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MODULES

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ABSTRACT. This paper studies lax higher dimensional structure over bicategories. The general notion of a module between two morphisms of bicategories is described. These modules together with their (multi-)2-cells, which we call modulations, organize themselves into a multi-bicategory. The usual notion of a module can be recovered from this general notion by simply choosing the domain bicategory to be the terminal or final bicategory.

The composite of two such modules need not exist. However, when the domain bicategory is small and the codomain bicategory is locally cocomplete then the composite of any two modules does exist and has a simple construction using the local colimits. These modules and their modulations then give rise to a bicategory.

Recall that neither transformations nor optransformations (respectively lax natural transformations and oplax natural transformations) between morphisms of bicategories give rise to a smooth 3-dimensional structure. However, there is a smooth 3-dimensional structure for modules, and both transformations and optransformations give rise to associated modules. Furthermore, the modulations between two modules associated with transformations can then be described directly as a new sort of modification between the transformations. This provides a locally full and faithful homomorphism from transformations and modifications into the bicategory of modules.

Finally, if each 1-cell component of a transformation is a left-adjoint then the right-adjoints provide an optransformation. In the module bicategory the module associated with this optransformation is right-adjoint to the module associated with the transformation. Therefore the inclusion of transformations whose 1-cells have left adjoints into the (multi-)bicategory of modules provides a source of proarrow equipment.

1. Introduction

The purpose of this paper is to provide an accessible exposition of the general theory of modules for bicategories. Various aspects of this theory are available in the literature, particularly in the recent papers [KLSS02, CKS03]. However, an exposition of the theory at the level of this paper is not available: both [KLSS02, CKS03] are more abstract (although in completely different directions). The authors feel, therefore, that this exposition is a useful addition to those expositions as the theory of modules is a substrate for

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a number of investigations which either touch upon or directly depend on this level of the theory.

The basic theory of modules is well-known and in particular encompasses the notion of (left R, right S) bimodule between rings (R and S). In fact, such bimodules may be viewed as morphisms between two monoidal functors from the terminal monoidal category $\mathbf{1}$ to the monoidal category of Abelian groups. Regarding monoidal categories as one-object bicategories, these bimodules easily generalize to a notion of module between morphisms of bicategories.

For morphisms of the form $\mathbf{1} \longrightarrow \mathcal{M}$, where $\mathcal{M}$ is an arbitrary bicategory, the generalization is quite familiar as such morphisms correspond to monads in $\mathcal{M}$. For example, taking $\mathcal{M} = \mathcal{V}\text{-}\mathbf{mat}$, the bicategory of sets and $\mathcal{V}$-valued matrices for cocomplete monoidal $\mathcal{V}$, such monads are simply $\mathcal{V}$-enriched categories and the modules between them are $\mathcal{V}$-enriched profunctors. Similarly, for $\mathcal{M} = \mathbf{spn}(\mathcal{S})$, the bicategory of spans in finitely complete $\mathcal{S}$, such monads are category objects in $\mathcal{S}$ and the modules are internal profunctors.

The modules which are the subject of this paper generalize this basic notion to modules between morphisms of arbitrary bicategories. The case when the domain bicategory has more than one object is less familiar. However, one special case which has been studied extensively is that in which considers morphisms $\mathcal{X} \longrightarrow \mathcal{M}$, where $\mathcal{X}$ is the locally discrete bicategory on the chaotic category determined by a set X. Such morphisms are precisely Bénabou's *polyads* [B67] and Walters' $\mathcal{M}$-enriched categories [W81], [W82] with X as their set of objects. Our notion of module, in the latter case, is precisely that of a module between such enriched categories.

There is also a completely different motivation for studying modules: morphisms of bicategories were introduced by Bénabou [B67] and are abundant, occurring surprisingly frequently in their full generality. The notion of transformation between morphisms of bicategories, however, is much more problematic. A natural choice is that which is often called a lax natural transformation, but which, following [S80], we shall call simply a transformation. This is not a canonical choice: there is a dual choice, namely, optransformations (often known as oplax natural transformations). At the next level there is a very natural notion of modification for either of these choices. One would like all this structure to fit together into a natural (weak) 3-dimensional structure. For fixed $\mathcal{W}$ and $\mathcal{M}$ one can form a bicategory $\mathbf{bicat}(\mathcal{W}, \mathcal{M})$ of morphisms, transformations, and modifications; however, it is well-known that right composition with a morphism $F\colon \mathcal{M} \longrightarrow \mathcal{B}$ does not yield a bicategory morphism from $\mathbf{bicat}(\mathcal{W}, \mathcal{M})$ to $\mathbf{bicat}(\mathcal{W}, \mathcal{B})$ that could serve as a reasonable generalization of the whiskering operation.

This means that the notion of transformation between morphisms of bicategories is rather unsatisfactory. Modules, as described here, do not suffer from this particular problem: one obtains a smooth three dimensional structure which allows whiskering whenever the composite is defined. The basic examples above suggest that the composition of these general modules should be a quotient induced by the effect of the actions in the middle. We describe this construction in section 2 below. However, notice that this composition requires that the codomain bicategory has local colimits, the size of these colimits being determined by the size of the domain bicategory. Thus these modules, in general, cannot be composed.

It may seem that one has simply traded one problem for another. Fortunately, this is not the case. The requirement that local colimits exist in $\mathcal{M}$ is already a strong requirement and it seems sensible to actually dispense with it altogether. Thus, we prefer to view modules, and the obvious morphisms between these, as providing multi-categorical structure, rather than bicategorical on the set of morphisms of bicategories from $\mathcal{W}$ to $\mathcal{M}$. The composition of two modules, when it exists, is completely determined by this multi-structure. This has been made precise by Claudio Hermida's notion of *representability* [H00] (see also [B71]).

The question of when the composite of two modules exists, while not being a central issue, is nonetheless of interest, and so the paper begins by describing which colimits must be present for module composites to exist. Next we observe that both transformations and optransformations, in the sense discussed above, give rise to modules and composition of these transformations corresponds to module composition. In fact we show that modules arising from transformations always have (right) composites defined while modules arising from optransformations always have (left) composites defined. This is very similar to the development at the end of [KLSS02].

This suggests that the view of transformations as special modules should be taken seriously: that is, we should view this situation as a source of proarrow equipment in the sense of [Wo82]. With this in mind it is reasonable, first, to consider the relation between modifications and morphisms of modules, which we call *modulations*; it turns out that a modification always gives rise to a modulation but that there are, in general, strictly more modulations between two transformations.

Modulations as morphisms of transformations are of quite independent interest: they were used, for example, in [LS02] to characterize the free Eilenberg-Moore completion of a bicategory. We therefore provide an explicit description of them and prove that the bicategory of transformations with modulations embeds locally fully-faithfully into the bicategory of modules and provides the source for a proarrow equipment.

This paper is a direct outgrowth of the paper [CKS03] by the first three authors, and indeed, much of the material here may be found developed in the poly-bicategory setting of that paper with a brief discussion of the multi-bicategorical situation. The fourth author noticed several simplifications which are incorporated here, but more importantly realised that this material could be usefully presented in a manner directly useful for bicategorical purposes. Although the present paper is self-contained, the reader interested in either the multi-bicategorical or poly-bicategorical presentation of these ideas is referred to [CKS03].

After we finished this paper we were made aware of [KLSS02] which provides a setting for modules more general than ours, but which remains within the context of bicategories. In [KLSS02] arrows between bicategories are introduced that are more general than the familiar "morphisms" of which we speak. In that paper an arrow, say $\mathcal{A}\colon \mathcal{V} \longrightarrow \mathcal{W}$ is viewed as a "category enriched from $\mathcal{V}$ to $\mathcal{W}$"; a complete treatment of such categories is provided in [KLSS02]. In particular, "modules" between categories are studied, as are also module morphisms. When the enriched categories of [KLSS02] are specialized to morphisms of bicategories, the modules of [KLSS02] are the same as ours.

Modules are but a small part of [KLSS02], whereas here they are our entire focus, so there is, as one would expect, a greater level of detail in this paper. In particular we provide a more explicit description of the colimit needed for composition of modules, making clear the universal property of module composition. It is not easy to extract all our results about modules between morphisms from the more general counterparts in [KLSS02]. The morphisms between modules, which we call "modulations", regarded as a generalization of the notion of a modification between lax transformations, recieve here a precise diagrammatic description. These modulations clarify the source of the crucial 2-cells which appeared in [LS02] in the EM-completion of a 2-category and have been useful in our ongoing work. We hope that the level of the exposition offered in the present paper may be useful to others as well who work in this domain.

2. Modules

2.1. To set up notation we recall the ingredients of a bicategory $\mathcal{W}$. There is firstly a set $|\mathcal{W}|$ of *objects* and typically these will be denoted by $W, X, Y, \cdots$. For each ordered pair of objects there is a category $\mathcal{W}(W,X)$ and arrows within these tend to get denoted by $\omega\colon w \longrightarrow x\colon W \longrightarrow X$. To the *vertical* structure provided by these categories is added horizontal structure (composition and identity cells) in the form of functors $\mathcal{W}(W,X) \times \mathcal{W}(X,Y) \longrightarrow \mathcal{W}(W,Y)$ and $\mathbf{1} \longrightarrow \mathcal{W}(W,W)$.

2.2. Definition. *For morphisms of bicategories $F, G\colon \mathcal{W} \longrightarrow \mathcal{M}$, a* module $M\colon F \longrightarrow\!\!\!\!\!\!+\;\; G$ from F to G *consists of a family of functors $M_{W,X}\colon \mathcal{W}(W,X) \longrightarrow \mathcal{M}(FW,GX)$ together with left and right actions (both denoted $\widetilde{M}$) as below*

$$\begin{tikzcd}
\mathcal{W}(W',W) \times \mathcal{W}(W,X) \arrow[r, "F_{W',W} \times M_{W,X}"] \arrow[d] \arrow[dr, phantom, "\widetilde{M}\Downarrow"] & \mathcal{M}(FW',FW) \times \mathcal{M}(FW,GX) \arrow[d] \\
\mathcal{W}(W',X) \arrow[r, "M_{W',X}"'] & \mathcal{M}(FW',GX)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{W}(W,X) \times \mathcal{W}(X,X') \arrow[r, "M_{W,X} \times G_{X,X'}"] \arrow[d] \arrow[dr, phantom, "\widetilde{M}\Downarrow"] & \mathcal{M}(FW,GX) \times \mathcal{M}(GX,GX') \arrow[d] \\
\mathcal{W}(W,X') \arrow[r, "M_{W,X'}"'] & \mathcal{M}(FW,GX')
\end{tikzcd}$$

where the unlabelled arrows are composition functors, satisfying the five familiar unit, associativity, and mutual associativity requirements.

For modules $M, N\colon F \to\!\!\!\!\!+ G\colon \mathcal{W} \longrightarrow \mathcal{M}$, *a* modulation $t\colon M \longrightarrow N$ from M to N *is a family of natural transformations* $t_{W,X}\colon M_{W,X} \longrightarrow N_{W,X}\colon \mathcal{W}(W,X) \longrightarrow \mathcal{M}(FW,GX)$ *which are equivariant in the sense that*

$$\begin{tikzcd} \bullet \arrow[r, bend left, "F\times M"] \arrow[r, bend right, "F\times N"'] \arrow[r, phantom, "\Downarrow F\times t"] \arrow[d] \arrow[dr, phantom, "\Downarrow\widetilde{N}"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend right, "N"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "F\times M"] \arrow[d] \arrow[dr, phantom, "\Downarrow\widetilde{M}"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow t"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left, "M\times G"] \arrow[r, bend right, "N\times G"'] \arrow[r, phantom, "\Downarrow t\times G"] \arrow[d] \arrow[dr, phantom, "\Downarrow\widetilde{N}"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend right, "N"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "M\times G"] \arrow[d] \arrow[dr, phantom, "\Downarrow\widetilde{M}"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow t"] & \bullet \end{tikzcd}$$

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The data defining a module amounts to naturally assigning the following data to any pair of arrows $W \xrightarrow{x} X \xrightarrow{y} Y$ in $\mathcal{W}$.

$$\begin{tikzcd} FW \arrow[r, "Mx"] \arrow[d, "Fx"'] \arrow[dr, "M(yx)" description] & GX \arrow[d, "Gy"] \arrow[dl, phantom, "\overset{\widetilde{M}}{\Leftarrow}"] \\ FX \arrow[r, "My"'] \arrow[ur, phantom, "\overset{\widetilde{M}}{\Rightarrow}"] & GY \end{tikzcd}$$

An example of a module is the "identity" module $M = F = G$, where $\widetilde{F}$ is part of the lax structure of a morphism of bicategories. This module will in fact be the identity for the "horizontal composition" we are about to define.

The equivariance conditions above for a modulation can be expressed by saying that, for all composable 1-cells $W \xrightarrow{x} X \xrightarrow{y} Y$ in $\mathcal{W}$, the diagrams

$$\begin{tikzcd} MyFx \arrow[r, "tyFx"] \arrow[d, "\widetilde{M}"'] & NyFx \arrow[d, "\widetilde{N}"] \\ M(yx) \arrow[r, "t(yx)"'] & N(yx) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} GyMx \arrow[r, "Gytx"] \arrow[d, "\widetilde{M}"'] & GyNx \arrow[d, "\widetilde{N}"] \\ M(yx) \arrow[r, "t(yx)"'] & N(yx) \end{tikzcd}$$

of 2-cells in $\mathcal{M}$ commute. Here and elsewhere, we find it convenient to drop some of the subscripts when they can be inferred from the context. Thus in the first square directly above, ty is the y-component of $t_{X,Y}$. Occasionally, we find it convenient to speak of 'equivariance in F' and 'equivariance in G' to distinguish the two conditions above.

For modules $M, N, P\colon F \to\!\!\!\!\!+ G\colon \mathcal{W} \longrightarrow \mathcal{M}$ and modulations $M \xrightarrow{t} N \xrightarrow{u} P$ we define $M \xrightarrow{u\cdot t} P$ by $(u \cdot t)_{W,X} = u_{W,X} \cdot t_{W,X}$, as natural transformations, and it is then clear from the diagrams above that $u \cdot t$ is a modulation from M to P.

2.3. The horizontal compositions for modules and modulations are, however, more involved. We start by considering the problem of horizontally composing two modules. To this end it is convenient to define a new notion of multi-modulation: for simplicity we make the details explicit in the binary case. With $\mathcal{W}$ and $\mathcal{M}$ fixed for the discussion and modules $M\colon F \nrightarrow G\colon \mathcal{W} \longrightarrow \mathcal{M}$, $N\colon G \nrightarrow H\colon \mathcal{W} \longrightarrow \mathcal{M}$, and $P\colon F \nrightarrow H\colon \mathcal{W} \longrightarrow \mathcal{M}$, we define a *bimodulation* $b\colon N, M \longrightarrow P$ as a family of natural transformations

$$\begin{tikzcd}
\mathcal{W}(W,X) \times \mathcal{W}(X,Y) \arrow[rr, "M_{W,X} \times N_{X,Y}"] \arrow[d] & {} \arrow[d, phantom, "{b_{W,X,Y}\Downarrow}"] & \mathcal{M}(FW,GX) \times \mathcal{M}(GX,HY) \arrow[d] \\
\mathcal{W}(W,Y) \arrow[rr, "P_{W,Y}"'] & {} & \mathcal{M}(FW,HY)
\end{tikzcd}$$

which are *equivariant in F, G, and H*. It is convenient to denote the (x,y) component of $b_{W,X,Y}$ by $b(y,x)$. Equivariance in F is as above for modulations, but adapted to disambiguate 3-fold composites. Thus we require, for all $W \xrightarrow{x} X \xrightarrow{y} Y \xrightarrow{z} Z$ in $\mathcal{W}$, commutativity of the left hexagon below, where as usual α denotes an associativity constraint isomorphism.

$$\begin{tikzcd}[column sep=tiny]
& P(zy)Fx \arrow[dr, "\widetilde{P}"] & \\
(NzMy)Fx \arrow[ur, "b(z{,}y)Fx"] \arrow[d, "\alpha"'] & & P((zy)x) \arrow[d, "P\alpha"] \\
Nz(MyFx) \arrow[dr, "Nz\widetilde{M}"'] & & P(z(yx)) \\
& NzM(yx) \arrow[ur, "b(z{,}yx)"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& N(zy)Mx \arrow[dr, "b(zy{,}x)"] & \\
(NzGy)Mx \arrow[ur, "\widetilde{N}Mx"] \arrow[d, "\alpha"'] & & P((zy)x) \arrow[d, "P\alpha"] \\
Nz(GyMx) \arrow[dr, "Nz\widetilde{M}"'] & & P(z(yx)) \\
& NzM(yx) \arrow[ur, "b(z{,}yx)"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& N(zy)Mx \arrow[dr, "b(zy{,}x)"] & \\
(HzNy)Mx \arrow[ur, "\widetilde{N}Mx"] \arrow[d, "\alpha"'] & & P((zy)x) \arrow[d, "P\alpha"] \\
Hz(NyMx) \arrow[dr, "Hzb(y{,}x)"'] & & P(z(yx)) \\
& HzP(yx) \arrow[ur, "\widetilde{P}"'] &
\end{tikzcd}$$

Similarly, equivariance in H is given by the right hexagon. However, equivariance in G, given by the centre hexagon, is the most important condition to be clear about in the description of composition of modules. Note that the obvious composite of a bimodulation $b\colon N, M \longrightarrow P$ and a modulation $t\colon P \longrightarrow Q$ yields a bimodulation $t \cdot b\colon N, M \longrightarrow Q$. We will construct, in the context above and given sufficient cocompleteness conditions, a module NM so that bimodulations $N, M \longrightarrow P$ are in bijective correspondence with modulations $NM \longrightarrow P$. More precisely, for modules M and N as above, we will construct a module $NM\colon F \nrightarrow H$ and a bimodulation $k\colon N, M \longrightarrow NM$ with the property that, for every bimodulation $b\colon N, M \longrightarrow P$, there exists a unique modulation $t\colon NM \longrightarrow P$, such that the following diagram commutes:

$$\begin{tikzcd}
N, M \arrow[r, "b"] \arrow[d, "k"'] & P \\
NM \arrow[ur, "t"'] &
\end{tikzcd}$$

When such a composite NM exists, we say the pair N, M is *representable* [H00]. One of the advantages of the multi- (or even poly-) bicategorical context is that although such composites don't always exist (the cocompleteness requirement is not trivial), one may nonetheless deal with bi- (and more generally multi- or poly-) modulations, rather as one might use bilinear maps even if tensors didn't exist. This perspective is the focus of [CKS03].

2.4. For modules $M: F \to\!\!\!\!\!\!+\;\; G$ and $N: G \to\!\!\!\!\!\!+\;\; H$ we define $NM: F \to\!\!\!\!\!\!+\;\; H$ via functors $(NM)_{W,Z}: \mathcal{W}(W, Z) \longrightarrow \mathcal{M}(FW, HZ)$ in terms of local colimits in $\mathcal{M}$. Thus it will suffice to define $(NM)(w)$ for $w: W \longrightarrow Z$, a typical 1-cell in $\mathcal{W}$. We shall assume until Theorem 2.7 that the bicategory is locally (small) cocomplete. That is that it has (small) local colimits and that all compositions with 1-cells preserve these colimits.

We define $(NM)(w)$ as a quotient $\sum_\beta NvMu \longrightarrow (NM)(w)$, where the sum is over all 2-cells of the form

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[rr, bend right=50, "w"'] & T \arrow[r, "v"] \arrow[d, phantom, "\Downarrow\beta"] & Z \\ & {} & \end{tikzcd}$$

The relations defining the quotient involve all commutative diagrams of the form

$$\begin{tikzcd} & lx \arrow[dr, "\gamma"] & \\ (zy)x \arrow[ur, "\lambda x"] \arrow[d, "\alpha"'] & & w \\ z(yx) \arrow[r, "z\rho"'] & zr \arrow[ur, "\delta"'] & \end{tikzcd}$$

(for *general* 2-cells $(\lambda, \gamma, \rho, \delta)$ of the kind indicated). It will be convenient to abuse notation slightly and refer to such a pentagon as $(\lambda, \gamma, \rho, \delta)$. Given $w: W \longrightarrow X$, consider, for each $(\lambda, \gamma, \rho, \delta)$, the following parallel 2-cells in $\mathcal{M}$:

$$\begin{tikzcd} (NzGy)Mx \arrow[r, "\widetilde{N}Mx"] \arrow[d, "\alpha"'] & N(zy)Mx \arrow[r, "N\lambda Mx"] & NlMx \arrow[r, "\iota_\gamma"] & \sum_\beta NvMu \\ Nz(GyMx) \arrow[r, "Nz\widetilde{M}"'] & NzM(yx) \arrow[r, "NzM\rho"'] & NzMr \arrow[ur, "\iota_\delta"'] & \end{tikzcd}$$

where we write ι_β for the β-injection into the sum. These parallel pairs collectively define a parallel pair out of the sum, over all $(\lambda, \gamma, \rho, \delta)$, of the $(NzGy)Mx$ and $NM(w)$ is defined as their coequalizer in the category $\mathcal{M}(FW, HX)$:

$$\sum_{(\lambda,\gamma,\rho,\delta)} (NzGy)Mx \rightrightarrows \sum_\beta NvMu \longrightarrow NM(w)$$

Functoriality of NM is straightforward. Let $\tau\colon w \longrightarrow w'\colon W \longrightarrow X$ be a 2-cell in $\mathcal{W}$ and write β' for a typical index of the 'generators' sum for $NM(w')$ and $(\lambda',\gamma',\rho',\delta')$ for a typical index of the 'relations' sum for $NM(w')$. Then, for every β as above, $\tau\cdot\beta$ is a β' and, for every $(\lambda,\gamma,\rho,\delta)$ as above, $(\lambda,\tau\cdot\gamma,\rho,\tau\cdot\delta)$ is a $(\lambda',\gamma',\rho',\delta')$. In the following evident symbolic diagram

$$\begin{tikzcd}
\sum\limits_{(\lambda,\gamma,\rho,\delta)}\bullet \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "R"'] & \sum\limits_{\beta}\bullet \arrow[d, "\Gamma"] \\
\sum\limits_{(\lambda',\gamma',\rho',\delta')}\bullet \arrow[r, shift left] \arrow[r, shift right] & \sum\limits_{\beta'}\bullet
\end{tikzcd}$$

Γ is defined by $\Gamma\cdot\iota_\beta = \iota_{\tau\cdot\beta}$ and R is defined by $R\cdot\iota_{(\lambda,\gamma,\rho,\delta)} = \iota_{(\lambda,\tau\cdot\gamma,\rho,\tau\cdot\delta)}$. The diagram serially commutes and thus defines $NM(\tau)\colon NM(w) \longrightarrow NM(w')$. Of course the colimits that define the $NM(w)$ need not be expressed in 'coequalizer of sums' format. For some purposes it is more convenient to think directly in terms of the diagram that the given coequalizer of sums encodes. In particular, those sums need not exist as long as the required colimit does.

2.5. We have defined a family of functors $(NM)_{W,X}\colon \mathcal{W}(W,X) \longrightarrow \mathcal{M}(FW,HX)$. To complete the definition of NM as a module $F \longrightarrow\!\!\!\!\!\!\!+\;\; H$ we require also actions of F and H. To this end we now assume that the colimits used to construct the $NM(w)$ are preserved by composition with 1-cells from either side. It suffices to describe the action of F on NM. The action of H is similar, and can be described by duality. Thus given $X' \xrightarrow{t} X \xrightarrow{w} Z$ in $\mathcal{W}$ we now describe 2-cells $\widetilde{NM}\colon NM(w)Ft \longrightarrow NM(wt)$. Since

$$\sum_{(\lambda,\gamma,\rho,\delta)}((NzGy)Mx)Ft \rightrightarrows \sum_{\beta}(NvMu)Ft \longrightarrow NM(w)Ft$$

is a coequalizer, consider the 2-cells

$$(NvMu)Ft \xrightarrow{\alpha} Nv(MuFt) \xrightarrow{Nv\widetilde{M}} NvM(ut) \xrightarrow{\iota_{\beta t\cdot\alpha^{-1}}} \sum_{\beta'} Nv'Mu'$$

for all β, where β' is a typical index of the generators sum for $NM(wt)$, and

$$((NzGy)Mx)Ft \xrightarrow{\alpha} (NzGy)(MxFt) \xrightarrow{(NzGy)\widetilde{M}} (NzGy)M(xt) \xrightarrow{\iota} \sum_{(\lambda',\gamma',\rho',\delta')}(Nz'Gy')Mx'$$

for all $(\lambda,\gamma,\rho,\delta)$, where $(\lambda',\gamma',\rho',\delta')$ is a typical index for the relations sum for $NM(wt)$ and ι is the sum injection $\iota_{(\lambda,\gamma t\cdot\alpha^{-1},\rho t\cdot\alpha^{-1},\delta t\cdot\alpha^{-1})}$. These give 2-cells

$$\sum_{\beta}(NvMu)Ft \longrightarrow \sum_{\beta'} Nv'Mu'$$

and

$$\sum_{(\lambda,\gamma,\rho,\delta)}((NzGy)Mx)Ft \longrightarrow \sum_{(\lambda',\gamma',\rho',\delta')}(Nz'Gy')Mx'$$

respectively, which serially commute with the parallel pairs of the coequalizer diagrams in question. Thus they define a 2-cell $NM(w)Ft \longrightarrow NM(wt)$ that we call $\widetilde{NM}$. It is clear from the description above that this putative action of F on NM is built from that of F on M, $\widetilde{M}$. The only ingredients other than $\widetilde{M}$ are associativity constraints and canonical colimit arrows so that it is straightforward to show that $\widetilde{NM}$ does indeed satisfy the requirements to be an action. Of course the action of H on NM is given in terms of the action of H on N.

2.6. We now describe the bimodulation $k: N, M \longrightarrow NM$. For composable 1-cells $W \xrightarrow{x} X \xrightarrow{y} Y$ in $\mathcal{W}$, we define $k(y,x): NyMx \longrightarrow NM(yx)$ to be the 1_{yx}-injection into the colimit. Equivariance of k in F and in H now follows from inspection of the first and third hexagons in 2.3, using the description of the $\widetilde{NM}$. Equivariance of k in G follows from inspection of the second hexagon in 2.3, using the description of functoriality of NM to provide $NM(\alpha)$.

Recall that a bicategory $\mathcal{M}$ is said to be *locally small-cocomplete* if each category $\mathcal{M}(A,B)$ has all small colimits and these are preserved by composition with all 1-cells $a: A' \longrightarrow A$ and with all 1-cells $b: B \longrightarrow B'$.

2.7. THEOREM. *For $\mathcal{W}$ a small bicategory and $\mathcal{M}$ a locally small-cocomplete bicategory, the constructions of 2.4, 2.5, and 2.6 satisfy the universal property of 2.3: For every bimodulation*

$$b: N, M \longrightarrow P: F \longrightarrow\!\!\!\!\!\!+\;\, H: \mathcal{W} \longrightarrow \mathcal{M}$$

there exists a unique modulation

$$t: NM \longrightarrow P: F \longrightarrow\!\!\!\!\!\!+\;\, H: \mathcal{W} \longrightarrow \mathcal{M}$$

such that $t \cdot k = b$.

PROOF. Given the $b(v,u): NvMu \longrightarrow P(vu)$, we must define, for each 1-cell $w: W \longrightarrow Z$ in $\mathcal{W}$ a 2-cell $tw: NM(w) \longrightarrow Pw$ in $\mathcal{M}$. To give such 2-cells is to give for each 2-cell

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[rr, bend right=50, "w"'] & T \arrow[r, "v"] \arrow[d, phantom, "\Downarrow\beta"] & Z \\ & {} & \end{tikzcd}$$

in $\mathcal{W}$, 2-cells $t_\beta\colon NuMv \longrightarrow Pw$, compatible with the relations defined by the $(\lambda, \gamma, \rho, \delta)$. These t_β we define as the composites:

$$\begin{tikzcd}
NvMu \arrow[r, "t_\beta"] \arrow[d, "b(v,u)"'] & Pw \\
P(vu) \arrow[ur, "P\beta"'] &
\end{tikzcd}$$

The required compatibility for a relations index $(\lambda, \gamma, \rho, \delta)$ is precisely commutativity of the outer figure below.

$$\begin{tikzcd}
 & & NlMx \arrow[dr, "b(l,x)"] \arrow[rrddd, bend left, "t_\gamma"] & & \\
 & N(zy)Mx \arrow[ur, "N\lambda Mx"] \arrow[dr, "b(zy,x)"] & & P(lx) \arrow[ddr, "P(\gamma)"] & \\
(NzGy)Mx \arrow[ur, "\widetilde{N}Mx"] \arrow[dd, "\alpha"'] & & P((zy)x) \arrow[ur, "P(\lambda x)"'] \arrow[dd, "P\alpha"] & & \\
 & & & & Pw \\
Nz(GyMx) \arrow[dr, "Nz\widetilde{M}"'] & & P(z(yx)) \arrow[dr, "P(z\rho)"] & & \\
 & NzM(yx) \arrow[ur, "b(z,yx)"'] \arrow[dr, "NzM\rho"'] & & P(zr) \arrow[uur, "P(\delta)"'] & \\
 & & NzMr \arrow[ur, "b(z,r)"'] \arrow[rruuu, bend right, "t_\delta"'] & &
\end{tikzcd}$$

The regions involving curved arrows commute by definition, the diamonds commute by naturality of b, the pentagon is P applied to the commutative $(\lambda, \gamma, \rho, \delta)$, and the hexagon expresses equivariance of b in G. All aspects of the Theorem follow easily from this observation. ∎

Since composition of modules satisfies a universal property [H00] that evidently extends to 'trimodulations' — defined in the obvious way — it follows that composition of modules is associative to within coherent isomorphism. Moreover for each homomorphism $F\colon \mathcal{W} \longrightarrow \mathcal{M}$, the data for F also defines a module $1_F\colon F \longrightarrow\!\!\!\!\!\!\!+\ F$ and the 1_F provide identities to within coherent isomorphism. It is not difficult to check that:

2.8. Theorem. *For $\mathcal{W}$ a small bicategory and $\mathcal{M}$ a locally small-cocomplete bicategory, the homomorphisms $\mathcal{W} \longrightarrow \mathcal{M}$, the modules between these, and the modulations between these last, together with the composites discussed, constitute a bicategory. We denote this bicategory by* $\mathbf{mod}(\mathcal{W}, \mathcal{M})$.

3. Optransformations and transformations

3.1. For morphisms $F, G\colon \mathcal{W} \longrightarrow \mathcal{M}$ we recall that a *transformation u from F to G*, $u\colon F \longrightarrow G\colon \mathcal{W} \longrightarrow \mathcal{M}$, consists of a family of 1-cells $uW\colon FW \longrightarrow GW$, indexed by the objects of $\mathcal{W}$, and a family of 2-cells

$$\begin{tikzcd} FW \arrow[r, "Fx"] \arrow[d, "uW"'] \arrow[dr, phantom, "\xrightarrow{ux}"] & FX \arrow[d, "uX"] \\ GW \arrow[r, "Gx"'] & GX \end{tikzcd}$$

indexed by the 1-cells $x\colon W \longrightarrow X$ of $\mathcal{W}$, which respect composition and identities. An *optransformation t from F to G*, $t\colon F \longrightarrow G\colon \mathcal{W} \longrightarrow \mathcal{M}$, is defined similarly except that the 2-cells are directed oppositely as in:

$$\begin{tikzcd} FW \arrow[r, "Fx"] \arrow[d, "tW"'] \arrow[dr, phantom, "\xleftarrow{tx}"] & FX \arrow[d, "tX"] \\ GW \arrow[r, "Gx"'] & GX \end{tikzcd}$$

Since we will be interested in those optransformations t for which each tW is a map, we need the following, almost trivial, Lemma about *mates*.

3.2. Lemma. *In any bicategory, if*

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\phi"] \arrow[d, "t_1"'] & \bullet \arrow[d, "t_2"] \\ \bullet \arrow[r, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\tau"] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, bend left, "f'"] \arrow[d, "t_1"'] & \bullet \arrow[d, "t_2"] \\ \bullet \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] \arrow[ur, phantom, "\Downarrow\tau'"] & \bullet \end{tikzcd}$$

and $t_1 \dashv t_1^*$, $t_2 \dashv t_2^*$, and $\widehat{\tau}$ is the mate of τ, $\widehat{\tau'}$ the mate of τ' under these adjunctions, then

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\phi"] & \bullet \\ \bullet \arrow[u, "t_1^*"] \arrow[r, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\widehat{\tau}"] & \bullet \arrow[u, "t_2^*"'] \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, bend left, "f'"] & \bullet \\ \bullet \arrow[u, "t_1^*"] \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] \arrow[ur, phantom, "\Downarrow\widehat{\tau}'"] & \bullet \arrow[u, "t_2^*"'] \end{tikzcd}$$

PROOF. It suffices, by the coherence theorem for bicategories, to prove the result for 2-categories. (In any event, only minor adjustments are needed to adapt this proof to a bicategory.) Recall that, in any 2-category, the mate of a 2-cell under adjunctions $\eta_1, \epsilon_1 \colon t_1 \dashv t_1^*$ and $\eta_2, \epsilon_2 \colon t_2 \dashv t_2^*$ is given by pasting the counit ϵ_1 and the unit η_2 as shown on both sides of the equation below. Thus in this case the conclusion follows immediately from associativity of pasting.

$$
\begin{tikzcd}
 & \bullet \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\phi"] \arrow[d, "t_1"] \arrow[dr, phantom, "\Downarrow\tau"] & \bullet \arrow[r, "1"] \arrow[d, "t_2"'] \arrow[dr, phantom, "\Downarrow\eta_2"] & \bullet \\
\bullet \arrow[ur, "t_1^*"] \arrow[r, "1"'] \arrow[r, phantom, "\Downarrow\epsilon_1", shift left=4ex] & \bullet \arrow[r, bend right, "g"'] & \bullet \arrow[ur, "t_2^*"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & \bullet \arrow[r, bend left, "f'"] \arrow[d, "t_1"] \arrow[dr, phantom, "\Downarrow\tau'"] & \bullet \arrow[r, "1"] \arrow[d, "t_2"'] \arrow[dr, phantom, "\Downarrow\eta_2"] & \bullet \\
\bullet \arrow[ur, "t_1^*"] \arrow[r, "1"'] \arrow[r, phantom, "\Downarrow\epsilon_1", shift left=4ex] & \bullet \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & \bullet \arrow[ur, "t_2^*"'] &
\end{tikzcd}
$$

■

3.3. PROPOSITION. *If $t \colon F \longrightarrow G \colon \mathcal{W} \longrightarrow \mathcal{M}$ is an optransformation for which each tW is a map, with right adjoint t^*W, then these together with the mates $\widehat{tx} \colon (Fx)(t^*W) \longrightarrow (t^*X)(Gx)$ of the tx under the adjunctions $tW \dashv t^*W$ and $tX \dashv t^*X$ constitute a transformation $\widehat{t} \colon G \longrightarrow F \colon \mathcal{W} \longrightarrow \mathcal{M}$.*

PROOF. At least the data makes sense. For the binary condition we must show that, for all composable 1-cells $W \xrightarrow{x} X \xrightarrow{y} Y$ in $\mathcal{W}$,

$$
\begin{tikzcd}
GW \arrow[rr, "G(yx)"] \arrow[d, "t^*W"'] & \arrow[d, phantom, "\Uparrow\widehat{t}(yx)"] & GY \arrow[d, "t^*Y"] \\
FW \arrow[rr, "F(yx)"] \arrow[dr, "Fx"'] & \arrow[d, phantom, "\Uparrow\widetilde{F}"] & FY \\
 & FX \arrow[ur, "Fy"'] &
\end{tikzcd}
=
\begin{tikzcd}
GW \arrow[r, "Gx"] \arrow[rr, bend left=50, "G(yx)"] \arrow[d, "t^*W"'] \arrow[dr, phantom, "\Uparrow\widehat{tx}"] & GX \arrow[r, "Gy"] \arrow[d, "t^*X"] \arrow[u, phantom, "\Uparrow\widetilde{G}"] \arrow[dr, phantom, "\Uparrow\widehat{ty}"] & GY \arrow[d, "t^*Y"] \\
FW \arrow[r, "Fx"'] & FX \arrow[r, "Fy"'] & FY
\end{tikzcd}
$$

This follows from the corresponding equation for t, using Lemma 3.2 and the fact that the mate of a paste composite is the paste composite of mates. The nullary condition is shown similarly. ■

3.4. PROPOSITION. *Optransformations $t \colon F \longrightarrow G \colon \mathcal{W} \longrightarrow \mathcal{M}$ give rise to modules $T = t_\sharp \colon F \longrightarrow\!\!\!\!\!\!\shortmid\;\; G \colon \mathcal{W} \longrightarrow \mathcal{M}$; similarly transformations $u \colon F \longrightarrow G \colon \mathcal{W} \longrightarrow \mathcal{M}$ give rise to modules $U = u^\sharp \colon F \longrightarrow\!\!\!\!\!\!\shortmid\;\; G \colon \mathcal{W} \longrightarrow \mathcal{M}$.*

Proof. The constructions of T and U are given by the following pasting diagrams, which display the data defining each module, given morphisms $W \xrightarrow{x} X \xrightarrow{y} Y$ in $\mathcal{W}$.

$$
\begin{tikzcd}
FW \arrow[r, "tW"] \arrow[d, "Fx"'] \arrow[dr, phantom, "\overset{tx}{\Rightarrow}"] & GW \arrow[r, "Gx"] \arrow[d, "Gx"] \arrow[dr, "G(yx)" description] & GX \arrow[d, "Gy"] \arrow[dl, phantom, near start, "\overset{\widetilde{G}}{\Leftarrow}"] \\
FX \arrow[r, "tX"'] & GX \arrow[r, "Gy"'] \arrow[ur, phantom, near start, "\overset{\widetilde{G}}{\Rightarrow}"] & GY
\end{tikzcd}
\qquad
\begin{tikzcd}
FW \arrow[r, "Fx"] \arrow[d, "Fx"'] \arrow[dr, "F(yx)" description] & FX \arrow[r, "uX"] \arrow[d, "Fy"] \arrow[dl, phantom, near start, "\overset{\widetilde{F}}{\Leftarrow}"] \arrow[dr, phantom, "\overset{uy}{\Leftarrow}"] & GX \arrow[d, "Gy"] \\
FX \arrow[r, "Fy"'] \arrow[ur, phantom, near start, "\overset{\widetilde{F}}{\Rightarrow}"] & FY \arrow[r, "uY"'] & GY
\end{tikzcd}
$$

Explicitly, for an *optransformation* $t\colon F \longrightarrow G$, $T_{W,X}\colon \mathcal{W}(W,X) \longrightarrow \mathcal{M}(FW,GX)$ is given by

$$\mathcal{W}(W,X) \xrightarrow{G_{W,X}} \mathcal{M}(GW,GX) \xrightarrow{\mathcal{M}(tW,-)} \mathcal{M}(FW,GX)$$

Thus with reference to the optransformation displayed in 3.1, Tx is the down-then-right composite $Gx.tW$ (where we use '.' to spare parentheses). For $y\colon X \longrightarrow Y$ in $\mathcal{W}$

$$GyTx = Gy(Gx.tW) \xrightarrow{\alpha^{-1}} (GyGx)tW \xrightarrow{\widetilde{G}tW} G(yx)tW$$

defines $\widetilde{T} : GyTx \longrightarrow T(yx)$ and for $w\colon V \longrightarrow W$ in $\mathcal{W}$,

$$(Gx.tW)Fw \xrightarrow{\alpha} Gx(tW.Fw) \xrightarrow{Gx.tw} Gx(Gw.tV) \xrightarrow{\alpha^{-1}} (GxGw)tV \xrightarrow{\widetilde{G}tV} G(xw)tV$$

defines $\widetilde{T} : TxFw \longrightarrow T(xw)$. It is routine to show that T together with the $\widetilde{T}$ defines a module $T\colon F \longrightarrow\!\!\!\!\!+ \; G$.

For a *transformation* $u\colon F \longrightarrow G$, define functors $U_{W,X}\colon \mathcal{W}(W,X) \longrightarrow \mathcal{M}(FW,GX)$ by

$$\mathcal{W}(W,X) \xrightarrow{F_{W,X}} \mathcal{M}(FW,FX) \xrightarrow{\mathcal{M}(-,uX)} \mathcal{M}(FW,GX)$$

So now referring to the transformation displayed in 3.1, Ux is the right-then-down composite $uX.Fx$. For $w\colon V \longrightarrow W$ in $\mathcal{W}$

$$UxFw = (uX.Fx)Fw \xrightarrow{\alpha} uX(FxFw) \xrightarrow{uX\widetilde{F}} uX.F(xw)$$

defines $\widetilde{U} : UxFw \longrightarrow U(xw)$ and for $y\colon X \longrightarrow Y$ in $\mathcal{W}$,

$$Gy(uX.Fx) \xrightarrow{\alpha^{-1}} (Gy.uX)Fx \xrightarrow{uy.Fx} (uY.Fy)Fx \xrightarrow{\alpha} uY(FyFx) \xrightarrow{uY\widetilde{F}} uY.F(yx)$$

defines $\widetilde{U} : GyUx \longrightarrow U(yx)$. Again, it is easy to show that U together with the $\widetilde{U}$ defines a module $U\colon F \longrightarrow\!\!\!\!\!+ \; G$. ∎

This is in fact a special case of more general (one-sided) actions of (op)transformations on modules, as described in the next Proposition, where we replace the identity modules above with arbitrary modules. Note that although in general modules need not compose without any smallness constraints we can in fact always pre-compose with modules of the form $t_\sharp$ and we can always post-compose with modules of the form $u^\sharp$.

3.5. PROPOSITION. *For an optransformation $t: F \longrightarrow G$ and a module $N: G \longrightarrow\!\!\!\!\!+\; H$, there is a module $N \cdot t: F \longrightarrow\!\!\!\!\!+\; H$. For such t, the module $t_\sharp$ is $1_G \cdot t$. The module composite $Nt_\sharp$ exists and is equal to $N \cdot t$. Dually, for a transformation $u: F \longrightarrow G$ and a module $M: K \longrightarrow\!\!\!\!\!+\; F$, there is a module $u \cdot M: K \longrightarrow\!\!\!\!\!+\; G$. For such u, the module $u^\sharp$ is $u \cdot 1_F$. The module composite $u^\sharp M$ exists and is equal to $u \cdot M$.*

PROOF. This is essentially the same as above; for $W \xrightarrow{x} X \xrightarrow{y} Y$ in $\mathcal{W}$ we have the following.

$$
\begin{tikzcd}
FW \arrow[r, "tW"] \arrow[d, "Fx"'] \arrow[dr, phantom, "\xrightarrow{tx}"] & GW \arrow[r, "Nx"] \arrow[d, "Gx"] \arrow[dr, "N(yx)" description] \arrow[r, phantom, shift right=3ex, "\xleftarrow{\widetilde{N}}"] & HX \arrow[d, "Hy"] \\
FX \arrow[r, "tX"'] & GX \arrow[r, "Ny"'] \arrow[r, phantom, shift left=3ex, "\xrightarrow{\widetilde{N}}"] & HY
\end{tikzcd}
\qquad
\begin{tikzcd}
KW \arrow[r, "Mx"] \arrow[d, "Kx"'] \arrow[dr, "M(yx)" description] \arrow[r, phantom, shift right=3ex, "\xleftarrow{\widetilde{M}}"] & FX \arrow[r, "uX"] \arrow[d, "Fy"] \arrow[dr, phantom, "\xleftarrow{uy}"] & GX \arrow[d, "Gy"] \\
KX \arrow[r, "My"'] \arrow[r, phantom, shift left=3ex, "\xrightarrow{\widetilde{M}}"] & FY \arrow[r, "uY"'] & GY
\end{tikzcd}
$$

So for example, $(N \cdot t)_{W,X} = \mathcal{W}(W, X) \xrightarrow{N_{W,X}} \mathcal{M}(GW, HX) \xrightarrow{\mathcal{M}(tW,-)} \mathcal{M}(FW, HX)$, and dually for $u \cdot M$. ■

In view of these results, for an optransformation t with maps as 1-cell components one would expect the modules $t_\sharp$ induced by t and $\widehat{t}^\sharp$ induced by its mate $\widehat{t}$ to be well-behaved, in fact to be adjoint. But there is a problem concerning the composite $t_\sharp\widehat{t}^\sharp$ that ought to be the domain of the counit: in general, this composite need not exist (*i.e.* the pair $t_\sharp, \widehat{t}^\sharp$ need not be representable). (The other composite $\widehat{t}^\sharp t_\sharp$ is defined, and is an example of the composites in the previous Proposition.) This problem can be circumvented, when we take seriously the idea of multi-bicategories (mentioned before). The notion of adjoints may be expressed in the multi setting, if just one of the necessary composites exists, as we see with $t_\sharp$ and $\widehat{t}^\sharp$, according to the following definition.

3.6. DEFINITION. (Adjointness in a multi-bicategory) *1-cells $f: A \longrightarrow B$ and $g: B \longrightarrow A$ are called* adjoint*, $f \dashv g$, if the composite gf is defined (*i.e.* g, f is representable), if there exists a 2-cell $\eta: 1_A \longrightarrow gf$, the* unit*, and if there exists a multi-2-cell $\epsilon: f, g \longrightarrow 1_B$, the* counit*, subject to the "usual" (obvious) adjointness requirements.*

Then in this context we do have the expected adjoint connection between the modules induced by t and $\widehat{t}$:

3.7. Proposition. *Any optransformation $t\colon F \longrightarrow G$ with maps as 1-cell components induces adjoint modules $t_\sharp \dashv \widehat{t}^\sharp$.*

Proof. The representability of $t_\sharp, \widehat{t}^\sharp$ follows by Proposition 3.5. The unit is induced by the units of the adjunctions tX and $\widehat{t}X$. Similarly, the counit bimodulation immediately results from the laxness of G and the counits of $tX \dashv \widehat{t}X$. The rest is straightforward. ∎

This, at least in the poly setting, is Theorem 5.8 of [CKS03]. Note that in that setting even the modest representability condition used here is unnecessary.

4. Modulations

4.1. For fixed bicategories $\mathcal{W}$ and $\mathcal{M}$ we have already introduced (*cf.* Theorem 2.8) the bicategory $\mathbf{mod}(\mathcal{W},\mathcal{M})$ of homomorphisms, modules and modulations. In particular, it contains 2-cells between modules induced by optransformations as well as 2-cells between modules induced by transformations according to Proposition 3.4. This suggests that the notion of modulation can be defined directly between transformations and between optransformations, respectively. In some sense this definition will be more intuitive for optransformations, as we will explain later, but it seems appropriate to give a detailed description in what is generally regarded as the standard variance.

4.2. Definition. *For transformations $u, v\colon F \longrightarrow G\colon \mathcal{W} \longrightarrow \mathcal{M}$, a* modulation *$\mu$ from u to v consists of a family of 2-cells*

$$\mu W\colon uW \longrightarrow (vW)(F1_W)$$

in $\mathcal{M}$, indexed by the objects of $\mathcal{W}$, such that for all $x\colon W \longrightarrow X$ in $\mathcal{W}$,

$$\begin{tikzcd}
FW \arrow[r, "F1_W"] \arrow[dr, "uW"'] \arrow[rr, bend left=50, "Fx"] \arrow[rr, phantom, bend left=25, "\Uparrow"] & FW \arrow[r, "Fx"] \arrow[d, "vW"] \arrow[dl, phantom, "\xrightarrow{\mu W}"] \arrow[dr, phantom, "\xrightarrow{vx}"] & FX \arrow[d, "vX"] \\
& GW \arrow[r, "Gx"'] & GX
\end{tikzcd}
=
\begin{tikzcd}
FW \arrow[r, "Fx"] \arrow[d, "uW"'] \arrow[rr, bend left=50, "Fx"] \arrow[rr, phantom, bend left=25, "\Uparrow"] \arrow[dr, phantom, "\xrightarrow{ux}"] & FX \arrow[r, "F1_X"] \arrow[d, "uX"] & FX \arrow[dl, "vX"] \arrow[dl, phantom, shift left=3, "\xrightarrow{\mu X}"] \\
GW \arrow[r, "Gx"'] & GX &
\end{tikzcd}$$

where the unlabelled transformation on the left side is $(Fx)(F1_W) \xrightarrow{\widetilde{F}} F(x1_W) \xrightarrow{F\rho} Fx$ while that on the right side is $(F1_X)(Fx) \xrightarrow{\widetilde{F}} F(1_W x) \xrightarrow{F\lambda} Fx$ and where here we use ρ and λ for unitary constraint isomorphisms.

Given transformations $u, v, w\colon F \longrightarrow G$ and modulations $\mu\colon u \longrightarrow v$ and $\nu\colon v \longrightarrow w$ we define a composite $\nu \cdot \mu\colon u \longrightarrow w$ by requiring that $(\nu \cdot \mu)W$ be the paste composite

$$
\begin{tikzcd}
FW \arrow[r, "F1_W"] \arrow[rr, bend left=60, "F1_W"] \arrow[dr, "uW"'] & FW \arrow[r, "F1_W"] \arrow[d, "vW"] \arrow[u, phantom, "\Uparrow"] \arrow[dl, phantom, "\xrightarrow{\mu W}"] \arrow[dr, phantom, "\xrightarrow{\nu W}"] & FW \arrow[dl, "wW"] \\
& GW &
\end{tikzcd}
$$

where the arrow $F1_W.F1_W \longrightarrow F1_W$ is that constructed from $\widetilde{F}$ and the constraint $1_W.1_W \cong 1_W$. It is easy to see, by shuffling pastings, that $\nu \cdot \mu$ is again a modulation, that composition is associative, and that composition is unitary for identities given by

$$
\begin{tikzcd}
FW \arrow[rr, "F1_W"] \arrow[rr, bend right, "1_{FW}"'] \arrow[rr, phantom, "\Uparrow F^{\circ}"] \arrow[dr, "uW"'] & & FW \arrow[dl, "uW"] \\
& GW \arrow[u, phantom, "\xrightarrow{\rho^{-1}}"] &
\end{tikzcd}
$$

Thus we have a category that will be denoted $\mathbf{trans}(\mathcal{W}, \mathcal{M})(F, G)$. Composition of transformations is given by pasting which is coherently associative and unitary up to isomorphism via constraints inherited from $\mathcal{M}$. To show that the $\mathbf{trans}(\mathcal{W}, \mathcal{M})(F, G)$ provide the hom-categories for a bicategory $\mathbf{trans}(\mathcal{W}, \mathcal{M})$ we show that composition of transformations is functorial with respect to modulations. Consider:

$$
\begin{tikzcd}
F \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\mu\Downarrow"] & G \arrow[r, bend left, "w"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\nu\Downarrow"] & H
\end{tikzcd}
$$

The whisker composites νu and $x\mu$ respectively are provided by the following diagrams:

$$
\begin{tikzcd}
FW \arrow[r, "F1_W"] \arrow[d, "uW"'] \arrow[dr, phantom, "\xrightarrow{u1_W}"] & FW \arrow[d, "uW"] \\
GW \arrow[r, "G1_W"] \arrow[d, "wW"'] \arrow[d, phantom, shift left=4ex, "\xrightarrow{\nu W}"] & GW \arrow[dl, "xW"] \\
HW &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
FW \arrow[r, "F1_W"] \arrow[d, "uW"'] \arrow[d, phantom, shift left=4ex, "\xrightarrow{\mu W}"] & FW \arrow[dl, "vW"] \\
GW \arrow[d, "xW"'] & \\
HW &
\end{tikzcd}
$$

It is easy to verify that $x\mu$ so defined is a modulation, a little harder to show that νu is so. What is needed to complete the verification that $\mathbf{trans}(\mathcal{W}, \mathcal{M})$ is a bicategory is commutativity of

$$\begin{tikzcd}
wu \arrow[r, "w\mu"] \arrow[d, "\nu u"'] & wv \arrow[d, "\nu v"] \\
xu \arrow[r, "x\mu"'] & xv
\end{tikzcd}$$

where the composites are to be understood in the sense of composition of modulations introduced above. In other words we require equality as below.

$$\begin{tikzcd}
FW \arrow[rr, bend left=50, "F1_W"] \arrow[r, "F1_W"] \arrow[d, "uW"'] \arrow[r, phantom, bend left=40, "\uparrow"] & FW \arrow[r, "F1_W"] \arrow[d, "uW"'] & FW \arrow[dl, "vW"] \\
GW \arrow[r, "G1_W"] \arrow[d, "wW"'] & GW \arrow[dl, "xW"] & \\
HW & &
\end{tikzcd}
\quad (\text{with } \xrightarrow{u1_W},\ \xrightarrow{\mu W},\ \xrightarrow{\nu W})
\quad = \quad
\begin{tikzcd}
FW \arrow[rr, bend left=50, "F1_W"] \arrow[r, "F1_W"] \arrow[d, "uW"'] & FW \arrow[r, "F1_W"] \arrow[dl, "vW"] & FW \arrow[dl, "vW"] \\
GW \arrow[r, "G1_W"] \arrow[d, "wW"'] & GW \arrow[dl, "xW"] & \\
HW & &
\end{tikzcd}
\quad (\text{with } \xrightarrow{\mu W},\ \xrightarrow{v1_W},\ \xrightarrow{\nu W})$$

This follows from the modulation equation for μ.

4.3. Recall that a *modification* $\mu: u \longrightarrow v$ between transformations $v, u: F \longrightarrow G$ is a family of arrows $\mu W: uW \longrightarrow vW$ satisfying the single 'cylinder' equation. Given a modification $\mu: u \longrightarrow v$ the following pasting gives a modulation.

$$\begin{tikzcd}
FW \arrow[r, bend left, "F1_W"] \arrow[r, bend right, "1_{FW}"'] \arrow[r, phantom, "\uparrow"] \arrow[d, bend right=40, "uW"'] \arrow[d, bend left=40, "vW"] \arrow[d, phantom, "\xrightarrow{\mu W}"] & FW \arrow[dl, bend left=40, "vW"] \arrow[dl, phantom, "\xrightarrow{\simeq}"] \\
GW &
\end{tikzcd}$$

The unlabelled transformations are the obvious ones. Write $\overline{\mu}$ for the modulation above. Observe that the identity modulation on $u: F \longrightarrow G$ is $\overline{1_u}$ and that if $u \xrightarrow{\mu} v \xrightarrow{\nu} w$ are composable modifications then $\overline{\nu \cdot \mu} = \overline{\nu} \cdot \overline{\mu}$. It follows directly that $\mathbf{bicat}(\mathcal{W}, \mathcal{M})$ is a subbicategory of $\mathbf{trans}(\mathcal{W}, \mathcal{M})$ consisting of the same objects, the same 1-cells, but a smaller class of 2-cells. This generalizes the situation in [LS02]. For $\mathcal{K}$ a bicategory (usually a 2-category), the bicategory $\mathbf{bicat}(\mathbf{1}, \mathcal{K})$ is $\mathbf{Mnd}\mathcal{K}$ as first introduced in [S72]. On the other hand $\mathbf{trans}(\mathbf{1}, \mathcal{K})$ is $\mathbf{EM}\mathcal{K}$, the free completion of $\mathcal{K}$ with respect to Eilenberg-Moore objects, which was introduced and studied in [LS02].

If F is a homomorphism then, for all G, each modulation $u \longrightarrow v: F \longrightarrow G$ is of the form $\overline{\mu}$ for a unique modification $\mu: u \longrightarrow v$.

4.4. Definition. *For optransformations $s, t: F \longrightarrow G: \mathcal{W} \longrightarrow \mathcal{M}$, a* modulation *$\mu$ from s to t consists of a family of 2-cells*

$$\mu W: sW \longrightarrow (G1_W)(tW)$$

in $\mathcal{M}$, indexed by the objects of $\mathcal{W}$, such that for all $x: W \longrightarrow X$ in $\mathcal{W}$,

$$\begin{tikzcd}
 & FW \arrow[r, "Fx"] \arrow[dl, "tW"'] \arrow[d, "sW"] \arrow[dl, phantom, shift left=4ex, "\overset{\Leftarrow}{\mu W}"] & FX \arrow[d, "sX"] \arrow[dl, phantom, shift right=2ex, "\overset{\Leftarrow}{sx}"] \\
GW \arrow[r, "G1_W"'] \arrow[rr, bend right=50, "Gx"'] \arrow[rr, phantom, bend right=25, "\downarrow"] & GW \arrow[r, "Gx"'] & GX
\end{tikzcd}
=
\begin{tikzcd}
FW \arrow[r, "Fx"] \arrow[d, "tW"'] \arrow[dr, phantom, "\overset{\Leftarrow}{tx}"] & FX \arrow[d, "tX"] \arrow[dr, "sX"] \arrow[dr, phantom, shift right=2ex, "\overset{\Leftarrow}{\mu X}"] & \\
GW \arrow[r, "Gx"'] \arrow[rr, bend right=50, "Gx"'] \arrow[rr, phantom, bend right=25, "\downarrow"] & GX \arrow[r, "G1_X"'] & GX
\end{tikzcd}$$

4.5. To better understand the definition of modulation between transformations it is at first simplest to consider modulations between optransformations, as above, in a very special case. Let $\mathcal{W} = \mathbf{1}$ and $\mathcal{M} = \mathbf{mat}$. Then, as noted before, F and G are just ordinary categories. We may as well write $|F|$ for F applied to the unique object $*$ of $\mathbf{1}$ and similarly $|G|$ for the effect of G on $*$. So $|F|$ is the set of objects of 'F' and $|G|$ is the set of objects of 'G'. Then write $F: |F| \longrightarrow |F|$ for F applied to the unique arrow of $\mathbf{1}$. It is the matrix of hom-sets of 'F' and similarly $G: |G| \longrightarrow |G|$ provides the arrows of 'G'. Identities and composition in 'G' are given, respectively by $G^{\circ}: 1_{|G|} \longrightarrow G$ and $\widetilde{G}: GG \longrightarrow G$. To give a functor $s: F \longrightarrow G$ is to give a function $|s|: |F| \longrightarrow |G|$, essentially a *map* $|F| \longrightarrow |G|$ in $\mathbf{mat}$, and a 2-cell $s: |s|F \longrightarrow G|t|$ satisfying the two equations of an optransformation. The single 2-cell encodes all the information of the effects-on-homs functions. Finally, and this is the point, a natural transformation $\mu: s \longrightarrow t$ is in the first instance a family of arrows of 'G', indexed by the objects of $|F|$, and the single 2-cell $\mu: |s| \longrightarrow G|t|$ that appears in the definition of modulation in the case at hand encodes all its components. The equation of Definition 4.4 expresses a family of equalities of composites, precisely all the commutative squares given by the familiar definition of natural transformation:

$$
\begin{tikzcd}
 & {|F|} \arrow[r, "F"] \arrow[dl, "{|t|}"'] \arrow[d, "{|s|}"] \arrow[dl, phantom, shift left=6, "\xleftarrow{\mu}"] & {|F|} \arrow[d, "{|s|}"] \arrow[dl, phantom, "\xleftarrow{s}"] \\
{|G|} \arrow[r, "G"'] \arrow[rr, bend right=50, "G"'] \arrow[rr, phantom, bend right=25, "\Downarrow"] & {|G|} \arrow[r, "G"'] & {|G|}
\end{tikzcd}
=
\begin{tikzcd}
{|F|} \arrow[r, "F"] \arrow[d, "{|t|}"'] \arrow[dr, phantom, "\xleftarrow{t}"] & {|F|} \arrow[d, "{|t|}"] \arrow[dr, "{|s|}"] \arrow[dr, phantom, shift right=6, "\xleftarrow{\mu}"] & \\
{|G|} \arrow[r, "G"'] \arrow[rr, bend right=50, "G"'] \arrow[rr, phantom, bend right=25, "\Downarrow"] & {|G|} \arrow[r, "G"'] & {|G|}
\end{tikzcd}
$$

Notice that, for general $\mathcal{W}$ and $\mathcal{M}$, if G is a homomorphism then, for all F, each modulation $s \longrightarrow t\colon F \longrightarrow G$ arises from a unique modification $s \longrightarrow t$. However, this observation, dual to that at the end of 4.3 is not germane to the case of morphisms $\mathbf{1} \longrightarrow \mathbf{mat}$. For if a category G, regarded as a morphism $G\colon \mathbf{1} \longrightarrow \mathbf{mat}$, has $\widetilde{G}\colon GG \longrightarrow G$ an isomorphism then every arrow $\phi\colon x \longrightarrow y$ in G has a *unique* factorization but for $x \neq y$ we have both $x \xrightarrow{1_x} x \xrightarrow{\phi} y$ and $x \xrightarrow{\phi} y \xrightarrow{1_y} y$.)

Morphisms, optransformations, and modulations, with composites defined similarly to those of $\mathbf{trans}(\mathcal{W}, \mathcal{M})$, form a bicategory that we denote by $\mathbf{optrans}(\mathcal{W}, \mathcal{M})$.

4.6. The reason that we have used the term 'modulation' in three different contexts is that they are unified in $\mathbf{mod}(\mathcal{W}, \mathcal{M})$. In proposition 3.4 we defined modules arising from optransformations and modules arising from transformations. Let $\mu\colon s \longrightarrow t\colon F \longrightarrow G\colon \mathcal{W} \longrightarrow \mathcal{M}$ be a modulation between optransformations and suppose that $T := t_\sharp$ is the module constructed from t and $S := s_\sharp$ from s as in 3.4. Define $m\colon S \longrightarrow T$ by

$$m_{W,X} : S_{W,X} \longrightarrow T_{W,X}\colon \mathcal{W}(W, X) \longrightarrow \mathcal{M}(FW, GX)$$

so that for $x\colon W \longrightarrow X$ in $\mathcal{W}$, $m_{W,X}x = mx\colon Gx.sW \longrightarrow Gx.tW$ is the paste composite

$$
\begin{tikzcd}
 & GW \arrow[dr, "Gx"] & \\
FW \arrow[ur, "sW"] \arrow[dr, "tW"'] \arrow[r, phantom, "\mu W \Downarrow"] & {} & GX \arrow[l, phantom, "\Downarrow \widetilde{G}"] \\
 & GW \arrow[uu, "G1_W"] \arrow[ur, "Gx"'] &
\end{tikzcd}
$$

Similarly, for $\mu\colon u \longrightarrow v\colon F \longrightarrow G \longrightarrow \mathcal{W} \longrightarrow \mathcal{M}$ a modulation between transformations, let $U := u^\sharp$ be the module constructed from u, and $V := v^\sharp$ from v, again as in 3.4; define $m\colon U \longrightarrow V$ by

$$m_{W,X} : U_{W,X} \longrightarrow V_{W,X}\colon \mathcal{W}(W, X) \longrightarrow \mathcal{M}(FW, GX)$$

so that for $x\colon W \longrightarrow X$ in $\mathcal{W}$, $m_{W,X}x = mx\colon uX.Fx \longrightarrow vX.Fx$ is the paste composite

$$\begin{tikzcd}
 & FX \arrow[dr, "uX"] \arrow[dd, "F1_X" description] & \\
FW \arrow[ur, "Fx"] \arrow[dr, "Fx"'] \arrow[r, phantom, "\widetilde{F}\Downarrow"] & & GX \arrow[l, phantom, "\Downarrow\mu X"] \\
 & FX \arrow[ur, "vX"'] &
\end{tikzcd}$$

4.7. Theorem. *For $\mathcal{W}$ a small bicategory and $\mathcal{M}$ a locally small-cocomplete bicategory, the assignments*

$$\mu\colon s \longrightarrow t\colon F \longrightarrow G \quad \mapsto \quad m\colon S \longrightarrow T\colon F \relbar\joinrel\mapstochar\joinrel\rightarrow G$$

and

$$\mu\colon u \longrightarrow v\colon F \longrightarrow G \quad \mapsto \quad m\colon U \longrightarrow V\colon F \relbar\joinrel\mapstochar\joinrel\rightarrow G$$

of 4.6 define, respectively, homomorphisms of bicategories

$$\mathbf{optrans}(\mathcal{W},\mathcal{M}) \longrightarrow \mathbf{mod}(\mathcal{W},\mathcal{M})$$

and

$$\mathbf{trans}(\mathcal{W},\mathcal{M}) \longrightarrow \mathbf{mod}(\mathcal{W},\mathcal{M})$$

which are given by the identity on objects and are locally fully faithful.

Proof. This result is mainly a matter of following the definitions. The local full faithfulness is a direct consequence of the definitions of the induced structure in $\mathbf{mod}(\mathcal{W},\mathcal{M})$. ∎

Note that local full faithfulness allows us to treat $\mathbf{optrans}(\mathcal{W},\mathcal{M})$ as a sub-bicategory of $\mathbf{mod}(\mathcal{W},\mathcal{M})$ with the same objects and the same 2-cells between the defining 1-cells. In other words, when modulations are taken as the morphisms between them, optransformations are just special modules. (Of course this interpretation is not possible if only *modifications* are allowed as morphisms between optransformations.)

4.8. Remark. In [CKS03] we gave a characterization (Proposition 5.9) of the modules induced by (op)transformations; in that setting, duality allowed a considerable simplification, but in the present setting, the gain is less notable. In essence, given a module whose basic structure is given by the basic data of a (op)transformation, one obtains the coherence requirements 'for free'. More precisely, suppose for a module $M\colon F \relbar\joinrel\mapstochar\joinrel\rightarrow G$ that there is a family of maps (1-cells with right adjoints) $tW\colon F(W) \longrightarrow G(W)$ indexed by 0-cells W of $\mathcal{W}$, with the property that for any 1-cell $x\colon W \longrightarrow X$, $M(x) = G(x).tW$ and for any 2-cell $\alpha\colon x \longrightarrow y$, $M(\alpha) = G(\alpha).tW$. Then there is an optransformation t so that $M = t_\sharp$. Conversely, it is clear that $t_\sharp$ must satisfy these conditions so that this actually characterizes modules which arise from transformations.

4.9. Now consider the locally full sub-bicategory of $\mathbf{optrans}(\mathcal{W}, \mathcal{M})$ determined by those optransformations t for which tW is a map in $\mathcal{M}$, for all W in $\mathcal{W}$. In light of the discussion in 4.5 it is suggestive to denote this bicategory by $\mathbf{fun}(\mathcal{W}, \mathcal{M})$. Similarly, we write $\mathbf{cofun}(\mathcal{W}, \mathcal{M})$ for the locally full sub-bicategory of $\mathbf{trans}(\mathcal{W}, \mathcal{M})$ determined by those u for which all uW have left adjoints. With this terminology we can sharpen the statement of Proposition 3.3

The following theorems assume that $\mathcal{W}$ is a small bicategory and $\mathcal{M}$ is a locally small-cocomplete bicategory. This may be relaxed if we give these results in a "multi" version, but the essential content may be more simply conveyed in the more familiar context.

4.10. Theorem. *Each choice of adjunctions* $tW \dashv t^*W$ *in* $\mathcal{M}$ *determines an involutive isomorphism of bicategories* $(-)^*\colon \mathbf{fun}(\mathcal{W}, \mathcal{M})^{\mathrm{coop}} \longrightarrow \mathbf{cofun}(\mathcal{W}, \mathcal{M})$ *(meaning that* $(-)^*$ *is a homomorphism with* $(-)^{**} = 1$*).*

4.11. Theorem. *The composite* $\mathbf{fun}(\mathcal{W}, \mathcal{M}) \longrightarrow \mathbf{trans}(\mathcal{W}, \mathcal{M}) \longrightarrow \mathbf{mod}(\mathcal{W}, \mathcal{M})$ *is* proarrow equipment *in the sense of Wo82, meaning that it is a locally fully faithful, identity on objects homomorphism of bicategories for which each 1-cell in* $\mathbf{fun}(\mathcal{W}, \mathcal{M})$ *has a right adjoint in* $\mathbf{mod}(\mathcal{W}, \mathcal{M})$.

Proof. For an optransformation t it suffices to show that the transformation $\widehat{t}$ of Proposition 3.3 is right adjoint to t in $\mathbf{mod}(\mathcal{W}, \mathcal{M})$. ∎

Since the usual inclusion of functors as profunctors, the paradigm for proarrow equipment, is a special case of Theorem 4.11 the result is not surprising but the formulation immediately allows a host of new questions and well-motivated definitions. For example, it is now clear what it means to say that a morphism of bicategories $F\colon \mathcal{W} \longrightarrow \mathcal{M}$ is *Cauchy complete*, that a parallel pair of morphisms are Morita equivalent, and so on. For further possibilities the reader is referred to [Wo82], [Wo85], and [RW].

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