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CHANGE OF BASE, CAUCHY COMPLETENESS AND REVERSIBILITY

ANNA LABELLA AND VINCENT SCHMITT

ABSTRACT. We investigate the effect on Cauchy complete objects of the change of base 2-functor $\mathcal{V}\text{-}Cat \to \mathcal{W}\text{-}Cat$ induced by a two-sided enrichment $\mathcal{V} \to \mathcal{W}$. We restrict our study to the case of locally partially ordered bases. The reversibility notion introduced in [Wal82] is extended to two-sided enrichments and Cauchy completion. We show that a reversible left adjoint two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ between locally partially ordered reversible bicategories induces an adjunction $F_\sim \dashv F^\sim : \mathcal{V}\text{-}SkCRcCat \rightharpoonup \mathcal{W}\text{-}SkCRcCat$ between sub-categories of skeletal and Cauchy-reversible complete enrichments. We give two applications: sheaves over locales and group actions.

1. Introduction

Our motivation for the study of the change of base for enrichments over bicategories originated with the two following facts. Sheaves over a locale have been characterised as Cauchy complete enrichments over bicategories [Wal81]. Any continuous map $f : X \to Y$ yields an adjunction between categories of sheaves $f_* \dashv f^* : Sh(Y) \rightharpoonup Sh(X)$. From this we hoped to find a categorical generalisation in terms of enrichments of the later adjunction.

Our first problem was to define a good notion of base morphism $F : \mathcal{V} \to \mathcal{W}$. We had in mind that such a morphism should induce a 2-adjunction between 2-categories of enrichments, say $F_@ \dashv F^@ : \mathcal{V}\text{-}Cat \rightharpoonup \mathcal{W}\text{-}Cat$. This question was largely answered in [KLSS99] with the introduction of the so-called two-sided enrichments. To explain partly these results, we should start from the definition of *MonCat*, the category of monoidal functors between monoidal categories [Ben63], and enrichments over them [EiKe66], [Law73]. A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces a 2-functor $F_@ : \mathcal{V}\text{-}Cat \to \mathcal{W}\text{-}Cat$. *MonCat* is equipped with a 2-categorical structure by defining 2-cells in it as monoidal natural transformations ([EiKe66]). The process $(-)_@$ of sending $\mathcal{V}$ to $\mathcal{V}\text{-}Cat$ and F to $F_@$ extends to a 2-functor from *MonCat* to 2-*Cat*. Adjunctions in *MonCat* were characterised in [Kel74]. Moving to the case of enrichments over bicategories, several notions of "morphism" between bicategories were proposed in the literature, but no analogous 2-functor $(-)_@$ was known in this situation.

Two-sided enrichments are a slight generalisation of monoidal functors and Bénabou's lax functors [Ben67]. With these new morphisms of bicategories one obtained a bicategory

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Base together with the expected pseudo-functor $(-)_@ : Base \to 2\text{-}Cat$. Introducing some 3-cells on *Base*, one eventually gets a tricategory *Caten* such that the 2-categories of enrichments are representable $\mathcal{V}\text{-}Cat \cong Caten(1, \mathcal{V})$ (1 is the unit bicategory). These new morphisms of bicategories could be seen as a two-sided version of the usual enrichments over bicategories, hence their name. Adjunctions in *Caten* — of particular interest in this paper — are as follows: left adjoint two-sided enrichments are exactly the pseudo functors with "local right adjoints". This notion of local adjunction is slightly more general than the one in [BePo88]. To complete the picture, let us recall the following results. The cartesian product of bicategories extends to a pseudo-functor $Caten \times Caten \to Caten$ that makes *Caten* into a monoidal tricategory. *Caten* has a closed structure $Caten(\mathcal{V}\times\mathcal{W}, \mathcal{Z}) \cong Caten(\mathcal{V}, Conv(\mathcal{W}, \mathcal{Z}))$. Eventually two-sided enrichments are also just enrichments since thus $Caten(\mathcal{V}, \mathcal{W}) \cong Conv(\mathcal{V}, \mathcal{W})\text{-}Cat$.

Our second problem was to understand the effect of change of base $F_@$ for a two-sided enrichment F, on the skeletality and Cauchy completeness of objects. This is the object of this paper. The idea is that the adjunction $\mathcal{V}\text{-}Cat \to \mathcal{W}\text{-}Cat$ given by an adjoint 2-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ should also "behave well" with respect to the skeletality and Cauchy completeness of objects. One may hope for some (categorical) connection between the sub-2-categories of skeletal and Cauchy complete objects respectively of $\mathcal{V}\text{-}Cat$ and $\mathcal{W}\text{-}Cat$.

Our investigation may be summarised as follows. We restrict our study to the case when the base bicategories are locally partially ordered. This is both for convenience and because our purpose was to give a satisfying categorical generalisation of the adjunction between categories of sheaves. First we look for conditions on a two-sided enrichment F that ensure that it preserves skeletality, Cauchy completion, Morita equivalence and so on... We introduce the "super" two-sided enrichments. They generalise the super monoidal functors used by one of the authors to code uniformly continuous maps from enriched functors [Sc01]. They are more general than left adjoint two-sided enrichments and enjoy nice properties. It was known from [KLSS99] that a two-sided enrichment F induces a normal lax functor between bicategories of modules $F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$. For any super F, $F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$ preserves adjoints and $F_@ : \mathcal{V}\text{-}Cat \to \mathcal{W}\text{-}Cat$ preserves the Morita equivalence. Eventually we extend the notion of reversibility defined for enrichments in [Wal82] to two-sided enrichments and to the Cauchy completion. Most familiar examples of enrichments over locally partially ordered bicategories (sheaves and group actions) enjoy a nice property of symmetry, called reversibility. As pointed out by Kasangian [BeWal82] a Cauchy completion of a reversible enrichment fails in general to be reversible. This fact forces us to define a completion for reversible enrichments that preserves reversibility. This is the so-called Cauchy-reversible completion. Thus we get the desired generalisation: an adjoint pair in *Caten* $F \dashv G : \mathcal{V} \rightharpoonup \mathcal{W}$ where F is reversible induces some adjunction between sub-categories of skeletal and Cauchy-reversible complete objects $\mathcal{V}\text{-}SkCRcCat \rightharpoonup \mathcal{W}\text{-}SkCRcCat$. An instance of this theorem is also the classical result in group theory that a group morphism $f : H \to G$ induces an adjunction between categories of group actions $f_* \dashv f^* : H\text{-}Set \rightharpoonup G\text{-}Set$.

This paper is organised as follows. Section 2 recalls basics of the theory of enrichments over locally partially ordered bicategories. This makes the paper self-contained and allows later comparisons with the reversible case. (Anyway, we refer the reader to [Kel82] for the fundamentals of enriched category theory). Section 3 treats the change of base for enrichments and the connection with two-sided enrichments in the case when the base categories are locally partially ordered. We begin by recalling some elements of the theory of two-sided enrichments developed in [KLSS99]: the definition of the tricategory *Caten*, the connection with enrichments, the characterisation of adjoints in *Caten*, and the change of base. Then we present the various new results of our investigation into the effect of the change of base bicategories on skeletality, and Cauchy completeness of enrichments. The super two-sided enrichments are introduced. Section 4 is devoted to the development of the theory in the reversible case. The notion of reversibility is defined for bicategories, (two-sided) enrichments, enriched modules, completion, etc. We retrace all the steps of the theory. Two detailed examples are given, namely sheaves on locales and group actions.

2. Basic enriched category theory

2.1. Definition. *A bicategory $\mathcal{V}$ is* locally partially ordered *(respectively* locally preordered*), when for any objects a and b, the homset $\mathcal{V}(a,b)$ is a partial order (respectively a preorder).*

Further on in this section, $\mathcal{V}$ will denote a locally partially ordered bicategory that is moreover:
- *biclosed*: i.e. for any arrow $f : u \to v$ and any object w of $\mathcal{V}$, the functors $f \circ - : \mathcal{V}(w,u) \to \mathcal{V}(w,v)$ and $- \circ f : \mathcal{V}(v,w) \to \mathcal{V}(u,w)$ are adjoints.
- *locally cocomplete*: i.e. each homset $\mathcal{V}(u,v)$ is cocomplete.
For any object v, I_v stands for the identity in v.

2.2. Definition. [$\mathcal{V}$-categories] *An* enrichment *A over $\mathcal{V}$, also called a* $\mathcal{V}$-category*, is a set $Obj(A)$ — the objects of A, with two mappings $(-)^A_+ : Obj(A) \longrightarrow Obj(\mathcal{V})$ (sometimes just written $(-)_+$) and $A(-,-) : Obj(A) \times Obj(A) \longrightarrow Arrows(\mathcal{V})$ that satisfy:*
- *$(enr - i)$ for any objects a,b of A, $A(a,b) : a_+ \longrightarrow b_+$;*
- *$(enr - ii)$ for any object a of A, $I_{a_+} \leq A(a,a)$;*
- *$(enr - iii)$ for any objects a,b,c of A, $A(b,c) \circ A(a,b) \leq A(a,c)$.*
Given $\mathcal{V}$-categories A and B, a $\mathcal{V}$-functor *f from A to B is a map $f : Obj(A) \longrightarrow Obj(B)$ such that:*
- *$(fun - i)$ $(-)^A_+ = (-)^B_+ \circ f$;*
- *$(fun - ii)$ for any objects a, b of A, $A(a,b) \leq B(fa, fb)$.*
Given $\mathcal{V}$-functors $f, g : A \to B$ there is a unique $\mathcal{V}$-natural transformation *from f to g, when for any object a of A, $I_{a_+} \leq B(fa, ga)$.*

2.3. Proposition. *$\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form a locally preordered 2-category denoted $\mathcal{V}$-Cat.*

For any object u in $\mathcal{V}$, $\hat{u}$ will denote the enrichment on $\mathcal{V}$, with one object say $*$, with $*_+ = u$ and $\hat{u}(*,*) = I_u$. If B is a $\mathcal{V}$-category, for any object b in B, $b : \widehat{b_+} \to B$ will denote the $\mathcal{V}$-functor sending $*$ to b.

2.4. Definition. [$\mathcal{V}$-modules] *Given two $\mathcal{V}$-categories A and B, a* $\mathcal{V}$-module *φ from A to B — denoted $\varphi : A \multimap B$ — is a map $\varphi : Obj(B) \times Obj(A) \to Arrows(\mathcal{V})$ such that:*
- $(mod - i)$ for all objects a of A, and b of B, $\varphi(b,a)$ is an arrow $b_+ \to a_+$ in $\mathcal{V}$.
- $(mod - ii)$ for all objects a, a' of A, and b of B, $A(a,a') \circ \varphi(b,a) \leq \varphi(b,a')$;
- $(mod - iii)$ for all objects $a \in Obj(A)$, and b, b' of B, $\varphi(b,a) \circ B(b',b) \leq \varphi(b',a)$.

For any two $\mathcal{V}$-modules $\varphi : A \multimap B$ and $\psi : B \multimap C$, their composite $\psi \bullet \varphi : A \multimap C$ is defined by $\psi \bullet \varphi(c,a) = \bigvee_{b \in Obj(B)} \varphi(b,a) \circ \psi(c,b)$. The set $\mathcal{V}$-$Mod(A,B)$ of $\mathcal{V}$-modules from A to B is partially ordered by $\varphi \leq \psi \Leftrightarrow \forall (b,a) \in Obj(B) \times Obj(A), \varphi(b,a) \leq \psi(b,a)$.

2.5. Proposition. *$\mathcal{V}$-categories and $\mathcal{V}$-modules with partial orders defined above, constitute a locally partially ordered bicategory denoted $\mathcal{V}$-Mod.*

In $\mathcal{V}$-Mod, the identity in A is the module with underlying map $A(-,-) : Obj(A) \times Obj(A) \to Arrows(\mathcal{V})$ sending (y,x) to $A(y,x)$. For any $\mathcal{V}$-categories A and B, a $\mathcal{V}$-module $\varphi : A \multimap B$ has right adjoint $\psi : B \multimap A$ if and only if $A(-,-) \leq \psi \bullet \varphi$ and $\varphi \bullet \psi \leq B(-,-)$. For any left adjoint module φ, $\tilde{\varphi}$ will denote its (unique!) right adjoint. Any $\mathcal{V}$-functor $f : A \to B$ corresponds to a pair of adjoint modules $f_\diamond \dashv f^\diamond$, $f_\diamond : A \multimap B$, $f^\diamond : B \multimap A$, as follows: for all objects a of A, and b of B, $f_\diamond(b,a) = B(b,fa)$, $f^\diamond(a,b) = B(fa,b)$.

Further on $\mathcal{V}$-$AMod$ will denote the locally partially ordered bicategory with:
- objects $\mathcal{V}$-categories,
- arrows: *left adjoint* $\mathcal{V}$-modules;
- 2-cells: partial orders inherited from $\mathcal{V}$-Mod.

There is a 2-functor $J_\mathcal{V} : \mathcal{V}\text{-}Cat \to \mathcal{V}\text{-}Mod$ as follows. It is the identity on objects, and is the map $(-)_\diamond$ on arrows sending any $\mathcal{V}$-functor f to the $\mathcal{V}$-module $f_\diamond$. Note that for any $\mathcal{V}$-categories A and B, for any $\mathcal{V}$-functors $f, g : A \to B$, $f \Rightarrow g$ if and only if $f_\diamond \leq g_\diamond$.

2.6. Definition. [Skeletality, Cauchy completeness] *Let B be a $\mathcal{V}$-category. B is called* skeletal *if and only if for any $\mathcal{V}$-category A, the map $(-)_\diamond : \mathcal{V}\text{-}Cat(A,B) \to \mathcal{V}\text{-}AMod(A,B)$ is injective. B is called* Cauchy complete *if and only if for any $\mathcal{V}$-category A, the map $(-)_\diamond : \mathcal{V}\text{-}Cat(A,B) \to \mathcal{V}\text{-}AMod(A,B)$ is surjective.*

2.7. Observation. *For any $\mathcal{V}$-category B, the following assertions are equivalent:*
- B is skeletal*;*
- For any object u of $\mathcal{V}$, the map $(-)_\diamond : \mathcal{V}\text{-}Cat(\hat{u},B) \to \mathcal{V}\text{-}AMod(\hat{u},B)$ is injective;
- For any objects a and b of B, if $a_+ = b_+$, $I_{a_+} \leq B(a,b)$ and, $I_{a_+} \leq B(b,a)$ then $a = b$.

2.8. Proposition. *A $\mathcal{V}$-category B is* Cauchy complete *when for any object u of $\mathcal{V}$, the map $(-)_\diamond : \mathcal{V}\text{-}Cat(\hat{u},B) \to \mathcal{V}\text{-}AMod(\hat{u},B)$ is surjective (i.e. for any object u of $\mathcal{V}$ and any left adjoint $\mathcal{V}$-module $\varphi : \hat{u} \multimap B$, $\varphi = b_\diamond$ for some object b of B with $b_+ = u$).*

Proof. Suppose that $\varphi : A \multimap B$ has right adjoint $\tilde{\varphi}$. Then for any object a of A, $\varphi \bullet a_\diamond : \widehat{a_+} \multimap B$ and $a^\diamond \bullet \tilde{\varphi} : B \multimap \widehat{a_+}$ are adjoints. For any objects a of A and b of B,

$$\varphi \bullet a_\diamond(b, *) = \bigvee_{a' \in Obj(A)} A(a', a) \circ \varphi(b, a') = \varphi(b, a)$$

and

$$a^\diamond \bullet \tilde{\varphi}(*, b) = \bigvee_{a' \in Obj(A)} \tilde{\varphi}(a', b) \circ A(a, a') = \tilde{\varphi}(a, b).$$

By assumption, for any object a of A, there is a $\mathcal{V}$-functor $f_a : \widehat{a_+} \to B$ such that $(f_a)_\diamond = \varphi \bullet a_\diamond$ and $(f_a)^\diamond = a^\diamond \bullet \tilde{\varphi}$. This means that for any objects a of A and b of B, $\varphi(b, a) = B(b, f_a(*))$ and $\tilde{\varphi}(a, b) = B(f_a(*), b)$. Let f denote the map from $Obj(A)$ to $Obj(B)$ sending a to $f_a(*)$. f defines a $\mathcal{V}$-functor from A to B since for any objects a,a' of A,

$$\begin{aligned} B(fa, fa') &= \textstyle\bigvee_{b \in Obj(B)} B(b, fa') \circ B(fa, b) \\ &= \textstyle\bigvee_{b \in Obj(B)} \varphi(b, a') \circ \tilde{\varphi}(a, b) \\ &= \tilde{\varphi} \bullet \varphi(a, a') \\ &\geq A(a, a'). \end{aligned}$$

■

We define the following locally partially ordered bicategories.

- $\mathcal{V}$-*SkCcCat*, with:
    - objects: skeletal and Cauchy complete $\mathcal{V}$-categories,
    - arrows and 2-cells: inherited from $\mathcal{V}$-*Cat*.

- $\mathcal{V}$-*SkCcAMod*, with:
    - objects: skeletal and Cauchy complete $\mathcal{V}$-categories,
    - arrows and 2-cells: inherited from $\mathcal{V}$-*AMod*.

2.9. Proposition. *The restriction of $J_\mathcal{V}$ on $\mathcal{V}$-SkCcCat is an isomorphism onto $\mathcal{V}$-SkCcAMod.*

2.10. Definition. [Cauchy completion] *Let A be a $\mathcal{V}$-category. The* Cauchy completion *of A is the $\mathcal{V}$-category $\bar{A}$ defined as follows. Its objects are the left adjoint $\mathcal{V}$-modules of the form $\varphi : \hat{u} \multimap A$, u ranging in $Obj(\mathcal{V})$. For any object φ of $\bar{A}$ with $\varphi : \hat{u} \multimap A$, $\varphi_+ = u$. For convenience we will consider modules in $Obj(\bar{A})$ as well as their adjoints, as maps with domains A. For any objects φ,ψ of $\bar{A}$, $\bar{A}(\varphi, \psi) = (\tilde{\varphi} \bullet \psi)(*, *)$.*

Let us check that the $\bar{A}$ of 2.10 is a well defined $\mathcal{V}$-category. $\bar{A}$ trivially satisfies $(enr - i)$. Let us show that it satisfies $(enr - ii)$. Let $\varphi : \hat{u} \multimap A$ be an object of $\bar{A}$, then

$$\begin{aligned} \bar{A}(\varphi, \varphi) &= \textstyle\bigvee_{a \in Obj(A)} \varphi(a) \circ \tilde{\varphi}(a) \\ &= \tilde{\varphi} \bullet \varphi(*, *) \\ &\geq I_u. \end{aligned}$$

Let us show that $\bar{A}$ satisfies $(enr - iii)$. Let φ, ψ, γ be objects of $\bar{A}$. Then

$$\begin{aligned}\bar{A}(\gamma,\psi)\circ\bar{A}(\varphi,\gamma) &= (\bigvee_{a\in Obj(A)}\psi(a)\circ\tilde{\gamma}(a))\circ(\bigvee_{a'\in Obj(A)}\gamma(a')\circ\tilde{\varphi}(a'))\\ &= \bigvee_{a,a'\in Obj(A)}\psi(a)\circ\tilde{\gamma}(a)\circ\gamma(a')\circ\tilde{\varphi}(a')\\ &\leq \bigvee_{a,a'\in Obj(A)}\psi(a)\circ A(a',a)\circ\tilde{\varphi}(a')\\ &= \bigvee_{a\in Obj(A)}\psi(a)\circ\tilde{\varphi}(a)\\ &= \bar{A}(\varphi,\psi).\end{aligned}$$

Let A be a $\mathcal{V}$-category with Cauchy completion $\bar{A}$. For any objects a of A and φ of $\bar{A}$, $\bar{A}(a_\diamond,\varphi) = \varphi(a)$ and $\bar{A}(\varphi, a_\diamond) = \tilde{\varphi}(a)$. The map $i_A : Obj(A) \to Obj(\bar{A})$, sending a to $a_\diamond$ is a $\mathcal{V}$-functor from A to $\bar{A}$ since for any object a of A, $(a_\diamond)_+ = a_+$, and for any objects a,b of A, $\bar{A}(a_\diamond, b_\diamond) = \bigvee_{c\in Obj(A)} A(c,b)\circ A(a,c) = A(a,b)$.

2.11. Remark. For any $\mathcal{V}$-category A,
- A is skeletal if and only if the map $i_A : Obj(A) \to Obj(\bar{A})$ is injective;
- A is Cauchy complete if and only the map i_A is onto.

2.12. Proposition. *For any $\mathcal{V}$-category A, $\bar{A}$ is skeletal.*

Proof. Suppose that φ, ψ are some objects of $\bar{A}$ such that $\varphi_+ = \psi_+ = u$ and, (i) $I_u \leq \bar{A}(\varphi,\psi)$ and, (ii) $I_u \leq \bar{A}(\psi,\varphi)$. The inequation (i) is equivalent to $(1_{\hat{u}})_\diamond \leq \tilde{\varphi}\bullet\psi$ which is equivalent to $\varphi \leq \psi$ since $\varphi \dashv \tilde{\varphi}$. Similarly, (ii) is equivalent to $\psi \leq \varphi$. Therefore $\varphi = \psi$. ■

2.13. Lemma. *Let A be a $\mathcal{V}$-category and, $\Gamma : \hat{u} \multimap \bar{A}$ be some $\mathcal{V}$-module where u is an object of $\mathcal{V}$. For any object φ of $\bar{A}$, $\tilde{\varphi}$ denoting its adjoint,* $\Gamma(\varphi) = \bigvee_{a\in Obj(A)}\Gamma(a_\diamond)\circ\tilde{\varphi}(a)$.

Proof. Let a be some object of A,

$$\begin{aligned}\Gamma(a_\diamond) &= \bigvee_{\psi\in Obj(\bar{A})}\Gamma(\psi)\circ\bar{A}(a_\diamond,\psi),\\ &= \bigvee_{\psi\in Obj(\bar{A})}\Gamma(\psi)\circ\psi(a).\end{aligned}$$

Therefore, for any object φ of $\bar{A}$,

$$\begin{aligned}\bigvee_{a\in Obj(A)}\Gamma(a_\diamond)\circ\tilde{\varphi}(a) &= \bigvee_{a\in Obj(A)}(\bigvee_{\psi\in Obj(\bar{A})}\Gamma(\psi)\circ\psi(a))\circ\tilde{\varphi}(a)\\ &= \bigvee_{\psi\in Obj(\bar{A})}\Gamma(\psi)\circ\bar{A}(\varphi,\psi)\\ &= \Gamma(\varphi).\end{aligned}$$

■

2.14. Proposition. *For any $\mathcal{V}$-category A, $\bar{A}$ is Cauchy complete.*

Proof. Given a left adjoint module $\Gamma : \hat{u} \multimap \bar{A}$ with u an object of $\mathcal{V}$. We define the maps $\gamma, \gamma' : Obj(A) \to Arrows(\mathcal{V})$ respectively by $\gamma(a) = \Gamma(a_\diamond)$ and $\gamma'(a) = \tilde{\Gamma}(a_\diamond)$. We are going to show that γ is an object of $\bar{A}$ and $\Gamma = \bar{A}(-,\gamma)$. It is immediate that γ and γ' are modules respectively from $\hat{u}$ to A and from A to $\hat{u}$. Let us show that $\gamma \dashv \gamma'$. According to 2.13 (and its dual!),

$$\begin{aligned}\textstyle\bigvee_{\varphi\in Obj(\bar{A})}\Gamma(\varphi)\circ\tilde{\Gamma}(\varphi) &= \textstyle\bigvee_{\varphi\in Obj(\bar{A})}\bigvee_{a,b\in Obj(A)}\Gamma(a_\diamond)\circ\tilde{\varphi}(a)\circ\varphi(b)\circ\tilde{\Gamma}(b_\diamond)\\ &\leq \textstyle\bigvee_{a,b\in Obj(A)}\Gamma(a_\diamond)\circ A(b,a)\circ\tilde{\Gamma}(b_\diamond)\\ &= \textstyle\bigvee_{a,b\in Obj(A)}\Gamma(a_\diamond)\circ\bar{A}(b_\diamond,a_\diamond)\circ\tilde{\Gamma}(b_\diamond)\\ &= \textstyle\bigvee_{a\in Obj(A)}\Gamma(a_\diamond)\circ\tilde{\Gamma}(a_\diamond).\end{aligned}$$

Since $\Gamma\dashv\tilde{\Gamma}$,

$$\begin{aligned}I_u &\leq \textstyle\bigvee_{\varphi\in Obj(\bar{A})}\Gamma(\varphi)\circ\tilde{\Gamma}(\varphi)\\ &= \textstyle\bigvee_{a\in Obj(A)}\Gamma(a_\diamond)\circ\tilde{\Gamma}(a_\diamond)\\ &= \textstyle\bigvee_{a\in Obj(A)}\gamma(a)\circ\gamma'(a)\end{aligned}$$

and for any objects a,b of A,

$$\begin{aligned}\gamma\bullet\gamma'(a,b) &= \gamma'(b)\circ\gamma(a)\\ &= \tilde{\Gamma}(b_\diamond)\circ\Gamma(a_\diamond)\\ &= \Gamma\circ\tilde{\Gamma}(a_\diamond,b_\diamond)\\ &\leq \bar{A}(a_\diamond,b_\diamond)\\ &= A(a,b).\end{aligned}$$

Now, since γ is an object of $\bar{A}$, 2.13 shows that $\Gamma=\bar{A}(-,\gamma)$. ■

2.15. Lemma. *For any $\mathcal{V}$-category A, $i_{A\diamond}: A \longrightarrow\!\!\circ\!\!\rightarrow \bar{A}$ and $i_A{}^\diamond: \bar{A} \longrightarrow\!\!\circ\!\!\rightarrow A$ are inverse modules.*

Proof. For any object φ of $\bar{A}$, the maps $\bar{A}(-,\varphi)$ and $\bar{A}(\varphi,-)$ — sending respectively an object ψ of $\bar{A}$ to $\bar{A}(\psi,\varphi)$ and to $\bar{A}(\varphi,\psi)$ — define modules. Therefore, for any objects a,b of A,

$$\begin{aligned}(i_A{}^\diamond\bullet i_{A\diamond})(a,b) &= \textstyle\bigvee_{\varphi\in Obj(\bar{A})}\bar{A}(\varphi,b_\diamond)\circ\bar{A}(a_\diamond,\varphi)\\ &= \bar{A}(a_\diamond,b_\diamond)\\ &= A(a,b).\end{aligned}$$

For any objects φ,ψ of $\bar{A}$,

$$\begin{aligned}(i_{A\diamond}\bullet i_A{}^\diamond)(\varphi,\psi) &= \textstyle\bigvee_{a\in Obj(A)}\bar{A}(a_\diamond,\psi)\circ\bar{A}(\varphi,a_\diamond)\\ &= \textstyle\bigvee_{a\in Obj(A)}\psi(a)\circ\tilde{\varphi}(a)\\ &= \bar{A}(\varphi,\psi).\end{aligned}$$

■

According to this,

2.16. Proposition. *$\mathcal{V}$-AMod and $\mathcal{V}$-SkCcAMod are equivalent. There is a 2-functor S : $\mathcal{V}$-AMod $\rightarrow$ $\mathcal{V}$-SkCcCat which is an equivalence of 2-categories. S is defined on objects by $S(A)=\bar{A}$ and on arrow by $S(\varphi)=f_\varphi$ where for any left adjoint $\mathcal{V}$-module $\varphi: A \longrightarrow\!\!\circ\!\!\rightarrow B$, f_φ is the unique $\mathcal{V}$-functor $f:\bar{A}\rightarrow\bar{B}$ satisfying $f_\diamond\bullet i_{A\diamond}=i_{B\diamond}\bullet\varphi$. The inclusion 2-functor $\mathcal{V}$-SkCcCat $\rightarrow$ $\mathcal{V}$-Cat is a right 2-adjoint.*

The left 2-adjoint of the inclusion $\mathcal{V}$-*SkCcCat* $\rightarrow$ $\mathcal{V}$-*Cat* sends a $\mathcal{V}$-category A to $\bar{A}$ and the unit takes value $i_A: A\rightarrow\bar{A}$ in A.

According to 2.12 and 2.15, the following is coherent

2.17. Definition. [Morita equivalence] *Two $\mathcal{V}$-categories A and B are* Morita equivalent *when one of the following equivalent assertions is satisfied:*
(i) Their Cauchy completions are isomorphic in $\mathcal{V}$-Cat;
(ii) They are isomorphic in $\mathcal{V}$-Mod.

3. Change of base

The study of the change of base for enrichments over bicategories yielded the concept of two-sided enrichments (see [KLSS99] for the general treatment). We will specify the theory in the particular and simpler case when the base bicategories for enrichments are locally partially ordered (3.1-3.5). We keep on using the original terminology of [KLSS99]. Section 3.8 presents a study of the effect on the change of base bicategories on skeletality and Cauchy completeness of enrichments. Super two-sided enrichments are introduced.

3.1. Two-sided enrichments. *Caten* is the tricategory as follows. Its objects are locally partially ordered bicategories. An arrow $A : \mathcal{V} \to \mathcal{W}$ in *Caten* is a *two-sided enrichment*. It consists in:
- a span in *Set*

$$\begin{tikzcd} & Obj(A) \arrow[dl, "(-)^A_-"'] \arrow[dr, "(-)^A_+"] & \\ Obj(\mathcal{V}) & & Obj(\mathcal{W}) \end{tikzcd}$$

where $Obj(A)$ is the set of "objects" of A,
- a collection of functors $A_{a,b} : \mathcal{V}(a_-, b_-) \to \mathcal{W}(a_+, b_+)$, (a, b) ranging in $Obj(A)^2$, that satisfy the coherence conditions (c),(u) below.
(c) For any objects a,b,c of A, $A_{a,b}(-) \circ A_{b,c}(?) \leq A_{a,c}(-\circ ?)$, i.e there exists a 2-cell in *Po*, the locally partially ordered bicategory of partial orders, monotonous maps with pointwise ordering, as follows

$$\begin{tikzcd} \mathcal{V}(a_-, b_-) \times \mathcal{V}(b_-, c_-) \arrow[r, "\circ^{\mathcal{V}}"] \arrow[d, "A_{a,b} \times A_{b,c}"'] & \mathcal{V}(a_-, c_-) \arrow[d, "A_{a,c}"] \\ \mathcal{W}(a_+, b_+) \times \mathcal{W}(b_+, c_+) \arrow[r, "\circ^{\mathcal{W}}"'] \arrow[ur, Rightarrow] & \mathcal{W}(a_+, c_+) \end{tikzcd}$$

(u) For any object a of A, $I^{\mathcal{W}}_{a_+} \leq A_{a,a}(I^{\mathcal{V}}_{a_-})$ which corresponds to the 2-cell in *Po*

$$\begin{tikzcd} 1 \arrow[r, "I^{\mathcal{V}}_{a_-}"] \arrow[d, equal] & \mathcal{V}(a_-, a_-) \arrow[d, "A_{a,a}"] \\ 1 \arrow[r, "I^{\mathcal{W}}_{a_+}"'] \arrow[ur, Rightarrow] & \mathcal{W}(a_+, a_+) \end{tikzcd}$$

For any two-sided enrichments $\mathcal{V}$ and $\mathcal{W}$, the bicategory $Caten(\mathcal{V},\mathcal{W})$ is locally partially ordered, as follows. A 2-cell $f : A \to B : \mathcal{V} \to \mathcal{W}$ in $Caten$ is called a *functor*, it consists in a map $f : Obj(A) \to Obj(B)$ such that:
- $(-)^A_- = (-)^B_- \circ f$;
- $(-)^A_+ = (-)^B_+ \circ f$;

$$\begin{tikzcd}
 & Obj(A) \arrow[dl, "(-)^A_-"'] \arrow[dd, "f"] \arrow[dr, "(-)^A_+"] & \\
Obj(\mathcal{V}) & & Obj(\mathcal{W}) \\
 & Obj(B) \arrow[ul, "(-)^B_-"] \arrow[ur, "(-)^B_+"'] &
\end{tikzcd}$$

- for any objects a,b of A, $A_{a,b} \leq B_{fa,fb} : \mathcal{V}(a_-, b_-) \to \mathcal{W}(a_+, b_+)$, i.e. there is a 2-cell in Po

$$\begin{tikzcd}
\mathcal{V}(a_-, b_-) \arrow[r, bend left, "A_{a,b}"] \arrow[r, bend right, "B_{fa,fb}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{W}(a_+, b_+)
\end{tikzcd}$$

2-cells in $Caten$ are ordered as follows. $f \leq g : A \to B : \mathcal{V} \to \mathcal{W}$ when for any object a of A, $I^{\mathcal{W}}_{a_+} \leq B_{fa,ga}(I^{\mathcal{V}}_{a_-}) : a_+ \to a_+$, which corresponds to the 2-cell in Po

$$\begin{tikzcd}
1 \arrow[r, "I^{\mathcal{V}}_{a_-}"] \arrow[d, equal] & \mathcal{V}(a_-, a_-) \arrow[d, "B_{fa,ga}"] \\
1 \arrow[r, "I^{\mathcal{W}}_{a_+}"'] \arrow[ur, Rightarrow] & \mathcal{W}(a_+, a_+)
\end{tikzcd}$$

The (vertical) composition of the 2-cells $f : A \to B, g : B \to C : \mathcal{V} \to \mathcal{W}$ is the composite of the maps on objects f and g. The identity 1_A of $A : \mathcal{V} \to \mathcal{V}$ has underlying map the identity map on $Obj(A)$.

Let $A : \mathcal{U} \to \mathcal{V}$ and $B : \mathcal{V} \to \mathcal{W}$ be two-sided enrichments. Their composite $B \circ A : \mathcal{U} \to \mathcal{W}$ has for span the composition of the spans of A and B: its objects are the pairs (a, b) in $Obj(A) \times Obj(B)$ such that $a_+ = b_-$ and the maps $(-)^{B\circ A}_-$ and $(-)^{B\circ A}_+$ are given by $(a, b)_- = a_-$ and $(a, b)_+ = b_+$. The functors $B \circ A_{(a,b),(a',b')}$ are the composites

$$\mathcal{U}(a_-, a'_-) \xrightarrow{A_{(a,a')}} \mathcal{V}(a_+, a'_-) = \mathcal{V}(b_-, b'_-) \xrightarrow{B_{(b,b')}} \mathcal{W}(b_+, b'_-)\ .$$

The horizontal composition of the 2-cells $f : A \to B : \mathcal{U} \to \mathcal{V}$ and $g : C \to D : \mathcal{V} \to \mathcal{W}$ is the map sending $(a, c) \in Obj(C \circ A)$ to (fa, gc).

3.2. A closed structure on *Caten*. The cartesian product of locally partially ordered bicategories extends straightforwardly to a pseudo functor $Caten \times Caten \to Caten$ that makes *Caten* into a monoidal tricategory. *Caten* has a biclosed structure:

$$Caten(\mathcal{U} \times \mathcal{V}, \mathcal{W}) \cong Caten(\mathcal{U}, Conv(\mathcal{V}, \mathcal{W})).$$

where $Conv(\mathcal{V}, \mathcal{W})$ is defined as follows:

3.3. Definition. [The bicategory Conv(V,W)] *Given two locally partially ordered bicategories $\mathcal{V}$, $\mathcal{W}$ the locally partially ordered bicategory $Conv(\mathcal{V}, \mathcal{W})$ is defined as follows. Its objects are ordered pairs $(v, w) \in Obj(\mathcal{V}) \times Obj(\mathcal{W})$. Its arrows $(v, w) \to (v', w')$ are functors $\mathcal{V}(v, v') \to \mathcal{W}(w, w')$, (i.e. monotonous maps). The partial order on arrows is pointwise: For any $F, G : (v, w) \to (v', w')$, $F \leq G$ when for any $h : v \to v'$ in $\mathcal{V}$, $F(h) \leq G(h) : w \to w'$ in $\mathcal{W}$. For any $F : (v, w) \to (v', w')$ and $G : (v', w') \to (v'', w'')$, the (horizontal) composite $G\bar{\circ}F : (v, w) \to (v'', w'')$ is the left Kan extension of*

$$\mathcal{V}(v, v') \times \mathcal{V}(v', v'') \xrightarrow{F \times G} \mathcal{W}(w, w') \times \mathcal{W}(w', w'') \xrightarrow{\circ^{\mathcal{W}}} \mathcal{W}(w, w'')$$

along $\mathcal{V}(v, v') \times \mathcal{V}(v', v'') \xrightarrow{\circ^{\mathcal{V}}} \mathcal{U}(v, v'')$, *where $\circ^{\mathcal{V}}$, $\circ^{\mathcal{W}}$ are the horizontal compositions respectively of $\mathcal{V}$ and $\mathcal{W}$. This means that for any arrow h in $\mathcal{V}(v, v'')$, $G\bar{\circ}F(h) = \bigvee\{G(g) \circ F(f) \mid g \circ f \leq h\}$. The identity $I_{(v,w)}$ in (v, w) is the left Kan extension of $I^{\mathcal{W}}_w : 1 \to \mathcal{W}(w, w)$ along $I^{\mathcal{V}}_v : 1 \to \mathcal{V}(v, v)$, thus for any f in $\mathcal{V}(v, v)$, $I_{(v,w)}(f) =$*
$$\begin{cases} I^{\mathcal{W}}_w & \text{if } f \geq I^{\mathcal{V}}_v, \\ \perp, & \text{the initial element of } \mathcal{W}(w, w), \text{ otherwise.} \end{cases}$$

For any bicategory $\mathcal{V}$, one has an isomorphism in 2-*Cat*

$$\mathcal{V} \cong Conv(1, \mathcal{V}).$$

Two-sided enrichments are also just enrichments since for any $\mathcal{V}$, $\mathcal{W}$

$$Caten(\mathcal{V}, \mathcal{W}) \cong Conv(\mathcal{V}, \mathcal{W})\text{-}Cat.$$

Thus one also gets that for any $\mathcal{V}$, the 2-category $\mathcal{V}$-*Cat* is representable:

$$\mathcal{V}\text{-}Cat \cong Caten(1, \mathcal{V}).$$

3.4. Adjoints in *Caten*. Adjoints $F \dashv G : \mathcal{V} \rightharpoonup \mathcal{W}$ in *Caten* are (up to isomorphism) as follows. $Obj(F) = Obj(G) = Obj(\mathcal{V})$, the span of F is a left adjoint in the bicategory of spans on *Set*

$$\begin{tikzcd} & Obj(\mathcal{V}) \arrow[dl, "1"'] \arrow[dr, "F"] & \\ Obj(\mathcal{V}) & & Obj(\mathcal{W}) \end{tikzcd}$$

the span of G is its right adjoint

$$\begin{tikzcd} & Obj(\mathcal{V}) \arrow[dl, "F"'] \arrow[dr, "1"] & \\ Obj(\mathcal{W}) & & Obj(\mathcal{V}) \end{tikzcd}$$

and the functors $F_{u,v} : \mathcal{V}(u,v) \to \mathcal{W}(Fu,Fv)$ have right adjoints $G_{u,v} : \mathcal{W}(Fu,Fv) \to \mathcal{V}(u,v)$. Actually, a left adjoint F in *Caten* may be seen as a 2-functor $\mathcal{V} \to \mathcal{W}$ with "local (right) adjoints". In this case, the above family of right adjoints satisfies some coherence properties. It can be shown that for any lax functor $F : \mathcal{V} \to \mathcal{W}$ such that for any objects u, v in $\mathcal{V}$, the component $F_{u,v} : \mathcal{V}(u,v) \to \mathcal{W}(Fu,Fv)$ has right adjoint $G_{u,v}$, these adjoints in their totality satisfy the coherence conditions, on composition:

$$\text{(1) for all objects } u,v,t \text{ of } \mathcal{V}, \text{ for all } f : Fu \to Fv \text{ and } g : Fv \to Ft,$$
$$G_{v,t}(g) \circ G_{u,v}(f) \leq G_{u,t}(g \circ f);$$

and on identities:

$$\text{(2) for all objects } u \text{ of } \mathcal{V},\ I_u \leq G_{u,u}(I_{Fu}).$$

if and only if F is a 2-functor.

3.5. The change of base for enrichments. Using the representability for $\mathcal{V}$-categories and the fact that *Caten* is a tricategory, one has the change of base theorem for enrichments over bicategories. The composition on the left by any two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, yields a 2-functor $F \circ - : \mathit{Caten}(1,\mathcal{V}) \to \mathit{Caten}(1,\mathcal{W})$, and by the representability result, it corresponds to a 2-functor $F_@ : \mathcal{V}\text{-}\mathit{Cat} \to \mathcal{W}\text{-}\mathit{Cat}$. In the same way, with any 2-cells $\sigma : F \to G$ of *Caten* yields a 2-natural transformation $\sigma_@ : F_@ \to G_@ : \mathcal{V}\text{-}\mathit{Cat} \to \mathcal{W}\text{-}\mathit{Cat}$. Indeed,

3.6. Theorem. *There is a pseudo-functor* $(-)_@$ *from Caten to* 2-*Cat (the 2-category of 2-categories, 2-functors and 2-natural transformations). It sends a bicategory* $\mathcal{V}$ *to the 2-category* $\mathcal{V}$-*Cat, its actions on two-sided enrichments and 2-cells are the ones described above.*

The above $F_@ : \mathcal{V}\text{-}\mathit{Cat} \to \mathcal{W}\text{-}\mathit{Cat}$ is now described in terms of enrichments. For any $\mathcal{V}$-category A, $F_@A$ is given by:
- $Obj(F_@A) = \{(a,x) \mid a \in Obj(A), x \in Obj(F), a_+ = x_-\}$;
- $(a,x)_+^{F_@A} = x_+$;
- $F_@A((a,x),(b,y)) = F_{x,y}(A(a,b))$.
For any $\mathcal{V}$-functor $f : A \to B$, $F_@(f)$ is the $\mathcal{W}$-functor with underlying map $(a,x) \mapsto (fa,x)$.

Now if F is moreover a 2-functor, then for any $\mathcal{V}$-category A, $F_@A$ is given by:
- $Obj(F_@A) = Obj(A)$;
- $(a)_+^{F_@A} = F(a_+)$;

- $(F_@ A)(a,b) = FA(a,b)$;
and for any $\mathcal{V}$-functor $f : A \to B$, $F_@(f)$ is the $\mathcal{W}$-functor with the same underlying map as f.

From an adjoint pair $F \dashv G$ in *Caten*, one gets the adjoint pair $F_@ \dashv G_@$ in 2-*Cat*. Precisely,

3.7. THEOREM. *If the 2-functor between locally partially ordered bicategories $F : \mathcal{V} \to \mathcal{W}$ admits local adjoints $G_{u,v}$, (u,v) ranging in $Obj(\mathcal{V})^2$ then the 2-functor $F_@$ admits a right 2-adjoint, denoted $F^@$ and defined on objects as follows.*
For any object B of $\mathcal{W}$-Cat, $F^@ B$ is given by:
- $Obj(F^@ B) = \{(b,v) \mid b \in Obj(B), v \in Obj(\mathcal{V}), b_+ = Fv\}$;
- $(b,v)_+ = v$;
- $F^@ B((b,v),(b',v')) = G_{v,v'}(B(b,b'))$.

In the case above $F^@$ is just $G_@$, the collection of local adjoints $G_{u,v}$ defining the right adjoint G to F in *Caten*.

3.8. CHANGE OF BASE, SKELETALITY AND CAUCHY COMPLETION. Further on we consider a two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ where $\mathcal{V}$ and $\mathcal{W}$ are locally cocomplete and biclosed.

3.9. LEMMA. *If F is left adjoint in Caten then for any $\mathcal{W}$-category B, if B is skeletal then also is $F^@ B$.*

PROOF. Suppose that $(b,v),(b',v) \in Obj(F^@ B)$ are such that $F^@ B((b,v),(b',v)) \geq I_v$, i.e. $G_{v,v}B(b,b') \geq I_v$. Since $F_{v,v} \dashv G_{v,v}$, $B(b,b') \geq FI_v = I_{Fv}$. By skeletality of B, $b = b'$. ■

3.10. PROPOSITION. *There is a lax normal[1] functor $F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$ that extends $F_@$ in the sense the diagram below commutes:*

$$\begin{tikzcd}
\mathcal{V}\text{-}Cat \arrow[r, "F_@"] \arrow[d, "J_{\mathcal{V}}"'] & \mathcal{W}\text{-}Cat \arrow[d, "J_{\mathcal{W}}"] \\
\mathcal{V}\text{-}Mod \arrow[r, "F_\sharp"'] & \mathcal{W}\text{-}Mod
\end{tikzcd}$$

For any $\mathcal{V}$-module $\varphi : A \multimap B$, $F_\sharp(\varphi) : F_@ A \multimap F_@ B$ is defined by: for all objects (a,x) of $F_@ A$ and (b,y) of $F_@ B$, $F_\sharp(\varphi)((b,y),(a,x)) = F_{y,x}(\varphi(b,a))$. This is depicted by the diagram in Cat below:

$$\begin{tikzcd}
 & 1 \arrow[dl, "\varphi(b,a)"'] \arrow[dr, "F_\sharp\varphi(b,a)"] & \\
\mathcal{V}(b_+, a_+) \arrow[d, equal] & & \mathcal{W}((b,y)_+, (a,x)_+) \arrow[d, equal] \\
\mathcal{V}(y_-, x_-) \arrow[rr, "F_{y,x}"'] & & \mathcal{W}(y_+, x_+)
\end{tikzcd}$$

[1]A lax functor is *normal* when it strictly preserves identities.

Proof. Let $\varphi : A \multimap B$. For any objects (a,x), (a',x') of $F_{@}A$ and (b,y) of $F_{@}B$,

$$\begin{aligned} F_{@}A((a,x),(a',x')) \circ F_{\sharp}\varphi((b,y),(a,x)) &= F_{x,x'}A(a,a') \circ F_{y,x}\varphi(b,a) \\ &\leq F_{y,x'}(A(a,a') \circ \varphi(b,a)) \\ &\leq F_{y,x'}(\varphi(b,a')) \\ &= F_{\sharp}\varphi((y,b),(a',x')). \end{aligned}$$

Similarly, for any objects (a,x) of $F_{@}A$ and (b,y), (b',y') of $F_{@}B$, $F_{\sharp}\varphi((b,y),(a,x)) \circ F_{@}B((b',y'),(b,y)) \leq F_{\sharp}\varphi((b',y'),(a,x))$.

Now given $\mathcal{V}$-module, $\varphi : A \multimap B$ and $\psi : B \multimap C$, for any objects (a,x) of $F_{@}A$ and (c,z) of $F_{@}C$,

$$\begin{aligned} (F_{\sharp}(\psi \bullet \varphi))((c,z),(a,x)) &= F_{z,x}(\psi \bullet \varphi)(c,a), \\ &= F_{z,x}(\textstyle\bigvee_{b \in Obj(B)} \varphi(b,a) \circ \psi(c,b)), \\ &\geq \textstyle\bigvee_{b \in Obj(B)} F_{z,x}(\varphi(b,a) \circ \psi(c,b)) \\ &\geq \textstyle\bigvee_{(b,y) \in Obj(F_{@}B)} F_{y,x}\varphi(b,a) \circ F_{z,y}\psi(c,b) \\ &= (F_{\sharp}\psi \bullet F_{\sharp}\varphi)((c,z),(a,x)). \end{aligned}$$

By definition $F_{\sharp}(f_{\diamond}) = (F_{@}f)_{\diamond}$ for any $\mathcal{V}$-functor f, thus for any $\mathcal{V}$-category A, $F_{\sharp}(1_{A\diamond}) = (F_{@}1_A)_{\diamond} = (1_{F_{@}A})_{\diamond}$. ∎

3.11. Lemma. *Let $f : A \to B$ be a $\mathcal{V}$-functor. Then:*
(i) $F_{\sharp}(f_{\diamond}) \dashv F_{\sharp}(f^{\diamond})$ in $\mathcal{W}$-Mod;
(ii) If $f_{\diamond}$ is an isomorphism in $\mathcal{V}$-Mod then $F_{\sharp}(f^{\diamond}) \bullet F_{\sharp}(f_{\diamond}) = 1$.

Proof. (i): Results from the facts that for any $\mathcal{V}$-functor f, $F_{\sharp}(f_{\diamond}) = (F_{@}(f))_{\diamond}$ and also $F_{\sharp}(f^{\diamond}) = (F_{@}(f))^{\diamond}$.
(ii): Suppose $f_{\diamond} \bullet f^{\diamond} = 1$ and $f^{\diamond} \bullet f_{\diamond} = 1$. Composing the second equality by $F_{\sharp}$ which is lax and normal, one gets $F_{\sharp}(f^{\diamond}) \bullet F_{\sharp}(f_{\diamond}) \leq F_{\sharp}(f^{\diamond} \bullet f_{\diamond}) = F_{\sharp}(1) \leq 1$. Thus $F_{\sharp}(f^{\diamond}) \bullet F_{\sharp}(f_{\diamond}) = 1$ since $F_{\sharp}(f_{\diamond}) \dashv F_{\sharp}(f^{\diamond})$. ∎

These later facts motivated the somewhat technical definition below, as we were looking for sufficient conditions on a two-sided enrichment F that ensures that $F_{@}$ preserves the Morita equivalence.

3.12. Definition. [Super two-sided enrichments] *A two-sided enrichment $G : \mathcal{V} \to \mathcal{W}$ is* super *when for any object x of G, and any family of pairs of arrows*

$$x_- \xrightarrow{f_i} u_i \xrightarrow{g_i} x_-$$

in $\mathcal{V}$, i ranging in $\mathcal{I}$, if $I_{x_-} \leq \bigvee_{i \in \mathcal{I}} g_i \circ f_i$ then

$$I_{x_+} \leq \bigvee_{\{y \in Obj(G) \mid \exists i \in \mathcal{I}, u_i = y_-\}} G_{y,x}(g_i) \circ G_{x,y}(f_i).$$

It is straightforward to check that the composition of super two-sided enrichments is super. Trivially identities are super.

3.13. Lemma. *If F is super then $F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$ preserves adjoints, i.e. for any $\varphi \dashv \tilde{\varphi}$ in $\mathcal{V}$-Mod, one has the ajoint pair $F_\sharp(\varphi) \dashv F_\sharp(\tilde{\varphi})$ in $\mathcal{W}$-Mod.*

Proof. Suppose that F is super. Consider a left adjoint $\varphi : A \mathrel{-\!\circ\!\!\to} B$ in $\mathcal{V}$-*Mod*. First let us show that $1 \le F_\sharp(\tilde{\varphi}) \bullet F_\sharp(\varphi)$. Note that for any object (a, x) of $F_@A$, since $I^{\mathcal{V}}_{a+} \le \bigvee_{b \in Obj(B)}(\varphi(b, a) \circ \tilde{\varphi}(a, b))$ one has $I^{\mathcal{W}}_{x+} \le \bigvee_{(b,z) \in Obj(F_@B)} F_{z,x}\varphi(b, a) \circ F_{x,z}\tilde{\varphi}(a, b)$. Thus for any objects $(a, x), (a', y)$ of $F_@A$.

$$
\begin{aligned}
F_@A((a,x),(a',y)) &= F_{x,y}A(a,a') \\
&\le F_{x,y}A(a,a') \circ (\textstyle\bigvee_{(b,z)\in Obj(F_@B)} F_{z,x}\varphi(b,a) \circ F_{x,z}\tilde{\varphi}(a,b)) \\
&= \textstyle\bigvee_{(b,z)\in Obj(F_@B)}((F_{x,y}A(a,a') \circ F_{z,x}\varphi(b,a)) \circ F_{x,z}\tilde{\varphi}(a,b)) \\
&\le \textstyle\bigvee_{(b,z)\in Obj(F_@B)}(F_{z,y}\varphi(b,a')) \circ F_{x,z}\tilde{\varphi}(a,b)) \\
&= (F_\sharp\tilde{\varphi} \bullet F_\sharp\varphi)((a,x),(a',y)).
\end{aligned}
$$

Now since $F_\sharp$ is lax and normal and $\varphi \bullet \tilde{\varphi} \le 1$, $F_\sharp(\varphi) \bullet F_\sharp(\tilde{\varphi}) \le F_\sharp(\varphi \bullet \tilde{\varphi}) \le F_\sharp(1) = 1$. ■

In particular the above $F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$ will preserve inverse pairs of modules, thus isomorphisms.

3.14. Corollary. *If F is super then $F_\sharp$ preserves the Morita equivalence, i.e. if the $\mathcal{V}$-categories A and B are Morita equivalent then also are $F_@A$ and $F_@B$.*

Further on, we suppose that F has right adjoint G in *Caten*, i.e. $F : \mathcal{V} \to \mathcal{W}$ is a 2-functor with local adjoints $G_{a,b}$, (a, b) ranging in $Obj(\mathcal{V})^2$.

3.15. Proposition. *$F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$ is a 2-functor.*

Proof. We already know that $F_\sharp$ is lax and normal. It remains to show that $F_\sharp$ preserves composition. Let $\varphi : A \mathrel{-\!\circ\!\!\to} B$ and $\psi : B \mathrel{-\!\circ\!\!\to} C$ be $\mathcal{V}$-modules. Since F has local right adjoints, it "preserves local least upper bounds", therefore for any objects a of A and c of C,

$$
\begin{aligned}
F_\sharp(\psi \bullet \varphi)(c,a) &= F(\textstyle\bigvee_{b\in Obj(B)} \varphi(b,a) \circ \psi(c,b)), \\
&= \textstyle\bigvee_{b\in Obj(B)} F(\varphi(b,a) \circ \psi(c,b)) \\
&= \textstyle\bigvee_{b\in Obj(B)} F\varphi(b,a) \circ F\psi(c,b) \\
&= (F_\sharp\psi \bullet F_\sharp\varphi)(c,a).
\end{aligned}
$$

■

We shall finish with technical lemmas. Their relevance will appear in the next section treating reversibility.

3.16. Lemma. *Let η be the unit of the adjunction $F_@ \dashv F^@$. For any left adjoint module $\varphi : A \mathrel{-\!\circ\!\!\to} B$ there is a 2-cell in $\mathcal{V}$-Mod:*

$$
\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\eta_{A\diamond}"'] & B \arrow[d, "\eta_{B\diamond}"] \arrow[dl, Rightarrow] \\
F^@F_@A \arrow[r, "G_\sharp F_\sharp\varphi"'] & F^@F_@B
\end{tikzcd}
$$

PROOF. Let $\varphi : A \longrightarrow\!\!\!\!\!\!\circ\;\; B$ be a right adjoint in $\mathcal{V}$-*Mod*. Consider some objects a of A and $(v,b) \in F^{@}F_{@}B$. Then

$$\begin{aligned}(\eta_{B_\diamond} \bullet \varphi)((v,b),a) &= \textstyle\bigvee_{b' \in Obj(B)} \varphi(b',a) \circ G_{v,b'_+} FB(b,b') \\ &\leq \textstyle\bigvee_{b' \in Obj(B)} G_{b'_+,a_+} F\varphi(b',a) \circ G_{v,b'_+} FB(b,b') \\ &\leq G_{v,a_+} F\varphi(b,a);\end{aligned}$$

On the other hand,

$$\begin{aligned}(G_\sharp F_\sharp \varphi \bullet \eta_{A_\diamond})((v,b),a) &= \textstyle\bigvee_{(a',v') \in Obj(F^{@}F_{@}A)} G_{v',a_+} FA(a',a) \circ G_{v,v'} F\varphi(b,a') \\ &\leq \textstyle\bigvee_{(a',v') \in Obj(F^{@}F_{@}A)} G_{v,a_+} F(A(a',a) \circ \varphi(b,a')) \\ &\leq G_{v,a_+} F\varphi(b,a)\end{aligned}$$

And since $(a, a_+) \in Obj(F^{@}F_{@}A)$, one has also

$$\begin{aligned}G_{v,a_+} F\varphi(b,a) &\leq G_{a_+,a_+} FI_{a_+} \circ G_{v,a_+} F\varphi(b,a) \\ &\leq G_{a_+,a_+} FA(a,a) \circ G_{v,a_+} F\varphi(b,a) \\ &\leq (G_\sharp F_\sharp \varphi \bullet \eta_{A_\diamond})((v,b),a)\end{aligned}$$

■

3.17. LEMMA. *Let A be a Cauchy complete $\mathcal{W}$-category. For any object u of $\mathcal{V}$ and any left adjoint $\mathcal{V}$-module $\varphi : \hat{u} \longrightarrow\!\!\!\!\!\!\circ\;\; F^{@}A$, there is a $\mathcal{V}$-functor $f : \hat{u} \to F^{@}A$ with $f_\diamond \geq \varphi$.*

PROOF. Let u be an object of $\mathcal{V}$ and $\varphi : \hat{u} \longrightarrow\!\!\!\!\!\!\circ\;\; F^{@}A$ be a left adjoint $\mathcal{V}$-module. According to Fact 3.15, $F_\sharp\varphi : F_{@}\hat{u} \longrightarrow\!\!\!\!\!\!\circ\;\; F_{@}F^{@}A$ is a left adjoint $\mathcal{W}$-module, therefore also is $\epsilon_{A_\diamond} \bullet F_\sharp\varphi : F_{@}\hat{u} \longrightarrow\!\!\!\!\!\!\circ\;\; A$, ϵ denoting the co-unit of the adjunction $F_{@} \dashv F^{@}$ (see diagram below).

$$\begin{tikzcd} F^{@}A \\ \hat{u} \arrow[u, "\varphi"'] \end{tikzcd} \qquad \begin{tikzcd} F_{@}F^{@}A \arrow[r, "\epsilon_{A_\diamond}"] & A \\ F_{@}\hat{u} \arrow[u, "F_\sharp\varphi"] \arrow[ur, dotted, "g_\diamond"'] & \end{tikzcd}$$

Since A is Cauchy complete there is some $\mathcal{V}$-functor $g : F_{@}\hat{u} \to A$ such that $g_\diamond = \epsilon_{A_\diamond} \bullet F_\sharp\varphi$. η denoting the unit of the adjunction $F_{@} \dashv F^{@}$, let us see that the $\mathcal{W}$-functor $f = F^{@}g \circ \eta_{\hat{u}} : \hat{u} \to F^{@}A$ — see diagram below — satisfies $f_\diamond \geq \varphi$.

$$\begin{tikzcd} \hat{u} \arrow[r, "\eta_{\hat{u}}"] \arrow[dr, "f"'] & F^{@}F_{@}\hat{u} \arrow[d, "F^{@}g"] \\ & F^{@}A \end{tikzcd} \qquad \begin{tikzcd} F_{@}\hat{u} \arrow[d, "g"] \\ A \end{tikzcd}$$

The required 2-cell results from the pasting of diagrams in $\mathcal{V}$-*Mod*:

$$\begin{tikzcd}
\hat{u} \arrow[r, "\varphi"] \arrow[d, "\eta_{\hat{u}\diamond}"'] \arrow[dr, phantom, "(I)"] & F^{@}A \arrow[d, "(\eta_{F^{@}A})_{\diamond}"] \\
F^{@}F_{@}\hat{u} \arrow[r, "G_{\sharp}F_{\sharp}\varphi"'] \arrow[dr, "(F^{@}g)_{\diamond}"'] \arrow[dr, phantom, shift left=4, "(II)"] & F^{@}F_{@}F^{@}A \arrow[d, "(F^{@}\epsilon_A)_{\diamond}"] \\
 & F^{@}A
\end{tikzcd}$$

2-cell (I) is given by 3.16, and 2-cell (II) is given by the laxity of $G_{\sharp}$. ∎

4. Reversibility

In this section we will restrict the theory of two-sided enrichments over locally partially ordered bicategories to the symmetric case. We extend the notion of reversibility of [Wal82] to two-sided enrichments, modules, and completion. Definitions will allow to rebuild the complete theory in this special case. They will be also illustrated by two examples: sheaves over locales and action groups. Eventually we study the effect of the change of bases on the completeness and skeletality of enrichments in the reversible context. The main result is theorem 4.34.

4.1. Reversible enrichments.

4.2. Definition. *A 2-category $\mathcal{V}$ is* reversible *when there are isomorphisms of categories $s_{u,v} : \mathcal{V}(u,v) \cong \mathcal{V}(v,u)$ for all objects u and v of $\mathcal{V}$, and these isomorphisms satisfy the conditions:*
- $rev-(i) : s_{u,v} \circ s_{v,u} = 1$, for any objects u and v;
- $rev-(ii)$: for any objects u,v,w the diagram (in Cat) below commutes:

$$\begin{tikzcd}
\mathcal{V}(v,w) \times \mathcal{V}(u,v) \arrow[r, "\circ_{u,v,w}"] \arrow[d, "\cong"'] & \mathcal{V}(u,w) \arrow[dd, "s_{u,w}"] \\
\mathcal{V}(u,v) \times \mathcal{V}(v,w) \arrow[d, "s_{u,v} \times s_{v,w}"'] & \\
\mathcal{V}(v,u) \times \mathcal{V}(w,v) \arrow[r, "\circ_{w,v,u}"'] & \mathcal{V}(w,u)
\end{tikzcd}$$

where $\circ$ is the horizontal composition in $\mathcal{V}$;
- $rev-(iii)$: for any object v the diagram below commutes

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "I_v"] \arrow[d, equal] & \mathcal{V}(v,v) \arrow[d, "s_{v,v}"] \\
\mathbf{1} \arrow[r, "I_v"'] & \mathcal{V}(v,v)
\end{tikzcd}$$

Further on $\mathcal{V}$ is a reversible locally partially ordered biclosed and cocomplete bicategory.

4.3. Definition. [reversible enrichments] *A $\mathcal{V}$-category A is* reversible *when for any objects a, b of A, $A(b,a) = s_{a_+,b_+}(A(a,b))$.*

Let A and B be $\mathcal{V}$-categories. If B is reversible then the preorder $\mathcal{V}\text{-}Cat(A,B)$, denoted here $\Rightarrow$, is an equivalence since for any $\mathcal{V}$-functors $f, g : A \to B$,

$f \Rightarrow g$ if and only if for all object a of A, $I_{a_+} \leq B(fa, ga)$,
- , $s(I_{a_+}) \leq s(B(fa, ga))$
- , $I_{a_+} \leq B(ga, fa)$
if and only if $g \Rightarrow f$.

4.4. Observation. *If B above is moreover skeletal then $\Rightarrow$ is the identity on $\mathcal{V}\text{-}Cat(A,B)$.*

4.5. Proposition. *Let A and B be reversible $\mathcal{V}$-categories.*
(1) *If $\varphi : A \multimap B$ is a $\mathcal{V}$-module then the map defined on $Obj(A) \times Obj(B)$ sending (a,b) to the arrow $s_{b_+,a_+}(\varphi(b,a))$ defines a $\mathcal{V}$-module $\varphi^s : B \multimap A$.*
(2) *If φ is left adjoint then φ^s is right adjoint with $(\tilde{\varphi})^s \dashv \varphi^s$.*

Proof. Assertion (1) is straightforward. Let us see (2) : Suppose that φ is left adjoint. From $1 \leq \tilde{\varphi} \bullet \varphi$, i.e. for any objects a, a' of A,

$$A(a,a') \leq \bigvee_{b \in Obj(B)} \varphi(b,a') \circ \tilde{\varphi}(a,b)$$

one gets by applying $s_{(a_+,a'_+)}$ on both sides of the inequality that for any a, a',

$$A(a',a) \leq \bigvee_{b \in Obj(A)} (\tilde{\varphi})^s(b,a) \circ \varphi^s(a',b),$$

i.e. $1 \leq \varphi^s \bullet (\tilde{\varphi})^s$. Similarly one gets from $(\tilde{\varphi})^s \bullet \varphi^s \leq 1$ that $\varphi \bullet \tilde{\varphi} \leq 1$. ∎

There are reversible enrichments with non-reversible Cauchy completions. A very simple example (due to S. Kasangian) may be found in [BeWal82]. This motivates the notion of Cauchy-reversible completeness defined further.

4.6. Definition. [Reversible modules] *A $\mathcal{V}$-module φ between reversible $\mathcal{V}$-categories is* reversible *when it has right adjoint φ^s.*

4.7. Remark. If f is a $\mathcal{V}$-functor between reversible $\mathcal{V}$-categories then the $\mathcal{V}$-module $f_\diamond$ is reversible.

4.8. Definition. [Cauchy-reversible completeness] *A reversible $\mathcal{V}$-category B is* Cauchy-reversible complete *when one of the following two equivalent assertions holds:*
(i) For any reversible $\mathcal{V}$-module $\varphi : A \multimap B$, $\varphi = f_\diamond$ for some $\mathcal{V}$-functor $f : A \to B$;
(ii) For any object u of $\mathcal{V}$ and any reversible $\mathcal{V}$-module $\varphi : \hat{u} \multimap B$, $\varphi = b_\diamond$ for some object b of B with $b_+ = u$.

Let us prove the equivalence of assertions (i) and (ii) of the previous definition.
$(i) \Rightarrow (ii)$: immediate.
$(ii) \Rightarrow (i)$: Note that for some reversible module $\varphi : A \multimap B$, for any object a of A, $\varphi \bullet a_\diamond \dashv a^\diamond \bullet \varphi^s = (\varphi \bullet a_\diamond)^s$. Then one concludes analogously as for 2.8.

Clearly, any reversible Cauchy complete $\mathcal{V}$-category is also a Cauchy-reversible complete one.

4.9. Definition. [Cauchy-reversible completion] *Let A be a reversible $\mathcal{V}$-category with Cauchy completion $\bar{A}$. A^s, the* Cauchy-reversible completion *of A is the reversible $\mathcal{V}$-category where $Obj(A^s) \subseteq Obj(\bar{A})$, is the set of reversible $\mathcal{V}$-modules of the form $\varphi : \hat{u} \multimap A$, u ranging in $Obj(\mathcal{V})$. The maps $A^s(-,-)$ and $(-)_+$ of A^s are the restrictions of the maps $\bar{A}(-,-)$ and $(-)_+$ of $\bar{A}$.*

From 2.10, it is immediate that the above data defines a $\mathcal{V}$-category A^s. A^s is moreover reversible as shown below. Let $\varphi : \hat{u} \multimap A$, $\psi : \hat{v} \multimap A$ be reversible adjoints, then

$$\begin{aligned}\bar{A}(\varphi,\psi) &= \textstyle\bigvee_{a\in Obj(A)} \psi(a)\circ\varphi^s(a)\\ &= \textstyle\bigvee_{a\in Obj(A)} s_{v,u}(\varphi(a)\circ\psi^s(a))\\ &= s_{v,u}(\textstyle\bigvee_{a\in Obj(A)} \varphi(a)\circ\psi^s(a))\\ &= s_{v,u}(\bar{A}(\psi,\varphi)).\end{aligned}$$

Let A, $\bar{A}$ and, A^s be as in 4.8. The map $i_A : Obj(A) \to Obj(\bar{A})$ takes values in $Obj(A^s)$. Thus it defines a $\mathcal{V}$-functor from A to A^s further on denoted j_A.

An immediate consequence of 2.12 is

4.10. Proposition. *For any reversible $\mathcal{V}$-category A, A^s is skeletal.*

Adapting the proof of 2.13 one shows

4.11. Lemma. *Let A be a reversible $\mathcal{V}$-category and, $\Gamma : \hat{u} \multimap A^s$ be some $\mathcal{V}$-module where u is an object of $\mathcal{V}$. For any object φ of A^s, $\Gamma(\varphi) = \bigvee_{a\in Obj(A)}\Gamma(a_\diamond)\circ\varphi^s(a)$.*

4.12. Proposition. *For any reversible $\mathcal{V}$-category A, A^s is Cauchy-reversible complete.*

Proof. Merely an adaptation of the proof of 2.14. Given $\Gamma : \hat{u} \multimap A^s$ a reversible $\mathcal{V}$-module. One checks that the maps $\gamma : Obj(A) \to Arrows(\mathcal{V})$ given by for any $a \in Obj(A)$,

$$\gamma(a) = \Gamma(a_\diamond)$$

defines a reversible module from $\hat{u}$ to A with adjoint γ' given by for any object a of A,

$$\gamma'(a) = \Gamma^s(a_\diamond).$$

Then according to 4.11, $\Gamma = A^s(-,\gamma)$. ∎

4.13. Lemma. *For any reversible $\mathcal{V}$-category A, $j_{A_\diamond} : A \multimap A^s$ and $j_A{}^\diamond : A^s \multimap A$ are inverse reversible modules.*

Proof. The proof that $(j_A)_\diamond$ and $(j_A)^\diamond$ are inverse modules is the one for 2.15. Since A^s is reversible, $j_A{}^\diamond = (j_{A_\diamond})^s$. ∎

We define the following 2-categories.

- $\mathcal{V}$-*RCat* with:
  - objects: reversible $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V}$-*Cat*.

- $\mathcal{V}$-*SkCRcCat* with:
  - objects: skeletal and Cauchy-reversible complete $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V}$-*Cat*.

- $\mathcal{V}$-*RMod* with:
  - objects: reversible $\mathcal{V}$-categories,
  - arrows: reversible $\mathcal{V}$-modules,
  - 2-cells: inherited from $\mathcal{V}$-*Mod*.

According to 4.13,

4.14. Proposition. *$\mathcal{V}$-SkCRcCat and $\mathcal{V}$-RMod are 2-equivalent. The 2-equivalence S' : $\mathcal{V}$-RMod $\to$ $\mathcal{V}$-SkCRcCat is defined on objects by $S'(A) = A^s$ and on arrows by $S'(\varphi) = f_\varphi$ where for any reversible $\mathcal{V}$-module $\varphi : A \multimap B$, $f_\varphi : A^s \to B^s$ is the unique $\mathcal{V}$-functor f satisfying $(f_\varphi)_\diamond \bullet j_A{}^\diamond = j_{B_\diamond} \bullet \varphi$. The inclusion 2-functor $\mathcal{V}$-SkCRcCat $\to$ $\mathcal{V}$-RCat has a left 2-adjoint.*

According to 4.4, $\mathcal{V}$-*SkCRcCat* and $\mathcal{V}$-*RMod* are locally discrete, i.e. the only 2-cells are identies. Therefore we shall sometimes consider them just as categories. Skeletal and Cauchy-reversible complete $\mathcal{V}$-categories form a reflective subcategory of the underlying category of $\mathcal{V}$-*RCat*. The left adjoint of the inclusion sends a reversible $\mathcal{V}$-category A to A^s and, the unit takes value $j_A : A \to A^s$ in A.

We may define,

4.15. Definition. [Morita reversible equivalence] *Two reversible $\mathcal{V}$-categories A and B are* Morita reversible equivalent *when one of the following equivalent assertions is satisfied:*
(i) *Their Cauchy-reversible completions are isomorphic in $\mathcal{V}$-Cat;*
(ii) *They are isomorphic in $\mathcal{V}$-RMod.*

Note that if two reversible $\mathcal{V}$-categories are Morita reversible equivalent then they are Morita equivalent.

4.16. Examples. Further on we give two examples of reversible enrichments on locally partially ordered bicategories. Note that for the second example, the reversibility isomorphisms are not identities.

4.17. Example. [Sheaves][Wal81]

To any locale L corresponds a locally partially ordered bicategory $\mathcal{C}_L$ given by:
- objects are the $u \in L$;
- arrows from u to v are the $w \in L$ with $w \leq u \wedge v$;
- the partial order on $\mathcal{C}_L(u,v)$ is that of L;
- the composition of arrows is the intersection;
- the unit in u is u.

Such a $\mathcal{C}_L$ is locally cocomplete and biclosed. It is also reversible where the isomorphisms $s_{u,v}$ are the identity maps $\mathcal{C}_L(u,v) \to \mathcal{C}_L(v,u)$.

If L is a locale, $Sh(L)$ will denote the category of sheaves on L. In [Wal81], the following result is proved.

4.18. Theorem. *Given a locale L, $Sh(L)$ is equivalent to the category of reversible skeletal Cauchy complete $\mathcal{C}_L$-enrichments.*

The equivalence J above is defined as follows. For any sheaf F on L, $J(F)$ is the enrichment over $\mathcal{C}_L$ where:
- $Obj(J(F))$ is the set of partial sections of F;
- For any section s, s_+ is its domain $dom(s)$;
- For any sections s, s', $J(F)(s,s') = \bigvee\{u \leq dom(s) \wedge dom(s') \mid s\lfloor u = s'\lfloor u\}$ — $s\lfloor u$ denotes the restriction of s to $u \leq dom(s)$. If $h : F \to G$ is a morphism of $Sh(L)$, $J(h)$ is the map that sends any partial section $s \in Fu$ to $h(s) \in Gu$.

In this case it happens that Cauchy-reversible complete enrichments are Cauchy complete. This is due to

4.19. Proposition. *For any locale L, any left adjoint $\mathcal{C}_L$-module is reversible.*

Proof. Let $u \in L$ and $\varphi : \hat{u} \multimap A$ be some left adjoint module with $\varphi \dashv \tilde{\varphi}$. Then the map ψ given by: $\psi(s) = \varphi(s) \wedge \tilde{\varphi}(s)$ defines a module $\psi : \hat{u} \multimap A$ with itself as adjoint. Since $\psi \leq \varphi$ and $\psi \leq \tilde{\varphi}$, $\varphi = \psi = \tilde{\varphi}$. ∎

So we may reformulate 4.18

4.20. Theorem. *Given a locale L, $Sh(L)$ is equivalent to the category of skeletal Cauchy-reversible complete $\mathcal{C}_L$-enrichments.*

4.21. Example. [Group actions]

Let α be an action of the group G on the set X. We note for such an action:
- $\langle\sigma, x\rangle_\alpha$ (or simply $\langle\sigma, x\rangle$), the image of $x \in X$ by $\alpha(\sigma)$ for $\sigma \in G$,
- "$\cdot$", the composition law of G,
- u, the neutral element of G.

Such an action will be denoted (G, X, α) or simply (X, α).

Given a group G, G-*Set* denotes the category with objects actions of G on sets and with arrows action preserving maps i.e. $f : (Z, \beta) \to (X, \alpha)$ is an arrow of G-*Set* when the underlying map $f : Z \to X$ makes the following diagram commute:

$$\begin{array}{ccc} G \times Z & \xrightarrow{\beta} & Z \\ {\scriptstyle 1\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times X & \xrightarrow[\alpha]{} & X \end{array}$$

If α is an action of G on X, the locally partially ordered bicategory $\mathcal{C}_\alpha$ is defined as follows:
- Objects are the $x \in X$;
- Arrows from x to x' are the subsets of $\{\sigma \in G \mid \langle \sigma, x\rangle = x'\}$;
- the partial order on $\mathcal{C}_\alpha(x, x')$ is the inclusion;
- the composition $L' \circ L$ of $L : x \to x'$ and $L' : x' \to x''$ is $\{\tau \cdot \sigma \mid \sigma \in L \ \textit{and} \ \tau \in L'\}$;
- the unit at x is $\{u\}$.

Clearly $\mathcal{C}_\alpha$ is locally cocomplete and biclosed. $\mathcal{C}_\alpha$ is reversible with isomorphisms s: $s_{x,x'}(L) = \{\sigma^{-1} \mid \sigma \in L\}$, for any $L : x \to x'$.

4.22. Lemma. *Let α be an action of a group G on a set X and A be a reversible enrichment on $\mathcal{C}_\alpha$.*

(1) *For any $a_0 \in Obj(A)$ and any $\sigma_0 \in G$, the map φ given by*

$$(*) \quad \forall a \in Obj(A), \varphi(a) = \{\sigma_0\} \circ A(a, a_0)$$

defines a left adjoint module $\varphi : \hat{x} \multimap A$ where $x = \langle \sigma_0, (a_0)_+\rangle$, with right adjoint $\tilde{\varphi}$ given by

$$\forall a \in Obj(A), \tilde{\varphi}(a) = A(a_0, a) \circ \{{\sigma_0}^{-1}\}.$$

Conversely for any $x \in X$ and any left adjoint module $\varphi : \hat{x} \multimap A$ there is some $a_0 \in Obj(A)$ and some $\sigma_0 \in G$ such that $x = \langle \sigma_0, (a_0)_+\rangle$ and the map φ is given by formulae $(*)$ above.

(2) *If $\varphi, \psi : \hat{x} \multimap A$ are left adjoint modules given by:*

$$\forall a \in Obj(A), \varphi(a) = \{\sigma_0\} \circ A(a, a_0)$$

and

$$\forall a \in Obj(A), \psi(a) = \{\tau_0\} \circ A(a, {a'}_0)$$

where $\sigma_0, \tau_0 \in G$ and $a_0, a'_0 \in Obj(A)$, then $\varphi = \psi$ if and only if $(\sigma_0, a_0) \sim (\tau_0, {a'}_0)$, $\sim$ being the equivalence relation on $G \times X$:

$$\forall \sigma, \sigma' \in G, \forall a, a' \in Obj(A), \ \ (\sigma, a) \sim (\sigma', a') \Leftrightarrow (\sigma')^{-1} \cdot \sigma \in A(a, a').$$

Proof. (1) : Let $a_0 \in Obj(A)$, $\sigma_0 \in G$, and φ be the map on $Obj(A)$ defined by $\varphi(a) = \{\sigma_0\} \circ A(a, a_0)$. For short, let $x = \langle \sigma_0, (a_0)_+\rangle$. Actually φ defines a module $\hat{x} \multimap A$. φ satisfies $(mod-i)$ since for any $a \in Obj(A)$, $\varphi(a)$ is an object of $\mathcal{C}_\alpha(a_+, x)$. φ satisfies $(mod-ii)$ since for any $a \in Obj(A)$, $\{u\} \circ \varphi(a) \subseteq \varphi(a)$. φ satisfies $(mod-iii)$, since for any $a, a' \in Obj(A)$,

$$\begin{aligned}\varphi(a)\circ A(a',a) &= \{\sigma_0\}\circ A(a,a_0)\circ A(a',a)\\ &\subseteq \{\sigma_0\}\circ A(a',a_0)\\ &= \varphi(a').\end{aligned}$$

Analogously one shows that the map sending any $a \in Obj(A)$ to $A(a_0,a)\circ\{{\sigma_0}^{-1}\}$ defines a module $\tilde{\varphi} : A \multimap \hat{x}$. $\tilde{\varphi}$ is right adjoint to φ as shown below.
Since $u \in \bigcup_{a\in Obj(A)} A(a,a_0)\circ A(a_0,a)$,

$$\begin{aligned}\{u\} &\subseteq \{a_0,a_0\}\\ &\subseteq \textstyle\bigcup_{a\in Obj(A)}\{\sigma_0\}\circ A(a,a_0)\circ A(a_0,a)\circ\{{\sigma_0}^{-1}\}\\ &= \textstyle\bigcup_{a\in Obj(A)}\varphi(a)\circ\tilde{\varphi}(a).\end{aligned}$$

For any $a,a' \in Obj(A)$,

$$\begin{aligned}\tilde{\varphi}(a)\circ\varphi(a') &= A(a_0,a)\circ\{{\sigma_0}^{-1}\}\circ\{\sigma_0\}\circ A(a',a_0)\\ &\subseteq A(a',a).\end{aligned}$$

Now consider some $x \in X$, and some left adjoint module $\varphi : \hat{x} \multimap A$ with right adjoint $\tilde{\varphi}$. Then $\{u\} \subseteq \bigcup_{a\in Obj(A)}\varphi(a)\circ\tilde{\varphi}(a)$. This implies that there are some $a_0 \in Obj(A)$ and some $\sigma_0 \in \varphi(a_0)$ with ${\sigma_0}^{-1} \in \tilde{\varphi}(a_0)$. For this a_0, $\langle\sigma_0,(a_0)_+\rangle = x$. Let $\psi : \hat{x} \multimap A$ be the left adjoint module defined by $\forall a \in Obj(A), \psi(a) = \{\sigma_0\}\circ A(a,a_0)$. Then for any $a \in Obj(A)$,

$$\begin{aligned}\psi(a) &= \{\sigma_0\}\circ A(a,a_0)\\ &\subseteq \varphi(a_0)\circ A(a,a_0)\\ &\subseteq \varphi(a).\end{aligned}$$

Thus $\psi \leq \varphi$. Analogously for any $a \in Obj(A)$, $A(a_0,a)\circ\{{\sigma_0}^{-1}\} \subseteq \tilde{\varphi}(a)$ i.e. $\tilde{\psi} \leq \tilde{\varphi}$. Therefore $\psi = \varphi$.

(2) : First let us check that $\sim$ is an equivalence relation. It is reflexive since for any object a of A, $u^{-1}\cdot u = u \in A(a,a)$. It is symmetric. If $(\sigma,a) \sim (\sigma',a')$ then ${\sigma'}^{-1}.\sigma \in A(a,a')$, thus $\sigma^{-1}.\sigma' \in A(a',a)$ since A is reversible. It is transitive. Suppose $(\sigma,a) \sim (\sigma',a')$ and $(\sigma',a') \sim (\sigma'',a'')$. Then $(\sigma')^{-1}\cdot\sigma \in A(a,a')$ and $(\sigma'')^{-1}\cdot\sigma' \in A(a',a'')$. Thus ${\sigma''}^{-1}\cdot\sigma \in A(a',a'')\circ A(a,a') \subseteq A(a,a'')$.

Consider further on $\varphi,\psi : \hat{x} \multimap A$, $\sigma_0,\sigma'_0 \in G$ and, $a_0,a'_0 \in Obj(A)$ such that $\forall a \in Obj(A)$, $\varphi(a) = \{\sigma_0\}\circ A(a,a_0)$ and $\psi(a) = \{\sigma'_0\}\circ A(a,a'_0)$. First, suppose $(\sigma_0,a_0) \sim (\sigma'_0,a'_0)$ i.e., ${\sigma'_0}^{-1}\cdot\sigma_0 \in A(a_0,a'_0)$. Then for any $a \in Obj(A)$,

$$\begin{aligned}\varphi(a) &= \{\sigma_0\}\circ A(a,a_0)\\ &= \{\sigma'_0\}\circ\{{\sigma'_0}^{-1}\cdot\sigma_0\}\circ A(a,a_0)\\ &\subseteq \{\sigma'_0\}\circ A(a_0,a'_0)\circ A(a,a_0)\\ &\subseteq \{\sigma'_0\}\circ A(a,a'_0)\\ &= \psi(a),\end{aligned}$$

then $\varphi \leq \psi$. Since $\sigma'_0 = \sigma_0\cdot\gamma^{-1}$ and then $\gamma^{-1} \in A(a'_0,a_0)$, one obtains analogously $\psi \leq \varphi$.

Now suppose $\varphi = \psi$. Then for any $a \in Obj(A)$, $\{\sigma_0\}\circ A(a,a_0) = \{\sigma'_0\}\circ A(a,a'_0)$. For $a = a_0$, we get $\{\sigma_0\} \subseteq \{\sigma'_0\}\circ A(a_0,a'_0)$, i.e. there is $\gamma \in A(a_0,a'_0)$, such that $\sigma_0 = \sigma'_0\cdot\gamma$. ∎

4.23. Corollary. *Let α be an action of the group G on a set X and A be a reversible enrichment on $\mathcal{C}_\alpha$. Any left adjoint module $\varphi : \hat{x} \longrightarrow\!\!\!\!\circ\!\!\!\to A$ has right adjoint φ^s.*

4.24. Proposition. *Let (X, α) be an action of the group G. The category of skeletal and Cauchy-reversible complete enrichments on $\mathcal{C}_\alpha$ is equivalent to the category G-Set $\downarrow \alpha$ of objects over α. Precisely the equivalence $J :$ G-Set $\downarrow \alpha \cong \mathcal{C}_\alpha$-SkCRcCat is given by:*
- For any $f : (Z, \beta) \to (X, \alpha)$ in G-Set, $J(f)$ is the enrichment with set of objects Z, and such that for any $z \in Z$, $z_+ = f(z)$, and for any $z, z' \in Z$, $J(f)(z, z')$ is the set $\{\sigma \in G \mid \langle\sigma, z\rangle_\beta = z'\}$;
- For any arrow $\langle f_1, h, f_2\rangle$ of G-Set $\downarrow \alpha$ with $f_1 : (Z_1, \beta_1) \to (X, \alpha)$ and $f_2 : (Z_2, \beta_2) \to (X, \alpha)$, $J(h)$ is the $\mathcal{C}_\alpha$-functor from $J(f_1)$ to $J(f_2)$ defined by the map $h : Z_1 \to Z_2$.
J has adjoint J' defined on objects as follows. For any skeletal Cauchy-reversible complete enrichment A on $\mathcal{C}_\alpha$, $J'(A)$ is the action $(Obj(A), \beta)$ where for any $\sigma \in G$ and any $a \in Obj(A)$, $\langle\sigma, a\rangle_\beta$ is the unique a' such that $\sigma \in A(a, a')$.

Proof. Let us check that given $f : (Z, \beta) \to (X, \alpha)$, $J(f)$ is a skeletal Cauchy-reversible complete $\mathcal{C}_\alpha$-enrichment.
- It satisfies $(enr - i)$: Since f is a morphism of *G-Set*, for any $z \in Z$ and any $\sigma \in G$, $\langle\sigma, f(z)\rangle_\alpha = f(\langle\sigma, z\rangle_\beta)$. Therefore for any $z, z' \in Z$,

$$\begin{aligned} J(f)(z, z') &= \{\sigma \in G \mid \langle\sigma, z\rangle_\beta = z'\} \\ &\subseteq \{\sigma \in G \mid \langle\sigma, f(z)\rangle_\alpha = f(z')\} \end{aligned}$$

i.e. $J(f)(z, z')$ is an object of $\mathcal{C}_\alpha(z_+, {z'}_+)$.
- $J(f)$ satisfies $(enr - ii)$ since for any $z \in Z$, $\{u\} \subseteq J(f)(z, z)$.
- It trivially satisfies $(enr - iii)$.
- $J(f)$ is trivially reversible.
- $J(f)$ is skeletal since for any $z, z' \in Z$, $u \in J(f)(z, z') \Rightarrow z = z'$.
- Let us show that it is Cauchy complete. Consider a module $\varphi : \hat{x} \longrightarrow\!\!\!\!\circ\!\!\!\to J(f)$ with right adjoint $\tilde{\varphi}$. According to Lemma 4.22 there are some $z_0 \in Z$ and some $\sigma_0 \in \varphi(z_0)$ such that $\varphi(z) = \{\sigma_0\} \circ J(f)(z, z_0)$. Let $z_1 = \langle\sigma_0, z_0\rangle_\beta$. Then $f(z_1) = \langle\sigma_0, f(z_0)\rangle = x$. We are going to show that $\varphi = {z_1}_\diamond$. ${z_1}_\diamond \le \varphi$ since for any $z \in Z$,

$$\begin{aligned} J(f)(z, z_1) &= \{\sigma_0\} \circ J(f)(z, z_0) \\ &\subseteq \varphi(z_0) \circ J(f)(z, z_0) \\ &\subseteq \varphi(z). \end{aligned}$$

Analogously ${z_1}^\diamond \le \tilde{\varphi}$ since for any $z \in Z$, $J(f)(z_1, z) = J(f)(z_0, z) \circ \{{\sigma_0}^{-1}\}$. Thus ${z_1}_\diamond = \varphi$.

Let us check that given some arrow $\langle f_1, h, f_2\rangle$ of *G-Set* $\downarrow (X, \alpha)$, $J(h)$ is a well defined $\mathcal{C}_\alpha$-functor. Suppose $f_1 : (Z_1, \beta_1) \to (X, \alpha)$ and $f_2 : (Z_2, \beta_2) \to (X, \alpha)$. h is an arrow of *G-Set* that makes the following diagram in *G-Set* commute:

$$\begin{tikzcd} (Z_1, \beta_1) \arrow[rr, "h"] \arrow[dr, "f_1"'] & & (Z_2, \beta_2) \arrow[dl, "f_2"] \\ & (X, \alpha) & \end{tikzcd}$$

Since h is an arrow of *G-Set*, for any $z, z' \in Z_1$, $J(f_1)(z, z') \subseteq J(f_2)(hz, hz')$ and the map $h : Z_1 \to Z_2$ defines a $\mathcal{C}_\alpha$-functor from $J(f_1)$ to $J(f_2)$.

Now it is straightforward to check that J as defined above is a functor from *G-Set* $\downarrow (X, \alpha)$ to $\mathcal{C}_\alpha$-*Cat*.

Let us check that given some skeletal reversible Cauchy complete enrichment A on $\mathcal{C}_\alpha$, $J'(A) = (Obj(A), \beta)$ is a well defined object of *G-Set* $\downarrow (X, \alpha)$.

First we need to prove the property $(*)$:

Given $x \in X$, $a_0 \in Obj(A)$ and $\sigma_0 \in G$ such that $\langle \sigma_0, (a_0)_+ \rangle = x$, there is a unique $a_1 \in Obj(A)$ with $\sigma_0 \in A(a_0, a_1)$. It satisfies $(a_1)_+ = x$.

Let x, a_0, and σ_0 be as above and $\varphi : \hat{x} \multimap A$ be the left adjoint module defined by $\varphi(a) = \{\sigma_0\} \circ A(a, a_0)$. Due to the Cauchy completeness of A, there is an $a_1 \in Obj(A)$ such that $\varphi = a_{1\diamond}$. Such an a_1 satisfies therefore $(a_1)_+ = x$ and $\sigma_0 \in A(a_0, a_1)$. Due to skeletality, such an a_1 is unique.

According to $(*)$, $\beta : G \times Obj(A) \to Obj(A)$ is a well defined map. β defines indeed an action of G on $Obj(A)$, as shown below (for short we omit the subscript β).

- For any $a \in Obj(A)$, $\langle u, a \rangle = a$ since $u \in A(a, a)$.
- For any $a \in Obj(A)$ and any $g_1, g_2 \in G$, $\{g_1\} \subseteq A(a, \langle g_1, a \rangle)$ and $\{g_2\} \subseteq A(\langle g_1, a \rangle, \langle g_2, \langle g_1, a \rangle\rangle)$. Therefore

$$\begin{aligned} g_2 \cdot g_1 &\in A(\langle g_1, a \rangle, \langle g_2, \langle g_1, a \rangle\rangle) \circ A(a, \langle g_1, a \rangle) \\ &\subseteq A(a, \langle g_2, \langle g_1, a \rangle\rangle), \end{aligned}$$

proving $\langle g_2 \cdot g_1, a \rangle = \langle g_2, \langle g_1, a \rangle\rangle$.

Since by definition for any $a, a' \in Obj(A)$, $A(a, a') = \{\sigma \in G \mid \langle \sigma, a \rangle_\beta = a'\}$, it is clear that $(-)_+$ defines an arrow of *G-Set* from $(Obj(A), \beta)$ above to (X, α).

In order to prove that J is an equivalence of category with adjoint J' we must prove $J' \circ J \cong 1$ and $J \circ J' \cong 1$. It is immediate to see that the two natural isomorphisms above are identities. ∎

We found convenient to present now the following result that we shall use later (4.38).

4.25. Lemma. *If A is a reversible enrichment on $\mathcal{C}_\alpha$ with (reversible) Cauchy completion $\bar{A}$, then $J(\bar{A})$ is isomorphic to the arrow of G-Set*

$$f : ((G \times Obj(A))/\sim, \delta) \to (X, \alpha)$$

where:

- $\sim$ is the equivalence on $G \times Obj(A)$ defined in Lemma 4.22-(2);

- G acts on $(G \times Obj(A))/\sim$ as follows:

$$\langle \tau, [(\sigma, z)] \rangle_\delta = [(\tau \cdot \sigma, z)];$$

- for any $a \in Obj(A)$ and any $\sigma \in G$,

$$f([(\sigma, a)]) = \langle \sigma, a_+ \rangle_\alpha$$

PROOF. First, let us show that the maps δ and f are well defined. Suppose $(\sigma, a) \sim (\sigma', a')$ for some $\sigma, \sigma' \in G$ and some $a, a' \in Obj(A)$. By definition ${\sigma'}^{-1} \cdot \sigma \in A(a, a')$. Thus for any $\tau \in G$, $(\tau \cdot \sigma')^{-1} \tau \cdot \sigma = {\sigma'}^{-1} \cdot \sigma \in A(a, a')$, showing $(\tau \cdot \sigma, a) \sim (\tau \cdot \sigma', a')$. Also $\langle {\sigma'}^{-1}\sigma, a_+ \rangle_\alpha = (a')_+$ and $\langle \sigma, a_+ \rangle_\alpha = \langle \sigma', (a')_+ \rangle_\alpha$. Trivially δ defines an action of G on $(G \times Obj(A))/\sim$ that is preserved by f.

Further on, let $\bar{A}$ denote the Cauchy completion of A. According to Lemma 4.22-(2), the map $h : (G \times X)/\sim \to Obj(\bar{A})$ sending $[(\sigma_0, a_0)]$ to φ such that $\forall a, \varphi(a) = \{\sigma_0\} \circ A(a, a_0)$ is well defined and is a bijection.

According to 4.24, in order to conclude, it is enough to show that h defines an isomorphism of $\mathcal{C}_\alpha$-*Cat* from $J(f)$ to $\bar{A}$. $f \circ h = (-)_+$ since for any $\varphi : \hat{x} \longrightarrow\!\!\!\!\circ\;\; A \;= h([(\sigma_0, a_0)])$, $\varphi_+ = x = \langle \sigma_0, (a_0)_+ \rangle_\alpha = f([(\sigma_0, a_0)])$. For any $\sigma, \sigma' \in G$ and any $a, a' \in Obj(A)$,

$$\begin{aligned} J(f)([\sigma, a], [\sigma', a']) &= \{\tau \in G \mid \langle \tau, [\sigma, a] \rangle_\delta = [\sigma', a']\} \\ &= \{\tau \in G \mid [\tau \cdot \sigma, a] = [\sigma', a']\} \\ &= \{\tau \in G \mid {\sigma'}^{-1} \cdot \tau \cdot \sigma \in A(a, a')\} \\ &= \{\sigma'\} \circ A(a, a') \circ \{\sigma^{-1}\} \\ &= \{\sigma'\} \circ \textstyle\bigvee_{b \in Obj(A)} (A(b, a') \circ A(a, b)) \circ \{\sigma^{-1}\} \\ &= \bar{A}(h([\sigma, a]), h([\sigma', a'])). \end{aligned}$$

■

4.26. CHANGE OF BASE, REVERSIBILITY AND COMPLETIONS. If $\mathcal{U}$ and $\mathcal{V}$ are reversible then $Conv(\mathcal{U}, \mathcal{V})$ may be provided with a reversible structure. The isomorphism $s_{(u,v),(u',v')}$: $Conv(\mathcal{U}, \mathcal{V})((u, v), (u', v')) \cong Conv(\mathcal{U}, \mathcal{V})((u', v'), (u, v))$ sends any functor $F : (u, v) \to (u', v')$ to the functor $F^s : (u', v') \to (u, v)$ such that

$$\begin{tikzcd} \mathcal{U}(u, u') \arrow[r, "F"] & \mathcal{V}(v, v') \arrow[d, "s^{\mathcal{V}}_{v,v'}"] \\ \mathcal{U}(u', u) \arrow[u, "s^{\mathcal{U}}_{u',u}"] \arrow[r, "F^s"'] & \mathcal{V}(v', v) \end{tikzcd}$$

and any $\sigma : F \to G : (u, v) \to (u', v')$ to $\sigma^s : F^s \to G^s = s^{\mathcal{V}}_{v,v'} \circ \sigma \circ s^{\mathcal{U}}_{u',u}$.

The $s_{(u,v),(u',v')}$ are trivially idempotent. To see that $Conv(\mathcal{U}, \mathcal{V})$ satisfies $rev-(ii)$ and $rev-(iii)$, we will need the following lemma. Its proof is left to the reader.

4.27. LEMMA. *Given functors* $M : \mathcal{A} \to \mathcal{B}$, $T : \mathcal{A} \to \mathcal{C}$, $M' : \mathcal{A}' \to \mathcal{B}'$, $T' : \mathcal{A}' \to \mathcal{C}'$ *and isomorphisms* $s_{\mathcal{A},\mathcal{A}'} : \mathcal{A} \to \mathcal{A}'$, $s_{\mathcal{B},\mathcal{B}'} : \mathcal{B} \to \mathcal{B}'$ *and,* $s_{\mathcal{C},\mathcal{C}'} : \mathcal{C} \to \mathcal{C}'$ *such that:*
- the left Kan extension L *of* T *along* M *and the left Kan extension* L' *of* T' *along* M' *exist,*
- $s_{\mathcal{B},\mathcal{B}'} \circ M = M' \circ s_{\mathcal{A},\mathcal{A}'}$,
- $s_{\mathcal{C},\mathcal{C}'} \circ T = T' \circ s_{\mathcal{A},\mathcal{A}'}$,
then $L' \circ s_{\mathcal{B},\mathcal{B}'} \cong s_{\mathcal{C},\mathcal{C}'} \circ L$.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[drr, "s_{\mathcal{C},\mathcal{C}'}"] & & \\
 & & & \mathcal{C}' \\
\mathcal{A} \arrow[uur, "T"] \arrow[rr, "M"] \arrow[dr, "s_{\mathcal{A},\mathcal{A}'}"'] & & \mathcal{B} \arrow[uul, "L"'] \arrow[dr, "s_{\mathcal{B},\mathcal{B}'}"] & \\
 & \mathcal{A}' \arrow[uurr, "T'"] \arrow[rr, "M'"'] & & \mathcal{B}' \arrow[uu, "L'"']
\end{tikzcd}$$

That $Conv(\mathcal{U},\mathcal{V})$ satisfies $rev-(ii)$ is equivalent to the fact that for any functors $F: \mathcal{U}(u,u') \to \mathcal{V}(v,v')$ and $G : \mathcal{U}(u',u'') \to \mathcal{V}(v',v'')$ the diagram below (in Cat) commutes

$$\begin{tikzcd}
\mathcal{U}(u,u'') \arrow[r, "G\bar{\circ}F"] \arrow[d, "s^{\mathcal{U}}_{u,u''}"'] & \mathcal{V}(v,v'') \arrow[d, "s^{\mathcal{V}}_{v,v''}"] \\
\mathcal{U}(u'',u) \arrow[r, "F^s\bar{\circ}G^s"'] & \mathcal{V}(v'',v)
\end{tikzcd}$$

To see that it holds, just apply 4.27 with:

- $M = \mathcal{U}(u,u') \times \mathcal{U}(u',u'') \xrightarrow{\circ^{\mathcal{U}}} \mathcal{U}(u,u'')$,
- $T = \mathcal{U}(u,u') \times \mathcal{U}(u',u'') \xrightarrow{F\times G} \mathcal{V}(v,v') \times \mathcal{V}(v',v'') \xrightarrow{\circ^{\mathcal{V}}} \mathcal{V}(v,v'')$,
- $M' = \mathcal{U}(u'',u') \times \mathcal{U}(u',u) \xrightarrow{\circ^{\mathcal{U}}} \mathcal{U}(u'',u)$,
- $T' = \mathcal{U}(u'',u') \times \mathcal{U}(u',u) \xrightarrow{G^s\times F^s} \mathcal{V}(v'',v') \times \mathcal{V}(v',v) \xrightarrow{\circ^{\mathcal{V}}} \mathcal{V}(v'',v)$,

and isomorphisms:

- $\mathcal{U}(u,u') \times \mathcal{U}(u',u'') \xrightarrow{\cong} \mathcal{U}(u',u'') \times \mathcal{U}(u,u') \xrightarrow{s_{u',u''}\times s_{u,u'}} \mathcal{U}(u'',u') \times \mathcal{U}(u',u)$,
- $\mathcal{U}(u,u'') \xrightarrow{s_{u,u''}} \mathcal{U}(u'',u)$,
- $\mathcal{V}(v,v'') \xrightarrow{s_{v,v''}} \mathcal{V}(v'',v)$.

That $Conv(\mathcal{U},\mathcal{V})$ satisfies $rev-(iii)$ is equivalent to the fact that for any objects u of $\mathcal{U}$ and v of $\mathcal{V}$ the following diagram commutes.

$$\begin{tikzcd}
\mathcal{U}(u,u) \arrow[r, "I_{(u,v)}"] \arrow[d, "s^{\mathcal{U}}_{u,u}"'] & \mathcal{V}(v,v) \arrow[d, "s^{\mathcal{V}}_{v,v}"] \\
\mathcal{U}(u,u) \arrow[r, "I_{(u,v)}"'] & \mathcal{V}(v,v)
\end{tikzcd} .$$

This results from applying 4.27 with:

- $M = 1 \xrightarrow{I^{\mathcal{U}}_u} \mathcal{U}(u,u)$,

- $T = 1 \xrightarrow{I_v^{\mathcal{V}}} \mathcal{V}(v,v)$,
- $M' = 1 \xrightarrow{I_u^{\mathcal{U}}} \mathcal{U}(u,u)$,
- $T' = 1 \xrightarrow{I_v^{\mathcal{V}}} \mathcal{V}(v,v)$,

and isomorphisms $1 : 1 \to 1$, $s_{u,u}$ and, $s_{v,v}$.

4.28. Definition. *A two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ between reversible locally partially ordered bicategories is* reversible *when for any objects x, y of F the following diagram commutes*

$$\begin{tikzcd} \mathcal{V}(x_-, y_-) \arrow[r, "F_{x,y}"] \arrow[d, "s^{\mathcal{V}}_{x_-,y_-}"'] & \mathcal{W}(x_+, y_+) \arrow[d, "s^{\mathcal{W}}_{x_+,y_+}"] \\ \mathcal{V}(y_-, x_-) \arrow[r, "F_{y,x}"'] & \mathcal{W}(y_+, x_+) \end{tikzcd}$$

By definition for reversible locally partially ordered bicategories $\mathcal{U}$ and $\mathcal{V}$, the isomorphism

$$Caten(\mathcal{U}, \mathcal{V}) \cong Conv(\mathcal{U}, \mathcal{V})\text{-}Cat$$

sends exactly reversible two-sided enrichments to reversible enrichments. Also, if $\mathcal{V}$ is reversible the composite isomorphism

$$\mathcal{V}\text{-}Cat \cong Conv(1, \mathcal{V})\text{-}Cat \cong Caten(1, \mathcal{V})$$

sends bijectively reversible enrichments onto reversible two-sided enrichments. Note that reversible enrichments form a subcategory of *Caten*. Thus for any reversible two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, the corresponding change of base $F_@ : \mathcal{V} \to \mathcal{W}$ preserves the reversibility of objects.

Further on $\mathcal{V}$ and $\mathcal{W}$ will denote two locally partially ordered reversible cocomplete and biclosed bicategories.

Using 3.13, it is straightforward to check that

4.29. Observation. *Given a reversible super two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, the lax functor $F_\sharp : \mathcal{V}\text{-}Mod \to \mathcal{W}\text{-}Mod$ defined in 3.15 preserves the reversibility of arrows: for any reversible $\mathcal{V}$-module φ, $F_\sharp(\varphi)$ is reversible.*

According to this and 4.15

4.30. Proposition. *For any super reversible two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, $F_@$ preserves the Morita reversible equivalence: if the reversible $\mathcal{V}$-categories A and B are Morita reversible equivalent then also are $F_@A$ and $F_@B$.*

Further on, we consider some adjunction $F \dashv G : \mathcal{V} \rightharpoonup \mathcal{W}$ in *Caten*, say F is a 2-functor with local right adjoints $G_{a,b} : \mathcal{W}(Fa, Fb) \to \mathcal{V}(a, b)$.

4.31. Lemma. *F is reversible if and only if G is.*

To prove this, we need

4.32. Lemma. *Given objects a, b of $\mathcal{V}$, if $s : \mathcal{V}(a,b) \to \mathcal{V}(b,a)$ and $s' : \mathcal{W}(Fa,Fb) \to \mathcal{W}(Fb,Fa)$ are functors then*

$$F_{b,a} \circ s \leq s' \circ F_{a,b} \Leftrightarrow s \circ G_{a,b} \leq G_{b,a} \circ s'.$$

Proof. ($\Rightarrow$) Consider some arrow of $\mathcal{W}$, $\alpha : Fa \to Fb$. Since $F_{a,b} \dashv G_{a,b}$, $F_{a,b}G_{a,b}(\alpha) \leq \alpha$. Therefore $s'F_{a,b}G_{a,b}(\alpha) \leq s'(\alpha)$. By assumption, $F_{b,a} \circ s \leq s' \circ F_{a,b}$ thus $F_{b,a}sG_{a,b}(\alpha) \leq s'F_{a,b}G_{a,b}(\alpha) \leq s'(\alpha)$. Then, since $F_{b,a} \dashv G_{Fb,Fa}$, $sG_{a,b}(\alpha) \leq G_{b,a}s'(\alpha)$ (see the diagram in $\mathcal{W}(Fb,Fa)$ below where subscripts a,b are omitted).

$$\begin{tikzcd}
s'(\alpha) & FGs'(\alpha) \arrow[l] \\
s'FG(\alpha) \arrow[u] & FsG(\alpha) \arrow[l] \arrow[u, dotted]
\end{tikzcd}$$

($\Leftarrow$) Consider some arrow of $\mathcal{V}$, $\alpha : a \to b$. Since $F_{a,b} \dashv G_{a,b}$, $\alpha \leq G_{a,b}F_{a,b}(\alpha)$. Therefore $s(\alpha) \leq sG_{a,b}F_{a,b}(\alpha)$. By assumption $s \circ G_{a,b} \leq G_{b,a} \circ s'$, thus $sG_{a,b}F_{a,b}(\alpha) \leq G_{b,a}s'F_{a,b}(\alpha)$. Then since $F_{b,a} \dashv G_{b,a}$, $F_{b,a}s(\alpha) \leq s'F_{a,b}(\alpha)$ (see the diagram in $\mathcal{V}(b,a)$ below).

$$\begin{tikzcd}
s(\alpha) \arrow[r] \arrow[d] & GFs(\alpha) \arrow[d, dotted] \\
sGF(\alpha) \arrow[r] & Gs'F(\alpha)
\end{tikzcd}$$

■

Proof of 4.31. As consequence of 4.32, one has that for any objects a, b of $\mathcal{V}$, the following diagram (in Cat) commutes

$$\begin{tikzcd}
\mathcal{W}(Fa,Fb) \arrow[r, "G_{a,b}"] \arrow[d, "s^{\mathcal{W}}_{Fa,Fb}"'] & \mathcal{V}(a,b) \arrow[d, "s^{\mathcal{V}}_{a,b}"] \\
\mathcal{W}(Fb,Fa) \arrow[r, "G_{b,a}"'] & \mathcal{V}(b,a)
\end{tikzcd}$$

if and only the diagram

$$\begin{tikzcd}
\mathcal{V}(a,b) \arrow[r, "F_{a,b}"] \arrow[d, "s^{\mathcal{V}}_{a,b}"'] & \mathcal{W}(Fa,Fb) \arrow[d, "s^{\mathcal{W}}_{Fa,Fb}"] \\
\mathcal{V}(b,a) \arrow[r, "F_{b,a}"'] & \mathcal{W}(Fb,Fa)
\end{tikzcd}$$

commutes. ■

Further on, we suppose that the left adjoint F is reversible.

4.33. LEMMA. *If A is a Cauchy-reversible complete $\mathcal{W}$-category then $F^@A$ is Cauchy-reversible complete.*

PROOF. First we can show that for any object v of $\mathcal{V}$ and any reversible $\mathcal{V}$-module $\varphi : \hat{v} \multimap F^@A$, there is a $\mathcal{V}$-functor $f : \hat{v} \to F^@A$ with $f_\diamond \geq \varphi$. The proof of this merely an adaptation of the proof of 3.17. Let v be an object of $\mathcal{V}$ and $\varphi : \hat{v} \multimap F^@A$ be a reversible $\mathcal{V}$-module. According to 4.29, $F_\sharp\varphi : F_@\hat{v} \multimap F_@F^@A$ is reversible. Then also is $\epsilon_{A\diamond} \bullet F_@\varphi : F_@\hat{v} \multimap A$. Due to the Cauchy-reversible completeness of A, there is a $\mathcal{V}$-functor $g : F_@\hat{v} \to A$ such that $g_\diamond = \epsilon_{A\diamond} \bullet F_\sharp\varphi$. Then one concludes as for 3.17. Indeed for the f determined above $f_\diamond = \varphi$, since the category $\mathcal{V}$-*RMod*$(\hat{v}, F^@A)$ is discrete. ■

Since $F_@$ and $F^@$ preserves the reversibility of objects and $F^@$ preserves the skeletality and the Cauchy-reversible completeness of objects (3.9, 4.33), one deduces from 3.7 and 4.14 the existence for any skeletal and Cauchy-reversible complete $\mathcal{V}$-category A of a natural isomorphism in B:

$$\mathcal{V}\text{-}SkCRcCat(A, F^@B) \cong_B \mathcal{W}\text{-}SkCRcCat((F_@A)^s, B).$$

Let us sum up

4.34. THEOREM. *For any reversible left adjoint $F : \mathcal{V} \to \mathcal{W}$ in Caten, the functor $F^\sim : \mathcal{W}\text{-}SkCRcCat \to \mathcal{V}\text{-}SkCRcCat$*

$$\begin{cases} B \mapsto F^@B, \\ f : B \to B' \quad \mapsto \quad F^@f : F^@B \to F^@B' \end{cases}$$

has a left adjoint $F_\sim$ defined on objects by $F_\sim A = (F_@A)^s$.

4.35. EXAMPLES (CONTINUED). Now we give two applications of theorem 4.34.

4.36. EXAMPLE. [Geometric morphisms]

Let $f : L \to H$ be a continuous map of locales, with corresponding frame morphisms $f^- : H \to L$. It corresponds to a 2-functor $F : \mathcal{C}_H \to \mathcal{C}_L$ that preserves reversibility. F sends any object u to f^-u; $F_{u,v}$ sends any morphisms $w : u \to v$ (i.e. $w \leq u \wedge v$) to $f^-w : f^-u \to f^-v$. $F_{u,v}$ has a right adjoint $G_{u,v}$, defined for any $w \leq f^-u \wedge f^-v$ by

$$G_{u,v}(w) = \bigvee\{w' \leq u \wedge v \mid f^-w' \leq w\}.$$

According to this and 4.20, we may apply 4.34 to retrieve the following well known result.

4.37. THEOREM. *Any continuous map of locales $f : L \to H$ yields an adjunction $f_* \dashv f^* : Sh(H) \rightharpoonup Sh(L)$, where f_* and f^* are respectively the* inverse *and* direct *image functors. f^* is the following. Let f^- denote the frame morphism corresponding to f. For any sheaf $A : L^{op} \to Set$, and any open v of H,*

$$(f^*A)(v) = A(f^-(v)).$$

For any sheaves $A, B : L^{op} \to Set$ and any natural transformation $h : A \to B$, f^*h is the natural transformation defined in each open v of H by

$$(f^*h)_v = h_{f^-v} : A(f^-(v)) \to B(f^-(v)).$$

To see this, it is enough to check that given such an $f : L \to H$, if $F^\sim$ denotes the right adjoint of 4.34 for the F defined above, then f^* is $F^\sim$ up to the equivalence of 4.20. Which is immediate.

4.38. Example. [Group actions]

Recall that any group morphism $\varphi : G \to H$ induces an adjoint pair $\varphi_* \dashv \varphi^* : G\text{-}Set \rightharpoonup H\text{-}Set$. φ_* is named the *extension* functor) and φ^*, the *restriction* functor. Actually φ^* has also a right adjoint named the *induction* functor.

For any object (Y, β) of H-*Set*, $\varphi^*((Y, \beta))$ is the set Y with action

$$\begin{cases} G \times Y \to Y, \\ (\sigma, y) \mapsto \langle \varphi(\sigma), y \rangle_\beta. \end{cases}$$

An arrow from (Y_1, β_1) to (Y_2, β_2) of H-*Set* with underlying map $p : Y_1 \to Y_2$ has image by φ^*, the arrow of G-*Set* (from $\varphi^*((Y_1, \alpha_1))$ to $\varphi^*((Y_2, \alpha_2))$) with underlying map p.

For any object (X, α) of G-*Set*, its image by φ_* is the *contracted product* $H \wedge^G (X, \alpha)$ defined as follows. Its underlying set is

$$(H \times X)/\sim$$

with $\sim$ the equivalence defined by

$$\forall \tau, \tau' \in H, \ \ \forall x, x' \in X, \ \ (\tau, x) \sim (\tau', x') \Leftrightarrow \exists \sigma \in G, \langle \sigma, x \rangle_\alpha = x', \tau = \tau' \cdot \varphi(\sigma).$$

Our purpose is to show the later adjunction may be retrieved as an instance of 4.34. Precisely, considering a category *Grp-Set* of group actions, described below, we will show that any arrow of *Grp-Set*, $(\varphi, f) : (G, X, \alpha) \to (H, Y, \beta)$ yields an adjunction $(\varphi, f)_\sim \dashv (\varphi, f)^\sim$ with $(\varphi, f)_\sim : G\text{-}Set \downarrow (G, X, \alpha) \to H\text{-}Set \downarrow (H, Y, \beta)$. The previous pair $\varphi_* \dashv \varphi^*$ corresponding then to the adjunct pair $(\varphi, 1_{\{*\}})_\sim \dashv (\varphi, 1_{\{*\}})^\sim$ with $(\varphi, 1_{\{*\}})_\sim : (G, \{*\}, !_G) \to (H, \{*\}, !_H)$, $\{*\}$ denoting the one-point set, $1_{\{*\}}$, the identity map on this set, and $!_G$ and $!_H$, the respective unique actions of G and H on this set.

The category *Grp-Set* of group actions is as follows. Objects are group actions. Arrows from (G, X, α) to (H, Y, β) are the pairs (φ, f) where $\varphi : G \to H$ is a group morphism, $f : X \to Y$ is a map and, the following diagram commutes:

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] \arrow[d, "\varphi\times f"'] & X \arrow[d, "f"] \\ H \times Y \arrow[r, "\beta"'] & Y \end{tikzcd}$$

Consider an arrow $(\varphi, f) : (G, X, \alpha) \to (H, Y, \beta)$. It corresponds to a 2-functor $F : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ that preserves reversibility. F sends any object x to $f(x)$ and, $F_{x,x'}$ sends any arrow $L : x \to x'$ — i.e. $L \subseteq \{\sigma \in G \mid \langle \sigma, x\rangle_\alpha = x'\}$ to the arrow

$$\varphi(L) = \{\varphi(\sigma) \mid \sigma \in L\} : f(x) \to f(x').$$

F has local right adjoints: for any $x, x' \in X$, $G_{x,x'}$, the right adjoint to $F_{x,x'}$, is defined for any $L \subseteq \{\tau \in H \mid \langle \tau, f(x)\rangle_\beta = f(x')\}$ by

$$G_{x,x'}(L) = \varphi^{-1}(L) \cap \{\sigma \in G \mid \langle \sigma, x\rangle_\alpha = x'\}.$$

Therefore we may apply 4.34 together with 4.24 and 4.25 to obtain the following result.

4.39. Theorem. *Any arrow $(\varphi, f) : (G, X, \alpha) \to (H, Y, \beta)$ of Grp-Set yields some adjunction $(\varphi, f)_\sim \dashv (\varphi, f)^\sim :$ G-Set $\downarrow \alpha \rightharpoonup$ H-Set $\downarrow \beta$, described below.*
- If $h : (Z, \gamma) \to (Y, \beta)$ is an arrow of H-Set, its image by $(\varphi, f)^\sim$ is the arrow of G-Set, $\pi^2 : (Z \times_{h,f} X, \delta) \to (X, \alpha)$, with

$$Z \times_{h,f} X = \{(z, x) \mid h(z) = f(x)\},$$

for any $(z, x) \in Z \times_{h,f} X$,

$$\pi^2((z, x)) = x,$$

and for any $\sigma \in G$ and any $(z, x) \in Z \times_{h,f} X$,

$$\langle \sigma, (z, x)\rangle_\delta = (\langle \varphi(\sigma), z\rangle_\gamma, \langle \sigma, x\rangle_\alpha).$$

- $(\varphi, f)^\sim$ is defined on arrows as follows. Considering the following commuting diagram in H-Set:

$$\begin{tikzcd}
(Z_1, \gamma_1) \arrow[rr, "p"] \arrow[dr, "h_1"'] & & (Z_2, \gamma_2) \arrow[dl, "h_2"] \\
& (Y, \beta) &
\end{tikzcd}.$$

the image by $(\varphi, f)^\sim$ of the arrow $\langle h_1, p, h_2\rangle$ of H-Set $\downarrow (Y, \beta)$ has underlying arrow in G-Set

$$p' : (Z_1 \times_{(h_1,f)} X, \delta_1) \to (Z_2 \times_{(h_2,f)} X, \delta_2)$$

where for any $(z, x) \in Z_1 \times_{(h_1,f)} X$,

$$p'((z, x)) = (p(z), x)$$

— see the commuting diagram in G-Set below where for $i = 1, 2$, $(\varphi, f)^\sim(h_i) = \pi^2 : (Z_i \times_{(h_i,f)} X, \delta_1) \to (X, \alpha)$.

$$\begin{tikzcd}
(Z_1 \times_{(h_1,f)} X, \delta_1) \arrow[rr, "p'"] \arrow[dr, "\pi^2"'] & & (Z_2 \times_{(h_2,f)} X, \delta_2) \arrow[dl, "\pi^2"] \\
& (X, \alpha) &
\end{tikzcd}$$

- $(\varphi, f)_\sim$ *is defined on objects of G-Set* $\downarrow (X, \alpha)$ *as follows. For any arrow* $g : (Z, \gamma) \to (X, \alpha)$ *of G-Set,* $(\varphi, f)_\sim(g)$ *is the arrow*

$$h : ((H \times Z)/ \sim, \delta) \to (Y, \beta)$$

of H-Set where $\sim$ *is the equivalence on* $H \times Z$ *defined by*

$$\forall \tau, \tau' \in H, \ \ \forall z, z' \in Z, \ \ (\ (\tau, z) \sim (\tau', z') \Leftrightarrow \exists \sigma \in G, \langle \sigma, z \rangle = z', \tau = \tau' \cdot \varphi(\sigma)\ )$$

for any $\tau, \kappa \in H$ *and* $z \in Z$,

$$\langle \kappa, [(\tau, z)] \rangle_\delta = [(\kappa \cdot \tau, z)]$$

for any $\tau \in H$ *and* $z \in Z$,

$$h([(\tau, z)]) = \langle \tau, f \circ g(z) \rangle_\beta.$$

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Dip. Scienze dell'Informazione, Universitá di Roma "La Sapienza",
via Salaria, 113, 00198 Roma, Italy,
and
Department of Mathematics and Computer Science,
University of Leicester, University Road,
Leicester LE1 7RH, England

Email: `labella@dsi.uniroma1.it`
`vs27@mcs.le.ac.uk`

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