are Hamiltonian with respect to both symplectic structures and, by Theorem 1, these vector fields are parallel with respect to both ∇_B^1 and ∇_B^2. Since the X_k's span the tangent space to the leaf, by the uniqueness results of Theorem 1 and the following remark, it may be concluded that the connections ∇_B^1 and ∇_B^2 coincide. ∎

Under the hypothesis of the theorem above, as a consequence of Proposition 6, we have that, in a neighborhood of every point, there exist Darboux–Nijenhuis coordinates. As a consequence, we can give an alternative proof of the theorem above that employs symplectic connections and Darboux–Nijenhuis coordinates. This approach requires constructing explicitly the symplectic connections associated with ω_1 and ω_2. This will be done in the proof of the following theorem.

Theorem 3. *Under the hypothesis above there are two torsion-free symplectic connections ∇^1 and ∇^2, symplectic with respect to ω_1 and ω_2, respectively, and such that*

1) *they preserve the Lagrangian distribution L associated to the foliation defined by the eigenvalues $\lambda_1, \dots, \lambda_n$,*
2) *the restrictions of ∇^1 and ∇^2 to L coincide with the Bott connections defined by $(\nabla^1_B)_X Y = \omega_1^\sharp \mathcal{L}_X(\omega_1^\flat Y)$ and $(\nabla^2_B)_X Y = \omega_2^\sharp \mathcal{L}_X(\omega_2^\flat Y)$, respectively,*
3) *the restrictions of ∇^1 and ∇^2 to L coincide, and thus $\nabla^1_B = \nabla^2_B$.*

In this approach the link between Magri's notion of compatibility and bi-affine compatibility follows immediately from Theorem 3 and is given in the following corollary.

Corollary 1. *If the bi-Lagrangian foliation is a fibration with compact connected fibers, then Magri's compatibility implies bi-affine compatibility.*

Proof of Theorem 3. (1) We construct the symplectic connections explicitly. The fact that ∇^1 and ∇^2 are symplectic and preserve L will follow from the construction. Take an atlas of M composed by Darboux–Nijenhuis charts. On each chart one can construct a torsion-free flat connection, symplectic with respect to ω_1 and preserving L, by taking the linear connection whose Christoffel symbols vanish identically in these coordinates. One can then obtain a global connection by using partitions of unity to "glue" the connections obtained in each Darboux–Nijenhuis chart. This construction gives the connection ∇^1. For a more detailed explanation of this process see the proof of Theorem 2 in [7].

We now explain the construction of ∇^2 in detail. Suppose we are in a Darboux–Nijenhuis chart. Let $\mathbf{z} = (\boldsymbol{\lambda}, \boldsymbol{\mu})$, then $\mathbf{e}_i = \frac{\partial}{\partial z^i}$ is a basis of tangent vectors. In these coordinates, the vanishing of the covariant derivative of ω_2 is

$$\left(\nabla^2_{\mathbf{e}_k} \omega_2\right)_{ij} = \frac{\partial}{\partial z^k}(\omega_2)_{ij} - \sum_l \Gamma^l_{ik}(\omega_2)_{lj} - \sum_l \Gamma^l_{jk}(\omega_2)_{il} = 0, \tag{3}$$

where the coefficients Γ^k_{ij} are called Christoffel symbols. Since in Darboux–Nijenhuis coordinates we have

$$\omega_2 = \begin{bmatrix} \mathbf{0}_n & \boldsymbol{\Lambda}_n^{-1} \\ -\boldsymbol{\Lambda}_n^{-1} & \mathbf{0}_n \end{bmatrix},$$

the term $\frac{\partial}{\partial z^k}(\omega_2)_{ij}$ is always zero except for

- $k = i$, $j = i + n$ with $1 \leq i \leq n$, in which case $\frac{\partial}{\partial \lambda i}(\omega_2)_{i(i+n)} = -\frac{1}{\lambda_i^2}$,
- $k = j$, $i = j + n$ with $1 \leq j \leq n$, in which case $\frac{\partial}{\partial \lambda j}(\omega_2)_{(j+n)j} = \frac{1}{\lambda_j^2}$.

Let us take all the Γ^k_{ij} to be zero except the ones of the form $\Gamma^i_{ii} = -\frac{1}{\lambda_i}$ for $1 \leq i \leq n$. With this choice the Christoffel symbols are symmetric, and thus the connection is torsion-free.

To verify that ∇^2 is symplectic with respect to ω_2 we need to check only equation (3) for a few values of i, j and k , since for all the other values the equation is trivially verified. We have

$$\frac{\partial}{\partial z^i}(\omega_2)_{i(i+n)} - \Gamma^i_{ii}(\omega_2)_{i(i+n)} = \frac{\partial}{\partial \lambda_i}(\omega_2)_{i(i+n)} + \frac{1}{\lambda_i^2} = -\frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^2} = 0,$$

where $1 \le i \le n$, and

$$\frac{\partial}{\partial z^j}(\omega_2)_{(j+n)j} - \Gamma^j_{jj}(\omega_2)_{(j+n)j} = \frac{\partial}{\partial \lambda_j}(\omega_2)_{(j+n)j} - \frac{1}{\lambda_j^2} = \frac{1}{\lambda_j^2} - \frac{1}{\lambda_j^2} = 0,$$

where $1 \le j \le n$. Therefore, the connection ∇^2 is symplectic with respect to ω_2.

To verify that ∇^2 parallelizes L we use the following coordinate formula

$$\nabla^2_{\mathbf{e}_j}\mathbf{u} = \left(\frac{\partial u^i}{\partial z^j} + u^k \Gamma^i_{jk}\right)\mathbf{e}_i,$$

where, in general, $\mathbf{u} = \sum_i u^i \mathbf{e}_i$ is a vector field in $\Gamma(TM)$. Now suppose $\mathbf{u} \in \Gamma(L)$, then $\mathbf{u} = \sum i = n + 1^{2n} u^i \mathbf{e}_i$. Let $\mathbf{v} = \nabla^2_{\mathbf{e}_j}\mathbf{u}$. Since $\Gamma^k_{ij} \neq 0$ if and only if $i = j = k$, it follows that $\mathbf{v}$ is of the form

$$\mathbf{v} = \sum_{i=n+1}^{2n} a^i \mathbf{e}_i.$$

Since $u^k \neq 0$ only if $k \ge n+1$ and $\Gamma^k_{ij} \neq 0$ only when $1 \le i \le n$, then

$$a^i = \begin{cases} \dfrac{\partial u^i}{\partial z^j}, & \text{if } \ i \ge n+1, \\ 0, & \text{in all other cases.} \end{cases}$$

Thus, $\mathbf{v} \in \Gamma(L)$, that is, ω_2 parallelizes L.

Now, taking an atlas covered with Darboux–Nijenhuis charts, passing to a locally finite refinement $(U_\alpha)_{\alpha \in A}$, denoting the corresponding family of linear connections constructed as above by $(\nabla^2_\alpha)_{\alpha \in A}$ and choosing a partition of unity $(\chi_\alpha)_{\alpha \in A}$ subordinate to the open covering $(U_\alpha)_{\alpha \in A}$, we can define

$$\nabla^2 = \sum_{\alpha \in A} \chi_\alpha \nabla^2_\alpha.$$

The conditions of parallelizing a given differential form, of parallelizing a given vector subbundle, and of being torsion-free are all local as well as affine. Thus, since each of the ∇^2_α parallelizes ω_2 and L, it follows that ∇^2 also parallelizes ω_2 and L.

(2) This part of the theorem follows immediately from Proposition 5.

(3) On each Darboux–Nijenhuis chart the restrictions of the connections ∇^1 and ∇^2 to the leaves of the foliation coincide, since for both connections the restriction of the Christoffel symbols will be of the form $\widetilde{\Gamma}^k_{ij} = \Gamma^{k+n}_{(i+n)(j+n)}$ with $1 \le i, j, k \le n$, and hence the Christoffel symbols $\widetilde{\Gamma}^k_{ij}$ of both connections vanish identically in these coordinates. ■

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References

[1] Bieliavsky P., Cahen M., Gutt S., Rawnsley J., Schwachhöfer L., Symplectic connections, *Int. J. Geom. Methods Mod. Phys.* **3** (2006), 375–420, math.SG/0511194.

[2] Bogoyavlenskij O.I., Theory of tensor invariants of integrable Hamiltonian systems. I. Incompatible Poisson structures, *Comm. Math. Phys.* **180** (1996), 529–586.

[3] Brouzet R., Systèmes bihamiltoniens et complète intégrabilité en dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 895–898.

[4] Falqui G., Pedroni M., Poisson pencils, algebraic integrability, and separation of variables, *Regul. Chaotic Dyn.* **16** (2011), 223–244.

[5] Fassò F., Ratiu T., Compatibility of symplectic structures adapted to noncommutatively integrable systems, *J. Geom. Phys.* **27** (1998), 199–220.

[6] Fernandes R.L., Completely integrable bi-Hamiltonian systems, *J. Dynam. Differential Equations* **6** (1994), 53–69.

[7] Forger M., Yepes S.Z., Lagrangian distributions and connections in multisymplectic and polysymplectic geometry, *Differential Geom. Appl.* **31** (2013), 775–807, arXiv:1202.5054.

[8] Gel'fand I.M., Dorfman I.Ja., Hamiltonian operators and algebraic structures related to them, *Funct. Anal. Appl.* **13** (1979), 248–262.

[9] Lee J.M., Introduction to smooth manifolds, *Graduate Texts in Mathematics*, Vol. 218, Springer-Verlag, New York, 2003.

[10] Magri F., A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978), 1156–1162.

[11] Magri F., Casati P., Falqui G., Pedroni M., Eight lectures on integrable systems, in Integrability of Nonlinear Systems, *Lecture Notes in Phys.*, Vol. 638, Editors Y. Kosmann-Schwarzbach, K.M. Tamizhmani, B. Grammaticos, Springer, Berlin, 2004, 209–250.

[12] Magri F., Morosi C., A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds, Quaderni del Dipartimento di Matematica, Università di Milano, 1984.

[13] Olver P.J., Canonical forms and integrability of bi-Hamiltonian systems, *Phys. Lett. A* **148** (1990), 177–187.

[14] Tondo G., Generalized Lenard chains and separation of variables, *Quad. Mat. Univ. Trieste* **573** (2006), 1–27.

[15] Turiel F.-J., Classification locale simultanée de deux formes symplectiques compatibles, *Manuscripta Math.* **82** (1994), 349–362.