Hypergeometric τ Functions of the q-Painlevé Systems of Types $A_4^{(1)}$ and $(A_1 + A_1')^{(1)}$

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Abstract. We consider q-Painlevé equations arising from birational representations of the extended affine Weyl groups of $A_4^{(1)}$- and $(A_1 + A_1)^{(1)}$-types. We study their hypergeometric solutions on the level of τ functions.

Key words: q-Painlevé equation; basic hypergeometric function; affine Weyl group; τ function

2010 Mathematics Subject Classification: 33D05; 33D15; 33E17; 39A13

1 Introduction

1.1 Purpose

The purpose of this paper is to construct the hypergeometric τ functions associated with q-Painlevé equations of $A_4^{(1)}$- and $A_6^{(1)}$-surface types in Sakai's classification [56]. As a corollary, we obtain the hypergeometric solutions of the corresponding q-Painlevé equations.

This work is motivated by the project to construct all possible hypergeometric τ functions associated with the multiplicative surface types in the Sakai's classification [56], that is, $A_0^{(1)}$-, $A_1^{(1)}$-, $A_2^{(1)}$-, $A_3^{(1)}$-, $A_4^{(1)}$-, $A_5^{(1)}$- and $A_6^{(1)}$-surface types. The corresponding symmetry groups are $W(E_8^{(1)})$, $\widetilde{W}(E_7^{(1)})$, $\widetilde{W}(E_6^{(1)})$, $\widetilde{W}(D_5^{(1)})$, $\widetilde{W}(A_4^{(1)})$, $\widetilde{W}((A_2 + A_1)^{(1)})$ and $\widetilde{W}((A_1 + A_1')^{(1)})$, respectively. The works for $W(E_8^{(1)})$-type [41], $\widetilde{W}(E_7^{(1)})$-type [40] and $\widetilde{W}((A_2 + A_1)^{(1)})$-type [43] have been done. In this paper, we consider the hypergeometric τ functions of $\widetilde{W}(A_4^{(1)})$- and $\widetilde{W}((A_1 + A_1')^{(1)})$-types.

1.2 Background

Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which include discrete analogues of the six Painlevé equations, and are classified by types of rational surfaces connected to affine Weyl groups [56]. They admit particular solutions, so called hypergeometric solutions, which are expressible in terms of the hypergeometric type functions, when some of the parameters take special values (see, for example, [30, 31, 33] and references therein). Together with the Painlevé equations, discrete Painlevé equations are now regarded as one of the most important classes of equations in the theory of integrable systems (see, e.g., [14, 35]).

It is well known that the τ functions play a crucial role in the theory of integrable systems [42], and it is also possible to introduce them in the theory of Painlevé systems [20, 21, 22, 45, 47, 48, 49, 50]. A representation of the affine Weyl groups can be lifted on the level of the τ functions [25, 26, 29, 32, 40, 41, 58, 59], which gives rise to various bilinear equations of Hirota type satisfied by the τ functions.

Usually, the hypergeometric solutions of discrete Painlevé equations are derived by reducing the bilinear equations to the Plücker relations by using the contiguity relations satisfied by the entries of determinants [16, 17, 23, 27, 28, 34, 36, 37, 38, 46, 55]. This method is elementary, but it encounters technical difficulties for discrete Painlevé equations with large symmetries. In order to overcome this difficulty, Masuda has proposed a method of constructing hypergeometric solutions under a certain boundary condition on the lattice where the τ functions live, so that they are consistent with the action of the affine Weyl groups. We call such hypergeometric solutions hypergeometric τ functions [40, 41, 43]. Although this requires somewhat complex calculations, the merit is that it is systematic and can be applied to the systems with large symmetries.

Some discrete Painlevé equations have been found in the studies of random matrices [11, 19, 51]. As one such example, let us consider the partition function of the Gaussian Unitary Ensemble of an $n \times n$ random matrix:

$$Z_n^{(2)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Delta(t_1, \ldots, t_n)^2 \prod_{i=1}^{n} e^{-g_1 t_i^2 - g_2 t_i^4} dt_i,$$

where $g_2 > 0$ and $\Delta(t_1, \ldots, t_n)$ is Vandermonde's determinant. Letting

$$R_n = \frac{Z_{n+1}^{(2)} Z_{n-1}^{(2)}}{\left(Z_n^{(2)}\right)^2},$$

we obtain the following difference equation [11, 13, 15, 53]

$$R_{n+1} + R_n + R_{n-1} = \frac{n}{4g_2} \frac{1}{R_n} - \frac{g_1}{2g_2}. \tag{1.1}$$

Equation (1.1) is known as a discrete analogue of the Painlevé I equation and also as a Bäcklund transformation of the Painlevé IV equation. The partition function $Z_n^{(2)}$ corresponds to hypergeometric τ functions. Such relations between discrete Painlevé equations and random matrices are well investigated. Moreover, in recent years, the relations between τ functions of Painlevé systems and a certain class of integrable partial difference equations introduced by Adler–Bobenko–Suris and Boll [1, 2, 8, 9, 10], which include a discrete analogue of the Korteweg–de Vries equation, are well investigated [7, 18, 24, 25, 26]. Throughout these relations and by using the hypergeometric τ functions, a discrete analogue of the power function was derived and its properties, such as discrete analogue of the Riemann surface and circle packing, were shown in [3, 4, 5, 6, 7, 44]. These results consolidate the importance of the studies of the hypergeometric τ function for applications of Painlevé systems.

In [16, 17], the hypergeometric solutions of the q-Painlevé equations (2.32) and (3.1) (or (3.4)) are constructed by solving the minimum required bilinear equations to obtain those equations. In this paper, we solve all bilinear equations arising from the actions of the translation subgroups of $\widetilde{W}\big(A_4^{(1)}\big)$ and $\widetilde{W}\big((A_1 + A_1')^{(1)}\big)$, that is, the hypergeometric τ functions given in Theorems 2.7 and 3.1 are for not only the hypergeometric solutions of the q-Painlevé equations (2.32) and (3.1) but also those of other q-Painlevé equations, e.g., (2.33), (3.2) and (3.3) (see Corollaries 2.9 and 3.2). Moreover, as mentioned above we can derive the various integrable partial difference equations from the τ functions of discrete Painlevé equations (see, for example, [18, 25, 26]). Therefore, the hypergeometric τ functions constructed in this paper also give the hypergeometric solutions of the partial difference equations appeared in [25, 26].

1.3   Plan of the paper

This paper is organized as follows: in Section 2, we first introduce τ functions with the representation of the affine Weyl group $\widetilde{W}\big(A_4^{(1)}\big)$. Next, we construct the hypergeometric τ functions

of $\widetilde{W}\big(A_4^{(1)}\big)$-type (see Theorem 2.7). Finally, we obtain the hypergeometric solutions of the q-Painlevé equations of $A_4^{(1)}$-surface type (see Corollary 2.9). In Section 3, we summarize the result for the $\widetilde{W}\big((A_1 + A_1')^{(1)}\big)$-type (or, $A_6^{(1)}$-surface type).

1.4 q-Special functions

We use the following conventions of q-analysis with $|p|, |q| < 1$ throughout this paper [12].

- q-Shifted factorials:

$$(a;q)_n = \prod_{i=0}^{n-1} \big(1 - q^i a\big), \qquad n = 1, 2, \ldots, \qquad (a;q)_\infty = \prod_{i=0}^{\infty} \big(1 - q^i a\big),$$

$$(a; p, q)_\infty = \prod_{i,j=0}^{\infty} \big(1 - q^i p^j a\big).$$

- Modified Jacobi theta function:

$$\Theta(a;q) = (a;q)_\infty \big(qa^{-1};q\big)_\infty.$$

- Elliptic gamma function:

$$\Gamma(a;p,q) = \frac{\big(pqa^{-1}; p, q\big)_\infty}{(a;p,q)_\infty}.$$

- Basic hypergeometric series:

$$_s\varphi_r \begin{pmatrix} a_1, \ldots, a_s \\ b_1, \ldots, b_r \end{pmatrix}; q, z \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_s; q)_n}{(b_1, \ldots, b_r; q)_n (q; q)_n} \big[(-1)^n q^{n(n-1)/2}\big]^{1+r-s} z^n,$$

  where

$$(a_1, \ldots, a_s; q)_n = \prod_{i=1}^{s} (a_i; q)_n.$$

We note that the following formulae hold

$$\frac{(q^n a; q)_\infty}{(a;q)_\infty} = \prod_{i=0}^{n-1} \frac{1}{1 - q^i a}, \qquad \frac{\Theta(q^n a; q)}{\Theta(a;q)} = (-1)^n \prod_{i=0}^{n-1} \frac{1}{q^i a},$$

$$\frac{(q^n a; p, q)_\infty}{(a; p, q)_\infty} = \prod_{i=0}^{n-1} \frac{1}{(q^i a; p)_\infty}, \qquad \frac{(p^n a; p, q)_\infty}{(a; p, q)_\infty} = \prod_{i=0}^{n-1} \frac{1}{(p^i a; q)_\infty},$$

$$\frac{\Gamma(q^n a; p, q)}{\Gamma(a; p, q)} = \prod_{i=0}^{n-1} \Theta\big(q^i a; p\big), \qquad \frac{\Gamma(p^n a; p, q)}{\Gamma(a; p, q)} = \prod_{i=0}^{n-1} \Theta\big(p^i a; q\big),$$

where $n \in \mathbb{Z}_{>0}$.

2   Hypergeometric τ functions of $\widetilde{W}\big(A_4^{(1)}\big)$-type

In this section, we construct the hypergeometric τ functions of $\widetilde{W}\big(A_4^{(1)}\big)$-type.

2.1 τ functions

Let us consider ten variables: $\tau_i^{(j)}$ $(i = 1, 2,\ j = 1, \ldots, 5)$ and six parameters: $a_0, \ldots, a_4, q \in \mathbb{C}^*$ with the following three relations for the variables

$$\tau_2^{(1)} = \frac{a_0 a_1 \left(a_3 \tau_1^{(3)} \tau_1^{(5)} + a_0 \tau_1^{(4)} \tau_2^{(3)}\right)}{a_2 a_3{}^2 \tau_2^{(5)}}, \tag{2.1a}$$

$$\tau_2^{(2)} = \frac{a_1 a_2 \left(a_4 \tau_1^{(1)} \tau_1^{(4)} + a_1 \tau_1^{(5)} \tau_2^{(4)}\right)}{a_3 a_4{}^2 \tau_2^{(1)}}, \tag{2.1b}$$

$$\tau_2^{(4)} = \frac{a_3 a_4 \left(a_1 \tau_1^{(1)} \tau_1^{(3)} + a_3 \tau_1^{(2)} \tau_2^{(1)}\right)}{a_0 a_1{}^2 \tau_2^{(3)}}, \tag{2.1c}$$

and the following condition for the parameters

$$a_0 a_1 a_2 a_3 a_4 = q.$$

The action of the transformation group $\langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle$ on the parameters is given by

$$s_i(a_j) = a_j a_i{}^{-a_{ij}}, \qquad \sigma(a_i) = a_{i+1},$$
$$\iota \colon\ (a_0, a_1, a_2, a_3, a_4) \mapsto \left(a_0{}^{-1}, a_4{}^{-1}, a_3{}^{-1}, a_2{}^{-1}, a_1{}^{-1}\right),$$

where $i, j \in \mathbb{Z}/5\mathbb{Z}$ and the symmetric 5×5 matrix

$$(a_{ij})_{i,j=0}^4 = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

is the Cartan matrix of type $A_4^{(1)}$. Moreover, the action on the variables is given by

$$s_i\left(\tau_1^{(i+5)}\right) = \tau_2^{(i+4)}, \qquad s_i\left(\tau_2^{(i+3)}\right) = \frac{a_{i+3} a_{i+4} \left(a_i a_{i+1} \tau_1^{(i+1)} \tau_1^{(i+3)} + a_{i+3} \tau_1^{(i+2)} \tau_2^{(i+1)}\right)}{a_{i+1}{}^2 \tau_1^{(i+5)}}, \tag{2.2a}$$

$$s_i\left(\tau_2^{(i+4)}\right) = \tau_1^{(i+5)}, \qquad s_i\left(\tau_2^{(i+5)}\right) = \frac{a_{i+4} \left(a_{i+2} \tau_1^{(i+2)} \tau_1^{(i+4)} + a_i a_{i+4} \tau_1^{(i+3)} \tau_2^{(i+2)}\right)}{a_i a_{i+1} a_{i+2}{}^2 \tau_1^{(i+5)}}, \tag{2.2b}$$

$$\sigma\left(\tau_1^{(i)}\right) = \tau_1^{(i+1)}, \qquad \sigma\left(\tau_2^{(i)}\right) = \tau_2^{(i+1)}, \tag{2.2c}$$

$$\iota \colon\ \left(\tau_1^{(1)}, \tau_1^{(2)}, \tau_1^{(3)}, \tau_1^{(4)}, \tau_2^{(1)}, \tau_2^{(2)}, \tau_2^{(3)}, \tau_2^{(5)}\right) \mapsto \left(\tau_1^{(4)}, \tau_1^{(3)}, \tau_1^{(2)}, \tau_1^{(1)}, \tau_2^{(2)}, \tau_2^{(1)}, \tau_2^{(5)}, \tau_2^{(3)}\right), \tag{2.2d}$$

where $i \in \mathbb{Z}/5\mathbb{Z}$. In general, for a function $F = F\left(a_i, \tau_j^{(k)}\right)$, we let an element $w \in \widetilde{W}\left(A_4^{(1)}\right)$ act as $w.F = F\left(w.a_i, w.\tau_j^{(k)}\right)$, that is, w acts on the arguments from the left. Note that $q = a_0 a_1 a_2 a_3 a_4$ is invariant under the action of $\langle s_0, s_1, s_2, s_3, s_4, \sigma \rangle$.

Proposition 2.1 ([26, 58]). *The group of birational transformations* $\langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle$, *denoted by* $\widetilde{W}\left(A_4^{(1)}\right)$, *forms the extended affine Weyl group of type* $A_4^{(1)}$. *Namely, the transformations satisfy the fundamental relations*

$$s_i{}^2 = 1, \qquad (s_i s_{i\pm 1})^3 = 1, \qquad (s_i s_j)^2 = 1, \qquad j \neq i \pm 1,$$
$$\sigma^5 = 1, \qquad \sigma s_i = s_{i+1}\sigma, \qquad \iota^2 = 1, \qquad \iota s_0 = s_0\iota, \qquad \iota s_1 = s_4\iota, \qquad \iota s_2 = s_3\iota,$$

where $i, j \in \mathbb{Z}/5\mathbb{Z}$.

To iterate each variable $\tau_i^{(j)}$, we need the translations T_i, $i = 0, \dots, 4$, defined by

$$T_0 = \sigma s_4 s_3 s_2 s_1, \qquad T_1 = \sigma s_0 s_4 s_3 s_2, \qquad T_2 = \sigma s_1 s_0 s_4 s_3, \qquad T_3 = \sigma s_2 s_1 s_0 s_4, \qquad (2.3a)$$

$$T_4 = \sigma s_3 s_2 s_1 s_0. \qquad (2.3b)$$

The action of translations on the parameters is given by

$$T_i(a_i) = q a_i, \qquad T_i(a_{i+1}) = q^{-1} a_{i+1},$$

where $i \in \mathbb{Z}/5\mathbb{Z}$. Note that T_i, $i = 0, \dots, 4$, commute with each other and

$$T_0 T_1 T_2 T_3 T_4 = 1.$$

We define τ functions by

$$\tau_{l_1}^{l_0, l_2, l_3} = T_0^{l_0} T_1^{l_1} T_2^{l_2} T_3^{l_3} \left(\tau_2^{(3)} \right), \qquad (2.4)$$

where $l_i \in \mathbb{Z}$. We note that

$$\tau_1^{(1)} = \tau_0^{1,0,1}, \qquad \tau_1^{(2)} = \tau_1^{1,0,1}, \qquad \tau_1^{(3)} = \tau_1^{1,1,1}, \qquad \tau_1^{(4)} = \tau_1^{1,1,2}, \qquad \tau_1^{(5)} = \tau_0^{0,0,1}, \quad (2.5a)$$

$$\tau_2^{(1)} = \tau_0^{1,1,1}, \qquad \tau_2^{(2)} = \tau_1^{1,0,2}, \qquad \tau_2^{(3)} = \tau_0^{0,0,0}, \qquad \tau_2^{(4)} = \tau_1^{2,1,2}, \qquad \tau_2^{(5)} = \tau_1^{0,0,1}. \quad (2.5b)$$

2.2   Hypergeometric τ functions

The aim of this section is to construct the hypergeometric τ functions of $\widetilde{W}\big(A_4^{(1)}\big)$-type.

Hereinafter, we consider the τ functions $\tau_{l_1}^{l_0, l_2, l_3}$ satisfying the following conditions:

(i) $\tau_{l_1}^{l_0, l_2, l_3}$ satisfy the action of the translation subgroup of $\widetilde{W}\big(A_4^{(1)}\big)$, $\langle T_0, T_1, T_2, T_3, T_4 \rangle$;

(ii) $\tau_{l_1}^{l_0, l_2, l_3}$ are functions in a_0, a_2 and a_4 consistent with the action of $\langle T_0, T_2, T_3 \rangle$, i.e., $\tau_{l_1}^{l_0, l_2, l_3} = \tau_{l_1}\big(q^{l_0} a_0, q^{l_2} a_2, q^{-l_3} a_4\big)$;

(iii) $\tau_{l_1}^{l_0, l_2, l_3}$ satisfy the following boundary conditions:

$$\tau_{l_1}^{l_0, l_2, l_3} = 0, \qquad (2.6)$$

for $l_1 < 0$;

under the conditions of parameters

$$a_0 a_1 = q. \qquad (2.7)$$

We here call such functions $\tau_{l_1}^{l_0, l_2, l_3}$ hypergeometric τ functions of $\widetilde{W}\big(A_4^{(1)}\big)$-type.

From the condition (i), every $\tau_{l_1}^{l_0, l_2, l_3}$ can be given by a rational function of ten variables $\tau_i^{(j)}$ (or, $\{\tau_0^{l_0, l_2, l_3}\}_{l_i \in \mathbb{Z}}$ and $\{\tau_1^{l_0, l_2, l_3}\}_{l_i \in \mathbb{Z}}$). Therefore, our purpose in this section is to obtain the explicit formulae for $\{\tau_0^{l_0, l_2, l_3}\}_{l_i \in \mathbb{Z}}$ and $\{\tau_1^{l_0, l_2, l_3}\}_{l_i \in \mathbb{Z}}$, satisfying the condition (ii) under the condition (iii) and construct the closed-form expressions of $\{\tau_{l_1}^{l_0, l_2, l_3}\}_{l_i \in \mathbb{Z}, l_1 \geq 2}$.

Step 1. Begin by preparing the equations necessary for the construction of the hypergeometric τ functions of $\widetilde{W}\big(A_4^{(1)}\big)$-type. From the actions (2.2) and the definitions (2.3), the actions of T_0, T_2 and T_3 and their inverses on ten variables $\tau_i^{(j)}$ are given by the following

$$T_0\big(\tau_1^{(4)}\big) = \tau_2^{(4)}, \qquad T_0\big(\tau_1^{(5)}\big) = \tau_1^{(1)}, \qquad T_0\big(\tau_2^{(5)}\big) = \tau_1^{(2)},$$

$$T_2\big(\tau_1^{(1)}\big) = \tau_2^{(1)}, \qquad T_2\big(\tau_1^{(2)}\big) = \tau_1^{(3)}, \qquad T_2\big(\tau_2^{(2)}\big) = \tau_1^{(4)},$$

$$T_3\big(\tau_1^{(2)}\big) = \tau_2^{(2)}, \qquad T_3\big(\tau_1^{(3)}\big) = \tau_1^{(4)}, \qquad T_3\big(\tau_2^{(3)}\big) = \tau_1^{(5)},$$

$$T_0\big(\tau_1^{(1)}\big) = \frac{qa_0{}^2a_4\big(a_3\tau_1^{(1)}T_0\big(\tau_1^{(3)}\big) + a_0a_1\tau_2^{(4)}T_0\big(\tau_2^{(3)}\big)\big)}{a_3\tau_1^{(3)}}, \tag{2.8a}$$

$$T_0\big(\tau_1^{(2)}\big) = \frac{a_0a_1\big(qa_0\tau_2^{(4)}T_0\big(\tau_2^{(3)}\big) + a_2a_3\tau_1^{(1)}T_0\big(\tau_1^{(3)}\big)\big)}{a_3{}^2\tau_2^{(1)}}, \tag{2.8b}$$

$$T_0\big(\tau_1^{(3)}\big) = \frac{a_3a_4\big(a_0a_1\tau_1^{(1)}\tau_1^{(3)} + a_3\tau_1^{(2)}\tau_2^{(1)}\big)}{a_1{}^2\tau_1^{(5)}}, \tag{2.8c}$$

$$T_0\big(\tau_2^{(1)}\big) = \frac{a_0a_1\big(qa_0\tau_2^{(4)}T_0\big(\tau_2^{(3)}\big) + a_3\tau_1^{(1)}T_0\big(\tau_1^{(3)}\big)\big)}{a_2a_3{}^2\tau_1^{(2)}}, \tag{2.8d}$$

$$T_0\big(\tau_2^{(2)}\big) = \frac{a_1a_2\big(q^{-1}a_1\tau_1^{(1)}T_0\big(\tau_2^{(4)}\big) + a_4\tau_2^{(4)}T_0\big(\tau_1^{(1)}\big)\big)}{qa_3a_4{}^2T_0\big(\tau_2^{(1)}\big)}, \tag{2.8e}$$

$$T_0\big(\tau_2^{(3)}\big) = \frac{a_3\big(a_1\tau_1^{(1)}\tau_1^{(3)} + a_3a_4\tau_1^{(2)}\tau_2^{(1)}\big)}{a_0a_1{}^2a_4\tau_1^{(4)}}, \tag{2.8f}$$

$$T_0\big(\tau_2^{(4)}\big) = \frac{a_3a_4\big(a_1T_0\big(\tau_1^{(1)}\big)T_0\big(\tau_1^{(3)}\big) + qa_3T_0\big(\tau_1^{(2)}\big)T_0\big(\tau_2^{(1)}\big)\big)}{a_0a_1{}^2T_0\big(\tau_2^{(3)}\big)}, \tag{2.8g}$$

$$T_0{}^{-1}\big(\tau_1^{(3)}\big) = \frac{a_0\big(a_3\tau_1^{(3)}\tau_1^{(5)} + a_0a_1\tau_1^{(4)}\tau_2^{(3)}\big)}{a_1a_2a_3{}^2\tau_1^{(1)}}, \tag{2.8h}$$

$$T_0{}^{-1}\big(\tau_1^{(4)}\big) = \frac{a_3\big(qa_1\tau_1^{(5)}T_0{}^{-1}\big(\tau_1^{(3)}\big) + a_3a_4\tau_2^{(5)}T_0{}^{-1}\big(\tau_2^{(1)}\big)\big)}{qa_0a_1{}^2a_4\tau_2^{(3)}}, \tag{2.8i}$$

$$T_0{}^{-1}\big(\tau_1^{(5)}\big) = \frac{a_3a_4\big(a_0a_1\tau_1^{(5)}T_0{}^{-1}\big(\tau_1^{(3)}\big) + a_3\tau_2^{(5)}T_0{}^{-1}\big(\tau_2^{(1)}\big)\big)}{q^2a_1{}^2\tau_1^{(3)}}, \tag{2.8j}$$

$$T_0{}^{-1}\big(\tau_2^{(1)}\big) = \frac{a_0a_1\big(a_2a_3\tau_1^{(3)}\tau_1^{(5)} + a_0\tau_1^{(4)}\tau_2^{(3)}\big)}{a_3{}^2\tau_1^{(2)}}, \tag{2.8k}$$

$$T_0{}^{-1}\big(\tau_2^{(2)}\big) = \frac{qa_1a_2\big(qa_1\tau_1^{(4)}T_0{}^{-1}\big(\tau_1^{(5)}\big) + a_4\tau_1^{(5)}T_0{}^{-1}\big(\tau_1^{(4)}\big)\big)}{a_3a_4{}^2T_0{}^{-1}\big(\tau_2^{(1)}\big)}, \tag{2.8l}$$

$$T_0{}^{-1}\big(\tau_2^{(3)}\big) = \frac{a_3a_4\big(qa_1\tau_1^{(5)}T_0{}^{-1}\big(\tau_1^{(3)}\big) + a_3\tau_2^{(5)}T_0{}^{-1}\big(\tau_2^{(1)}\big)\big)}{qa_0a_1{}^2\tau_1^{(4)}}, \tag{2.8m}$$

$$T_0{}^{-1}\big(\tau_2^{(5)}\big) = \frac{a_0a_1\big(q^{-1}a_0T_0{}^{-1}\big(\tau_1^{(4)}\big)T_0{}^{-1}\big(\tau_2^{(3)}\big) + a_3T_0{}^{-1}\big(\tau_1^{(3)}\big)T_0{}^{-1}\big(\tau_1^{(5)}\big)\big)}{a_2a_3{}^2T_0{}^{-1}\big(\tau_2^{(1)}\big)}, \tag{2.8n}$$

$$T_2\big(\tau_1^{(3)}\big) = \frac{qa_2{}^2a_1\big(a_0\tau_1^{(3)}T_2\big(\tau_1^{(5)}\big) + a_2a_3\tau_2^{(1)}T_2\big(\tau_2^{(5)}\big)\big)}{a_0\tau_1^{(5)}}, \tag{2.9a}$$

$$T_2\big(\tau_1^{(4)}\big) = \frac{a_2a_3\big(qa_2\tau_2^{(1)}T_2\big(\tau_2^{(5)}\big) + a_4a_0\tau_1^{(3)}T_2\big(\tau_1^{(5)}\big)\big)}{a_0{}^2\tau_2^{(3)}}, \tag{2.9b}$$

$$T_2\big(\tau_1^{(5)}\big) = \frac{a_0a_1\big(a_2a_3\tau_1^{(3)}\tau_1^{(5)} + a_0\tau_1^{(4)}\tau_2^{(3)}\big)}{a_3{}^2\tau_1^{(2)}}, \tag{2.9c}$$

$$T_2\big(\tau_2^{(3)}\big) = \frac{a_2 a_3\big(q a_2 \tau_2^{(1)} T_2\big(\tau_2^{(5)}\big) + a_0 \tau_1^{(3)} T_2\big(\tau_1^{(5)}\big)\big)}{a_4 a_0{}^2 \tau_1^{(4)}}, \tag{2.9d}$$

$$T_2\big(\tau_2^{(4)}\big) = \frac{a_3 a_4\big(q^{-1} a_3 \tau_1^{(3)} T_2\big(\tau_2^{(1)}\big) + a_1 \tau_2^{(1)} T_2\big(\tau_1^{(3)}\big)\big)}{q a_0 a_1{}^2 T_2\big(\tau_2^{(3)}\big)}, \tag{2.9e}$$

$$T_2\big(\tau_2^{(5)}\big) = \frac{a_0\big(a_3 \tau_1^{(3)} \tau_1^{(5)} + a_0 a_1 \tau_1^{(4)} \tau_2^{(3)}\big)}{a_2 a_3{}^2 a_1 \tau_1^{(1)}}, \tag{2.9f}$$

$$T_2\big(\tau_2^{(1)}\big) = \frac{a_0 a_1\big(a_3 T_2\big(\tau_1^{(3)}\big) T_2\big(\tau_1^{(5)}\big) + q a_0 T_2\big(\tau_1^{(4)}\big) T_2\big(\tau_2^{(3)}\big)\big)}{a_2 a_3{}^2 T_2\big(\tau_2^{(5)}\big)}, \tag{2.9g}$$

$$T_2{}^{-1}\big(\tau_1^{(5)}\big) = \frac{a_2\big(a_0 \tau_1^{(5)} \tau_1^{(2)} + a_2 a_3 \tau_1^{(1)} \tau_2^{(5)}\big)}{a_3 a_4 a_0{}^2 \tau_1^{(3)}}, \tag{2.9h}$$

$$T_2{}^{-1}\big(\tau_1^{(1)}\big) = \frac{a_0\big(q a_3 \tau_1^{(2)} T_2{}^{-1}\big(\tau_1^{(5)}\big) + a_0 a_1 \tau_2^{(2)} T_2{}^{-1}\big(\tau_2^{(3)}\big)\big)}{q a_2 a_3{}^2 a_1 \tau_2^{(5)}}, \tag{2.9i}$$

$$T_2{}^{-1}\big(\tau_1^{(2)}\big) = \frac{a_0 a_1\big(a_2 a_3 \tau_1^{(2)} T_2{}^{-1}\big(\tau_1^{(5)}\big) + a_0 \tau_2^{(2)} T_2{}^{-1}\big(\tau_2^{(3)}\big)\big)}{q^2 a_3{}^2 \tau_1^{(5)}}, \tag{2.9j}$$

$$T_2{}^{-1}\big(\tau_2^{(3)}\big) = \frac{a_2 a_3\big(a_4 a_0 \tau_1^{(5)} \tau_1^{(2)} + a_2 \tau_1^{(1)} \tau_2^{(5)}\big)}{a_0{}^2 \tau_1^{(4)}}, \tag{2.9k}$$

$$T_2{}^{-1}\big(\tau_2^{(4)}\big) = \frac{q a_3 a_4\big(q a_3 \tau_1^{(1)} T_2{}^{-1}\big(\tau_1^{(2)}\big) + a_1 \tau_1^{(2)} T_2{}^{-1}\big(\tau_1^{(1)}\big)\big)}{a_0 a_1{}^2 T_2{}^{-1}\big(\tau_2^{(3)}\big)}, \tag{2.9l}$$

$$T_2{}^{-1}\big(\tau_2^{(5)}\big) = \frac{a_0 a_1\big(q a_3 \tau_1^{(2)} T_2{}^{-1}\big(\tau_1^{(5)}\big) + a_0 \tau_2^{(2)} T_2{}^{-1}\big(\tau_2^{(3)}\big)\big)}{q a_2 a_3{}^2 \tau_1^{(1)}}, \tag{2.9m}$$

$$T_2{}^{-1}\big(\tau_2^{(2)}\big) = \frac{a_2 a_3\big(q^{-1} a_2 T_2{}^{-1}\big(\tau_1^{(1)}\big) T_2{}^{-1}\big(\tau_2^{(5)}\big) + a_0 T_2{}^{-1}\big(\tau_1^{(5)}\big) T_2{}^{-1}\big(\tau_1^{(2)}\big)\big)}{a_4 a_0{}^2 T_2{}^{-1}\big(\tau_2^{(3)}\big)}, \tag{2.9n}$$

$$T_3\big(\tau_1^{(4)}\big) = \frac{q a_3{}^2 a_2\big(a_1 \tau_1^{(4)} T_3\big(\tau_1^{(1)}\big) + a_3 a_4 \tau_2^{(2)} T_3\big(\tau_2^{(1)}\big)\big)}{a_1 \tau_1^{(1)}}, \tag{2.10a}$$

$$T_3\big(\tau_1^{(5)}\big) = \frac{a_3 a_4\big(q a_3 \tau_2^{(2)} T_3\big(\tau_2^{(1)}\big) + a_0 a_1 \tau_1^{(4)} T_3\big(\tau_1^{(1)}\big)\big)}{a_1{}^2 \tau_2^{(4)}}, \tag{2.10b}$$

$$T_3\big(\tau_1^{(1)}\big) = \frac{a_1 a_2\big(a_3 a_4 \tau_1^{(4)} \tau_1^{(1)} + a_1 \tau_1^{(5)} \tau_2^{(4)}\big)}{a_4{}^2 \tau_1^{(3)}}, \tag{2.10c}$$

$$T_3\big(\tau_2^{(4)}\big) = \frac{a_3 a_4\big(q a_3 \tau_2^{(2)} T_3\big(\tau_2^{(1)}\big) + a_1 \tau_1^{(4)} T_3\big(\tau_1^{(1)}\big)\big)}{a_0 a_1{}^2 \tau_1^{(5)}}, \tag{2.10d}$$

$$T_3\big(\tau_2^{(5)}\big) = \frac{a_4 a_0\big(q^{-1} a_4 \tau_1^{(4)} T_3\big(\tau_2^{(2)}\big) + a_2 \tau_2^{(2)} T_3\big(\tau_1^{(4)}\big)\big)}{q a_1 a_2{}^2 T_3\big(\tau_2^{(4)}\big)}, \tag{2.10e}$$

$$T_3\big(\tau_2^{(1)}\big) = \frac{a_1\big(a_4 \tau_1^{(4)} \tau_1^{(1)} + a_1 a_2 \tau_1^{(5)} \tau_2^{(4)}\big)}{a_3 a_4{}^2 a_2 \tau_1^{(2)}}, \tag{2.10f}$$

$$T_3\big(\tau_2^{(2)}\big) = \frac{a_1 a_2\big(a_4 T_3\big(\tau_1^{(4)}\big) T_3\big(\tau_1^{(1)}\big) + q a_1 T_3\big(\tau_1^{(5)}\big) T_3\big(\tau_2^{(4)}\big)\big)}{a_3 a_4{}^2 T_3\big(\tau_2^{(1)}\big)}, \tag{2.10g}$$

$$T_3{}^{-1}\big(\tau_1^{(1)}\big) = \frac{a_3\big(a_1\tau_1^{(1)}\tau_1^{(3)} + a_3a_4\tau_1^{(2)}\tau_2^{(1)}\big)}{a_4a_0a_1{}^2\tau_1^{(4)}}, \tag{2.10h}$$

$$T_3{}^{-1}\big(\tau_1^{(2)}\big) = \frac{a_1\big(qa_4\tau_1^{(3)}T_3{}^{-1}\big(\tau_1^{(1)}\big) + a_1a_2\tau_2^{(3)}T_3{}^{-1}\big(\tau_2^{(4)}\big)\big)}{qa_3a_4{}^2a_2\tau_2^{(1)}}, \tag{2.10i}$$

$$T_3{}^{-1}\big(\tau_1^{(3)}\big) = \frac{a_1a_2\big(a_3a_4\tau_1^{(3)}T_3{}^{-1}\big(\tau_1^{(1)}\big) + a_1\tau_2^{(3)}T_3{}^{-1}\big(\tau_2^{(4)}\big)\big)}{q^2a_4{}^2\tau_1^{(1)}}, \tag{2.10j}$$

$$T_3{}^{-1}\big(\tau_2^{(4)}\big) = \frac{a_3a_4\big(a_0a_1\tau_1^{(1)}\tau_1^{(3)} + a_3\tau_1^{(2)}\tau_2^{(1)}\big)}{a_1{}^2\tau_1^{(5)}}, \tag{2.10k}$$

$$T_3{}^{-1}\big(\tau_2^{(5)}\big) = \frac{qa_4a_0\big(qa_4\tau_1^{(2)}T_3{}^{-1}\big(\tau_1^{(3)}\big) + a_2\tau_1^{(3)}T_3{}^{-1}\big(\tau_1^{(2)}\big)\big)}{a_1a_2{}^2T_3{}^{-1}\big(\tau_2^{(4)}\big)}, \tag{2.10l}$$

$$T_3{}^{-1}\big(\tau_2^{(1)}\big) = \frac{a_1a_2\big(qa_4\tau_1^{(3)}T_3{}^{-1}\big(\tau_1^{(1)}\big) + a_1\tau_2^{(3)}T_3{}^{-1}\big(\tau_2^{(4)}\big)\big)}{qa_3a_4{}^2\tau_1^{(2)}}, \tag{2.10m}$$

$$T_3{}^{-1}\big(\tau_2^{(3)}\big) = \frac{a_3a_4\big(q^{-1}a_3T_3{}^{-1}\big(\tau_1^{(2)}\big)T_3{}^{-1}\big(\tau_2^{(1)}\big) + a_1T_3{}^{-1}\big(\tau_1^{(1)}\big)T_3{}^{-1}\big(\tau_1^{(3)}\big)\big)}{a_0a_1{}^2T_3{}^{-1}\big(\tau_2^{(4)}\big)}. \tag{2.10n}$$

Moreover, by using the action of T_1, we obtain the following lemma.

Lemma 2.2. *The following discrete Toda type bilinear equations hold*

$$\tau_{l_1+1}^{l_0,l_2,l_3}\tau_{l_1-1}^{l_0,l_2,l_3} = q^{3l_1-l_2-l_3}\frac{a_0a_1}{a_2{}^2a_3}\big(-1 + q^{-l_0+l_1}a_1\big)\big(\tau_{l_1}^{l_0,l_2,l_3}\big)^2$$
$$+ q^{4(-l_0+l_1)}a_1{}^4\tau_{l_1}^{l_0+1,l_2,l_3}\tau_{l_1}^{l_0-1,l_2,l_3}, \tag{2.11a}$$

$$\tau_{l_1+1}^{l_0,l_2,l_3}\tau_{l_1-1}^{l_0,l_2,l_3} = q^{-l_0+4l_1-l_2-l_3}\frac{a_0a_1{}^2}{a_2{}^2a_3}\big(1 - q^{-l_1+l_2}a_2\big)\big(\tau_{l_1}^{l_0,l_2,l_3}\big)^2$$
$$+ q^{4(l_1-l_2)}a_2{}^{-4}\tau_{l_1}^{l_0,l_2+1,l_3}\tau_{l_1}^{l_0,l_2-1,l_3}, \tag{2.11b}$$

$$\tau_{l_1+1}^{l_0,l_2,l_3}\tau_{l_1-1}^{l_0,l_2,l_3} = q^{-l_0+3l_1-l_2}\frac{a_1}{a_2{}^2a_3a_4}\big(-1 + q^{l_1-l_3}a_0a_1a_4\big)\big(\tau_{l_1}^{l_0,l_2,l_3}\big)^2$$
$$+ q^{4(l_1-l_3)}a_0{}^4a_1{}^4a_4{}^4\tau_{l_1}^{l_0,l_2,l_3+1}\tau_{l_1}^{l_0,l_2,l_3-1}. \tag{2.11c}$$

Proof. The actions of T_0, $T_1{}^{-1}$ and $T_2{}^{-1}$ on $\tau_1^{(1)}$ are given by

$$T_0\big(\tau_1^{(1)}\big) = \frac{qa_0{}^2a_3a_4{}^2\tau_1^{(1)}\big(a_0a_1\tau_1^{(1)}\tau_1^{(3)} + a_3\tau_1^{(2)}\tau_2^{(1)}\big)}{a_1{}^2\tau_1^{(3)}\tau_1^{(5)}}$$
$$+ \frac{qa_0{}^2\tau_2^{(4)}\big(a_1\tau_1^{(1)}\tau_1^{(3)} + a_3a_4\tau_1^{(2)}\tau_2^{(1)}\big)}{a_1\tau_1^{(3)}\tau_1^{(4)}}, \tag{2.12}$$

$$T_1{}^{-1}\big(\tau_1^{(1)}\big) = \frac{\tau_2^{(1)}\big(a_3a_4\tau_1^{(1)}\tau_1^{(4)} + a_1\tau_1^{(5)}\tau_2^{(4)}\big)}{qa_2{}^2a_3a_4\tau_1^{(3)}\tau_1^{(4)}}$$
$$+ \frac{a_1\tau_1^{(1)}\big(a_4\tau_1^{(1)}\tau_1^{(4)} + a_1a_2\tau_1^{(5)}\tau_2^{(4)}\big)}{qa_2{}^3a_3{}^2a_4{}^2\tau_1^{(2)}\tau_1^{(4)}}, \tag{2.13}$$

$$T_2{}^{-1}\big(\tau_1^{(1)}\big) = \frac{a_2{}^2\tau_1^{(2)}\big(a_3a_4\tau_1^{(1)}\tau_1^{(4)} + a_1\tau_1^{(5)}\tau_2^{(4)}\big)}{qa_3a_4\tau_1^{(3)}\tau_1^{(4)}}$$
$$+ \frac{a_1a_2{}^2\tau_1^{(1)}\big(a_4\tau_1^{(1)}\tau_1^{(4)} + a_1\tau_1^{(5)}\tau_2^{(4)}\big)}{qa_3{}^2a_4{}^2\tau_2^{(1)}\tau_1^{(4)}}, \tag{2.14}$$

respectively. Eliminating the terms $\tau_1^{(3)}$, $\tau_1^{(4)}$, $\tau_2^{(1)}$ and $\tau_2^{(4)}$ from equations (2.12) and (2.13), we obtain

$$\tau_1^{(2)} T_1^{-1}(\tau_1^{(1)}) = q^{-1} \frac{a_0 a_1}{a_2{}^2 a_3} (-1 + q^{-1} a_1)(\tau_1^{(1)})^2 + q^{-4} a_1{}^4 T_0(\tau_1^{(1)}) \tau_1^{(5)}, \tag{2.15}$$

which is equivalent to equation (2.11a). Furthermore, eliminating the terms $\tau_1^{(3)}$, $\tau_1^{(4)}$, $\tau_1^{(5)}$ and $\tau_2^{(4)}$ from equations (2.13) and (2.14), we obtain

$$\tau_1^{(2)} T_1^{-1}(\tau_1^{(1)}) = q^{-2} \frac{a_0 a_1{}^2}{a_2{}^2 a_3} (1 - a_2)(\tau_1^{(1)})^2 + a_2{}^{-4} \tau_2^{(1)} T_2^{-1}(\tau_1^{(1)}), \tag{2.16}$$

which is equivalent to equation (2.11b). Eliminating the term $\tau_1^{(2)} T_1^{-1}(\tau_1^{(1)})$ from equations (2.15) and (2.16), we obtain

$$T_0(\tau_1^{(1)}) \tau_1^{(5)} = q^2 \frac{a_0}{a_1{}^2 a_2 a_3} (-1 + a_4 a_0 a_3)(\tau_1^{(1)})^2 + a_4{}^4 a_0{}^4 a_3{}^4 \tau_2^{(1)} T_2^{-1}(\tau_1^{(1)}). \tag{2.17}$$

Applying the transformation σ on equation (2.17), we obtain

$$T_1(\tau_1^{(2)}) \tau_1^{(1)} = q^2 \frac{a_1}{a_2{}^2 a_3 a_4} (-1 + a_0 a_1 a_4)(\tau_1^{(2)})^2 + a_0{}^4 a_1{}^4 a_4{}^4 \tau_2^{(2)} T_3^{-1} \tau_1^{(2)}, $$

which is equivalent to equation (2.11c). Therefore, we have completed the proof. ∎

Step 2. In this step, we get the explicit formulae for $\tau_0^{l_0, l_2, l_3}$ and $\tau_1^{l_0, l_2, l_3}$. Letting

$$\tau_1^{l_0, l_2, l_3} = \tau_0^{l_0, l_2, l_3} H_{l_0, l_2, l_3}, \tag{2.18}$$

where

$$H_{l_0, l_2, l_3} = H(q^{l_0} a_0, q^{l_2} a_2, q^{-l_3} a_4),$$

we obtain the following lemma.

Lemma 2.3. *A solution of the system of the equations* (2.1) *and* (2.8)–(2.10) *are given by the solution of the following system under the condition* (2.7):

$$\tau_0^{0,0,0} \tau_0^{0,1,1} + \frac{a_0 a_4}{q a_2} \tau_0^{0,1,0} \tau_0^{0,0,1} = 0, \tag{2.19a}$$

$$\tau_0^{0,0,0} \tau_0^{1,1,1} - q a_4{}^2 \tau_0^{0,0,1} \tau_0^{1,1,0} = 0, \tag{2.19b}$$

$$\tau_0^{0,0,0} \tau_0^{1,1,1} - \frac{1}{q a_2{}^2} \tau_0^{1,0,1} \tau_0^{0,1,0} = 0, \tag{2.19c}$$

$$\tau_0^{0,0,0} \tau_0^{1,1,1} - \frac{q}{a_0{}^2} \tau_0^{0,1,1} \tau_0^{1,0,0} = 0, \tag{2.19d}$$

$$\tau_0^{1,0,0} \tau_0^{-1,0,0} - \frac{a_0{}^4 a_4 (1 - q a_0{}^{-1})}{q^3 a_2} (\tau_0^{0,0,0})^2 = 0, \tag{2.19e}$$

$$\tau_0^{0,1,0} \tau_0^{0,-1,0} + \frac{q^2 a_2{}^3 a_4 (1 - a_2)}{a_0} (\tau_0^{0,0,0})^2 = 0, \tag{2.19f}$$

$$\tau_0^{0,0,1} \tau_0^{0,0,-1} - \frac{1 - q a_4}{q^3 a_0 a_2 a_4{}^4} (\tau_0^{0,0,0})^2 = 0, \tag{2.19g}$$

$$H_{0,0,0} = q^2 a_4 H_{1,1,0} + q(1 - q a_4) H_{1,1,1}, \tag{H01}$$

$$H_{0,0,0} = -q^2 a_2 a_4 H_{0,1,0} + q^2 a_4 (1 - q^{-1} a_0) H_{1,1,0}, \tag{H02}$$

$$H_{0,0,0} = -q^3 a_0{}^{-1} a_2 a_4 H_{0,1,0} - q^2 a_0{}^{-1} (1 - q^{-1} a_0)(1 - q a_4) H_{1,1,1}, \tag{H03}$$

$$H_{0,0,0} = -a_0 a_2{}^{-1} H_{1,0,0} - a_2{}^{-1} a_4{}^{-1}(1 - qa_4)H_{1,0,1}, \tag{H04}$$

$$H_{0,0,0} = -q^2 a_0{}^{-1} a_2 H_{0,1,0} - qa_0{}^{-1} a_4{}^{-1}(1 - qa_4)H_{0,1,1}, \tag{H05}$$

$$H_{0,0,0} = -qa_0 a_4 H_{1,0,0} + q^2 a_4(1 - a_2)H_{1,1,0}, \tag{H06}$$

$$H_{0,0,0} = -qa_0 a_2{}^{-1} a_4 H_{1,0,0} - qa_2{}^{-1}(1 - a_2)(1 - qa_4)H_{1,1,1}, \tag{H07}$$

$$H_{0,0,0} = -a_2{}^{-1} a_4{}^{-1} H_{0,0,1} + qa_2{}^{-1}\big(1 - q^{-1} a_0\big)H_{1,0,1}, \tag{H08}$$

$$H_{0,0,0} = q^2 a_0{}^{-1} H_{0,1,1} - q^2 a_0{}^{-1}\big(1 - q^{-1} a_0\big)H_{1,1,1}, \tag{H09}$$

$$a_4(1 - a_0)H_{2,1,1}H_{0,0,0} = qa_4 H_{1,1,0}H_{1,0,1} - a_0 a_4 H_{1,0,0}H_{1,1,1}. \tag{H10}$$

Proof. By using notation (2.4) and relation (2.18), equations (2.1) and (2.8)–(2.10) can be classified by the type of contiguity relations of function H_{l_0,l_2,l_3} (see Figs. 1–4) as the following table:

Type	Equation number
Type 1	(2.1a), (2.8d), (2.8n), (2.9g), (2.9m)
Type 2	(2.1b), (2.8c), (2.8e), (2.8j), (2.8l), (2.10b), (2.10g), (2.10k), (2.10m)
Type 3	(2.1c), (2.8g), (2.8m), (2.9e), (2.9l), (2.10d), (2.10n)
Type 4	(2.8a), (2.8h), (2.9d), (2.9f), (2.9i), (2.9n)
Type 5	(2.8b), (2.8f), (2.8i), (2.8k), (2.9c), (2.9j), (2.10a), (2.10h)
Type 6	(2.9a), (2.9h)
Type 7	(2.9b), (2.9k)
Type 8	(2.10c), (2.10j)
Type 9	(2.10f), (2.10i)
Type 10	(2.10e), (2.10l)

Under the condition (2.7), comparing the coefficients of H_{l_0,l_2,l_3} in the same types, for example (2.1a) $\equiv$ (2.8d), and substituting the boundary condition (2.6) in equations (2.11) with $l_1 = 0$, we obtain equations (2.19). Moreover, by using the relations (2.19), the equations of Type 1, ..., Type 10 are given by equations (H01)–(H10), respectively. Therefore, we have completed the proof. ∎

We can easily verify the following lemma by the direct calculation.

Lemma 2.4. *A solution of system* (2.19) *is given by*

$$\tau_0^{l_0,l_2,l_3} = \big(q^{l_0} a_0; q, q\big)_\infty \big(q^{l_2+1} a_2; q, q\big)_\infty \big(q^{l_3} a_4{}^{-1}; q, q\big)_\infty K_{l_0,l_2,l_3},$$

where

$$K_{l_0,l_2,l_3} \tag{2.20}$$
$$= \frac{\big(\Gamma\big(q^{l_0+l_2+1} a_0 a_2; q, q\big)\Gamma\big(q^{l_2+l_3+1} a_2 a_4{}^{-1}; q, q\big)\Gamma\big(q^{l_3+l_0} a_4{}^{-1} a_0; q, q\big)\big)^2}{\Gamma\big(q^{l_0+l_2+l_3+1} a_0 a_2 a_4{}^{-1}; q, q\big)\big(\Gamma\big(q^{l_0+1/3} a_0; q, q\big)\Gamma\big(q^{l_2+4/3} a_2; q, q\big)\Gamma\big(q^{l_3+1/3} a_4{}^{-1}; q, q\big)\big)^6}.$$

We consider a solution of system of the equations (H01)–(H10). First, we get the essential relations for the function H_{l_0,l_2,l_3}.

Lemma 2.5. *If the function H_{l_0,l_2,l_3} satisfies equations* (H02), (H06) *and* (H08) *and the following three-term relation*

$$qa_0 a_4(a_0 - q)H_{1,0,0} + (a_0 - qa_2 + qa_0 a_2 a_4)H_{0,0,0} + qa_2 H_{-1,0,0} = 0, \tag{H11}$$

then it also satisfies equations (H01), (H03), (H04), (H05), (H07), (H09) *and* (H10).
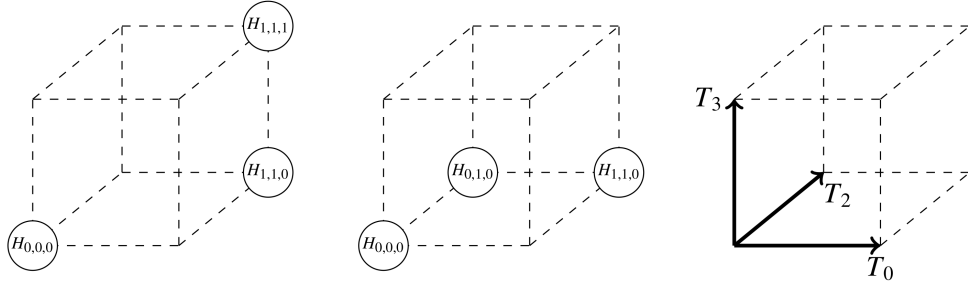
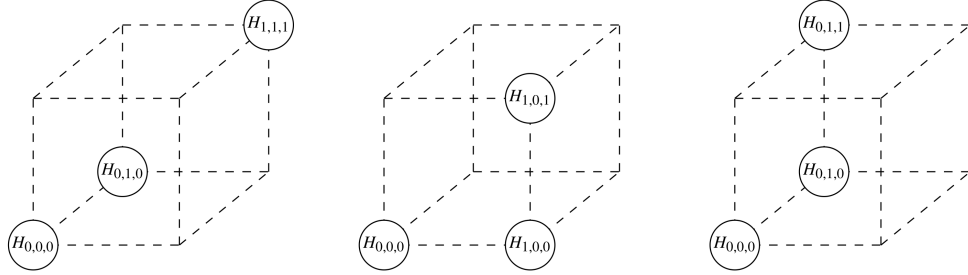**Figure 1.** Left: Type 1, center: Type 2, right: directions.

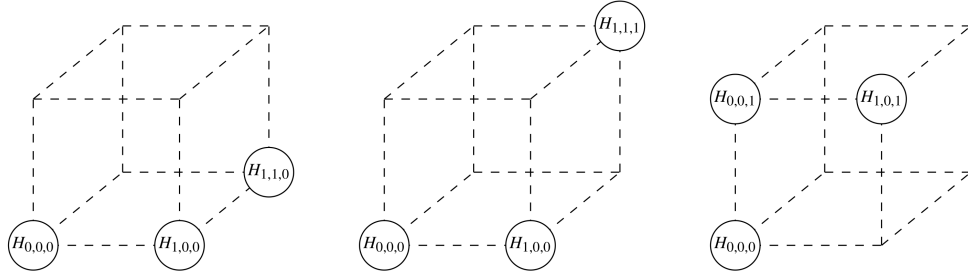**Figure 2.** Left: Type 3, center: Type 4, right: Type 5.

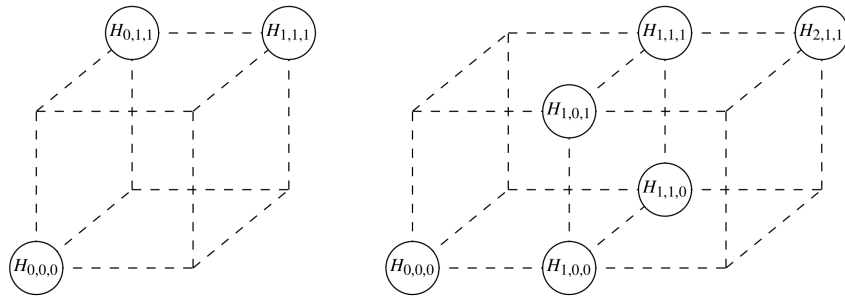**Figure 3.** Left: Type 6, center: Type 7, right: Type 8.

Figure 4. Left: Type 9, right: Type 10.

Proof. Equation (H04) can be obtained by using equations (H08) and (H11) as follows. Erasing the term $H_{1,0,1}$ from equations (H08) and (H11)$_3$, we obtain

$$a_0 a_4 H_{0,0,0} + (q - a_0 a_4) H_{0,0,1} - q H_{-1,0,1} = 0. \tag{H12}$$

We note that a subscript i of equation number means T_i-shifted corresponding equation. Moreover, erasing the term $H_{0,0,1}$ from equations (H12)$_0$ and (H08), we obtain equation (H04). This procedure is described in Fig. 5.

In a similar manner, we can derive equations (H01), (H03), (H07), (H05) and (H09) as shown in Figs. 6–10, respectively. On the other hand, we can prove equation (H10) by reducing it to equation (H06) with equations (H02), (H04) and (H07) as shown in Fig. 11. Therefore, we have completed the proof. ∎

Figure 5. Derivation of equation (H04). The black points are removed.

Figure 6. Derivation of equation (H01).

Figure 7. Derivation of equation (H03).

Figure 8. Derivation of equation (H07).

Figure 9. Derivation of equation (H05).

Figure 10. Derivation of equation (H09).

Figure 11. Reduction from equation (H10) to equation (H06).

Next, we solve the essential relations for the function H_{l_0,l_2,l_3}, that is, equations (H02), (H06), (H08) and (H11).

Lemma 2.6. *The solution of the system of equations* (H02), (H06), (H08) *and* (H11) *are given by*

$$H_{l_0,l_2,l_3} = q^{2/3} \left(q^{-l_0-l_2-l_3+2} \frac{a_4}{a_0 a_2} \right)^{1/2} \left(q^{l_0-1} a_0{}^{-1}; q \right)_\infty G_{l_0,l_2,l_3},$$

where

$$G_{l_0,l_2,l_3} = C_1 q^{(-2l_2+l_3)/2} a_2{}^{-1} a_4{}^{-1/2} \frac{\left(q^{-l_2+l_3} a_2{}^{-1} a_4{}^{-1}; q \right)_\infty}{\left(q^{-l_2+1} a_2{}^{-1}; q \right)_\infty} \frac{\Theta \left(q^{-l_0+l_2-1/2} a_0{}^{-1} a_2; q \right)}{\Theta \left(q^{-l_0} a_0{}^{-1}; q \right)}$$

$$\times \, {}_1\varphi_1\left(\begin{matrix} q^{-l_2+1}a_2^{-1} \\ q^{-l_2+l_3}a_2^{-1}a_4^{-1} \end{matrix}; q, q^{-l_0+l_3}a_0^{-1}a_4^{-1}\right)$$

$$+ C_2 q^{(l_2-2l_3)/2}a_2^{1/2}a_4 \frac{(q^{l_2-l_3+2}a_2a_4; q)_\infty}{(q^{-l_3+2}a_4; q)_\infty} \frac{\Theta(q^{-l_0+l_3-3/2}a_0^{-1}a_4^{-1}; q)}{\Theta(q^{-l_0}a_0^{-1}; q)}$$

$$\times \, {}_1\varphi_1\left(\begin{matrix} q^{-l_3+2}a_4 \\ q^{l_2-l_3+2}a_2a_4 \end{matrix}; q, q^{-l_0+l_2+1}a_0^{-1}a_2\right). \tag{2.21}$$

Here, $C_i = C_i(l_0, l_2, l_3)$, $i = 1, 2$, are periodic functions of period one for $l_0, l_2, l_3 \in \mathbb{Z}$, i.e.,

$$C_i(l_0 + 1, l_2, l_3) = C_i(l_0, l_2 + 1, l_3) = C_i(l_0, l_2, l_3 + 1) = C_i(l_0, l_2, l_3).$$

Proof. By letting

$$H_{l_0, l_2, l_3} = q^{2/3}\left(q^{-l_0-l_2-l_3+2}\frac{t}{\alpha\beta}\right)^{1/2}(q^{l_0-1}t^{-1}; q)_\infty G(q^{-l_0}t, q^{l_2}\alpha, q^{l_3}\beta),$$

where

$$t = a_0^{-1}, \qquad \alpha = a_2, \qquad \beta = a_4^{-1},$$

equations (H02), (H06), (H08) and (H11) can be rewritten as the following

$$\beta G(qt, \alpha, \beta) - qG(t, q\alpha, \beta) + q^{3/2}\alpha G(qt, q\alpha, \beta) = 0, \tag{2.22}$$

$$\beta(1 - q^2t)G(qt, \alpha, \beta) + q^3(1 - \alpha)tG(t, q\alpha, \beta) - q^{3/2}G(t, \alpha, \beta) = 0, \tag{2.23}$$

$$q^{1/2}\alpha G(qt, \alpha, \beta) - q^{1/2}G(t, \alpha, q\beta) + \beta G(qt, \alpha, q\beta) = 0, \tag{2.24}$$

$$\alpha\beta(q^3t - 1)G(q^2t, \alpha, \beta) - \left(q^{5/2}\alpha\beta t - q^{1/2}(q\alpha + \beta)\right)G(qt, \alpha, \beta) - q^2 G(t, \alpha, \beta) = 0, \tag{2.25}$$

respectively. Substituting

$$G(t, \alpha, \beta) = \sum_{n=0}^\infty c_n t^{n+\rho},$$

where $c_n = c_n(\alpha, \beta)$, in equation (2.25), we obtain

$$G(t, \alpha, \beta) = A(\alpha, \beta)\frac{\Theta(q^{-1/2}\alpha t; q)}{\Theta(t; q)} \, {}_1\varphi_1\left(\begin{matrix} q\alpha^{-1} \\ \alpha^{-1}\beta \end{matrix}; q, \beta t\right)$$

$$+ B(\alpha, \beta)\frac{\Theta(q^{-3/2}\beta t; q)}{\Theta(t; q)} \, {}_1\varphi_1\left(\begin{matrix} q^2\beta^{-1} \\ q^2\alpha\beta^{-1} \end{matrix}; q, q\alpha t\right), \tag{2.26}$$

where $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are arbitrary functions. Moreover, substituting (2.26) in equations (2.22), (2.23) and (2.24), we obtain the following relations

$$A(q\alpha, \beta) = \frac{1 - q^{-1}\alpha^{-1}\beta}{q(1 - \alpha^{-1})}A(\alpha, \beta), \qquad A(\alpha, q\beta) = \frac{q^{1/2}}{1 - \alpha^{-1}\beta}A(\alpha, \beta), \tag{2.27a}$$

$$B(q\alpha, \beta) = \frac{q^{1/2}}{1 - q^2\alpha\beta^{-1}}B(\alpha, \beta), \qquad B(\alpha, q\beta) = \frac{1 - q\alpha\beta^{-1}}{q(1 - q\beta^{-1})}B(\alpha, \beta), \tag{2.27b}$$

which can be solved by

$$A(\alpha, \beta) = \alpha^{-1}\beta^{1/2}\frac{(\alpha^{-1}\beta; q)_\infty}{(q\alpha^{-1}; q)_\infty}, \qquad B(\alpha, \beta) = \alpha^{1/2}\beta^{-1}\frac{(q^2\alpha\beta^{-1}; q)_\infty}{(q^2\beta^{-1}; q)_\infty}.$$

To obtain the relations (2.27), we used the following recurrence relations of hypergeometric series ${}_1\varphi_1$:

$$
{}_1\varphi_1\begin{pmatrix} a \\ b \end{pmatrix}; q, z\end{pmatrix} - \frac{q+b-q^2z}{q}\,{}_1\varphi_1\begin{pmatrix} a \\ b \end{pmatrix}; q, qz\end{pmatrix} - \frac{q^2az-b}{q}\,{}_1\varphi_1\begin{pmatrix} a \\ b \end{pmatrix}; q, q^2z\end{pmatrix} = 0,
$$
$$
{}_1\varphi_1\begin{pmatrix} a \\ b \end{pmatrix}; q, z\end{pmatrix} = {}_1\varphi_1\begin{pmatrix} a \\ b \end{pmatrix}; q, qz\end{pmatrix} + \frac{(a-1)z}{1-b}\,{}_1\varphi_1\begin{pmatrix} qa \\ qb \end{pmatrix}; q, qz\end{pmatrix},
$$
$$
{}_1\varphi_1\begin{pmatrix} a \\ b \end{pmatrix}; q, z\end{pmatrix} = \frac{1}{1-b}\,{}_1\varphi_1\begin{pmatrix} a \\ qb \end{pmatrix}; q, z\end{pmatrix} - \frac{b}{1-b}\,{}_1\varphi_1\begin{pmatrix} a \\ qb \end{pmatrix}; q, qz\end{pmatrix},
$$

which can be verified by the direct calculation. Therefore, we have completed the proof. ∎

Step 3. In this final step, we give the hypergeometric τ functions of $\widetilde{W}\big(A_4^{(1)}\big)$-type. Substituting

$$
\tau_{l_1}^{l_0,l_2,l_3} = q^{2l_1{}^3/3}\left(q^{-l_0-l_2-l_3+2}\frac{a_4}{a_0a_2}\right)^{l_1{}^2/2}\frac{(q^{l_0-l_1}a_0{}^{-1};q,q)_\infty}{(q^{l_0}a_0{}^{-1};q,q)_\infty}\tau_0^{l_0,l_2,l_3}\Phi_{l_1}^{l_0,l_2,l_3},
$$

in equation (2.11a), we obtain the following bilinear equation

$$
\Phi_{l_1+1}^{l_0,l_2,l_3}\Phi_{l_1-1}^{l_0,l_2,l_3} = (\Phi_{l_1}^{l_0,l_2,l_3})^2 - \Phi_{l_1}^{l_0+1,l_2,l_3}\Phi_{l_1}^{l_0-1,l_2,l_3}. \tag{2.28}
$$

In general, equation (2.28) admits a solution expressed in terms of Jacobi–Trudi type determinant

$$
\Phi_{l_1}^{l_0,l_2,l_3} = \det(c_{l_0+i-j,l_2,l_3})_{i,j=1,\dots,l_1},
$$

where $l_1 \in \mathbb{Z}_{>1}$, under the boundary conditions

$$
\Phi_{l_1}^{l_0,l_2,l_3} = 0 \qquad (l_1 < 0), \qquad \Phi_0^{l_0,l_2,l_3} = 1, \qquad \Phi_1^{l_0,l_2,l_3} = c_{l_0,l_2,l_3},
$$

where c_{l_0,l_2,l_3} is an arbitrary function. Therefore, we obtain the following theorem.

Theorem 2.7. *The hypergeometric τ functions of $\widetilde{W}(A_4^{(1)})$-type are given by the following*

$$
\tau_0^{l_0,l_2,l_3} = \big(q^{l_0}a_0; q, q\big)_\infty\big(q^{l_2+1}a_2; q, q\big)_\infty\big(q^{l_3}a_4{}^{-1}; q, q\big)_\infty K_{l_0,l_2,l_3},
$$
$$
\tau_{l_1}^{l_0,l_2,l_3} = q^{2l_1{}^3/3}\left(q^{-l_0-l_2-l_3+2}\frac{a_4}{a_0a_2}\right)^{l_1{}^2/2}\frac{\big(q^{l_0-l_1}a_0{}^{-1}; q, q\big)_\infty}{\big(q^{l_0}a_0{}^{-1}; q, q\big)_\infty}\tau_0^{l_0,l_2,l_3}\Phi_{l_1}^{l_0,l_2,l_3},
$$

where $l_0, l_2, l_3 \in \mathbb{Z}$, $l_1 \in \mathbb{Z}_{>0}$ and the functions $\big\{\Phi_{l_1}^{l_0,l_2,l_3}\big\}_{l_i\in\mathbb{Z}, l_1>0}$ are given by the following $l_1 \times l_1$ determinants

$$
\Phi_{l_1}^{l_0,l_2,l_3} = \begin{vmatrix} G_{l_0,l_2,l_3} & G_{l_0+1,l_2,l_3} & \cdots & G_{l_0+l_1-1,l_2,l_3} \\ G_{l_0-1,l_2,l_3} & G_{l_0,l_2,l_3} & \cdots & G_{l_0+l_1-2,l_2,l_3} \\ \vdots & \vdots & \cdots & \vdots \\ G_{l_0-l_1+1,l_2,l_3} & G_{l_0-l_1+2,l_2,l_3} & \cdots & G_{l_0,l_2,l_3} \end{vmatrix}. \tag{2.29}
$$

Here, the functions K_{l_0,l_2,l_3} and G_{l_0,l_2,l_3} are given in equations (2.20) and (2.21), respectively.

2.3   Discrete Painlevé equations

Let us define the ten f-variables by

$$f_1^{(j)} = \frac{\tau_1^{(j+1)}\tau_2^{(j)}}{\tau_1^{(j)}\tau_1^{(j+2)}}, \qquad f_2^{(j)} = \frac{a_j a_{j+1}}{a_{j+3}^2} \frac{\tau_1^{(j+1)}\left(a_{j+2}a_{j+3}\tau_1^{(j)}\tau_1^{(j+3)} + a_j\tau_1^{(j+4)}\tau_2^{(j+3)}\right)}{\tau_1^{(j)}\tau_1^{(j+2)}\tau_2^{(j+1)}}, \qquad (2.30)$$

where $j \in \mathbb{Z}/5\mathbb{Z}$. From the definition above and conditions (2.1), the following eight relations hold

$$f_2^{(j)} = \frac{a_j a_{j+1}\left(a_{j+2}a_{j+3} + a_j f_1^{(j+3)}\right)}{a_{j+3}^2 f_1^{(j+1)}}, \qquad j \in \mathbb{Z}/5\mathbb{Z},$$

$$a_4 a_0^2 f_1^{(2)} f_1^{(3)} = a_2 a_3\left(a_0 + a_2 f_1^{(5)}\right), \qquad a_0 a_1^2 f_1^{(3)} f_1^{(4)} = a_3 a_4\left(a_1 + a_3 f_1^{(1)}\right),$$

$$a_2 a_3^2 f_1^{(5)} f_1^{(1)} = a_0 a_1\left(a_3 + a_0 f_1^{(3)}\right).$$

Therefore, the f-variables are essentially two. The action of $\widetilde{W}\left(A_4^{(1)}\right)$ on these variables $f_i^{(j)}$ is given by the following lemma, which follows from the actions (2.2).

Lemma 2.8. *The action of $\widetilde{W}\left(A_4^{(1)}\right)$ on variables $f_i^{(j)}$ is given by*

$$s_j\left(f_1^{(j+3)}\right) = f_2^{(j+3)}, \qquad s_j\left(f_2^{(j+3)}\right) = f_1^{(j+3)},$$

$$s_j\left(f_1^{(j)}\right) = \frac{a_{j+4}}{a_j a_{j+1}a_{j+2}^2} \frac{a_{j+2} + a_j a_{j+4}f_1^{(j+2)}}{f_1^{(j+4)}},$$

$$s_j\left(f_2^{(j+2)}\right) = \frac{a_j a_{j+3}a_{j+4}}{a_{j+1}} \frac{a_{j+2} + a_j a_{j+4}f_1^{(j+2)} + a_j a_{j+1}a_{j+2}f_1^{(j+4)}}{f_1^{(j+4)} f_2^{(j+3)}},$$

$$s_j\left(f_2^{(j+4)}\right) = \frac{a_j a_{j+1}a_{j+2}^2}{a_{j+4}} \frac{f_1^{(j+4)} f_1^{(j)} f_2^{(j+4)}}{a_{j+2} + a_j a_{j+4}f_1^{(j+2)}},$$

$$s_j\left(f_2^{(j)}\right) = \frac{a_j a_{j+1}a_{j+4} + a_{j+3}a_{j+4}f_1^{(j+1)} + a_j a_{j+1}^2 a_{j+2}f_1^{(j+4)}}{a_j a_{j+1}a_{j+3}f_1^{(j+1)} f_1^{(j+4)}},$$

$$\pi\left(f_i^{(j)}\right) = f_i^{(j+1)}, \qquad \iota\left(f_1^{(j)}\right) = f_1^{(3-j)}, \qquad \iota\left(f_2^{(j)}\right) = \frac{a_{2-j}\left(a_{5-j} + a_{2-j}a_{3-j}f_1^{(5-j)}\right)}{a_{3-j}a_{4-j}a_{5-j}^2 f_1^{(2-j)}},$$

where $i = 1, 2$ and $j \in \mathbb{Z}/5\mathbb{Z}$.

It is well known that the translation part of $\widetilde{W}\left(A_4^{(1)}\right)$ give discrete Painlevé equations [56]. Let

$$X_{l_1}^{(i)} = T_1^{l_1}\left(f_1^{(i)}\right), \qquad Y_{l_1}^{(i)} = T_1^{l_1}\left(f_2^{(i)}\right), \qquad \alpha_{l_1}^{(i)} = T_1^{l_1}(a_i) = \begin{cases} q^{l_1}a_1 & \text{if } i = 1, \\ q^{-l_1}a_2 & \text{if } i = 2, \\ a_i & \text{otherwise.} \end{cases} \qquad (2.31)$$

The action of T_i:

$$T_i\colon (a_i, a_{i+1}) \mapsto \left(qa_i, q^{-1}a_{i+1}\right),$$

$$T_i\left(f_1^{(i+3)}\right)f_1^{(i+3)} = \frac{a_{i+3}}{a_i^2 a_{i+1}^2 a_{i+4}} \frac{\left(a_{i+1} + a_{i+3}a_{i+4}f_1^{(i+1)}\right)\left(a_{i+1} + a_{i+3}f_1^{(i+1)}\right)}{a_i a_{i+1} + a_{i+3}f_1^{(i+1)}},$$

$$T_i^{-1}\big(f_1^{(i+1)}\big)f_1^{(i+1)} = \frac{a_i a_{i+1}{}^3}{a_{i+3}{}^2}\frac{\big(a_{i+2}a_{i+3}+a_i f_1^{(i+3)}\big)\big(a_{i+3}+a_i f_1^{(i+3)}\big)}{a_{i+3}+a_i a_{i+1}f_1^{(i+3)}},$$

where $i \in \mathbb{Z}/5\mathbb{Z}$, lead a q-discrete analogue of Painlevé V equation [56]

$$T_i\big(X_{l_1}^{(i+3)}\big)X_{l_1}^{(i+3)} = \frac{\alpha_{l_1}^{(i+3)}}{\big(\alpha_{l_1}^{(i)}\big)^2\big(\alpha_{l_1}^{(i+1)}\big)^2\alpha_{l_1}^{(i+4)}}$$
$$\times\frac{\big(\alpha_{l_1}^{(i+1)}+\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}X_{l_1}^{(i+1)}\big)\big(\alpha_{l_1}^{(i+1)}+\alpha_{l_1}^{(i+3)}X_{l_1}^{(i+1)}\big)}{\alpha_{l_1}^{(i)}\alpha_{l_1}^{(i+1)}+\alpha_{l_1}^{(i+3)}X_{l_1}^{(i+1)}},$$

$$T_i^{-1}\big(X_{l_1}^{(i+1)}\big)X_{l_1}^{(i+1)} = \frac{\alpha_{l_1}^{(i)}\big(\alpha_{l_1}^{(i+1)}\big)^3}{\big(\alpha_{l_1}^{(i+3)}\big)^2}\frac{\big(\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}+\alpha_{l_1}^{(i)}X_{l_1}^{(i+3)}\big)\big(\alpha_{l_1}^{(i+3)}+\alpha_{l_1}^{(i)}X_{l_1}^{(i+3)}\big)}{\alpha_{l_1}^{(i+3)}+\alpha_{l_1}^{(i)}\alpha_{l_1}^{(i+1)}X_{l_1}^{(i+3)}}. \quad (2.32)$$

Moreover, the action of $T_{23}^{(i)} = T_{i+2}T_{i+3}$:

$$T_{23}^{(i)}\colon\ (a_{i+2},a_{i+4}) \mapsto \big(qa_{i+2},q^{-1}a_{i+4}\big),$$
$$\bigg(T_{23}^{(i)}\big(f_2^{(i+2)}\big)f_1^{(i+3)}-\frac{a_{i+2}a_{i+3}a_{i+4}}{a_i}\bigg)\bigg(f_2^{(i+2)}f_1^{(i+3)}-\frac{a_{i+2}a_{i+3}a_{i+4}}{a_i}\bigg)$$
$$=\frac{a_{i+2}{}^3 a_{i+3}a_{i+4}}{a_i{}^2}\frac{\big(a_{i+3}+a_i f_1^{(i+3)}\big)\big(a_{i+3}+a_i a_{i+1}f_1^{(i+3)}\big)}{a_{i+2}a_{i+3}+a_i f_1^{(i+3)}},$$
$$\bigg(f_2^{(i+2)}f_1^{(i+3)}-\frac{a_{i+2}a_{i+3}a_{i+4}}{a_i}\bigg)\bigg(f_2^{(i+2)}T_{23}^{(i)-1}\big(f_1^{(i+3)}\big)-\frac{a_{i+2}a_{i+3}a_{i+4}}{a_i}\bigg)$$
$$=\frac{a_{i+2}a_{i+3}}{a_i{}^2 a_{i+1}a_{i+4}}\frac{\big(a_{i+1}a_{i+2}a_{i+4}+f_2^{(i+2)}\big)\big(a_{i+2}a_{i+4}+f_2^{(i+2)}\big)}{a_{i+2}+a_i f_2^{(i+2)}},$$

where $i \in \mathbb{Z}/5\mathbb{Z}$, and that of $T_{13}^{(i)} = T_{i+1}T_{i+3}$:

$$T_{13}^{(i)}\colon\ (a_{i+1},a_{i+2},a_{i+3},a_{i+4}) \mapsto \big(qa_{i+1},q^{-1}a_{i+2},qa_{i+3},q^{-1}a_{i+4}\big),$$
$$\bigg(T_{13}^{(i)}\big(f_1^{(i+1)}\big)f_1^{(i+2)}-\frac{a_{i+1}a_{i+2}}{a_{i+3}a_{i+4}}\bigg)\bigg(f_1^{(i+1)}f_1^{(i+2)}-\frac{a_{i+1}a_{i+2}}{a_{i+3}a_{i+4}}\bigg)$$
$$=\frac{a_{i+1}{}^3 a_{i+2}}{a_{i+3}a_{i+4}{}^2}\frac{\big(a_{i+2}+a_i a_{i+4}f_1^{(i+2)}\big)\big(a_{i+2}+a_{i+4}f_1^{(i+2)}\big)}{a_{i+1}a_{i+2}+a_{i+4}f_1^{(i+2)}},$$
$$\bigg(f_1^{(i+1)}f_1^{(i+2)}-\frac{a_{i+1}a_{i+2}}{a_{i+3}a_{i+4}}\bigg)\bigg(f_1^{(i+1)}T_{13}^{(i)-1}\big(f_1^{(i+2)}\big)-\frac{a_{i+1}a_{i+2}}{a_{i+3}a_{i+4}}\bigg)$$
$$=\frac{a_{i+1}a_{i+2}}{a_i a_{i+3}{}^2 a_{i+4}{}^2}\frac{\big(a_{i+1}+a_{i+3}f_1^{(i+1)}\big)\big(a_i a_{i+1}+a_{i+3}f_1^{(i+1)}\big)}{a_{i+1}+a_{i+3}a_{i+4}f_1^{(i+1)}},$$

where $i \in \mathbb{Z}/5\mathbb{Z}$, respectively give the systems

$$\bigg(T_{23}^{(i)}\big(Y_{l_1}^{(i+2)}\big)X_{l_1}^{(i+3)}-\frac{\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}{\alpha_{l_1}^{(i)}}\bigg)\bigg(Y_{l_1}^{(i+2)}X_{l_1}^{(i+3)}-\frac{\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}{\alpha_{l_1}^{(i)}}\bigg)$$
$$=\frac{\alpha_{l_1}^{(i+2)}{}^3\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}{\big(\alpha_{l_1}^{(i)}\big)^2}\frac{\big(\alpha_{l_1}^{(i+3)}+\alpha_{l_1}^{(i)}X_{l_1}^{(i+3)}\big)\big(\alpha_{l_1}^{(i+3)}+\alpha_{l_1}^{(i)}\alpha_{l_1}^{(i+1)}X_{l_1}^{(i+3)}\big)}{\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}+\alpha_{l_1}^{(i)}X_{l_1}^{(i+3)}}, \quad (2.33a)$$

$$\left(Y_{l_1}^{(i+2)}X_{l_1}^{(i+3)} - \frac{\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}{\alpha_{l_1}^{(i)}}\right)\left(Y_{l_1}^{(i+2)}T_{23}^{(i)\,-1}\left(X_{l_1}^{(i+3)}\right) - \frac{\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}{\alpha_{l_1}^{(i)}}\right)$$

$$= \frac{\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+3)}}{\left(\alpha_{l_1}^{(i)}\right)^2\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+4)}}\frac{\left(\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+4)} + Y_{l_1}^{(i+2)}\right)\left(\alpha_{l_1}^{(i+2)}\alpha_{l_1}^{(i+4)} + Y_{l_1}^{(i+2)}\right)}{\alpha_{l_1}^{(i+2)} + \alpha_{l_1}^{(i)}Y_{l_1}^{(i+2)}}, \quad \text{(2.33b)}$$

and

$$\left(T_{13}^{(i)}\left(X_{l_1}^{(i+1)}\right)X_{l_1}^{(i+2)} - \frac{\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)}}{\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}\right)\left(X_{l_1}^{(i+1)}X_{l_1}^{(i+2)} - \frac{\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)}}{\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}\right)$$

$$= \frac{\left(\alpha_{l_1}^{(i+1)}\right)^3\alpha_{l_1}^{(i+2)}}{\alpha_{l_1}^{(i+3)}\left(\alpha_{l_1}^{(i+4)}\right)^2}\frac{\left(\alpha_{l_1}^{(i+2)} + \alpha_{l_1}^{(i)}\alpha_{l_1}^{(i+4)}X_{l_1}^{(i+2)}\right)\left(\alpha_{l_1}^{(i+2)} + \alpha_{l_1}^{(i+4)}X_{l_1}^{(i+2)}\right)}{\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)} + \alpha_{l_1}^{(i+4)}X_{l_1}^{(i+2)}}, \quad \text{(2.34a)}$$

$$\left(X_{l_1}^{(i+1)}X_{l_1}^{(i+2)} - \frac{\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)}}{\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}\right)\left(X_{l_1}^{(i+1)}T_{13}^{(i)\,-1}\left(X_{l_1}^{(i+2)}\right) - \frac{\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)}}{\alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}}\right)$$

$$= \frac{\alpha_{l_1}^{(i+1)}\alpha_{l_1}^{(i+2)}}{\alpha_{l_1}^{(i)}\left(\alpha_{l_1}^{(i+3)}\right)^2\left(\alpha_{l_1}^{(i+4)}\right)^2}\frac{\left(\alpha_{l_1}^{(i+1)} + \alpha_{l_1}^{(i+3)}X_{l_1}^{(i+1)}\right)\left(\alpha_{l_1}^{(i)}\alpha_{l_1}^{(i+1)} + \alpha_{l_1}^{(i+3)}X_{l_1}^{(i+1)}\right)}{\alpha_{l_1}^{(i+1)} + \alpha_{l_1}^{(i+3)}\alpha_{l_1}^{(i+4)}X_{l_1}^{(i+1)}}. \quad \text{(2.34b)}$$

Systems (2.33) and (2.34) are also known as q-discrete analogues of Painlevé V equation [57].

From equation (2.5), definitions (2.30) and (2.31) and Theorem 2.7, we obtain the following corollary.

Corollary 2.9. *Under the condition* (2.7), *the hypergeometric solutions of q-Painlevé equations* (2.32), (2.33) *and* (2.34) *are given by*

$$X_{l_1}^{(1)} = q^{l_1+1/2}\frac{\Phi_{l_1}^{1,1,1}\Phi_{l_1+1}^{1,0,1}}{\Phi_{l_1}^{1,0,1}\Phi_{l_1+1}^{1,1,1}}, \qquad X_{l_1}^{(2)} = -\frac{qa_2}{a_0a_4}\frac{\Phi_{l_1+1}^{1,0,2}\Phi_{l_1+1}^{1,1,1}}{\Phi_{l_1+1}^{1,0,1}\Phi_{l_1+1}^{1,1,2}},$$

$$X_{l_1}^{(3)} = \frac{1-a_4}{q^{l_1+1/2}a_0a_2a_4{}^2}\frac{\Phi_{l_1}^{0,0,0}\Phi_{l_1+1}^{1,1,2}}{\Phi_{l_1}^{0,0,1}\Phi_{l_1+1}^{1,1,1}}, \qquad X_{l_1}^{(4)} = -\frac{a_0a_4}{q^{l_1+1/2}a_2}\frac{\Phi_{l_1}^{0,0,1}\Phi_{l_1+1}^{2,1,2}}{\Phi_{l_1}^{1,0,1}\Phi_{l_1+1}^{1,1,2}},$$

$$X_{l_1}^{(5)} = \frac{q^{l_1+1}-a_0}{q^{1/2}}\frac{\Phi_{l_1}^{1,0,1}\Phi_{l_1+1}^{0,0,1}}{\Phi_{l_1}^{0,0,1}\Phi_{l_1+1}^{1,0,1}}, \qquad Y_{l_1}^{(1)} = \frac{1}{q^{1/2}a_2}\frac{\Phi_{l_1+1}^{1,0,1}\Phi_{l_1+1}^{2,1,2}}{\Phi_{l_1+1}^{1,0,2}\Phi_{l_1+1}^{1,1,1}}\left(\frac{\Phi_{l_1}^{0,0,1}}{\Phi_{l_1}^{1,0,1}} - \frac{q^{1/2}}{a_4}\frac{\Phi_{l_1+1}^{1,1,2}}{\Phi_{l_1+1}^{2,1,2}}\right),$$

$$Y_{l_1}^{(2)} = \frac{q^{1/2}a_2a_4{}^2}{1-a_4}\frac{\Phi_{l_1}^{1,0,1}\Phi_{l_1+1}^{1,1,1}}{\Phi_{l_1}^{0,0,0}\Phi_{l_1+1}^{1,1,2}}\left(\frac{\Phi_{l_1}^{0,0,1}}{\Phi_{l_1}^{1,0,1}} + \frac{a_2\left(q^{l_1+1}-a_0\right)}{q^{l_1+1/2}a_0a_4}\frac{\Phi_{l_1+1}^{0,0,1}}{\Phi_{l_1+1}^{1,0,1}}\right),$$

$$Y_{l_1}^{(3)} = -\frac{q^{1/2}a_0}{a_4}\frac{\Phi_{l_1}^{1,1,1}\Phi_{l_1+1}^{1,1,2}}{\Phi_{l_1}^{0,0,1}\Phi_{l_1+1}^{2,1,2}}\left(\frac{\Phi_{l_1}^{1,0,1}}{\Phi_{l_1}^{1,1,1}} + \frac{1}{q^{1/2}a_2a_4}\frac{\Phi_{l_1+1}^{1,0,1}}{\Phi_{l_1+1}^{1,1,1}}\right),$$

$$Y_{l_1}^{(4)} = \frac{q^{2l_1+1/2}a_4}{a_2\left(q^{l_1+1}-a_0\right)}\frac{\Phi_{l_1}^{0,0,1}\Phi_{l_1+1}^{1,0,2}}{\Phi_{l_1}^{1,0,1}\Phi_{l_1+1}^{0,0,1}}\left(\frac{\Phi_{l_1+1}^{1,0,1}}{\Phi_{l_1+1}^{1,0,2}} - \frac{\Phi_{l_1+1}^{1,1,1}}{\Phi_{l_1+1}^{1,1,2}}\right),$$

$$Y_{l_1}^{(5)} = \frac{a_2(1-a_4)}{q^{l_1}}\frac{\Phi_{l_1}^{1,0,1}\Phi_{l_1+1}^{1,1,2}}{\Phi_{l_1}^{1,1,1}\Phi_{l_1+1}^{1,0,1}}\left(\frac{\Phi_{l_1}^{0,0,0}}{\Phi_{l_1}^{0,0,1}} + \frac{q^{1/2}a_2a_4}{1-a_4}\frac{\Phi_{l_1+1}^{1,1,1}}{\Phi_{l_1+1}^{1,1,2}}\right),$$

where the functions $\left\{\Phi_{l_1}^{l_0,l_2,l_3}\right\}_{l_1\in\mathbb{Z}_{\geq 0}}$ *are defined by* (2.29). *Note that the actions of translations* T_i, $i = 0, \ldots, 4$, *on these solutions are given by the following*

$$T_0\colon \left(a_0, a_2, a_4, l_1, q, \Phi_{l_1}^{l_0,l_2,l_3}\right) \mapsto \left(qa_0, a_2, a_4, l_1, q, \Phi_{l_1}^{l_0+1,l_2,l_3}\right),$$

$$T_1\colon \ \left(a_0, a_2, a_4, l_1, q, \Phi_{l_1}^{l_0,l_2,l_3}\right) \mapsto \left(a_0, q^{-1}a_2, a_4, l_1+1, q, \Phi_{l_1+1}^{l_0,l_2,l_3}\right),$$
$$T_2\colon \ \left(a_0, a_2, a_4, l_1, q, \Phi_{l_1}^{l_0,l_2,l_3}\right) \mapsto \left(a_0, qa_2, a_4, l_1, q, \Phi_{l_1}^{l_0,l_2+1,l_3}\right),$$
$$T_3\colon \ \left(a_0, a_2, a_4, l_1, q, \Phi_{l_1}^{l_0,l_2,l_3}\right) \mapsto \left(a_0, a_2, q^{-1}a_4, l_1, q, \Phi_{l_1}^{l_0,l_2,l_3+1}\right),$$
$$T_4\colon \ \left(a_0, a_2, a_4, l_1, q, \Phi_{l_1}^{l_0,l_2,l_3}\right) \mapsto \left(q^{-1}a_0, a_2, qa_4, l_1-1, q, \Phi_{l_1-1}^{l_0-1,l_2-1,l_3-1}\right).$$

3 Hypergeometric τ functions of $\widetilde{W}\big((A_1 + A_1')^{(1)}\big)$-type

In this section, we construct the hypergeometric τ functions of $\widetilde{W}\big((A_1 + A_1')^{(1)}\big)$-type.

3.1 τ functions

The action of the transformation group $\widetilde{W}\big((A_1 + A_1')^{(1)}\big) = \langle s_0, s_1, w_0, w_1, \pi \rangle$ on the parameters a_0, a_1 and b are given by

$$s_0\colon \ (a_0, a_1, b) \mapsto \left(\frac{1}{a_0}, a_0{}^2 a_1, \frac{b}{a_0}\right), \qquad s_1\colon \ (a_0, a_1, b) \mapsto \left(a_0 a_1{}^2, \frac{1}{a_1}, a_1 b\right),$$
$$w_0\colon \ (a_0, a_1, b) \mapsto \left(\frac{1}{a_0}, \frac{1}{a_1}, \frac{b}{a_0}\right), \qquad w_1\colon \ (a_0, a_1, b) \mapsto \left(\frac{1}{a_0}, \frac{1}{a_1}, \frac{b}{a_0{}^2 a_1}\right),$$
$$\pi\colon \ (a_0, a_1, b) \mapsto \left(\frac{1}{a_1}, \frac{1}{a_0}, \frac{b}{a_0 a_1}\right),$$

while its actions on the variables τ_i, $i = -3, \dots, 3$, are given by

$$s_0\colon \ (\tau_{-3}, \tau_{-1}, \tau_1) \mapsto \left(\frac{a_0 \tau_1 \tau_{-2}{}^2 + \tau_{-1}\tau_0\tau_{-2} + \tau_{-3}\tau_0{}^2}{a_0\tau_{-1}\tau_1}, \frac{a_0\tau_0{}^2 + b\tau_{-2}\tau_2}{a_0\tau_1}, \frac{b\tau_{-2}\tau_2 + \tau_0{}^2}{\tau_{-1}}\right),$$
$$s_1\colon \ (\tau_{-2}, \tau_0) \mapsto \left(\frac{a_0 a_1 \tau_{-1}{}^2 + b\tau_{-3}\tau_1}{a_0 a_1 \tau_0}, \frac{a_0\tau_{-1}{}^2 + b\tau_{-3}\tau_1}{a_0\tau_{-2}}\right),$$
$$w_0\colon \ (\tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_1) \mapsto (\tau_3, \tau_2, \tau_1, \tau_{-1}),$$
$$w_1\colon \ (\tau_{-3}, \tau_{-2}, \tau_0, \tau_1) \mapsto (\tau_1, \tau_0, \tau_{-2}, \tau_{-3}),$$
$$\pi\colon \ (\tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1) \mapsto (\tau_2, \tau_1, \tau_0, \tau_{-1}, \tau_{-2}),$$

where

$$\tau_2 = \frac{a_0\left(\tau_{-1}\tau_0 + \tau_{-2}\tau_1\right)}{b\tau_{-3}}, \qquad \tau_3 = \frac{\tau_0\tau_1 + \tau_{-1}\tau_2}{b\tau_{-2}}.$$

For each element $w \in \widetilde{W}\big((A_1 + A_1')^{(1)}\big)$ and function $F = F(a_i, b, \tau_j)$, we use the notation $w.F$ to mean $w.F = F(w.a_i, w.b, w.\tau_j)$, that is, w acts on the arguments from the left. We note that the group of transformations $\widetilde{W}\big((A_1 + A_1')^{(1)}\big)$ forms the extended affine Weyl group of type $(A_1 + A_1)^{(1)}$ [25]. Namely, the transformations satisfy the fundamental relations

$$s_0{}^2 = s_1{}^2 = (s_0 s_1)^\infty = 1, \qquad w_0{}^2 = w_1{}^2 = (w_0 w_1)^\infty = 1,$$
$$\pi^2 = 1, \qquad \pi s_0 = s_1 \pi, \qquad \pi w_0 = w_1 \pi,$$

and the action of $W\big(A_1^{(1)}\big) = \langle s_0, s_1 \rangle$ and that of $W\big(A_1^{(1)\prime}\big) = \langle w_0, w_1 \rangle$ commute. We note that the relation $(ww')^\infty = 1$ for transformations w and w' means that there is no positive integer N such that $(ww')^N = 1$.

To iterate each variable τ_i, we need the translations T_i, $i = 1, 2, 3$, defined by

$$T_1 = w_0 w_1, \qquad T_2 = \pi s_1 w_0, \qquad T_3 = \pi s_0 w_0.$$

Note that T_i, $i = 1, 2, 3$, commute with each other and $T_1 T_2 T_3 = 1$. The actions of these on the parameters are given by

$$T_1\colon\ (a_0, a_1, b) \mapsto (a_0, a_1, qb), \qquad T_2\colon\ (a_0, a_1, b) \mapsto (qa_0, q^{-1}a_1, b),$$
$$T_3\colon\ (a_0, a_1, b) \mapsto (q^{-1}a_0, qa_1, q^{-1}b),$$

where the parameter $q = a_0 a_1$ is invariant under the action of translations. We define τ functions by

$$\tau_{l_2}^{l_1} = T_1^{l_1} T_2^{l_2} (\tau_{-3}),$$

where $l_1, l_2 \in \mathbb{Z}$. We note that

$$\tau_{-3} = \tau_0^0, \qquad \tau_{-2} = \tau_1^1, \qquad \tau_{-1} = \tau_0^1, \qquad \tau_0 = \tau_1^2, \qquad \tau_1 = \tau_0^2, \qquad \tau_2 = \tau_1^3, \qquad \tau_3 = \tau_0^3.$$

3.2 Discrete Painlevé equations

Let

$$f_0 = \frac{\tau_{-2}\tau_1}{\tau_{-1}\tau_0}, \qquad f_1 = \frac{\tau_{-3}\tau_0}{\tau_{-2}\tau_{-1}}, \qquad f_2 = \frac{(\tau_{-1})^2}{\tau_{-3}\tau_1},$$

where

$$f_0 f_1 f_2 = 1.$$

The action of $\widetilde{W}\big((A_1 + A_1')^{(1)}\big)$ on the variables f_i, $i = 0, 1, 2$, is given by

$$s_0\colon\ (f_0, f_1, f_2) \mapsto \left(\frac{f_0(a_0 f_0 + a_0 + f_1)}{f_0 + f_1 + 1},\ \frac{f_1(a_0 f_0 + f_1 + 1)}{a_0(f_0 + f_1 + 1)}, \right.$$
$$\left. \frac{a_0 f_2(f_0 + f_1 + 1)^2}{(a_0 f_0 + a_0 + f_1)(a_0 f_0 + f_1 + 1)} \right),$$
$$s_1\colon\ (f_0, f_1) \mapsto \left(\frac{f_0(a_0 a_1 + b f_0 f_1)}{a_1(a_0 + b f_0 f_1)},\ \frac{a_1 f_1(a_0 + b f_0 f_1)}{a_0 a_1 + b f_0 f_1} \right),$$
$$w_0\colon\ (f_0, f_1, f_2) \mapsto \left(\frac{a_0(f_0 + 1)}{b f_0 f_1},\ \frac{a_0 f_0 + a_0 + b f_0 f_1}{a_0 b f_0(f_0 + 1)},\ \frac{b^2 f_0}{f_2(a_0 f_0 + a_0 + b f_0 f_1)} \right),$$
$$w_1\colon\ (f_0, f_1) \mapsto (f_1, f_0),$$
$$\pi\colon\ (f_1, f_2) \mapsto \left(\frac{a_0(f_0 + 1)}{b f_0 f_1},\ \frac{b f_1}{a_0(f_0 + 1)} \right).$$

By letting

$$f_{l_2}^{(0)} = T_2^{l_2}(f_0), \qquad f_{l_2}^{(1)} = T_2^{l_2}(f_1), \qquad f_{l_2}^{(2)} = T_2^{l_2}(f_2),$$

the actions of T_i, $i = 1, 2, 3$:

$$T_1(f_1)f_1 = \frac{a_0(f_0 + 1)}{b f_0}, \qquad T_1(f_0)f_0 = \frac{T_1(f_1) + 1}{b T_1(f_1)},$$
$$T_2(f_2)f_2 = \frac{b}{q f_1(f_1 + 1)}, \qquad T_2(f_1)f_1 = \frac{a_0(b + q T_2(f_2))}{T_2(f_2)(q a_0 T_2(f_2) + b)},$$
$$T_3(f_0)f_0 = \frac{a_1 b + q f_2}{f_2(b + q f_2)}, \qquad T_3(f_2)f_2 = \frac{a_1 b}{q T_3(f_0)(T_3(f_0) + 1)},$$

lead the following q-Painlevé equations

$$T_1\big(f^{(1)}_{l_2}\big)f^{(1)}_{l_2} = \frac{q^{l_2}a_0\big(f^{(0)}_{l_2}+1\big)}{bf^{(0)}_{l_2}}, \qquad T_1\big(f^{(0)}_{l_2}\big)f^{(0)}_{l_2} = \frac{T_1\big(f^{(1)}_{l_2}\big)+1}{bT_1\big(f^{(1)}_{l_2}\big)}, \tag{3.1}$$

$$T_2\big(f^{(2)}_{l_2}\big)f^{(2)}_{l_2} = \frac{b}{qf^{(1)}_{l_2}\big(f^{(1)}_{l_2}+1\big)}, \qquad T_2\big(f^{(1)}_{l_2}\big)f^{(1)}_{l_2} = \frac{q^{l_2}a_0\big(b+qT_2\big(f^{(2)}_{l_2}\big)\big)}{T_2\big(f^{(2)}_{l_2}\big)\big(q^{l_2+1}a_0T_2\big(f^{(2)}_{l_2}\big)+b\big)}, \tag{3.2}$$

$$T_3\big(f^{(0)}_{l_2}\big)f^{(0)}_{l_2} = \frac{a_1b+q^{l_2+1}f^{(2)}_{l_2}}{q^{l_2}f^{(2)}_{l_2}\big(b+qf^{(2)}_{l_2}\big)}, \qquad T_3\big(f^{(2)}_{l_2}\big)f^{(2)}_{l_2} = \frac{a_1b}{q^{l_2+1}T_3\big(f^{(0)}_{l_2}\big)\big(T_3\big(f^{(0)}_{l_2}\big)+1\big)}. \tag{3.3}$$

We note that equation (3.1) is known as a q-discrete analogue of Painlevé II equation [39] and can be rewritten as the following single second-order ordinary difference equation [52, 54, 56]:

$$\left(T_1\big(f^{(0)}_{l_2}\big)f^{(0)}_{l_2}-\frac{1}{b}\right)\left(T_1^{-1}\big(f^{(0)}_{l_2}\big)f^{(0)}_{l_2}-\frac{q}{b}\right) = \frac{a_1}{q^{l_2}b}\frac{f^{(0)}_{l_2}}{1+f^{(0)}_{l_2}}. \tag{3.4}$$

3.3 Hypergeometric τ functions

We here define hypergeometric τ functions of $\widetilde{W}\big((A_1+A_1')^{(1)}\big)$-type by $\tau^{l_1}_{l_2}$ satisfying the following conditions:

(i) $\tau^{l_1}_{l_2}$ satisfy the action of the translation subgroup of $\widetilde{W}\big((A_1+A_1')^{(1)}\big)$, $\langle T_1,T_2,T_3\rangle$;

(ii) $\tau^{l_1}_{l_2}$ are functions in b consistent with the action of T_1, i.e., $\tau^{l_1}_{l_2} = \tau_{l_2}(q^{l_1}b)$;

(iii) $\tau^{l_1}_{l_2}$ satisfy the following boundary conditions: $\tau^{l_1}_{l_2} = 0$, for $l_2 < 0$;

under the conditions of parameters

$$a_0 = 1, \qquad a_1 = q. \tag{3.5}$$

In a similar manner as Section 2.2, we obtain the following theorem.

Theorem 3.1. *The hypergeometric τ functions of $\widetilde{W}\big((A_1+A_1')^{(1)}\big)$-type are given by the following*

$$\tau^{l_1}_0 = \Gamma\big(q^{l_1}b;q,q\big), \qquad \tau^{l_1}_{l_2} = \frac{\Gamma\big(q^{l_1}b;q,q\big)}{\Theta\big(q^{l_1}b;q\big)^{l_2}}\psi^{l_1}_{l_2},$$

where $l_1 \in \mathbb{Z}$, $l_2 \in \mathbb{Z}_{>0}$ and the functions $\{\psi^{l_1}_{l_2}\}_{l_1\in\mathbb{Z},\,l_2\in\mathbb{Z}_{>0}}$ are given by the following $l_2 \times l_2$ determinants

$$\psi^{l_1}_{l_2} = \begin{vmatrix} F_{l_1} & F_{l_1+1} & \cdots & F_{l_1+l_2-1} \\ F_{l_1-1} & F_{l_1} & \cdots & F_{l_1+l_2-2} \\ \vdots & \vdots & \ddots & \vdots \\ F_{l_1-l_2+1} & F_{l_1-l_2+2} & \cdots & F_{l_1} \end{vmatrix}.$$

Here, the function F_n is given by

$$F_n = \frac{\Theta\big(-q^{(2n-3)/4}b^{1/2};q^{1/2}\big)}{q^n b}\left(A_n\,{}_1\varphi_1\begin{pmatrix}0\\-q^{1/2}\end{pmatrix};q^{1/2},-q^{(2n-5)/4}b^{1/2}\right)$$
$$+ B_n e^{\pi i \log b/\log q}\,{}_1\varphi_1\begin{pmatrix}0\\-q^{1/2}\end{pmatrix};q^{1/2},q^{(2n-5)/4}b^{1/2}\bigg)\bigg),$$

where A_n and B_n are periodic functions of period one with respect to n, that is,

$$A_{n+1} = A_n, \qquad B_{n+1} = B_n.$$

Moreover, Theorem 3.1 leads the following corollary.

Corollary 3.2. *Under the condition* (3.5), *the hypergeometric solutions of q-Painlevé equations* (3.1)–(3.4) *are given by*

$$f_{l_2}^{(0)} = -\frac{1}{qb}\frac{\psi_{l_2+1}^1\psi_{l_2}^2}{\psi_{l_2}^1\psi_{l_2+1}^2}, \qquad f_{l_2}^{(1)} = q^{l_2+1}\frac{\psi_{l_2}^0\psi_{l_2+1}^2}{\psi_{l_2+1}^1\psi_{l_2}^1}, \qquad f_{l_2}^{(2)} = -\frac{b}{q^{l_2}}\frac{(\psi_{l_2}^1)^2}{\psi_{l_2}^0\psi_{l_2}^2}.$$

Note that the actions of translations T_i, $i = 1, 2, 3$, on these solutions are given by the following

$$T_1\colon\ \left(b, q, \psi_{l_2}^{l_1}\right) \mapsto \left(qb, q, \psi_{l_2}^{l_1+1}\right), \qquad T_2\colon\ \left(b, q, \psi_{l_2}^{l_1}\right) \mapsto \left(b, q, \psi_{l_2+1}^{l_1}\right),$$

$$T_3\colon\ \left(b, q, \psi_{l_2}^{l_1}\right) \mapsto \left(q^{-1}b, q, \psi_{l_2-1}^{l_1-1}\right).$$

Acknowledgements

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