Hypergeometric Differential Equation and New Identities for the Coefficients of Nørlund and Bühring⋆

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Abstract. The fundamental set of solutions of the generalized hypergeometric differential equation in the neighborhood of unity has been built by Nørlund in 1955. The behavior of the generalized hypergeometric function in the neighborhood of unity has been described in the beginning of 1990s by Bühring, Srivastava and Saigo. In the first part of this paper we review their results rewriting them in terms of Meijer's G-function and explaining the interconnections between them. In the second part we present new formulas and identities for the coefficients that appear in the expansions of Meijer's G-function and generalized hypergeometric function around unity. Particular cases of these identities include known and new relations for Thomae's hypergeometric function and forgotten Hermite's identity for the sine function.

Key words: generalized hypergeometric function; hypergeometric differential equation; Meijer's G-function; Bernoulli polynomials; Nørlund's coefficients; Bühring's coefficients

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1 Introduction

We will use standard notation $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ to denote integer, natural and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The hypergeometric differential equation ($D = z\frac{d}{dz}$)

$$\{(D - a_1)(D - a_2) \cdots (D - a_p) - z(D + 1 - b_1)(D + 1 - b_2) \cdots (D + 1 - b_p)\} y = 0 \quad (1.1)$$

for $p = 2$ was first considered by Euler and later studied by Gauss, Kummer, Riemann, Papperitz and Schwarz, among others, see [2, Section 2.3]. For general $p > 2$ it was probably first investigated by Thomae [42]. Note that our choice of parameters differs slightly from that of [3, (16.8.3)]. As will become apparent in the sequel, this choice is more convenient if the solution is to be built in terms of Meijer's G-functions and not generalized hypergeometric functions. Equation (1.1) is of Fuchsian type and has three regular singularities located at the points 0, 1, ∞. The local exponents read [5, (2.6)–(2.8)]

$$
\begin{array}{ll}
a_1, a_2, \ldots, a_p & \text{at } z = 0, \\
0, 1, 2, \ldots, p - 2, \sum (b_i - a_i) - 1 & \text{at } z = 1, \\
1 - b_1, 1 - b_2, \ldots, 1 - b_p & \text{at } z = \infty.
\end{array}
$$

The fundamental sets of solutions around the points $z = 0$ and $z = \infty$ were found by Thomae in [42] and are expressed in terms of the generalized hypergeometric series

$$
{}_pF_q\left(\begin{matrix}\mathbf{a}\\\mathbf{b}\end{matrix}\middle| z\right) = {}_pF_q\left(\mathbf{a};\mathbf{b};z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!}z^n,
$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the rising factorial and $\mathbf{a} = (a_1,\ldots,a_p)$, $\mathbf{b} = (b_1,\ldots,b_q)$ are complex vectors such that $-b_i \notin \mathbb{N}_0$ for $i = 1,\ldots,q$. If the components of $\mathbf{a}$ are distinct modulo $\mathbb{Z}$, then the basis of solutions near $z = 0$ is given by [33, (1.13)]

$$
z^{a_k}{}_pF_{p-1}\left(\begin{matrix}1-\mathbf{b}+a_k\\1-\mathbf{a}_{[k]}+a_k\end{matrix}\middle| z\right), \qquad k = 1,\ldots,p,
$$

where $\mathbf{a}_{[k]}$ signifies the vector $\mathbf{a}$ with the element a_k omitted and $\mathbf{a}+\alpha$ is understood as $(a_1+\alpha, \ldots, a_p+\alpha)$. If the components of $\mathbf{b}$ are distinct modulo $\mathbb{Z}$ then the fundamental system of solutions near $z = \infty$ is given by [33, (1.14)]

$$
z^{b_k-1}{}_pF_{p-1}\left(\begin{matrix}1+\mathbf{a}-b_k\\1+\mathbf{b}_{[k]}-b_k\end{matrix}\middle| \frac{1}{z}\right), \qquad k = 1,\ldots,p.
$$

Finally, the fundamental set of solutions around the point $z = 1$ was found by the remarkable Danish mathematician Niels Erik Nørlund in his milestone work [33]. These facts are well-known and have been frequently cited in the literature. It seems to be less known that the fundamental solutions constructed by Nørlund can be expressed in terms of G-function introduced some 15 years earlier by Meijer, see [30]. In fact, Meijer himself studied the hypergeometric differential equation more general than (1.1) and built the basis of solutions in the neighborhood of zero and infinity in terms of G-functions. The connection between Nørlund's solutions and Meijer's G-function was observed by Marichev and Kalla [27] and Marichev [26] but these two papers remained largely unnoticed. The fact that each solution around $z = 1$ must equal to a linear combination of fundamental solutions around $z = 0$ is reflected for $p = 2$ in the following identity due to Gauss [2, Theorem 2.3.2]

$$
{}_2F_1\left(\begin{matrix}\alpha_1,\alpha_2\\\beta\end{matrix};1-z\right) = \frac{\Gamma(\beta)\Gamma(\beta-\alpha_1-\alpha_2)}{\Gamma(\beta-\alpha_1)\Gamma(\beta-\alpha_2)}{}_2F_1\left(\begin{matrix}\alpha_1,\alpha_2\\\alpha_1+\alpha_2-\beta+1\end{matrix};z\right)
$$
$$
+ \frac{\Gamma(\beta)\Gamma(\alpha_1+\alpha_2-\beta)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}z^{\beta-\alpha_1-\alpha_2}{}_2F_1\left(\begin{matrix}\beta-\alpha_1,\beta-\alpha_2\\\beta-\alpha_1-\alpha_2+1\end{matrix};z\right). \qquad (1.2)
$$

The fact that each solution around $z = 0$ must equal to a linear combination of fundamental solutions around $z = 1$ leads to essentially the same identity with z replaced by $1 - z$. For $p > 2$, however the above connection formula has *two different generalizations*. One of them is the expansion (2.4) of $G_{p,p}^{p,0}$ (a solution around $z = 1$) into a sum of ${}_pF_{p-1}$ (fundamental solutions around $z = 0$) which can also be obtained by applying the residue theorem to the definition of G-function and is found in standard references on G-function, see, for instance, [3, (16.17.2)]. The other one is the expansion of ${}_pF_{p-1}$ into a sum of $G_{p,p}^{p,0}$ with $p-1$ instances of $G_{p,p}^{2,p}$ as given by formula (2.3) below. This expansion, discovered by Nørlund without any mentioning of G-function has been reproduced in [27] in a slightly different form but seems to be forgotten afterwards. A related expansion, again with no reference to G-function, has been then found in [7, 8] by Bühring whose goal was to describe the behavior of ${}_pF_{p-1}(z)$ near $z = 1$. The logarithmic cases have been studied by Saigo and Srivastava in [37]. Let us also mention that the monodromy group of the generalized hypergeometric differential equation has been constructed by Beukers and Heckman in [5].

This paper is organized as follows. Section 2 is of survey nature: we review the results of Nørlund and Bühring and rewrite them in terms of G-function. We also reveal connections between Nørlund's and Bühring's expansions and relate them to various results obtained in statistics literature. In Section 3 we present some new formulas for Nørlund's coefficients and utilize the above mentioned connections to derive various new identities for Nørlund's and Bühring's coefficients which reduce to hypergeometric and trigonometric identities for small p. The most striking of these identities,

$$\sum_{k=1}^{p} \frac{\prod_{i=1}^{p} \sin(\beta_i - \alpha_k)}{\prod_{\substack{i=1 \\ i \neq k}}^{p} \sin(\alpha_i - \alpha_k)} = \sin\left(\sum_{k=1}^{p} (\beta_k - \alpha_k)\right), \tag{1.3}$$

can be viewed as a generalization of Ptolemy's theorem: for a quadrilateral inscribed in a circle the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides. This theorem can be written in trigonometric form which yields precisely the above identity for $p = 2$. For $p = 3$ formula (1.3) was first presented without proof by Glaisher at a 1880 conference and then proved in full generality by Hermite in his 1885 paper [20]. The same paper contains a number of other amazing identities completely forgotten until the recent article [21] by Johnson, where many interesting mathematical and historical details can be found. Furthermore, if we substitute $\sin(z)$ by z in every occurrence of sin in (1.3), we obtain an identity discovered by Gosper, Ismail and Zhang in [19] and known as non-local derangement identity. It has been recently used by Feng, Kuznetsov and Yang to find new formulas for sums of products of generalized hypergeometric functions [18].

When this paper was nearly finished E. Scheidegger published a preprint [38] that also deals with the hypergeometric differential equation (1.1) and builds largely on the works of Nørlund [33] and Bühring [8]. Scheidegger suggests a new basis of solutions around 1 and presents its series expansion. Main emphasis in [38] is on the logarithmic cases (parameter differences are integers). Scheidegger's work is motivated by applications in certain one-parameter families of Calabi–Yau manifolds, known as the mirror quartic and the mirror quintic.

2 Results of Nørlund and Bühring revisited

2.1 Fundamental solutions around unity

The simple observation that nevertheless seems to be largely overlooked in the literature (except for [26, 27]) lies in the fact that Nørlund's solutions to (1.1) can be expressed in terms of Meijer's G-function. Let us remind its definition first. Suppose $0 \leq m \leq q$, $0 \leq n \leq p$ are integers and $\mathbf{a}$, $\mathbf{b}$ are arbitrary complex vectors, such that $a_i - b_j \notin \mathbb{N}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Meijer's G-function is defined by the Mellin–Barnes integral of the form (see [17, Section 5.3], [25, Chapter 1], [36, Section 8.2] or [3, Section 16.17]),

$$G_{p,q}^{m,n}\left(z \middle| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\right) := \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(b_1 + s) \cdots \Gamma(b_m + s)\Gamma(1 - a_1 - s) \cdots \Gamma(1 - a_n - s)z^{-s}}{\Gamma(a_{n+1} + s) \cdots \Gamma(a_p + s)\Gamma(1 - b_{m+1} - s) \cdots \Gamma(1 - b_q - s)} ds, \tag{2.1}$$

where the contour $\mathcal{L}$ is a simple loop that separates the poles of the integrand of the form $b_{jl} = -b_j - l$, $l \in \mathbb{N}_0$, leaving them on the left from the poles of the form $a_{ik} = 1 - a_i + k$, $k \in \mathbb{N}_0$, leaving them on the right [25, Section 1.1]. It may have one of the three forms $\mathcal{L}_-$, $\mathcal{L}_+$ or $\mathcal{L}_{i\gamma}$ described below. Choose any

$$\varphi_1 < \min\{-\Im b_1, \ldots, -\Im b_m, \Im(1 - a_1), \ldots, \Im(1 - a_n)\},$$
$$\varphi_2 > \max\{-\Im b_1, \ldots, -\Im b_m, \Im(1 - a_1), \ldots, \Im(1 - a_n)\}$$

and arbitrary real γ. The contour $\mathcal{L}_-$ is a left loop lying in the horizontal strip $\varphi_1 \leq \Im s \leq \varphi_2$. It starts at the point $-\infty + i\varphi_1$, terminates at the point $-\infty + i\varphi_2$ and coincides with the boundary of the strip for sufficiently large $|s|$. Similarly, the contour $\mathcal{L}_+$ is a right loop lying in the same strip, starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$. Finally, the contour $\mathcal{L}_{i\gamma}$ starts at $\gamma - i\infty$, terminates at $\gamma + i\infty$ and coincides with the line $\Re s = \gamma$ for all sufficiently large $|s|$. The power function z^{-s} is defined on the Riemann surface of the logarithm, so that

$$z^{-s} = \exp(-s\{\log|z| + i\arg(z)\})$$

and $\arg(z)$ is allowed to take any real value. Hence, $G_{p,q}^{m,n}(z)$ is also defined on the Riemann surface of the logarithm. In order that the above definition be consistent one needs to prove that the value of the integral remains intact if convergence takes place for several different contours. Alternatively, one may split the parameter space into nonintersecting subsets and stipulate which contour should be used in each subset. Another key issue that must be addressed with regard to the above definition is whether the integral in (2.1) equals the sum of residues of the integrand and, if yes, on which side of $\mathcal{L}$ the residues are to be counted. This is important for both theoretical considerations (expressing G-function in terms of hypergeometric functions) and especially for actually computing the value of G-function (although numerical contour integration can also be employed). In this paper we will only need the G-function of the form $G_{p,p}^{m,n}$. For this particular type of G-function the solutions to the above problems seem to be rather complete. We placed further details regarding the definition and answers to the above questions for $G_{p,p}^{m,n}$ in Appendix A.

A simple property of Meijer's G-function implied by its definition (2.1) which will be frequently used without further mentioning is given by [36, (8.2.2.15)]

$$z^\alpha G_{p,q}^{m,n}\left(z \left|\begin{matrix}\mathbf{a}\\\mathbf{b}\end{matrix}\right.\right) = G_{p,q}^{m,n}\left(z \left|\begin{matrix}\mathbf{a}+\alpha\\\mathbf{b}+\alpha\end{matrix}\right.\right), \tag{2.2}$$

where $\alpha \in \mathbb{C}$ and $\mathbf{a} + \alpha$ is understood as $(a_1 + \alpha, \ldots, a_p + \alpha)$.

The functions $G_{p,p}^{p,0}$ and $G_{p,p}^{2,p}$ will play a particularly important role in this paper, so that we found it useful to cite the known explicit expressions for small p. If $p = 1$ then [36, formula (8.4.2.3)]

$$G_{1,1}^{1,0}\left(z \left|\begin{matrix}b\\a\end{matrix}\right.\right) = \frac{z^a(1-z)_+^{b-a-1}}{\Gamma(b-a)},$$

where $(x)_+ = x$ for $x \geq 0$ and 0 otherwise. If $p = 2$ we have [36, formula (8.4.49.22)]

$$G_{2,2}^{2,0}\left(z \left|\begin{matrix}b_1, b_2\\a_1, a_2\end{matrix}\right.\right) = \frac{z^{a_2}(1-z)_+^{b_1+b_2-a_1-a_2-1}}{\Gamma(b_1+b_2-a_1-a_2)}\, {}_2F_1\left(\begin{matrix}b_1-a_1, b_2-a_1\\b_1+b_2-a_1-a_2\end{matrix};1-z\right),$$

and [36, formula (8.4.49.20)]

$$G_{2,2}^{2,2}\left(z \left|\begin{matrix}b_1, b_2\\a_1, a_2\end{matrix}\right.\right) = \frac{z^{a_1}\Gamma(1+a_1-b_1)\Gamma(1+a_1-b_2)\Gamma(1+a_2-b_2)\Gamma(1+a_2-b_1)}{\Gamma(2+a_1+a_2-b_1-b_2)}$$
$$\times\, {}_2F_1\left(\begin{matrix}1+a_1-b_1, 1+a_1-b_2\\2+a_1+a_2-b_1-b_2\end{matrix};1-z\right).$$

If $p = 3$ we have [36, formula (8.4.51.2)]

$$G_{3,3}^{3,0}\left(z \left|\begin{matrix}b_1, b_2, b_3\\a_1, a_2, a_3\end{matrix}\right.\right) = \frac{z^{a_1+a_2-b_1-1}(1-z)_+^{b_1+b_2+b_3-a_1-a_2-a_3-1}}{\Gamma(b_1+b_2+b_3-a_1-a_2-a_3)}$$
$$\times F_3(b_1-a_2, b_3-a_3; b_1-a_1, b_2-a_3; b_1+b_2+b_3-a_1-a_2-a_3; 1-1/z, 1-z),$$

where F_3 is Appell's hypergeometric function of two variables [36, index of functions].

Having made these preparations we can formulate Nørlund's result regarding a fundamental solution of (1.1) in the neighborhood of 1. First, introduce the following notation:

$$\psi_m = \sum_{i=1}^{m}(b_i - a_i), \qquad 1 \le m \le p,$$

$$\mathbf{a} = (a_1, \ldots, a_p), \qquad \mathbf{b} = (b_1, \ldots, b_p), \qquad \mathbf{a}_{[k]} = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_p),$$

$$\sin(\mathbf{a}) = \sin a_1 \sin a_2 \cdots \sin a_p, \qquad \Gamma(\mathbf{a}) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)$$

and let $\mathbf{a}_{[k,s]}$ denote the vector $\mathbf{a}$ with the elements a_k and a_s removed. We write $\Re(\mathbf{a}) > 0$ for $\Re(a_i) > 0$, $i = 1, \ldots, p$. The next theorem is implicit in [33].

Theorem 2.1 (Nørlund). *Suppose $k, s \in \{1, \ldots, p\}$ and*

$$u_s(z) = G_{p,p}^{p,0}\left(z \,\bigg|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right), \qquad u_k(z) = G_{p,p}^{2,p}\left(z \,\bigg|\, \begin{matrix} \mathbf{b} \\ a_k, a_s, \mathbf{a}_{[k,s]} \end{matrix}\right) \qquad for \quad k \ne s.$$

Then the set $\{u_k(z)\}_{k=1}^{p}$ forms the fundamental system of solutions of (1.1) in the neighborhood of $z = 1$ and

$$\sin(\pi\psi_p)\frac{z^{a_s}\Gamma(1 - \mathbf{b} + a_s)}{\Gamma(1 - \mathbf{a}_{[s]} + a_s)}\,_pF_{p-1}\left(\begin{matrix} 1 - \mathbf{b} + a_s \\ 1 - \mathbf{a}_{[s]} + a_s \end{matrix}; z\right)$$

$$= \pi G_{p,p}^{p,0}\left(z \,\bigg|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) - \frac{1}{\pi}\sum_{\substack{k=1 \\ k \ne s}}^{p}\frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k,s]} - a_k))}G_{p,p}^{2,p}\left(z \,\bigg|\, \begin{matrix} \mathbf{b} \\ a_k, a_s, \mathbf{a}_{[k,s]} \end{matrix}\right). \tag{2.3}$$

Proof. Formula (2.3) is a rewriting of [33, (5.40)]. Indeed, first compare [33, (2.44)] with the definition of G-function (2.1) to see how Nørlund's ξ_n is expressed by $G_{p,p}^{p,0}$. Alternatively, this connection follows on comparing the Mellin transforms (A.3) and [33, (2.18)]. Further, use [33, (5.45)] to expand Nørlund's φ_s and [33, (5.7)] to express $y_{k,s}$ in terms of $G_{p,p}^{2,p}$ (see also (2.10) below). ∎

Remark 2.2. Formula (2.3) can now be viewed as the reflection of the fact that any $p + 1$ solutions must be linearly dependent so that any solution in the neighborhood of 0 can be expressed in terms of the fundamental set of solutions around 1. This formula extends the connection formula (1.2) for the Gauss hypergeometric function to which it reduces when $p = 2$. A closely related formula has been also discovered by Bühring in two papers [7] (for $p = 3$) and [8] (for general p). See formula (2.19) below.

Remark 2.3. Marichev and Kalla reproduced formula [33, (5.40)] in [27, (11)] and gave expressions for its components in terms of G-function in [27, (27), (33), (36)].

Expansion of the solution $G_{p,p}^{p,0}$ in terms of the fundamental solutions around $z = 0$ that compliments (2.3) coincides with the well-known expansion obtained from the definition of G-function by the residue theorem. Namely, if the elements of the vector $\mathbf{a}$ are different modulo integers, formula (A.1) applied to the function $G_{p,p}^{p,0}$ takes the form (see also [26, formula (34)] and [36, 8.2.2.3]):

$$G_{p,p}^{p,0}\left(z \,\bigg|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) = \sum_{k=1}^{p} z^{a_k}\frac{\Gamma(\mathbf{a}_{[k]} - a_k)}{\Gamma(\mathbf{b} - a_k)}\,_pF_{p-1}\left(\begin{matrix} 1 - \mathbf{b} + a_k \\ 1 - \mathbf{a}_{[k]} + a_k \end{matrix} \,\bigg|\, z\right). \tag{2.4}$$

According to Theorem A.1(a) this formula holds for $|z| < 1$.

2.2 Expansion of $G_{p,p}^{p,0}$ in the neighborhood of unity

Among many other results contained in [33] Nørlund showed that the series

$$G_{p,p}^{p,0}\left(z\,\bigg|\begin{matrix}\mathbf{b}\\\mathbf{a}\end{matrix}\right) = \frac{z^{a_k}(1-z)^{\psi_p-1}}{\Gamma(\psi_p)}\sum_{n=0}^{\infty}\frac{g_p^k(n)}{(\psi_p)_n}(1-z)^n \tag{2.5}$$

represents a solution in the neighborhood of $z = 1$ corresponding to the local exponent $\psi_p - 1$ (see (2.1) for the definition of ψ_p) if this number is not a negative integer. Formula (2.5) holds in the disk $|1 - z| < 1$ for all $-\psi_p \notin \mathbb{N}_0$ and each $k = 1, 2, \ldots, p$. For $-\psi_p = l \in \mathbb{N}_0$ we have by taking limit in (2.5) (see [33, (1.34)]):

$$G_{p,p}^{p,0}\left(z\,\bigg|\begin{matrix}\mathbf{b}\\a\end{matrix}\right) = z^{a_k}\sum_{n=0}^{\infty}\frac{g_p^k(n+l+1)}{n!}(1-z)^n, \qquad k = 1, 2, \ldots, p. \tag{2.6}$$

The value of k in (2.5) and (2.6) can be chosen arbitrarily from the set $\{1, 2, \ldots, p\}$. This choice affects the first factor z^{a_k} and the coefficients $g_p^k(n)$ while the left-hand side is of course independent of k.

Let us present Nørlund's formulas for the coefficients $g_p^k(n)$. First, by applying the Frobenius ansatz to the differential equation (1.1) Nørlund demonstrated that $g_p^k(n)$ satisfy the p-th order difference equation with polynomial coefficients given by

$$\sum_{i=0}^{p-1}P_{p-i}(n\,|\,\mathbf{a},\mathbf{b})g_p^k(n+i) = 0, \qquad n = 1, 2, \ldots, \tag{2.7}$$

where $P_m(z\,|\,\mathbf{a},\mathbf{b})$ is a polynomial in z of degree m with coefficients dependent on the parameters $\mathbf{a}$ and $\mathbf{b}$. It is expressed in terms of the polynomials

$$Q(z) = \prod_{i=1}^{p}(z - a_i), \qquad R(z) = \prod_{i=1}^{p}(z + 1 - b_i)$$

as follows [33, (1.28)]:

$$P_1(z\,|\,\mathbf{a},\mathbf{b}) = p - 1 + z,$$
$$P_j(z\,|\,\mathbf{a},\mathbf{b}) = \frac{(-1)^j}{(p-j-1)!}\Delta^{p-j-1}Q(\psi_p + a_k + z) - \frac{(-1)^j}{(p-j)!}\Delta^{p-j}R(\psi_p - 1 + a_k + z)$$

for $j = 2, \ldots, p - 1$, and

$$P_p(z\,|\,\mathbf{a},\mathbf{b}) = -(-1)^p R(\psi_p - 1 + a_k + z).$$

Here the difference is understood as the forward difference $\Delta Q(z) = Q(z+1) - Q(z)$, $\Delta^m Q(z) = \Delta(\Delta^{m-1}Q(z))$. The initial values $g_p^k(0)$, $g_p^k(1)$, $\ldots$, $g_p^k(p-1)$ for the recurrence (2.7) are found by solving the next triangular system:

$$g_p^k(0) = 1,$$
$$g_p^k(1) + P_2(2 - p\,|\,\mathbf{a},\mathbf{b})g_p^k(0) = 0,$$
$$2g_p^k(2) + P_2(3 - p\,|\,\mathbf{a},\mathbf{b})g_p^k(1) + P_3(3 - p\,|\,\mathbf{a},\mathbf{b})g_p^k(0) = 0,$$
$$3g_p^k(3) + P_2(4 - p\,|\,\mathbf{a},\mathbf{b})g_p^k(2) + P_3(4 - p\,|\,\mathbf{a},\mathbf{b})g_p^k(1) + P_4(4 - p\,|\,\mathbf{a},\mathbf{b})g_p^k(0) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(p-1)g_p^k(p-1) + P_2(0\,|\,\mathbf{a},\mathbf{b})g_p^k(p-2) + P_3(0\,|\,\mathbf{a},\mathbf{b})g_p^k(p-3) + \cdots + P_p(0\,|\,\mathbf{a},\mathbf{b})g_p^k(0) = 0.$$

Another method to compute the coefficients $g_p^k(n)$ discovered by Nørlund is the following recurrence in p [33, (2.7)]:

$$g_1^1(n) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1, \end{cases} \qquad g_p^p(n) = \sum_{j=0}^{n} \frac{(b_p - a_k)_{n-j}}{(n-j)!}(\psi_{p-1} + j)_{n-j} g_{p-1}^k(n). \qquad (2.8)$$

The last formula can be applied for any $k = 1, 2, \ldots, p-1$ without affecting the left-hand side. The value of $g_p^k(n)$ for $k \neq p$ can then be obtained by exchanging the roles of a_p and a_k in the resulting expression for $g_p^p(n)$. Alternatively, Nørlund gives the following connection formula [33, (1.35)]:

$$g_p^k(n) = \sum_{j=0}^{n} \frac{(a_k - a_l)_{n-j}}{(n-j)!}(\psi_p + j)_{n-j} g_p^l(j), \qquad k, l = 1, \ldots, p, \qquad k \neq l.$$

Furthermore, he solved the above recurrence to obtain [33, (2.11)]:

$$g_p^p(n) = \sum_{0 \leq j_1 \leq j_2 \leq \cdots \leq j_{p-2} \leq n} \prod_{m=1}^{p-1} \frac{(\psi_m + j_{m-1})_{j_m - j_{m-1}}}{(j_m - j_{m-1})!}(b_{m+1} - a_m)_{j_m - j_{m-1}}, \qquad (2.9)$$

where, as before, $\psi_m = \sum_{i=1}^{m}(b_i - a_i)$ and $j_0 = 0$, $j_{p-1} = n$. This formula shows explicitly that $g_p^p(n)$ does not depend on a_p. Observe that

- the coefficient $g_p^k(n)$ is a symmetric polynomial in the components of $\mathbf{b}$ and $\mathbf{a}_{[k]}$ (separately);

- the summation in (2.9) is over all Young diagrams that fit $n \times (p-2)$ box.

Symmetry follows from expansion (2.5) and the invariance of G-function with respect to permutation the elements of $\mathbf{a}$ and $\mathbf{b}$. In Section 3 below we find another way to compute the coefficients $g_p^k(n)$ in terms of generalized Bernoulli polynomials and we further give explicit formulas for $g_p^k(n)$, $n = 1, 2, 3$. Connection of these coefficients to combinatorics deserves further investigation.

Remark 2.4. Formulas (2.3) and (2.5) have been reproduced by Marichev [26, (4), (17)] and Marichev and Kalla in [27, (12), (27)]. Surprisingly, these important formulas seem to have been largely overlooked in the special function literature and have never been included in any textbook on hypergeometric functions, except for the reference book [36] by Prudnikov, Brychkov and Marichev. However, even in this book the description of the behavior of $G_{p,p}^{p,0}(z)$ in the neighborhood of $z = 1$ contains an incorrect assertion in the case of non-positive integer ψ_p, see [36, Section 8.2.2.59]. None of the identities presented in Section 3 below are contained in [26, 27, 36].

Remark 2.5. Another very prolific line of research that involves G-function forms an important part of the statistics literature and began with 1932 paper of Wilks [44], where he observed that the moments of many likelihood ratio criteria in multivariate hypothesis testing are expressed in terms of product ratios of gamma functions. Wilks introduced two types of integral equations, of which "type B" is essentially equation (A.3) of the Appendix. He also noticed that the solution of "type B integral equation" represents the probability density of the product of independent beta distributed random variables. Wilks' ideas were elaborated in dozens of papers that followed, mainly concerned with calculating and approximating the solution of "type B integral equation". We will just mention a few key contributions, where an interested reader may find further references. In his 1939 paper [32] Nair derived the differential equation (1.1) satisfied

by the solution of "type B integral equation" thus demonstrating that $G_{p,p}^{p,0}$ satisfies (1.1). This happened long before Nørlund wrote his paper [33] and was also independent of Meijer's work, although Meijer already introduced G-function as a linear combination of hypergeometric functions in 1936. Nair also was the first (among researchers in statistics) to apply the inverse Mellin transform to the right-hand side of (A.3) – the approach further developed by Consul in a series of papers between 1964 and 1969, where the connection to Meijer's G-function was first observed. See [13] and references therein. Mellin transform technique was then utilized in a number of papers by Mathai who later rediscovered Nørlund's coefficients in a form similar to (2.9) in [29]. Mathai's and other contributions until 1973 are described in his survey paper [28]. In the same period Springer and Thompson independently expressed the densities of products and ratios of gamma and beta distributed random variables in terms of Meijer's G-function, see [39]. Davis [14] presented the matrix form of the differential equation for $G_{p,p}^{p,0}$ and suggested the series solution similar to (2.5) with coefficients found by certain recursive procedure. Another noticeable contribution is due to Gupta and Tang who found two series expansions for $G_{p,p}^{p,0}$ (again using "type B integral equation" terminology), one of them equivalent to (2.5), and rediscovered recurrence relation (2.8). See [40, 41] and references there. This line of research continues until today, as evidenced, for example, by a series of papers by Carlos Coelho with several co-authors, see [12] and references there. Furthermore, Charles Dunkl rediscovered Nørlund's recurrence (2.7) in his preprint [16] dated 2013, where he again considers the probability density of a product of beta distributed random variables. Independently, probability distribution with G-function density has been found to be the stationary distribution of certain Markov chains considered, for example, in actuarial science and is known as Dufresne law in this context. See [11, 15] and references therein. Let us also mention that G-function popped up recently in the random matrix theory as correlation kernel of a determinantal point process that governs singular values of products of M rectangular random matrices with independent complex Gaussian entries [1]. Curiously enough, none of these authors cited Nørlund's work. Let us also mention that in our recent paper [23] conditions are given under which $G_{p,p}^{p,0}(e^{-x})$ is infinitely divisible distribution on $[0,\infty)$.

2.3   Expansion of $G_{p,p}^{2,p}$ in the neighborhood of unity

Formula [33, (5.7)] shows that Nørlund's function $y_{1,2}(x)$ defined by [33, formula (5.2)] is expressed in terms of Meijer's G-function as follows

$$y_{1,2}(x) = G_{p,p}^{2,p}\left(x \middle| \begin{matrix} 1-\alpha_1, \ldots, 1-\alpha_p \\ \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p \end{matrix} \right). \tag{2.10}$$

To define $y_{k,s}(x)$ the roles of γ_1, γ_2 are exchanged with those of γ_k, γ_s. Nørlund found several expansions of the functions $y_{k,s}(x)$ in hypergeometric polynomials which we cite below. Changing Nørlund's notation to ours according to the rule $x \mapsto z$, $\gamma_i \mapsto a_i$, $\alpha_i \mapsto 1 - b_i$, formula [33, (5.3)] takes the form

$$\sin\pi(a_s - a_i)G_{p,p}^{2,p}\left(z \middle| \begin{matrix} \mathbf{b} \\ a_s, a_i, \mathbf{a}_{[s,i]} \end{matrix} \right) + \sin\pi(a_i - a_k)G_{p,p}^{2,p}\left(z \middle| \begin{matrix} \mathbf{b} \\ a_i, a_k, \mathbf{a}_{[i,k]} \end{matrix} \right)$$
$$+ \sin\pi(a_k - a_s)G_{p,p}^{2,p}\left(z \middle| \begin{matrix} \mathbf{b} \\ a_k, a_s, \mathbf{a}_{[k,s]} \end{matrix} \right) = 0 \tag{2.11}$$

for any distinct values of $s, i, k \in \{1, \ldots, p\}$, where $\mathbf{a}_{[k,s]}$ denotes the vector $\mathbf{a}$ with elements a_k and a_s removed. Expansions [33, (5.20), (5.22), (5.23), (5.31)] written in terms of G-function take the form

$$G_{p,p}^{2,p}\left(z \middle| \begin{matrix} \mathbf{b} \\ a_k, a_s, \mathbf{a}_{[k,s]} \end{matrix} \right) = \frac{z^{a_k}\Gamma(1 - \mathbf{b} + a_k)\Gamma(1 - b_1 + a_s)\Gamma(1 - b_2 + a_s)}{\Gamma(1 - \mathbf{a}_{[k,s]} + a_k)\Gamma(2 + a_k + a_s - b_1 - b_2)}$$

$$\times \sum_{n=0}^{\infty} \frac{(1-b_1+a_k)_n(1-b_2+a_k)_n}{(2+a_k+a_s-b_1-b_2)_n} \, {}_{p-1}F_{p-2}\left(\begin{matrix} -n, 1-\mathbf{b}_{[1,2]}+a_k \\ 1-\mathbf{a}_{[k,s]}+a_k \end{matrix}\middle| z\right) \tag{2.12}$$

$$= \frac{z^{a_k}\Gamma(1-a_k+a_s)\Gamma(1-\mathbf{b}+a_k)}{\Gamma(1-\mathbf{a}_{[k,s]}+a_k)} \sum_{n=0}^{\infty} \frac{(1-b_1+a_k)_n}{(1-b_1+a_s)_{n+1}} \, {}_{p}F_{p-1}\left(\begin{matrix} -n, 1-\mathbf{b}_{[1]}+a_k \\ 1, 1-\mathbf{a}_{[k,s]}+a_k \end{matrix}\middle| z\right) \tag{2.13}$$

$$= \frac{z^{a_k}\Gamma(1-b_1+a_s)\Gamma(1-\mathbf{b}_{[1]}+a_k)}{\Gamma(1-\mathbf{a}_{[k,s]}+a_k)} \sum_{n=0}^{\infty} \frac{(1-a_s+a_k)_n}{n!(1-b_1+a_k+n)}$$

$$\times {}_{p}F_{p-1}\left(\begin{matrix} -n, 1-\mathbf{b}_{[1]}+a_k \\ 1-\mathbf{a}_{[k]}+a_k \end{matrix}\middle| z\right) \tag{2.14}$$

$$= \frac{z^{a_k}\pi(a_k-a_s)\Gamma(1-\mathbf{b}+a_k)}{\sin(\pi(a_k-a_s))\Gamma(1-\mathbf{a}_{[k]}+a_k)} \sum_{n=0}^{\infty} \frac{(1-a_s+a_k)_n}{(n+1)!} \, {}_{p+1}F_p\left(\begin{matrix} -n, 1-\mathbf{b}+a_k \\ 1, 1-\mathbf{a}_{[k]}+a_k \end{matrix}\middle| z\right). \tag{2.15}$$

The series in (2.12) converges in the disk $|z-1| < 1$ if $\Re(1-\mathbf{b}_{[1,2]}+a_s) > 0$, the series in (2.13), (2.14) converge in the same disk if $\Re(1-\mathbf{b}_{[1]}+a_s) > 0$ and, finally, (2.15) converges in $|z-1| < 1$ if $\Re(1-\mathbf{b}+a_s) > 0$. In all cases, it is also required that none of the gamma functions in the numerator had poles. Further, Nørlund found two expansions of his function $y_{k,s}(x)$ in powers of $1-z$, which we will need below. Written in our notation, formulas [33, (5.35), (5.36)] read

$$G_{p,p}^{2,p}\left(z\middle|\begin{matrix} \mathbf{b} \\ a_k, a_s, \mathbf{a}_{[k,s]} \end{matrix}\right) = z^{a_s} \sum_{n=0}^{\infty} D_n^{[k,s]}(1-z)^n, \tag{2.16}$$

where the coefficients $D_n^{[k,s]}$ are given by

$$D_n^{[k,s]} = \frac{\Gamma(1-\mathbf{b}+a_k)\Gamma(1-b_1+a_s+n)\Gamma(1-b_2+a_s+n)}{\Gamma(1-\mathbf{a}_{[k,s]}+a_k)\Gamma(2+a_k+a_s-b_1-b_2+n)n!}$$

$$\times \sum_{j=0}^{\infty} \frac{(1-b_1+a_k)_j(1-b_2+a_k)_j}{j!(2+a_k+a_s-b_1-b_2+n)_j} \, {}_{p-1}F_{p-2}\left(\begin{matrix} -j, 1-\mathbf{b}_{[1,2]}+a_k \\ 1-\mathbf{a}_{[k,s]}+a_k \end{matrix}\right) \tag{2.17}$$

$$= \frac{\Gamma(1-\mathbf{b}+a_k)\Gamma(1-a_k+a_s+n)}{\Gamma(1-\mathbf{a}_{[k,s]}+a_k)n!} \sum_{j=0}^{\infty} \frac{(1-b_1+a_k)_j}{(1-b_1+a_s+n)_{j+1}}$$

$$\times {}_{p}F_{p-1}\left(\begin{matrix} -j, 1-\mathbf{b}_{[1]}+a_k \\ 1, 1-\mathbf{a}_{[k,s]}+a_k \end{matrix}\right). \tag{2.18}$$

Here and below ${}_{p}F_{p-1}$ without an argument is understood as ${}_{p}F_{p-1}(1)$. The series (2.16) converges in $|z-1| < 1$ if $\Re(1-\mathbf{b}_{[1,2]}+a_s) > 0$. The same condition suffices for convergence of (2.17), while (2.18) converges if $\Re(1-\mathbf{b}_{[1]}+a_s) > 0$. Uniqueness of power series coefficients implies equality of the coefficients in (2.17) and (2.18). For $p=2$ this equality is nothing but Gauss formula for ${}_2F_1(1)$. For $p=3$ we get after some simplifications and renaming variables

$$_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}\right) = \frac{\Gamma(\beta_2-\alpha_1-\alpha_2+1)\Gamma(\beta_2)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)} \sum_{j=0}^{\infty} \frac{(\alpha_1)_j}{(\beta_2-\alpha_2)_{j+1}} \, {}_3F_2\left(\begin{matrix} -j, \alpha_2, \beta_1-\alpha_3 \\ \beta_1, 1 \end{matrix}\right).$$

2.4 Connection to Bühring expansion of ${}_pF_{p-1}$

In his 1992 paper Bühring found the representation [8, Theorem 2]

$$\frac{\Gamma(\boldsymbol{\alpha})}{\Gamma(\boldsymbol{\beta})} \, {}_pF_{p-1}\left(\begin{matrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta}; z \end{matrix}\right) = (1-z)^\nu \sum_{n=0}^{\infty} f_p(n|\boldsymbol{\alpha},\boldsymbol{\beta})(1-z)^n + \sum_{n=0}^{\infty} h_p(n|\boldsymbol{\alpha},\boldsymbol{\beta})(1-z)^n, \tag{2.19}$$

where $\nu = \sum_{k=1}^{p-1} \beta_k - \sum_{k=1}^{p} \alpha_k$. He also derived explicit formulas for the coefficients $f_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $h_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$ to be given below. On setting $\boldsymbol{\alpha} = 1 - \mathbf{b} + a_s$, $\boldsymbol{\beta} = 1 - \mathbf{a}_{[s]} + a_s$, $\nu = \psi_p - 1$ and denoting

$$f_p^s(n) \equiv f_p(n|1 - \mathbf{b} + a_s, 1 - \mathbf{a}_{[s]} + a_s),$$
$$h_p^s(n) \equiv h_p(n|1 - \mathbf{b} + a_s, 1 - \mathbf{a}_{[s]} + a_s),$$

Bühring's formula takes the form

$$\frac{\Gamma(1 - \mathbf{b} + a_s)}{\Gamma(1 - \mathbf{a}_{[s]} + a_s)} {}_pF_{p-1}\left(\begin{array}{c} 1 - \mathbf{b} + a_s \\ 1 - \mathbf{a}_{[s]} + a_s \end{array}; z\right)$$
$$= (1 - z)^{\psi_p - 1} \sum_{n=0}^{\infty} f_p^s(n)(1 - z)^n + \sum_{n=0}^{\infty} h_p^s(n)(1 - z)^n. \tag{2.20}$$

Expansion of the form (2.19) is unique as long as ν is not an integer. Hence, in view of (2.4) and (2.17), identity (2.20) is equivalent to formula (2.3). From (2.3) we conclude that

$$(1 - z)^{\psi_p - 1} \sum_{n=0}^{\infty} f_p^s(n)(1 - z)^n = \frac{\pi z^{-a_s}}{\sin(\pi \psi_p)} G_{p,p}^{p,0}\left(z \left| \begin{array}{c} \mathbf{b} \\ \mathbf{a} \end{array}\right.\right) \qquad \text{and} \tag{2.21}$$

$$\sum_{n=0}^{\infty} h_p^s(n)(1 - z)^n = \frac{-z^{-a_s}}{\pi \sin(\pi \psi_p)} \sum_{\substack{k=1 \\ k \neq s}}^{p} \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k,s]} - a_k))} G_{p,p}^{2,p}\left(z \left| \begin{array}{c} \mathbf{b} - a_s \\ a_k, a_s, \mathbf{a}_{[k,s]} \end{array}\right.\right) \tag{2.22}$$

for each $s \in \{1, \ldots, p\}$. Comparing (2.21) with (2.5) we arrive at

$$f_p^s(n) = \frac{\Gamma(1 - \psi_p)}{(\psi_p)_n} g_p^s(n), \tag{2.23}$$

where Euler's reflection formula has been used. Thus, Bühring coefficients $f_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$ (denoted by $g_n(s)$ in [8]) are essentially the same as Nørlund coefficients $g_p^s(n)$ (denoted by $c_{n,p}^{(s)}$ in [33]). Explicit representation for the coefficients $f_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$ found in [8, (2.9), (2.16)] after some renaming of variables and setting $s = p$ can be put into the form

$$f_p^p(n) = \frac{\Gamma(1 - \psi_p)}{(\psi_p)_n} \sum_{j_{p-2}=0}^{n} \frac{(-1)^{j_{p-2}}(\psi_p + a_p - b_{p-1} + j_{p-2})_{n-j_{p-2}}}{(n - j_{p-2})!}$$

$$\times \sum_{j_{p-3}=0}^{j_{p-2}} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \prod_{m=1}^{p-1} \frac{(\psi_m + j_{m-1})_{j_m - j_{m-1}}}{(j_m - j_{m-1})!} \prod_{m=1}^{p-2} (b_m - a_{m+1})_{j_m - j_{m-1}}$$

$$= \frac{\Gamma(1 - \psi_p)}{(\psi_p)_n} \sum_{j_1, j_2, \ldots, j_{p-2}=0}^{\infty} \frac{(\psi_p + a_p - b_{p-1})_n}{(\psi_p + a_p - b_{p-1})_{j_{p-2}}} \prod_{m=1}^{p-1} \frac{(\psi_m)_{j_m}(-j_m)_{j_{m-1}}}{(\psi_m)_{j_{m-1}} j_m!}$$

$$\times \prod_{m=1}^{p-2} \frac{(b_m - a_{m+1})_{j_m}}{(1 - b_m + a_{m+1} - j_m)_{j_{m-1}}},$$

where $\psi_m = \sum_{i=1}^{m}(b_i - a_i)$ and $j_0 = 0$, $j_{p-1} = n$. In view of (2.9), formula (2.23) then leads to the following (presumably new) transformation for multiple hypergeometric series

$$\sum_{j_{p-2}=0}^{n} \frac{(\psi_p + a_p - b_{p-1} + j_{p-2})_{n-j_{p-2}}}{(-1)^{j_{p-2}}(n - j_{p-2})!}$$

$$\times \sum_{j_{p-3}=0}^{j_{p-2}} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \prod_{m=1}^{p-1} \frac{(\psi_m + j_{m-1})_{j_m - j_{m-1}}}{(j_m - j_{m-1})!} \prod_{m=1}^{p-2} (b_m - a_{m+1})_{j_m - j_{m-1}}$$

$$= \sum_{j_{p-2}=0}^{n} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \prod_{m=1}^{p-1} \frac{(\psi_m + j_{m-1})_{j_m - j_{m-1}}}{(j_m - j_{m-1})!} (b_{m+1} - a_m)_{j_m - j_{m-1}},$$

where $j_0 = 0$ and $j_{p-1} = n$. For $p = 3$ this formula reduces to the identity

$$_3F_2 \left(\begin{matrix} -n, \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{matrix} \right) = \frac{(\beta_2 - \alpha_2)_n}{(\beta_2)_n} \, _3F_2 \left(\begin{matrix} -n, \beta_1 - \alpha_1, \alpha_2 \\ \beta_1, 1 - \beta_2 + \alpha_2 - n \end{matrix} \right),$$

which is a guise of Sheppard's transformation [2, Corollary 3.3.4], also rediscovered by Bühring [9, (4.1)]. For $p = 4$ the corresponding result is given by [9, (4.2)]

$$\sum_{k=0}^{n} \frac{(-k)_n (\alpha_1)_k (\alpha_2)_k}{(\beta_1)_k (\beta_2)_k k!} \, _3F_2 \left(\begin{matrix} -k, \gamma_1, \gamma_2 \\ \alpha_1, \alpha_2 \end{matrix} \right)$$

$$= \frac{(\beta_2 - \alpha_2)_n}{(\beta_2)_n} \sum_{k=0}^{n} \frac{(-k)_n (\alpha_2)_k (\beta_1 - \alpha_1)_k}{(\beta_1)_k (1 + \alpha_2 - \beta_2 - n)_k k!} \, _3F_2 \left(\begin{matrix} -k, \gamma_1, \gamma_2 \\ \alpha_2, 1 + \alpha_1 - \beta_1 - k \end{matrix} \right).$$

In his paper [8] Bühring mentioned that the structure of (2.19) "was already given in the classic paper by Nørlund [33], but the coefficients were not all known". In his subsequent joint work [10] with Srivastava is it said that the coefficients $f_p^s(n)$ can be computed using Nørlund's recurrence, "but it is desirable to get an explicit representation" which is indeed derived in [10] in terms of some limit relations. Explicit formulas for the coefficients $h_p^s(n)$ have been first found in [8]. Adopted to our notation Bühring's formulas [8, (2.9), (2.16)], [10, (2.13), (2.15)] read

$$h_p^p(n) = -\frac{\Gamma(\psi_p)\Gamma(1 - b_p + a_p + n)\Gamma(1 - b_{p-1} + a_p + n)}{(1 - \psi_p)_{n+1} \Gamma(\psi_{p-1}) \Gamma(\psi_p - b_{p-1} + a_p) n!}$$

$$\times \sum_{j_{p-2}=0}^{\infty} \frac{(\psi_p - n - 1)_{j_{p-2}}}{(\psi_{p-1})_{j_{p-2}} (\psi_p - b_{p-1} + a_p)_{j_{p-2}}}$$

$$\times \sum_{j_{p-3}=0}^{j_{p-2}} \cdots \sum_{j_1=0}^{j_2} \prod_{m=1}^{p-2} \frac{(\psi_m + j_{m-1})_{j_m - j_{m-1}}}{(j_m - j_{m-1})!} (b_m - a_{m+1})_{j_m - j_{m-1}}, \tag{2.24}$$

where, as before, $\psi_m = \sum_{i=1}^{m} (b_i - a_i)$ and $j_0 = 0$. Conditions for convergence for the outer series above have also been found in [8] and [10] and are given by

$$\Re(1 - b_i + a_p + n) > 0 \qquad \text{for} \quad i = 1, 2, \ldots, p - 2.$$

Bühring and Srivastava found expressions for the coefficients $f_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$, $h_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$ as limits of hypergeometric polynomials [10, (3.13)]. They computed this limit for $p = 3, 4, 5$ yielding [10, (3.15), (3.17)]

$$h_3^s(n) \equiv h_3(n|1 - \mathbf{b} + a_s, 1 - \mathbf{a}_{[s]} + a_s)$$

$$= -\frac{\Gamma(\psi_3)\Gamma(1 - b_2 + a_s + n)\Gamma(1 - b_3 + a_s + n)}{(1 - \psi_3)_{n+1} n! \Gamma(\psi_3 - b_2 + a_s)\Gamma(\psi_3 - b_3 + a_s)} \, _3F_2 \left(\begin{matrix} \psi_3 - 1 - n, b_1 - \mathbf{a}_{[s]} \\ \psi_3 - \mathbf{b}_{[1]} + a_s \end{matrix} \right), \tag{2.25}$$

$$h_4^s(n) = -\frac{\Gamma(\psi_4)\Gamma(1 - b_3 + a_s + n)\Gamma(1 - b_4 + a_s + n)}{(1 - \psi_4)_{n+1} n! \Gamma(\psi_4 - b_3 + a_s)\Gamma(\psi_4 - b_4 + a_s)}$$

$$\times \sum_{k=0}^{\infty} (\psi_4 - 1 - n)_k \frac{(b_1 + b_2 - a_{i_1} - a_{i_2})_k (b_1 + b_2 - a_{i_1} - a_{i_3})_k}{(\psi_4 - b_3 + a_s)_k (\psi_4 - b_4 + a_s)_k}$$

$$\times \, {}_3F_2 \left(\begin{matrix} -k, b_1 - a_{i_1}, b_2 - a_{i_1} \\ b_1 + b_2 - a_{i_1} - a_{i_2}, b_1 + b_2 - a_{i_1} - a_{i_3} \end{matrix} \right),$$

where $\{a_{i_1}, a_{i_2}, a_{i_3}\} = \mathbf{a}_{[s]}$ (i.e., if $s = 1$ then $\{a_{i_1}, a_{i_2}, a_{i_3}\} = \{a_2, a_3, a_4\}$, if $s = 2$ then $\{a_{i_1}, a_{i_2}, a_{i_3}\} = \{a_1, a_3, a_4\}$, etc.). Expression for $p = 5$ is quite cumbersome and can be found in [10, (3.19)]. In spite of their non-symmetric appearance, both formulas are invariant with respect to permutation of the elements of $\mathbf{b}$.

Let us now cite the formulas for the coefficients $g_p^s(n)$ for $p = 2, 3, 4$. For $p = 2$ the corresponding formulas can be read off formula (1.2). For both $p = 2$ and $p = 3$ expressions for $g_p^s(n)$ have been found by Nørlund, see [33, (2.10)]. They are

$$g_2^s(n) = \frac{(b_1 - a_{3-s})_n (b_2 - a_{3-s})_n}{n!}, \qquad s = 1, 2,$$

and

$$g_3^s(n) = \frac{(\psi_3 - b_2 + a_s)_n (\psi_3 - b_3 + a_s)_n}{n!} \, {}_3F_2 \left(\begin{matrix} -n, b_1 - a_{i_1}, b_1 - a_{i_2} \\ \psi_3 - b_2 + a_s, \psi_3 - b_3 + a_s \end{matrix} \right), \qquad s = 1, 2, 3,$$

where $\{a_{i_1}, a_{i_2}\} = \mathbf{a}_{[s]}$. Notwithstanding the non-symmetric appearance, the last formula is symmetric with respect to the elements of $\mathbf{b}$. For $p = 4$ we convert the expression for $f_p^s(n)$ calculated in [10, (3.17)] into expressions for $g_p^s(n)$ using (2.23) and the necessary renaming of parameters. This yields

$$g_4^s(n) = \frac{(\psi_4 - b_3 + a_s)_n (\psi_4 - b_4 + a_s)_n}{n!}$$

$$\times \sum_{k=0}^{n} \frac{(-n)_k (b_1 + b_2 - a_{i_1} - a_{i_2})_k (b_1 + b_2 - a_{i_1} - a_{i_3})_k}{(\psi_4 - b_3 + a_s)_k (\psi_4 - b_4 + a_s)_k}$$

$$\times \, {}_3F_2 \left(\begin{matrix} -k, b_1 - a_{i_1}, b_2 - a_{i_1} \\ b_1 + b_2 - a_{i_1} - a_{i_2}, b_1 + b_2 - a_{i_1} - a_{i_3} \end{matrix} \right),$$

where $\{a_{i_1}, a_{i_2}, a_{i_3}\} = \mathbf{a}_{[s]}$. This formula is also invariant with respect to permutation of the elements of $\mathbf{b}$.

3   Main results

Having made these preparations, we are ready to formulate our main results. First, we give explicit formulas for the Nørlund's coefficients $g_p^s(1)$, $g_p^s(2)$, $g_p^s(3)$ with arbitrary p. Note that formula (2.9) contains $p - 2$ summations even for small n. Combined with (2.5) and $g_p^p(0) = 1$ our formulas essentially provide first four terms in asymptotic expansion of $G_{p,p}^{p,0}$ as $z \to 1$. Then we derive a new way to compute the coefficients $g_p^s(n)$ for all n in terms of generalized Bernoulli polynomials in Theorem 3.3. Theorems 3.5 and 3.10 contain new identities for the coefficients $g_p^s(n)$, $h_p^s(n)$ and $D_n^{[k,s]}$ defined by expansions (2.5), (2.16) and (2.20), respectively. For small values of n these identities lead to new and known relations involving hypergeometric and trigonometric functions, in particular, to identity (1.3).

Theorem 3.1. *The first four coefficients of the expansion* (2.5) *are given by*

$$g_p^p(0) = 1, \qquad g_p^p(1) = \sum_{m=1}^{p-1} (b_{m+1} - a_m)\psi_m, \tag{3.1}$$

$$g_p^p(2) = \frac{1}{2} \sum_{m=1}^{p-1} (b_{m+1} - a_m)_2 (\psi_m)_2 + \sum_{k=2}^{p-1} (b_{k+1} - a_k)(\psi_k + 1) \sum_{m=1}^{k-1} (b_{m+1} - a_m)\psi_m \qquad (3.2)$$

and

$$g_p^p(3) = \frac{1}{6} \sum_{m=1}^{p-1} (b_{m+1} - a_m)_3 (\psi_m)_3 + \frac{1}{2} \sum_{k=2}^{p-1} (b_{k+1} - a_k)(\psi_k + 2) \sum_{m=1}^{k-1} (b_{m+1} - a_m)_2 (\psi_m)_2$$

$$+ \frac{1}{2} \sum_{k=2}^{p-1} (\psi_k + 1)_2 (b_{k+1} - a_k)_2 \sum_{m=1}^{k-1} (b_{m+1} - a_m)\psi_m$$

$$+ \sum_{n=3}^{p-1} (b_{n+1} - a_n)(\psi_n + 2) \sum_{k=2}^{n-1} (b_{k+1} - a_k)(\psi_k + 1) \sum_{m=1}^{k-1} (b_{m+1} - a_m)\psi_m, \qquad (3.3)$$

where, as before, $\psi_m = \sum_{j=1}^{m} (b_j - a_j).$

Proof. We note that for $n = 1$ summation in (2.9) is over the index sets of the form

$$\{j_0, j_1, \dots, j_{p-1}\} = \{0, \dots, 0, 1, \dots, 1\},$$

where the number of ones changes from 1 to $p - 1$. In view of this observation rearrangement of the formula (2.9) yields (3.1). Next, we prove (3.3). Summation in (2.9) is over all Young diagrams that fit $3 \times (p - 2)$ box. We break all possible diagrams in four disjoint groups as follows (by definition $j_0 = 0$, $j_{p-1} = 3$):

(1) $j_0 = j_1 = \cdots = j_{m-1} = 0, \qquad j_m = \cdots = j_{p-1} = 3, \qquad m \in \{1, 2, \dots, p - 1\},$

(2) $j_0 = j_1 = \cdots = j_{m-1} = 0, \qquad j_m = \cdots = j_{k-1} = 2, \qquad j_k = \cdots = j_{p-1} = 3,$
$\quad k \in \{2, \dots, p - 1\}, \qquad m \in \{1, \dots, k - 1\}, \qquad m < k,$

(3) $j_0 = j_1 = \cdots = j_{m-1} = 0, \qquad j_m = \cdots = j_{k-1} = 1, \qquad j_k = \cdots = j_{p-1} = 3,$
$\quad k \in \{2, \dots, p - 1\}, \qquad m \in \{1, \dots, k - 1\}, \qquad m < k,$

(4) $j_0 = \cdots = j_{m-1} = 0, \qquad j_m = \cdots = j_{k-1} = 1, \qquad j_k = \cdots = j_{n-1} = 2,$
$\quad j_n = \cdots = j_{p-1} = 3, \qquad n \in \{3, \dots, p - 1\}, \qquad k \in \{2, \dots, n - 1\},$
$\quad m \in \{1, \dots, k - 1\}, \qquad m < k < n.$

Summation over the first type of diagrams leads to the first term (3.3). Similarly, summation over the i-th type of diagrams leads to the i-th term (3.3) for $i = 2, 3, 4$. Analogous considerations lead to (3.2). ∎

Remark 3.2. Exchanging the roles of a_p and a_s formulas (3.1), (3.2) and (3.3) lead to expressions for $g_p^s(1)$, $g_p^s(2)$, $g_p^s(3)$. Since each coefficient $g_p^s(n)$ is a symmetric polynomial in the elements of $\mathbf{a}_{[s]}$ and $\mathbf{b}$, such expressions can be expanded in terms of some basis of symmetric polynomials. In particular, denoting by $e_k(x_1, \dots, x_m)$ the k-th elementary symmetric polynomial of $x_1, \dots, x_m$, we get

$$g_p^s(1) = e_2(\mathbf{b}) - e_2(\mathbf{a}_{[s]}) + e_1(\mathbf{a}_{[s]})(e_1(\mathbf{a}_{[s]}) - e_1(\mathbf{b}))$$

and (with the help of *Mathematica*)

$$g_p^s(2) = \frac{1}{2} e_1(\mathbf{a}_{[s]})^4 - e_1(\mathbf{a}_{[s]})^3 + \frac{1}{2} e_1(\mathbf{a}_{[s]})^2 + \frac{3}{2} e_1(\mathbf{a}_{[s]}) e_2(\mathbf{a}_{[s]}) + \frac{1}{2} e_2(\mathbf{a}_{[s]})^2 - \frac{1}{2} e_2(\mathbf{a}_{[s]})$$

$$- e_1(\mathbf{a}_{[s]})^2 e_2(\mathbf{a}_{[s]}) - \frac{1}{2} e_3(\mathbf{a}_{[s]}) + \frac{1}{2} e_2(\mathbf{b})^2 + \frac{1}{2} e_2(\mathbf{b}) + \frac{1}{2} e_3(\mathbf{b}) - \frac{1}{2} e_1(\mathbf{a}_{[s]}) e_1(\mathbf{b})$$

$$+ \frac{3}{2} e_1(\mathbf{a}_{[s]})^2 e_1(\mathbf{b}) - e_1(\mathbf{a}_{[s]})^3 e_1(\mathbf{b}) - \frac{1}{2} e_1(\mathbf{a}_{[s]}) e_1(\mathbf{b})^2 + \frac{1}{2} e_1(\mathbf{a}_{[s]})^2 e_1(\mathbf{b})^2$$

$$+ e_1(\mathbf{a}_{[s]})^2 e_2(\mathbf{b}) + \frac{1}{2} e_1(\mathbf{b}) e_2(\mathbf{b}) - e_1(\mathbf{a}_{[s]}) e_1(\mathbf{b}) e_2(\mathbf{b}) - e_2(\mathbf{a}_{[s]}) e_2(\mathbf{b}) - e_1(\mathbf{b}) e_2(\mathbf{a}_{[s]})$$

$$+ e_1(\mathbf{a}_{[s]}) e_2(\mathbf{a}_{[s]}) e_1(\mathbf{b}) - e_1(\mathbf{a}_{[s]}) e_2(\mathbf{b}).$$

Next theorem gives a presumably new expression for Nørlund's coefficients $g_p^k(n)$ in terms of the generalized Bernoulli polynomials. Let us start by recalling that the Bernoulli–Nørlund (or the generalized Bernoulli) polynomial $\mathcal{B}_k^{(\sigma)}(x)$ is defined by the generating function [34, (1)]:

$$\frac{t^\sigma e^{xt}}{(e^t - 1)^\sigma} = \sum_{k=0}^{\infty} \mathcal{B}_k^{(\sigma)}(x) \frac{t^k}{k!}.$$

In particular, $\mathcal{B}_k^{(1)}(x) = \mathcal{B}_k(x)$ is the classical Bernoulli polynomial.

Theorem 3.3. *Coefficients $g_p^k(n)$, defined in* (2.5), *are given by any of the following formulas*

$$g_p^k(n) = \sum_{r=0}^{n} \frac{(-1)^{n-r}(r+1)_{n-r}}{(n-r)!} \tilde{l}_r \mathcal{B}_{n-r}^{(n+1)}(2 - a_k - \psi_p)$$

$$= \sum_{r=0}^{n} \frac{(-1)^{n-r}(\psi_p + r)_{n-r}}{(n-r)!} l_r \mathcal{B}_{n-r}^{(n+\psi_p)}(1 - a_k). \tag{3.4}$$

Here $\tilde{l}_0 = 1$ and $\tilde{l}_r$, $r \geq 1$, are found from the recurrence

$$\tilde{l}_r = \frac{1}{r} \sum_{m=1}^{r} \tilde{q}_m \tilde{l}_{r-m}, \tag{3.5}$$

where

$$\tilde{q}_m = \frac{(-1)^{m+1}}{m+1} \left[\mathcal{B}_{m+1}(a_k + \psi_p - 1) - \mathcal{B}_{m+1}(a_k) + \sum_{j=1}^{p} (\mathcal{B}_{m+1}(a_j) - \mathcal{B}_{m+1}(b_j)) \right].$$

Similarly, the coefficients l_r satisfy the recurrence relation

$$l_r = \frac{1}{r} \sum_{m=1}^{r} q_m l_{r-m} \qquad with \quad l_0 = 1 \tag{3.6}$$

and

$$q_m = \frac{(-1)^{m+1}}{m+1} \sum_{j=1}^{p} (\mathcal{B}_{m+1}(a_j) - \mathcal{B}_{m+1}(b_j)).$$

Remark 3.4. It is known [22, Lemma 1] that the recurrences (3.5) and (3.6) can be solved to give the following explicit expressions for l_r:

$$l_r = \sum_{k_1 + 2k_2 + \cdots + rk_r = r} \frac{q_1^{k_1}(q_2/2)^{k_2} \cdots (q_r/r)^{k_r}}{k_1! k_2! \cdots k_r!} = \sum_{n=1}^{r} \frac{1}{n!} \sum_{k_1 + k_2 + \cdots + k_n = r} \prod_{i=1}^{n} \frac{q_{k_i}}{k_i}.$$

Similar formula is of course true for $\tilde{l}_r$ once we write $\widetilde{q}_m$ instead of q_m. Moreover, Nair [32, Section 8] found a determinantal expression for such solution which in our notation takes the form

$$l_r = \frac{\det(\Omega_r)}{r!}, \qquad \Omega_r = [\omega_{i,j}]_{i,j=1}^r, \qquad \omega_{i,j} = \begin{cases} q_{i-j+1}(i-1)!/(j-1)!, & i \geq j, \\ -1, & i = j-1, \\ 0, & i < j-1. \end{cases}$$

Proof. The theorem is a corollary of an expansion of the H-function of Fox found in our recent paper [24]. Since Meijer's G-function is a particular case of Fox's H-function, formula (3.4) is a particular case of [24, Theorem 1] once we set $p = q$, $\mathbf{A} = \mathbf{B} = (1, \ldots, 1)$, $\nu = 1$, $\mu = \psi_p$, $\theta = a_k - 1$ in that theorem. ∎

As before, we use the shorthand notation

$$\sin(\pi(\mathbf{b} - a_k)) = \prod_{j=1}^p \sin(\pi(b_j - a_k)), \qquad \sin(\pi(\mathbf{a}_{[k]} - a_k)) = \prod_{\substack{j=1 \\ j \neq k}}^p \sin(\pi(a_j - a_k)),$$

for the products and $[a]_j = a(a-1)\cdots(a-j+1)$ for the falling factorial.

Theorem 3.5. *For each nonnegative integer m the following identities holds*

$$\sum_{j=0}^m \frac{(-1)^j}{j!} \sum_{k=1}^p [a_k]_j h_p^k(m-j) \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k]} - a_k))} = 0, \tag{3.7}$$

where the numbers $h_p^p(n)$ are defined by (2.20) and given explicitly by (2.24), $h_p^k(n)$ is obtained from $h_p^p(n)$ by exchanging the roles of a_p and a_k; furthermore,

$$\sum_{j=0}^m \frac{(-1)^j}{(\psi_p)_{m-j} j!} \left\{ [a_s]_j g_p^s(m-j) \sin(\pi\psi_p) - \sum_{k=1}^p [a_k]_j g_p^k(m-j) \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k]} - a_k))} \right\} = 0, \tag{3.8}$$

where $s \in \{1, \ldots, p\}$ is chosen arbitrarily and the numbers $g_p^k(n)$ are defined by expansion (2.5) and solve the recurrence (2.7) in n and the recurrence (2.8) in p. They are given explicitly by (2.9).

Proof. Assume that the components of the vector $\mathbf{a}$ are distinct modulo 1. Substituting expansion (2.20) into formula (2.4) and taking account of (2.23) we obtain

$$G_{p,p}^{p,0}\left(z \middle| \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) = (1-z)^{\psi_p-1} \sum_{k=1}^p z^{a_k} \frac{\Gamma(\mathbf{a}_{[k]} - a_k)\Gamma(1 - \mathbf{a}_{[k]} + a_k)}{\Gamma(\mathbf{b} - a_k)\Gamma(1 - \mathbf{b} + a_k)} \sum_{n=0}^\infty \frac{\Gamma(1 - \psi_p)}{(\psi_p)_n} g_p^k(n)(1-z)^n$$

$$+ \sum_{k=1}^p z^{a_k} \frac{\Gamma(\mathbf{a}_{[k]} - a_k)\Gamma(1 - \mathbf{a}_{[k]} + a_k)}{\Gamma(\mathbf{b} - a_k)\Gamma(1 - \mathbf{b} + a_k)} \sum_{n=0}^\infty h_p^k(n)(1-z)^n$$

$$= \frac{1}{\pi}(1-z)^{\psi_p-1}\Gamma(1-\psi_p) \sum_{n=0}^\infty \frac{(1-z)^n}{(\psi_p)_n} \sum_{k=1}^p z^{a_k} \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k]} - a_k))} g_p^k(n)$$

$$+ \frac{1}{\pi} \sum_{n=0}^\infty (1-z)^n \sum_{k=1}^p z^{a_k} \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k]} - a_k))} h_p^k(n),$$

where we applied Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. Further, substitute Nørlund's expansion (2.5) in place of G-function on the left-hand side and rearrange terms to get

$$(1-z)^{\psi_p-1}\sum_{n=0}^{\infty}(1-z)^n\left\{\frac{z^{a_s}g_p^s(n)}{\Gamma(\psi_p+n)} - \frac{\Gamma(1-\psi_p)}{(\psi_p)_n}\sum_{k=1}^{p}\frac{z^{a_k}\sin(\pi(\mathbf{b}-a_k))}{\pi\sin(\pi(\mathbf{a}_{[k]}-a_k))}g_p^k(n)\right\}$$

$$= \sum_{n=0}^{\infty}(1-z)^n\sum_{k=1}^{p}\frac{z^{a_k}\sin(\pi(\mathbf{b}-a_k))}{\pi\sin(\pi(\mathbf{a}_{[k]}-a_k))}h_p^k(n),$$

where $s \in \{1,\ldots,p\}$ can be chosen arbitrarily. Denote for brevity

$$\gamma_{n,s}(z) = \frac{z^{a_s}g_p^s(n)}{\Gamma(\psi_p+n)} - \frac{\Gamma(1-\psi_p)}{(\psi_p)_n}\sum_{k=1}^{p}\frac{z^{a_k}\sin(\pi(\mathbf{b}-a_k))}{\pi\sin(\pi(\mathbf{a}_{[k]}-a_k))}g_p^k(n),$$

$$\chi_n(z) = \sum_{k=1}^{p}\frac{z^{a_k}\sin(\pi(\mathbf{b}-a_k))}{\pi\sin(\pi(\mathbf{a}_{[k]}-a_k))}h_p^k(n),$$

so that the above equality reduces to

$$(1-z)^{\psi_p-1}\sum_{n=0}^{\infty}(1-z)^n\gamma_{n,s}(z) = \sum_{n=0}^{\infty}(1-z)^n\chi_n(z). \tag{3.9}$$

Assume for a moment that ψ_p is not an integer. We know that both series in (3.9) converge in $|z-1| < 1$. Furthermore, all functions $\gamma_{n,s}(z)$ and $\chi_n(z)$ are analytic the same disk. This implies that (3.9) is only possible for all z in a disk centered at 1 if

$$\sum_{n=0}^{\infty}(1-z)^n\gamma_{n,s}(z) \equiv 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty}(1-z)^n\chi_n(z) \equiv 0. \tag{3.10}$$

In terms of the functions $\chi_n(z)$ the claimed identity (3.7) takes the form

$$\sum_{j=0}^{m}\frac{(-1)^j}{j!}\chi_{m-j}^{(j)}(1) = 0, \tag{3.11}$$

which we prove by induction in m.

Letting $z \to 1$ in the second identity in (3.10) we get $\chi_0(1) = 0$ which establishes our claim for $m = 0$. Next, suppose (3.11) holds for $m = 0, 1, 2, \ldots, r-1$. Divide the second identity in (3.10) by $(1-z)^r$ and expand each $\chi_n(z)$, $n = 0, 1, \ldots, r$, in Taylor series around $z = 1$:

$$0 = \sum_{n=0}^{r}(1-z)^{n-r}\{\chi_n(1) + \chi_n'(1)(z-1) + \cdots$$

$$+ \chi_n^{(r-n)}(z-1)^{r-n}/(r-n)! + O((z-1)^{r-n+1})\} + \sum_{n=r+1}^{\infty}(1-z)^{n-r}\chi_n(z)$$

$$= \sum_{k=0}^{r}(1-z)^{-k}\sum_{i=0}^{r-k}\frac{(-1)^i}{i!}\chi_{r-k-i}^{(i)}(1) + O(z-1) = \sum_{i=0}^{r}\frac{(-1)^i}{i!}\chi_{r-i}^{(i)}(1) + O(z-1),$$

where the last equality is by induction hypothesis. We now obtain (3.11) for $m = r$ on letting $z \to 1$ in this formula. It is immediate to check that the claimed identity (3.8) takes the form

$$\sum_{j=0}^{m}\frac{(-1)^j}{j!}\gamma_{m-j,s}^{(j)}(1) = 0,$$

which can be demonstrated in a similar fashion starting with the first identity in (3.10).

Finally, we remove the assumption that ψ_p is not an integer. The left-hand sides of (3.7) and (3.8) are analytic functions of, say, parameter b_1 except for possible poles. Identities (3.7) and (3.8) for non-integer ψ_p then clearly imply by analytic continuation that these poles are removable and both identities hold for all ψ_p. ∎

Corollary 3.6. *Suppose for some $i \in \{1, 2, 3\}$ inequality $\Re(b_i) < \Re(a_k+1)$ holds for $k \in \{1, 2, 3\}$. Then*

$$\sum_{k=1}^{3} \frac{\sin(\pi(\mathbf{b} - a_k))\Gamma(1 - \mathbf{b}_{[i]} + a_k)}{\sin(\pi(\mathbf{a}_{[k]} - a_k))\Gamma(\psi - \mathbf{b}_{[i]} + a_k)} \, {}_3F_2 \left(\begin{matrix} \psi - 1, b_i - \mathbf{a}_{[k]} \\ \psi - \mathbf{b}_{[i]} + a_k \end{matrix} \right) = 0 \tag{3.12}$$

and

$$\sum_{k=1}^{3} \frac{\sin(\pi(\mathbf{b} - a_k))\Gamma(1 - \mathbf{b}_{[i]} + a_k)}{\sin(\pi(\mathbf{a}_{[k]} - a_k))\Gamma(\psi - \mathbf{b}_{[i]} + a_k)} \left\{ (1 - \mathbf{b}_{[i]} + a_k) \, {}_3F_2 \left(\begin{matrix} \psi - 2, b_i - \mathbf{a}_{[k]} \\ \psi - \mathbf{b}_{[i]} + a_k \end{matrix} \right) \right.$$
$$\left. - a_k(2 - \psi) \, {}_3F_2 \left(\begin{matrix} \psi - 1, b_i - \mathbf{a}_{[k]} \\ \psi - \mathbf{b}_{[i]} + a_k \end{matrix} \right) \right\} = 0, \tag{3.13}$$

where $\psi = \sum\limits_{k=1}^{3} (b_k - a_k)$ and ${}_pF_{p-1}$ without argument is understood as ${}_pF_{p-1}(1)$.

Proof. Put $p = 3$. Identities (3.12) and (3.13) are now reformulations of (3.7) for $m = 0$ and $m = 1$, respectively. The coefficients $h_3^k(0)$ and $h_3^k(1)$ have been computed by formula (2.25). ∎

The next corollary is a rewriting of (3.8) for $m = 0$ in view of $g_p^k(0) = 1$.

Corollary 3.7. *For any complex vectors $\mathbf{a}$, $\mathbf{b}$ the following identity holds*

$$\sum_{k=1}^{p} \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k]} - a_k))} = \sin(\pi\psi_p). \tag{3.14}$$

The right-hand side gives a continuous extension of the left-hand side if $\mathbf{a}_{[k]} - a_k$ contains integers.

Remark 3.8. For $p = 2$ this is equivalent to Ptolemy's theorem: if a quadrilateral is inscribed in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides, which can be written as

$$\sin(\theta_3 - \theta_1)\sin(\theta_4 - \theta_2) = \sin(\theta_2 - \theta_1)\sin(\theta_4 - \theta_3) + \sin(\theta_4 - \theta_1)\sin(\theta_3 - \theta_2).$$

Further details regarding the history behind the identity (3.14) can be found in the introduction and [21].

Corollary 3.9. *For each $m \in \mathbb{N}_0$ and each $p \in \mathbb{N}$ the function*

$$F_{p,m}(\mathbf{a}, \mathbf{b}) = \sum_{j=0}^{m} \frac{(-1)^j}{j!} [a_k]_j [\psi_p + m - 1]_j g_p^k(m - j)$$
$$= \sum_{j=0}^{m} \sum_{r=0}^{m-j} \frac{(-1)^{m-j}(\psi_p + r)_{m-r}}{j!(m - j - r)!} l_r [a_k]_j \mathcal{B}_{m-j-r}^{(m-j+\psi_p)}(1 - a_k) \tag{3.15}$$

is independent of k and represents a symmetric polynomial in the components of $\mathbf{a}$ and $\mathbf{b}$ (separately). Here l_r is defined by the recurrence relation (3.6).

Proof. Indeed, the first formula in (3.15) follows from (3.8) once we open the braces and apply the obvious relation $1/(\psi_p)_{m-j} = [\psi_p + m - 1]_j/(\psi_p)_m$. Substitution of (3.4) for g_p^k leads to the second formula. ∎

The following theorem can be viewed as a new method for computing the coefficients $h_p(n|\boldsymbol{\alpha}, \boldsymbol{\beta})$ in expansion (2.19) given by the multiple sum (2.24) by relating them to the numbers $D_n^{[k,s]}$ given by the single sums (2.17) and (2.18).

Theorem 3.10. *For each nonnegative integer n and arbitrary $s \in \{1, \ldots, p\}$ the following identity holds*

$$h_p^s(n) = -\frac{1}{\pi \sin(\pi\psi_p)} \sum_{\substack{k=1 \\ k \neq s}}^p \frac{\sin(\pi(\mathbf{b} - a_k))}{\sin(\pi(\mathbf{a}_{[k,s]} - a_k))} D_n^{[k,s]}, \tag{3.16}$$

where $h_p^s(n)$ is defined by expansion (2.20) and given explicitly by (2.24), while $D_n^{[k,s]}$ are given by (2.17) or (2.18). Moreover, for arbitrary distinct integers s, i, k from the set $\{1, 2, \ldots, p\}$ the following identity holds

$$\sum_{j=0}^n \frac{(-1)^j}{j!} \left([a_i]_j \sin(\pi(a_s - a_i)) D_{n-j}^{[s,i]} + [a_k]_j \sin(\pi(a_i - a_k)) D_{n-j}^{[i,k]} \right.$$

$$\left. + [a_s]_j \sin(\pi(a_k - a_s)) D_{n-j}^{[k,s]} \right) = 0. \tag{3.17}$$

Proof. To prove (3.16) it suffices to substitute expansion (2.16) into formula (2.22) and equate coefficients. Identity (3.17) is a direct consequence of (2.11). ∎

Corollary 3.11. *For each $n \in \mathbb{N}_0$ the following identity holds true*

$$\frac{\Gamma(a_3 - a_1)}{\Gamma(\mathbf{b} - a_1)\Gamma(2 + a_1 + a_2 - b_1 - b_2 + n)} \, {}_3F_2 \left(\begin{array}{c} 1 + a_1 - b_1, 1 + a_1 - b_2, b_3 - a_3 \\ 1 + a_1 - a_3, 2 + a_1 + a_2 - b_1 - b_2 + n \end{array} \right)$$

$$+ \frac{\Gamma(a_1 - a_3)}{\Gamma(\mathbf{b} - a_3)\Gamma(2 + a_2 + a_3 - b_1 - b_2 + n)} \, {}_3F_2 \left(\begin{array}{c} 1 + a_3 - b_1, 1 + a_3 - b_2, b_3 - a_1 \\ 1 + a_3 - a_1, 2 + a_2 + a_3 - b_1 - b_2 + n \end{array} \right)$$

$$= \frac{1}{\Gamma(2 - \psi + n)\Gamma(\psi + a_2 - b_1)\Gamma(\psi + a_2 - b_2)} \, {}_3F_2 \left(\begin{array}{c} b_3 - a_1, b_3 - a_3, \psi - 1 - n \\ \psi + a_2 - b_1, \psi + a_2 - b_2 \end{array} \right).$$

Proof. For $p = 3$, $s = 2$ formula (3.16) takes the form

$$h_3^2(n) = -\frac{1}{\pi \sin(\pi\psi)} \left(\frac{\sin(\pi(\mathbf{b} - a_1))}{\sin(\pi(a_3 - a_1))} D_n^{[1,2]} + \frac{\sin(\pi(\mathbf{b} - a_3))}{\sin(\pi(a_1 - a_3))} D_n^{[3,2]} \right). \tag{3.18}$$

Now, write $h_3^2(n)$ according to (2.25) and exchange the roles of b_1 and b_3. Next, apply Chu–Vandermonde identity on the right-hand side of (2.17) to get

$$D_n^{[k,s]} = \frac{\Gamma(1 - \mathbf{b} + a_k)\Gamma(1 - b_1 + a_s + n)\Gamma(1 - b_2 + a_s + n)}{\Gamma(1 - \mathbf{a}_{[k,s]} + a_k)\Gamma(2 + a_k + a_s - b_1 - b_2 + n)n!}$$

$$\times {}_3F_2 \left(\begin{array}{c} 1 - b_1 + a_k, 1 - b_2 + a_k, b_3 - \mathbf{a}_{[k,s]} \\ 2 + a_k + a_s - b_1 - b_2 + n, 1 - \mathbf{a}_{[k,s]} + a_k \end{array} \right).$$

Substituting this into (3.18), applying Euler's reflection formula for the gamma function and rearranging we get the claimed identity. ∎

A Definition of Meijer's G-function revisited

Meijer's G-function has been defined in the introduction, where we mentioned various aspects that need to be clarified in order that this definition be consistent. Most accurate information with proofs regarding G-function's definition is contained, in our opinion, in the series of papers of Meijer himself [30], the paper [6] by Braaksma and in the first chapters of the books [35] and [25]. Further facts are scattered in the literature with most comprehensive collection being [36, Chapter 8], [3] and especially [31]. An accessible introduction to G-function can be found in a nice recent survey by Beals and Szmigielski [4]. In this paper we only deal with the function $G^{m,n}_{p,p}$. For convenience, we have gathered all the necessary information regarding its definition in the following theorem.

Theorem A.1. Denote $a^* = m + n - p$, $\psi = \sum\limits_{k=1}^{p} (b_k - a_k)$ and

$$\mathcal{G}(s) = \frac{\Gamma(b_1 + s) \cdots \Gamma(b_m + s)\Gamma(1 - a_1 - s) \cdots \Gamma(1 - a_n - s)}{\Gamma(a_{n+1} + s) \cdots \Gamma(a_p + s)\Gamma(1 - b_{m+1} - s) \cdots \Gamma(1 - b_p - s)}.$$

(a) If $|z| < 1$ then the integral in (2.1) converges for $\mathcal{L} = \mathcal{L}_-$ and

$$G^{m,n}_{p,p} \left(z \left| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \right. \right) = \sum_{j=1}^{m} \sum_{l=0}^{\infty} \operatorname*{res}_{s=b_{jl}} \mathcal{G}(s) z^{-s}, \qquad b_{jl} = -b_j - l. \tag{A.1}$$

If, in addition, $a^* > 0$ and $|\arg(z)| < a^*\pi$ or $a^* = 0$, $\Re(\psi) < 0$ and $0 < z < 1$ then the integral in (2.1) also converges for $\mathcal{L} = \mathcal{L}_{i\gamma}$ and has the same value.

(b) If $|z| > 1$ then the integral in (2.1) converges for $\mathcal{L} = \mathcal{L}_+$ and

$$G^{m,n}_{p,p} \left(z \left| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \right. \right) = -\sum_{i=1}^{n} \sum_{k=0}^{\infty} \operatorname*{res}_{s=a_{ik}} \mathcal{G}(s) z^{-s}, \qquad a_{ik} = 1 - a_i + k.$$

If, in addition, $a^* > 0$ and $|\arg(z)| < a^*\pi$ or $a^* = 0$, $\Re(\psi) < 0$ and $z > 1$ then the integral in (2.1) also converges for $\mathcal{L} = \mathcal{L}_{i\gamma}$ and has the same value.

Remark A.2. If $|z| = 1$ and $\Re(\psi) < -1$ then according to [25, Theorem 1.1] both integrals over $\mathcal{L}_-$ and over $\mathcal{L}_+$ exist. If, in addition, $a^* > 0$ and $|\arg(z)| < a^*\pi$, then the integral over $\mathcal{L}_{i\gamma}$ also exists. We found no proof in the literature that these integrals are equal.

Proof. The claims regarding the contours $\mathcal{L}_+$ and $\mathcal{L}_-$ have been demonstrated in [25, Theorems 1.1, 1.2]. They follow in a relatively straightforward manner from Stirling's asymptotic formula for the gamma function and have been observed by Meijer himself in [30, Section 1]. If $a^* = 0$, $\Re\psi < 0$ the result was proved by Kilbas and Saigo [25, Theorem 3.3]. For $a^* > 0$ and $|z| < 1$ [30, Theorem D] states that the integrals over $\mathcal{L}_-$ and $\mathcal{L}_{i\gamma}$ coincide for any real γ which covers case (a) of the theorem. For the proof Meijer refers to [43, Section 14.51], where it is essentially demonstrated that

$$\lim_{R \to \infty} \int_{l_\gamma^-(R)} \frac{\prod\limits_{j=1}^{m} \Gamma(s + b_j) \prod\limits_{i=1}^{n} \Gamma(1 - s - a_i)}{\prod\limits_{i=n+1}^{p} \Gamma(s + a_i) \prod\limits_{j=m+1}^{p} \Gamma(1 - s - b_j)} z^{-s} ds = 0, \tag{A.2}$$

where $l_\gamma^-(R)$ is the shortest arc of the circle $|s| = R$ which connects the contours $\mathcal{L}_-$ and $\mathcal{L}_{i\gamma}$ (in fact, Whittaker and Watson proved a particular case, but the proof for the general case goes

along exactly the same lines). Since $\mathcal{G}(s)$ has no poles in the domain bounded by $\mathcal{L}_-$ and $\mathcal{L}_{i\gamma}$ this leads to equality of the integrals over $\mathcal{L}_-$ and $\mathcal{L}_{i\gamma}$ stated by Meijer. It remains to consider the case $|z| > 1$, $\mathcal{L} = \mathcal{L}_{i\gamma}$. Denote by $l_\gamma^+(R)$ the reflection of $l_\gamma^-(R)$ with respect to $\mathcal{L}_{i\gamma}$. For $|z| > 1$ we get by changing s to $-s$

$$
\int_{l_\gamma^+(R)} \frac{\prod\limits_{j=1}^{m} \Gamma(s+b_j) \prod\limits_{i=1}^{n} \Gamma(1-s-a_i) z^{-s}}{\prod\limits_{i=n+1}^{p} \Gamma(s+a_i) \prod\limits_{j=m+1}^{p} \Gamma(1-s-b_j)} ds
$$

$$
= -\int_{l_\gamma^-(R)} \frac{\prod\limits_{j=1}^{m} \Gamma(-s+b_j) \prod\limits_{i=1}^{n} \Gamma(1+s-a_i) z^{s}}{\prod\limits_{i=n+1}^{p} \Gamma(-s+a_i) \prod\limits_{j=m+1}^{p} \Gamma(1+s-b_j)} ds
$$

$$
= -\int_{l_\gamma^-(R)} \frac{\prod\limits_{i=1}^{n} \Gamma(s+b_i') \prod\limits_{j=1}^{m} \Gamma(1-s-a_j')}{\prod\limits_{j=m+1}^{p} \Gamma(s+a_j') \prod\limits_{i=n+1}^{p} \Gamma(1-s-b_i')} (1/z)^{-s} ds.
$$

Here $b_i' = 1 - a_i$, $a_j' = 1 - b_j$. In view of (A.2), we immediately conclude that for $|z| > 1$

$$
\lim_{R \to \infty} \int_{l_\gamma^+(R)} \frac{\prod\limits_{j=1}^{m} \Gamma(s+b_j) \prod\limits_{i=1}^{n} \Gamma(1-s-a_i)}{\prod\limits_{i=n+1}^{p} \Gamma(s+a_i) \prod\limits_{j=m+1}^{p} \Gamma(1-s-b_j)} z^{-s} ds = 0,
$$

which implies that the integrals over $\mathcal{L}_+$ and $\mathcal{L}_{i\gamma}$ coincide.   ■

Remark A.3. If $p > q$ ($q > p$) the integral in (2.1) exists for $\mathcal{L} = \mathcal{L}_+$ ($\mathcal{L} = \mathcal{L}_-$) and all complex $z \neq 0$ and is equal to the corresponding sum of residues [25, Theorems 1.1 and 1.2]. At the same time if $a^* > 0$ and $|\arg(z)| < a^*\pi$ or $a^* = 0$ and $z > 0$, $z \neq 1$, the integral in (2.1) also exists for $\mathcal{L} = \mathcal{L}_{i\gamma}$. Most authors assume in this situation that $\mathcal{L}_{i\gamma}$ can be deformed into $\mathcal{L} = \mathcal{L}_+$ if $p > q$ or $\mathcal{L} = \mathcal{L}_-$ if $q > p$ without altering the value of the integral. However, we were unable to find any proof of this claim in the literature.

Remark A.4. It follows from the above theorem that $G_{p,p}^{m,n}(z)$ is analytic in the sector $|\arg(z)| < a^*\pi$ if $a^* > 0$ (since the integral converges uniformly in z for $\mathcal{L} = \mathcal{L}_{i\gamma}$), while for $a^* \leq 0$ we get two different analytic functions – one defined inside and the other outside of the unit circle, see [36, (8.2.2.7)].

In the proof of the above theorem we essentially used the next well-known reflection property of G-function

$$
G_{p,q}^{m,n} \left(\frac{1}{z} \middle| \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \right) = G_{q,p}^{n,m} \left(z \middle| \begin{matrix} 1 - \mathbf{b} \\ 1 - \mathbf{a} \end{matrix} \right).
$$

It is important to note that by Theorem A.1(b)

$$
G_{p,p}^{p,0} \left(z \middle| \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix} \right) = 0 \qquad \text{for} \quad |z| > 1,
$$

which is, of course, different from the analytic continuation of the right-hand side of (2.4). The Mellin transform of $G_{p,p}^{p,0}$ exists if either $\Re(\psi) > 0$ or $\psi = -m \in \mathbb{N}_0$. In the former case [25, Theorem 2.2]

$$\int_0^\infty x^{s-1} G_{p,p}^{p,0}\left(x\,\bigg|\,\begin{matrix}\mathbf{b}\\\mathbf{a}\end{matrix}\right) dx = \int_0^1 x^{s-1} G_{p,p}^{p,0}\left(x\,\bigg|\,\begin{matrix}\mathbf{b}\\\mathbf{a}\end{matrix}\right) dx = \frac{\Gamma(\mathbf{a}+s)}{\Gamma(\mathbf{b}+s)} \tag{A.3}$$

for $\Re(s) > -\Re(\mathbf{a})$. If $\psi = -m \in \mathbb{N}_0$ then [33, (2.28)]

$$\int_0^\infty x^{s-1} G_{p,p}^{p,0}\left(x\,\bigg|\,\begin{matrix}\mathbf{b}\\\mathbf{a}\end{matrix}\right) dx = \int_0^1 x^{s-1} G_{p,p}^{p,0}\left(x\,\bigg|\,\begin{matrix}\mathbf{b}\\\mathbf{a}\end{matrix}\right) dx = \frac{\Gamma(\mathbf{a}+s)}{\Gamma(\mathbf{b}+s)} - q(s)$$

for $\Re(s) > -\Re(\mathbf{a})$. Here $q(s)$ is a polynomial of degree m given by

$$q(s) = \sum_{j=0}^m g_p^k(m-j)(s+a_k-j)_j, \qquad k = 1, 2, \ldots, p,$$

where the coefficients $g_p^k(n)$ are defined in expansion (2.5) and are given explicitly by (2.9). Note that $g_p^k(n)$ depends on k while the polynomial $q(s)$ is the same for each $k \in \{1, \ldots, p\}$.

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