Orthogonal Polynomials on the Unit Ball and Fourth-Order Partial Differential Equations*

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Abstract. The purpose of this work is to analyse a family of mutually orthogonal polynomials on the unit ball with respect to an inner product which includes an additional term on the sphere. First, we will get connection formulas relating classical multivariate orthogonal polynomials on the ball with our family of orthogonal polynomials. Then, using the representation of these polynomials in terms of spherical harmonics, algebraic and differential properties will be deduced.

Key words: multivariate orthogonal polynomials; unit ball; partial differential equations

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1 Introduction

In 1940 (see [8]), H.L. Krall classified orthogonal polynomials satisfying fourth-order differential equations. Apart from the classical orthogonal polynomials we can only find three other families, the now called Jacobi-type, Legendre-type and Laguerre-type polynomials. The corresponding orthogonality measures are some particular cases of classical measures modified by the addition of a Dirac delta at the end points of the interval of orthogonality. Further details on those families of polynomials were given later by A.M. Krall [7], L.L. Littlejohn [10] and T.H. Koornwinder [6].

Our main goal in this work is to extend the study of orthogonal polynomials satisfying fourth-order differential equations to a multidimensional context. In particular, we consider a nonclassical weight function supported on the d-dimensional unit ball and we will show that the corresponding orthogonal polynomials will satisfy a fourth-order partial differential equation.

In our study, classical orthogonal polynomials on the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$ play an essential role. They are orthogonal with respect to the inner product

$$\langle f,g\rangle_\mu := \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx,$$

where $W_\mu(x)$ is the weight function given by

$$W_\mu(x) := \left(1-\|x\|^2\right)^{\mu-1/2}$$

on $\mathbb{B}^d$, $\mu > -1/2$, and ω_μ is a normalizing constant such that $\langle 1,1\rangle_\mu = 1$. The condition $\mu > -1/2$ is necessary for the inner product to be well defined.

In the present paper, we will consider orthogonal polynomials with respect to the inner product

$$\langle f,g\rangle_\mu^\lambda = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx + \frac{\lambda}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}} f(x)g(x)d\sigma, \tag{1.1}$$

*This paper is a contribution to the Special Issue on Orthogonal Polynomials, Special Functions and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/OPSFA2015.html where $\lambda > 0$, $d\sigma$ denotes the surface measure on the unit sphere $\mathbb{S}^{d-1}$, and σ_{d-1} denotes the area of $\mathbb{S}^{d-1}$. The presence of the extra spherical term in (1.1) changes the importance of the values on the sphere of a given function f when approximating by means of the corresponding Fourier series.

Using spherical harmonics and polar coordinates we shall construct explicitly a sequence of mutually orthogonal polynomials with respect to $\langle f, g\rangle_\mu^\lambda$, which depends on a family of polynomials of one variable. The latter ones can be expressed in terms of Jacobi polynomials. A similar construction was used in [2] and [11] to obtain a sequence of mutually orthogonal polynomials with respect to a Sobolev inner product. In [2], the inner product includes the outward normal derivatives over the sphere and aside from getting the connection with classical polynomials on the ball, some asymptotic properties are obtained. On the other hand, the Sobolev inner product in [11] was defined in terms of the standard gradient operator.

Then, in the case $\mu = 1/2$, we will show that the real space of polynomials in d variables admits a mutually orthogonal base whose elements are eigenfunctions of a fourth-order linear partial differential operator where the coefficients are independent of the degree of the polynomials. The corresponding eigenvalues depend on the different indices of the polynomials. Thus, we get a non-trivial example of a family of multivariate polynomials orthogonal with respect a measure and satisfying a fourth-order differential equation.

Multivariate polynomials satisfying fourth-order differential equations were constructed previously in [4, Section 6.2]. In this work, the author uses also polar coordinates and spherical harmonics to construct a basis of polynomials which are eigenfunctions of a differential operator where the polynomial coefficients are independent of the indexes of the polynomials. The eigenvalues in [4] depend only on the total degree of the polynomials. However, the polynomials are orthogonal with respect to an inner product involving the spherical Laplacian, that is, a Sobolev product (see [4, equation (6.8)]). Our approach is different, we start from the standard inner product defined in (1.1) and we find a fourth-order partial differential operator having the orthogonal polynomials as eigenfunctions.

The paper is organized as follows. In the next section, we state some materials on orthogonal polynomials on the unit ball and spherical harmonics that we will need later. In Section 3, using spherical harmonics, a basis of mutually orthogonal polynomials associated to (1.1) is constructed. In Section 4, in the case $\mu = 1/2$, we consider the differential properties of the polynomials in the radial parts of the polynomials in the previous section. Finally, in Section 5, we show that these polynomials are eigenfunctions of a fourth-order differential operator where the coefficients do not depend on the degree of the polynomials.

2 Classical orthogonal polynomials on the ball

In this section we describe background materials on orthogonal polynomials and spherical harmonics. The first subsection collects properties on the Jacobi polynomials that we shall need later. The second subsection is devoted to the Jacobi-type orthogonal polynomials, associated to a weight function obtained as a linear combination of the classical Jacobi weight function and a delta function at 1. The third subsection recalls the basic results on spherical harmonics and classical orthogonal polynomials on the unit ball.

2.1 Classical Jacobi polynomials

We collect some properties of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$, all of them can be found in [12]. For $\alpha, \beta > -1$, these polynomials are orthogonal with respect to the Jacobi inner product

$$(f,g)_{\alpha,\beta} = \int_{-1}^{1} f(t)g(t)(1-t)^{\alpha}(1+t)^{\beta}dt.$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ is normalized by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}.$$

The derivative of a Jacobi polynomial is again a Jacobi polynomial

$$\frac{d}{dt}P_n^{(\alpha,\beta)}(t) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(t). \tag{2.1}$$

The polynomials $P_n^{(\alpha,\beta)}(t)$ satisfy the second-order differential equation

$$\left(1-t^2\right)y''(t) + [\beta-\alpha-(\alpha+\beta+2)t]y'(t) = -n(n+\alpha+\beta+1)y(t), \tag{2.2}$$

where $y(t) = P_n^{(\alpha,\beta)}(t)$.

2.2 Orthogonal polynomials on the unit ball and spherical harmonics

For a multi-index $\nu \in \mathbb{N}_0^d$, $\nu = (\nu_1, \ldots, \nu_d)$ and $x = (x_1, \ldots, x_d)$, a monomial in the variables $x_1, \ldots, x_d$ is a product

$$x^{\nu} = x_1^{\nu_1} \cdots x_d^{\nu_d}.$$

The number $|\nu| = \nu_1 + \cdots + \nu_d$ is called the total degree of x^{ν}. A polynomial P in d variables is a finite linear combination of monomials.

Let Π^d denote the space of polynomials in d real variables. For a given nonnegative integer n, let Π_n^d denote the linear space of polynomials in several variables of (total) degree at most n and let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n.

The unit ball and the unit sphere in $\mathbb{R}^d$ are denoted, respectively, by

$$\mathbb{B}^d := \left\{x \in \mathbb{R}^d \colon \|x\| \leqslant 1\right\} \qquad \text{and} \qquad \mathbb{S}^{d-1} := \left\{\xi \in \mathbb{R}^d \colon \|\xi\| = 1\right\}.$$

where $\|x\|$ denotes as usual the Euclidean norm of x.

For $\mu \in \mathbb{R}$, the weight function $W_{\mu}(x) = (1-\|x\|^2)^{\mu-1/2}$ is integrable on the unit ball if $\mu > -1/2$. Let us denote the normalization constant of W_{μ} by ω_{μ},

$$\omega_{\mu} := \int_{\mathbb{B}^d} W_{\mu}(x)dx = \frac{\pi^{d/2}\Gamma(\mu+1/2)}{\Gamma(\mu+(d+1)/2)}, \tag{2.3}$$

and consider the inner product

$$\langle f,g\rangle_{\mu} = \frac{1}{\omega_{\mu}}\int_{\mathbb{B}^d} f(x)g(x)W_{\mu}(x)dx,$$

which is normalized so that $\langle 1,1\rangle_{\mu} = 1$.

A polynomial $P \in \Pi_n^d$ is called *orthogonal* with respect to W_{μ} on the ball if $\langle P,Q\rangle_{\mu} = 0$ for all $Q \in \Pi_{n-1}^d$, that is, if it is orthogonal to all polynomials of lower degree. Let $\mathcal{V}_n^d(W_{\mu})$ denote the space of orthogonal polynomials of total degree n with respect to W_{μ}.

For $n \geq 0$, let $\{P_{\nu}^n(x)\colon |\nu| = n\}$ denote a basis of $\mathcal{V}_n^d(W_{\mu})$. Notice that every element of $\mathcal{V}_n^d(W_{\mu})$ is orthogonal to polynomials of lower degree. If the elements of the basis are also orthogonal to each other, that is, $\langle P^n_\nu, P^n_\eta\rangle_\mu = 0$ whenever $\nu \neq \eta$, we call the basis *mutually orthogonal*. If, in addition, $\langle P^n_\nu, P^n_\nu\rangle_\mu = 1$, we call the basis *orthonormal*.

Harmonic polynomials of degree n in d-variables are polynomials in $\mathcal{P}^d_n$ that satisfy the Laplace equation $\Delta Y = 0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the usual Laplace operator.

Let $\mathcal{H}^d_n$ denote the space of harmonic polynomials of degree n. It is well known that

$$a^d_n := \dim \mathcal{H}^d_n = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}.$$

Spherical harmonics are the restriction of harmonic polynomials to the unit sphere. If $Y \in \mathcal{H}^d_n$, then in spherical-polar coordinates $x = r\xi$, $r > 0$ and $\xi \in \mathbb{S}^{d-1}$, we get

$$Y(x) = r^n Y(\xi),$$

so that Y is uniquely determined by its restriction to the sphere. We shall also use $\mathcal{H}^d_n$ to denote the space of spherical harmonics of degree n.

Let $d\sigma$ denote the surface measure and σ_{d-1} denote the surface area,

$$\sigma_{d-1} := \int_{\mathbb{S}^{d-1}} d\sigma = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{2.4}$$

Spherical harmonics of different degrees are orthogonal with respect to the inner product

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} := \frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi).$$

If $Y(x)$ is a harmonic polynomial of degree n, by Euler's equation for homogeneous polynomials, we deduce

$$\langle x, \nabla\rangle Y(x) = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} Y(x) = nY(x).$$

In spherical-polar coordinates $x = r\xi$, $r > 0$ and $\xi \in \mathbb{S}^{d-1}$, the differential operators Δ and $\langle x, \nabla\rangle$ can be decomposed as follows (cf. [1]):

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{2.5}$$

$$\langle x, \nabla\rangle = r\frac{\partial}{\partial r}. \tag{2.6}$$

The operator Δ_0, which is the spherical part of the Laplacian, is called the Laplace–Beltrami operator and it has spherical harmonics as its eigenfunctions. More precisely, it holds (cf. [1])

$$\Delta_0 Y(\xi) = -n(n+d-2)Y(\xi), \qquad \forall\, Y \in \mathcal{H}^d_n, \qquad \xi \in \mathbb{S}^{d-1}. \tag{2.7}$$

In spherical-polar coordinates $x = r\xi$, $r > 0$ and $\xi \in \mathbb{S}^{d-1}$, a mutually orthogonal basis of $\mathcal{V}^d_n(W_\mu)$ can be given in terms of the Jacobi polynomials and spherical harmonics (see, for instance, [3]).

Proposition 2.1. *For $n \in \mathbb{N}_0$ and $0 \leqslant k \leqslant n/2$, let $\{Y^{n-2k}_\nu(x)\colon 1 \leqslant \nu \leqslant a^d_{n-2k}\}$ denote an orthonormal basis for $\mathcal{H}^d_{n-2k}$. Define*

$$P^n_{k,\nu}(x) = P_k^{(\mu-\frac{1}{2},\beta_k)}\big(2\|x\|^2 - 1\big) Y^{n-2k}_\nu(x), \tag{2.8}$$

where $\beta_k = n - 2k + \frac{d-2}{2}$. Then the set $\{P^n_{k,\nu}(x)\colon 0 \leqslant k \leqslant n/2,\ 1 \leqslant \nu \leqslant a^d_{n-2k}\}$ is a mutually orthogonal basis of $\mathcal{V}^d_n(W_\mu)$.

It is known that orthogonal polynomials with respect to W_μ are eigenfunctions of a second-order differential operator $\mathcal{D}_\mu$. More precisely, we have

$$\mathcal{D}_\mu P = -(n+d)(n+2\mu-1)P, \qquad \forall\, P \in \mathcal{V}_n^d(W_\mu), \tag{2.9}$$

where

$$\mathcal{D}_\mu := \Delta - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} x_j \left[2\mu - 1 + \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}\right].$$

3 An inner product on the ball with an extra spherical term

Let us define the inner product

$$\langle f, g\rangle_\mu^\lambda = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx + \frac{\lambda}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\sigma,$$

where $\lambda > 0$, $d\sigma$ denotes the surface measure on the unit sphere $\mathbb{S}^{d-1}$, σ_{d-1} denotes the area of $\mathbb{S}^{d-1}$ and ω_μ is the normalizing constant (2.3).

As a consequence of the central symmetry of the inner product, we can use a construction analogous to (2.8) to obtain a basis of mutually orthogonal polynomials with respect to $\langle f, g\rangle_\mu^\lambda$. This time, the radial parts constitute a family of polynomials in one variable related to Jacobi polynomials.

Theorem 3.1. *For $n \in \mathbb{N}_0$ and $0 \le k \le n/2$, let $\{Y_\nu^{n-2k} \colon 1 \le \nu \le a_{n-2k}^d\}$ denote an orthonormal basis for $\mathcal{H}_{n-2k}^d$. Let $\beta_k = n - 2k + (d-2)/2$ and let $q_k^{(\mu-1/2,\beta_k,\lambda)}(t)$ be the k-th orthogonal polynomial with respect to*

$$(f,g)_{\mu-1/2,\beta_k}^\lambda = \frac{\sigma_{d-1}}{\omega_\mu}\frac{1}{2^{\mu+\beta_k+3/2}}\int_{-1}^{1} f(t)g(t)(1-t)^{\mu-1/2}(1+t)^{\beta_k}dt + \lambda f(1)g(1). \tag{3.1}$$

Then the polynomials

$$Q_{k,\nu}^n(x) = q_k^{(\mu-1/2,\beta_k,\lambda)}\big(2\|x\|^2 - 1\big)Y_\nu^{n-2k}(x),$$

with $1 \le k \le n/2$, $1 \le \nu \le a_{n-2k}^d$ constitute a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu, \lambda)$ the linear space of orthogonal polynomials of degree exactly n with respect to $\langle\cdot,\cdot\rangle_\mu^\lambda$.

Proof. The proof of this theorem uses the following well known identity

$$\int_{\mathbb{B}^d} f(x)dx = \int_0^1 r^{d-1}\int_{\mathbb{S}^{d-1}} f(r\xi)d\sigma(\xi)dr \tag{3.2}$$

that arises from the spherical-polar coordinates $x = r\xi$, $\xi \in \mathbb{S}^{d-1}$.

In order to check the orthogonality, we need to compute the product

$$\langle Q_{j,\nu}^n, Q_{k,\eta}^m\rangle_\mu^\lambda = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} Q_{j,\nu}^n(x)Q_{k,\eta}^m(x)W_\mu(x)dx + \frac{\lambda}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}} Q_{j,\nu}^n(\xi)Q_{k,\eta}^m(\xi)d\sigma(\xi). \tag{3.3}$$

Let us start with the computation of the first integral

$$I_1 = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} Q_{j,\nu}^n(x)Q_{k,\eta}^m(x)W_\mu(x)dx.$$

Using polar coordinates, relation (3.2), and the orthogonality of the spherical harmonics we obtain

$$
\begin{aligned}
I_1 &= \frac{\sigma_{d-1}}{\omega_\mu}\int_0^1 q_j^{(\mu-1/2,\beta_j,\lambda)}\big(2r^2-1\big)q_k^{(\mu-1/2,\beta_k,\lambda)}\big(2r^2-1\big)\big(1-r^2\big)^{\mu-1/2}r^{n-2j+m-2k+d-1}dr \\
&\quad\times \delta_{n-2j,m-2k}\delta_{\nu\eta} \\
&= \frac{\sigma_{d-1}}{\omega_\mu}\int_0^1 q_j^{(\mu-1/2,\beta_j,\lambda)}\big(2r^2-1\big)q_k^{(\mu-1/2,\beta_j,\lambda)}\big(2r^2-1\big)\big(1-r^2\big)^{\mu-1/2}r^{2(n-2j)+d-1}dr \\
&\times \delta_{n-2j,m-2k}\delta_{\nu\eta}.
\end{aligned}
$$

Finally, the change of variables $t = 2r^2-1$ moves the integral to the interval $[-1,1]$,

$$
I_1 = \frac{1}{2^{\beta_j+\mu+3/2}}\frac{\sigma_{d-1}}{\omega_\mu}\int_{-1}^1 q_j^{(\mu,\beta_j,\lambda)}(t)q_k^{(\mu,\beta_j,\lambda)}(t)(1-t)^{\mu-1/2}(1+t)^{\beta_j}dt\times\delta_{n-2j,m-2k}\delta_{\nu\eta}. \tag{3.4}
$$

Let us now compute the second integral in (3.3),

$$
\begin{aligned}
I_2 &= \frac{\lambda}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}} Q_{j,\nu}^n(\xi)Q_{k,\eta}^m(\xi)d\sigma(\xi) \\
&= \frac{\lambda}{\sigma_{d-1}}q_j^{(\mu-1/2,\beta_j,\lambda)}(1)q_k^{(\mu-1/2,\beta_k,\lambda)}(1)\int_{\mathbb{S}^{d-1}} Y_\nu^{n-2j}(\xi)Y_\eta^{n-2k}(\xi)d\sigma(\xi) \\
&= \lambda q_j^{(\mu-1/2,\beta_j,\lambda)}(1)q_k^{(\mu-1/2,\beta_k,\lambda)}(1)\delta_{n-2j,m-2k}\delta_{\nu,\eta}.
\end{aligned} \tag{3.5}
$$

To end the proof, we just have to put together (3.4) and (3.5) to get the value of (3.3) in terms of the inner product (3.1) as

$$
\langle Q_{j,\nu}^n, Q_{k,\eta}^m\rangle_\mu^\lambda = \big(q_j^{(\mu-1/2,\beta_j,\lambda)}, q_k^{(\mu-1/2,\beta_k,\lambda)}\big)^\lambda_{\mu-1/2,\beta_j}\delta_{n-2j,m-2k}\delta_{\nu,\eta}.
$$

And the result follows from the orthogonality of the univariate polynomial $q_k^{(\mu-1/2,\beta_k,\lambda)}$. ■

Remark 3.2. Of course, Theorem 3.1 can be formulated if we replace the weight function $W_\mu(x)$ by any rotation invariant orthogonality measure on the unit ball.

Following Koornwinder [6, Theorem 3.1], orthogonal polynomials with respect to (3.1) can be written in terms of Jacobi polynomials as we show in the following theorem.

Theorem 3.3. *Let* $\{q_k^{(\alpha,\beta,\lambda)}(t)\}_{k\geqslant 0}$ *be the sequence of polynomials defined by*

$$
q_k^{(\alpha,\beta,\lambda)}(t) = \left[a_k - (1+t)\frac{d}{dt}\right]P_k^{(\alpha,\beta)}(t), \tag{3.6}
$$

where

$$
a_k = \frac{1}{\lambda}\frac{\Gamma(\alpha+\frac{d}{2}+1)}{\Gamma(\frac{d}{2})}\frac{k!}{(\alpha+1)_k}\frac{\Gamma(\beta+k+1)}{\Gamma(\alpha+\beta+k+1)} + \frac{k(k+\alpha+\beta+1)}{\alpha+1}.
$$

Then they are orthogonal with respect to the inner product

$$
(f,g)^\lambda_{\alpha,\beta} = \frac{\Gamma(\alpha+\frac{d}{2}+1)}{\Gamma(\frac{d}{2})\Gamma(\alpha+1)}\frac{1}{2^{\alpha+\beta+1}}\int_{-1}^1 f(t)g(t)(1-t)^\alpha(1+t)^\beta dt + \lambda f(1)g(1), \tag{3.7}
$$

and satisfy the normalization

$$
q_k^{(\alpha,\beta,\lambda)}(1) = \frac{1}{\lambda}\frac{\Gamma(\alpha+\frac{d}{2}+1)}{\Gamma(\frac{d}{2})}\frac{\Gamma(\beta+k+1)}{\Gamma(\alpha+\beta+k+1)}.
$$

Proof. The polynomials defined in (3.6) are of exact degree k and their orthogonality can be deduced using the basis $\{(1-t)^j\}_{0\leqslant j\leqslant k-1}$. From (2.1) and the orthogonality of the Jacobi polynomials we easily deduce

$$\big(q_k^{(\alpha,\beta,\lambda)}(t),(1-t)^j\big)_{\alpha,\beta}^{\lambda}=0,\qquad j=1,\dots,k-1.$$

Now consider the case $j=0$. Then

$$\begin{aligned}\big(q_k^{(\alpha,\beta,\lambda)},1\big)_{\alpha,\beta}^{\lambda}&=\frac{\Gamma(\alpha+\frac{d}{2}+1)}{\Gamma(\frac{d}{2})\Gamma(\alpha+1)}\frac{1}{2^{\alpha+\beta+1}}\int_{-1}^{1}q_k^{(\alpha,\beta,\lambda)}(t)(1-t)^{\alpha}(1+t)^{\beta}dt+\lambda q_k^{(\alpha,\beta,\lambda)}(1)\\&=-\frac{\Gamma(\alpha+\frac{d}{2}+1)}{\Gamma(\frac{d}{2})\Gamma(\alpha+1)}\frac{1}{2^{\alpha+\beta+1}}\int_{-1}^{1}\frac{d}{dt}P_k^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta+1}dt\\&\quad+\lambda\big[a_kP_k^{(\alpha,\beta)}(1)-(k+\alpha+\beta+1)P_{k-1}^{(\alpha+1,\beta+1)}(1)\big].\end{aligned}$$

Next, an iterated integration by parts reduces the integral in the last expression to a beta integral which finally gives

$$\int_{-1}^{1}\frac{d}{dt}P_k^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta+1}dt=2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)},$$

and the result follows from (3.7). ■

The partial differential equation (2.9) satisfied by classical orthogonal polynomials on the ball $\{P_{k,\nu}^n(x)\}$ can be deduced from the second-order differential equation corresponding to the Jacobi polynomials in the radial parts and the fact that spherical harmonics are the eigenfunctions of the Laplace–Beltrami operator.

In the case $\mu=1/2$, we are going to identify the polynomials in the radial parts of $\{Q_{k,\nu}^n(x)\}$ as a family of polynomials satisfying a fourth-order differential equation. Next, using polar coordinates and spherical harmonics we will obtain the explicit expression of a fourth-order partial differential operator having the multivariate orthogonal polynomials as eigenfunctions.

3.1 The case $\mu = 1/2$

In this case, the inner product reduces to

$$\langle f,g\rangle_{1/2}^{\lambda}:=\frac{1}{\omega_{1/2}}\int_{\mathbb{B}^d}f(x)g(x)dx+\frac{\lambda}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}}f(\xi)g(\xi)d\sigma(\xi).$$

The mutually orthogonal basis is given by

$$Q_{k,\nu}^n(x)=q_k^{(0,\beta_k,\lambda)}\big(2\|x\|^2-1\big)Y_\nu^{n-2k}(x),$$

with $1\le k\le n/2$ and $\{Y_\nu^{n-2k}(x)\colon 1\le\nu\le a_{n-2k}^d\}$ an orthonormal basis of spherical harmonics.

The polynomials in the radial part $q_k^{(0,\beta_k,\lambda)}$ are orthogonal with respect to the inner product

$$(f,g)_{0,\beta_k}^{\lambda}=\frac{\sigma_{d-1}}{\omega_{1/2}}\frac{1}{2^{\beta_k+2}}\int_{-1}^{1}f(t)g(t)(1+t)^{\beta_k}dt+\lambda f(1)g(1).$$

Finally, using (2.3) and (2.4) we get $\sigma_{d-1}/\omega_{1/2}=d$, and writing $M=\frac{d}{2\lambda}$ we conclude that they are orthogonal with respect to

$$(f,g)_{\beta_k}^{M}=\frac{1}{2^{\beta_k+1}}\int_{-1}^{1}f(t)g(t)(1+t)^{\beta_k}dt+\frac{1}{M}f(1)g(1).$$

Therefore, we can recognize $q_k^{(0,\beta_k,\lambda)}$ as the polynomials in one of the families studied by H.L. Krall in 1940: the Jacobi-type polynomials.

4 Krall's Jacobi-type orthogonal polynomials

Let $\alpha = 0$, $\beta > -1$ and $M > 0$. We introduce the *Jacobi-type polynomials* as follows

$$q_k^{\beta,M}(t) := \left[M - (1+t)\frac{d}{dt} + k(k+\beta+1)\right] P_k^{(0,\beta)}(t), \qquad k = 0, 1, \dots. \tag{4.1}$$

From Theorem 3.3 we can easily deduce their orthogonality properties. In fact, the polynomials $\{q_k^{\beta,M}(t)\}_{k\geqslant 0}$ are orthogonal with respect to the inner product

$$(f,g)_\beta^M = \frac{1}{2^{\beta+1}}\int_{-1}^1 f(t)g(t)(1+t)^\beta dt + \frac{1}{M}f(1)g(1),$$

and satisfy the normalization $q_k^{\beta,M}(1) = M$.

Using the differential equation for the Jacobi polynomials $\{P_k^{(0,\beta)}(t)\}_{k\geqslant 0}$ (2.2), relation (4.1) can be written as

$$q_k^{\beta,M}(t) = \left[M - \left(1-t^2\right)\frac{d^2}{dt^2} - (\beta+1)(1-t)\frac{d}{dt}\right] P_k^{(0,\beta)}(t). \tag{4.2}$$

Therefore, both families of orthogonal polynomials are connected by means of a second-order linear differential operator whose polynomial coefficients are independent of the degree of the polynomials.

If we denote by $d\mu_\beta$ the measure defined by

$$d\mu_\beta = \frac{1}{2^{\beta+1}}(1+t)^\beta dt + \frac{1}{M}\delta(t-1),$$

integrating by parts we can deduce that for arbitrary polynomials f and g we get

$$\begin{aligned}\int_{-1}^1 g(t)&\left[M - \left(1-t^2\right)\frac{d^2}{dt^2} - (\beta+1)(1-t)\frac{d}{dt}\right] f(t)d\mu_\beta(t) \\ &= \frac{1}{2^{\beta+1}}\int_{-1}^1 f(t)\left[M+\beta+1 - \left(1-t^2\right)\frac{d^2}{dt^2} - (\beta-1-(\beta+3)t)\frac{d}{dt}\right] g(t)(1+t)^\beta dt.\end{aligned} \tag{4.3}$$

Therefore, using the orthogonality properties of $P_k^{(0,\beta)}(t)$ and $q_k^{\beta,M}(t)$, from (4.2) and (4.3) we deduce

$$\begin{aligned}&\left[M+\beta+1 - \left(1-t^2\right)\frac{d^2}{dt^2} - (\beta-1-(\beta+3)t)\frac{d}{dt}\right] q_k^{\beta,M}(t) \\ &\qquad = (M + k(k+\beta))(M + (k+1)(k+\beta+1))P_k^{(0,\beta)}(t),\end{aligned} \tag{4.4}$$

where the coefficient of $P_k^{(0,\beta)}(t)$ has been obtained identifying the leading coefficients in both sides of (4.4).

Finally, if we combine (4.2) and (4.4) we deduce that the polynomials $q_k^{\beta,M}(t)$ are the eigenfunctions of a fourth-order differential operator with polynomials coefficients not depending on k. The quantities $(M + k(k+\beta))(M + (k+1)(k+\beta+1))$ are the corresponding eigenvalues.

Summarizing, we have got the following proposition.

Proposition 4.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ the linear operators defined by*

$$\mathcal{L}_1(f)(t) = \left[M - \left(1-t^2\right)\frac{d^2}{dt^2} - (\beta+1)(1-t)\frac{d}{dt}\right] f(t),$$

$$\mathcal{L}_2(f)(t) = \left[M+\beta+1 - \left(1-t^2\right)\frac{d^2}{dt^2} - (\beta-1-(\beta+3)t)\frac{d}{dt}\right] f(t).$$

Then, the Jacobi-type polynomials $q_k^{\beta,M}(t)$ *satisfy*

$$\mathcal{L}_1 P_k^{(0,\beta)}(t) = q_k^{\beta,M}(t),$$
$$\mathcal{L}_2 q_k^{\beta,M}(t) = (M + k(k+\beta))(M + (k+1)(k+\beta+1))P_k^{(0,\beta)}(t),$$

and the fourth-order differential equation

$$\mathcal{L}_1\mathcal{L}_2 q_k^{\beta,M}(t) = (M + k(k+\beta))(M + (k+1)(k+\beta+1))q_k^{\beta,M}(t).$$

5 A fourth-order partial differential equation

Let us return to the d-dimensionall ball. For $n \in \mathbb{N}_0$ and $0 \leqslant k \leqslant n/2$, let $\{Y_\nu^{n-2k}(x)\colon 1 \leqslant \nu \leqslant a_{n-2k}^d\}$ denote an orthonormal basis for $\mathcal{H}_{n-2k}^d$ and $\beta_k = n - 2k + \frac{d-2}{2}$. According to Proposition 2.1, in spherical polar coordinates $x = r\xi$ the polynomials

$$P_{k,\nu}^n(x) = P_k^{(0,\beta_k)}\big(2r^2-1\big) r^{n-2k} Y_\nu^{n-2k}(\xi),$$

with $0 \leqslant k \leqslant n/2$, $1 \leqslant \nu \leqslant a_{n-2k}^d$ is a mutually orthogonal basis of $\mathcal{V}_n^d(W_{1/2})$. From Theorem 3.1, we deduce that the polynomials

$$Q_{k,\nu}^n(x) = q_k^{\beta_k,M}\big(2r^2-1\big) r^{n-2k} Y_\nu^{n-2k}(\xi), \tag{5.1}$$

with $0 \leqslant k \leqslant n/2$, $1 \leqslant \nu \leqslant a_{n-2k}^d$ constitute a mutually orthogonal basis of $\mathcal{V}_n^d(W_{1/2}, \frac{d}{2M})$.

First, we use the change of variable $t = 2r^2 - 1$ to write relation (4.2) as

$$q_k^{\beta_k,M}\big(2r^2-1\big) = \left[M - \frac{1-r^2}{4}\left(\frac{(2\beta_k+1)}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right)\right] P_k^{(0,\beta_k)}\big(2r^2-1\big), \tag{5.2}$$

and relation (4.4) gives

$$\begin{aligned}\left[M + \beta_k + 1 - \frac{1-r^2}{4}\left(\frac{(2\beta_k+1)}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right) + r\frac{d}{dr}\right] q_k^{\beta_k,M}\big(2r^2-1\big)& \\ = (M + k(k+\beta_k))(M + (k+1)(k+\beta_k+1))P_k^{(0,\beta_k)}\big(2r^2-1\big).&\end{aligned} \tag{5.3}$$

Next, we introduce the factor r^{n-2k} in (5.2) and (5.3). After a tedious but straightforward computation we obtain the linear operators

$$\mathcal{M}_1 = M - \frac{1-r^2}{4}\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{(n-2k+d-2)(n-2k)}{r^2}\right),$$
$$\mathcal{M}_2 = M - \frac{1-r^2}{4}\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{(n-2k+d-2)(n-2k)}{r^2}\right) + \frac{d}{2} + r\frac{d}{dr},$$

which satisfy

$$\mathcal{M}_1\big(r^{n-2k}P_k^{(0,\beta_k)}(2r^2-1)\big) = r^{n-2k}q_k^{\beta_k,M}\big(2r^2-1\big), \tag{5.4}$$
$$\mathcal{M}_2\big(r^{n-2k}q_k^{\beta_k,M}(2r^2-1)\big) = \Lambda_{n,k} r^{n-2k}P_k^{(0,\beta_k)}\big(2r^2-1\big), \tag{5.5}$$

where

$$\begin{aligned}\Lambda_{n,k} &= (M + k(k+\beta_k))(M + (k+1)(k+\beta_k+1)) \\ &= \left(M + k\left(n - k + \frac{d-2}{2}\right)\right)\left(M + (k+1)\left(n-k+\frac{d}{2}\right)\right).\end{aligned}$$

Now, we can recognize the value $-(n-2k+d-2)(n-2k)$ as the eigenvalue of the Laplace–Beltrami operator Δ_0 corresponding to the spherical harmonic $Y_\nu^{n-2k}(\xi)$. Therefore, if we multiply in (5.4) and (5.5) by $Y_\nu^{n-2k}(\xi)$, relations (2.5), (2.6), and the change of variable $x = r\xi$ give

$$\left[M - \frac{1}{4}(1-\|x\|^2)\Delta\right] P_{k,\nu}^n(x) = Q_{k,\nu}^n(x),$$
$$\left[M + \frac{d}{2} - \frac{1}{4}(1-\|x\|^2)\Delta + \langle x, \nabla\rangle\right] Q_{k,\nu}^n(x) = \Lambda_{n,k} P_{k,\nu}^n(x).$$

Finally, if we combine the previous relations we deduce that the polynomials $Q_{k,\nu}^n(x)$ are the eigenfunctions of a fourth-order partial differential operator where the polynomial coefficients do not depend on n or k. The quantities $\Lambda_{n,k}$ are the corresponding eigenvalues.

In conclusion, we have shown the following theorem.

Theorem 5.1. *Let $Q_{k,\nu}^n(x) = q_k^{\beta_k,M}(2r^2-1)r^{n-2k}Y_\nu^{n-2k}(\xi)$ be the polynomials defined in (5.1) for $0 \leqslant k \leqslant n/2$, $1 \leqslant \nu \leqslant a_{n-2k}^d$, wich constitute a mutually orthogonal basis of $\mathcal{V}_n^d(W_{1/2}, \frac{d}{2M})$. Then, the polynomials $Q_{k,\nu}^n(x)$ satisfy the relations*

$$\left[M - \frac{1}{4}(1-\|x\|^2)\Delta\right] P_{k,\nu}^n(x) = Q_{k,\nu}^n(x),$$
$$\left[M + \frac{d}{2} - \frac{1}{4}(1-\|x\|^2)\Delta + \langle x, \nabla\rangle\right] Q_{k,\nu}^n(x) = \Lambda_{n,k} P_{k,\nu}^n(x),$$

and the fourth-order partial differential equation

$$\left[M - \frac{1}{4}(1-\|x\|^2)\Delta\right]\left[M + \frac{d}{2} - \frac{1}{4}(1-\|x\|^2)\Delta + \langle x, \nabla\rangle\right] Q_{k,\nu}^n(x) = \Lambda_{n,k} Q_{k,\nu}^n(x),$$

where

$$\Lambda_{n,k} = \left(M + k\left(n - k + \frac{d-2}{2}\right)\right)\left(M + (k+1)\left(n - k + \frac{d}{2}\right)\right).$$

6 Conclusions

The central symmetry of the measure defining the inner product (1.1) is the key to construct explicitly a sequence of mutually orthogonal polynomials using spherical harmonics and polar coordinates. These orthogonal polynomials depend on a family of polynomials in one variable. The latter ones are orthogonal with respect to a modification of the measure that is the sum of a classical Jacobi measure plus one point mass located at the end point of the interval.

In the case $\mu = 1/2$, the polynomials in the radial parts of $\{Q_{k,\nu}^n(x)\}$ are identified as the Krall's Jacobi-type polynomials, which satisfy a fourth-order differential equation. Next, using some properties of spherical harmonics we obtained the explicit expression of a fourth-order partial differential operator having the multivariate orthogonal polynomials as eigenfunctions. The corresponding eigenvalues depend on the polynomials indices.

In the general case, the polynomials in the radial part are a particular case of the polynomials studied by J. Koekoek and R. Koekoek in [5]. These authors proved that the polynomials satisfy a linear differential equation of a specific form. They gave explicit expressions for the coefficients and showed that this differential equation is always of infinite order except if $\mu = \alpha + 1/2$, with α a nonnegative integer, then the order is equal to $2\alpha + 4$. Closely connected is the work by K.H. Kwon, L.L. Littlejohn and G.J. Yoon in [9].

These results suggest the possible existence of higher-order partial differential operator having the multivariate polynomials $\{Q_{k,\nu}^n(x)\}$ as eigenfunctions in the case $\mu = \alpha + 1/2$, with α a nonnegative integer. This fact deserves further research.

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