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GENERALIZED BROWN REPRESENTABILITY IN HOMOTOPY CATEGORIES

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ABSTRACT. Brown representability approximates the homotopy category of spectra by means of cohomology functors defined on finite spectra. We will show that if a model category $\mathcal{K}$ is suitably determined by λ-small objects then its homotopy category $\mathrm{Ho}(\mathcal{K})$ is approximated by cohomology functors defined on those λ-small objects. In the case of simplicial sets, we have $\lambda = \omega_1$, i.e., λ-small means countable.

1. Introduction

There are two versions of Brown representability for a triangulated category $\mathcal{T}$: the first one says that every cohomological functor $\mathcal{T}^{\mathrm{op}} \to \mathbf{Ab}$ is representable and the second one deals with the representability of cohomological functors $\mathcal{T}_0^{\mathrm{op}} \to \mathbf{Ab}$ defined on the full subcategory $\mathcal{T}_0$ of small objects. The first version is often called Brown representability for cohomology while the second one is called Brown representability for homology (see [10]). We will consider a whole hierarchy of Brown representabilities by asking whether every cohomological functor defined on λ-small objects (where λ is a regular cardinal) is representable. We will show that, for every combinatorial stable model category $\mathcal{K}$, the triangulated category $\mathrm{Ho}(\mathcal{K})$ satisfies one of these λ-Brown representabilities. The consequence is that $\mathrm{Ho}(\mathcal{K})$ satisfies Brown representability for cohomology. It fits in similar results proved in [16], [35] and [29]. In fact, A. Neeman uses his new concept of well generated triangulated categories in his proof and we will show that every combinatorial model category $\mathcal{K}$ has $\mathrm{Ho}(\mathcal{K})$ well generated (cf. 6.10).

Moreover, we can extend our framework from triangulated categories to a general homotopy category $\mathrm{Ho}(\mathcal{K})$ of a model category $\mathcal{K}$. Brown representabilities then deal with weakly continuous functors $\mathrm{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}} \to \mathbf{Set}$ and ask whether they are representable. Let us stress that this was (for $\lambda = \omega$) the original setting considered by E. M. Brown [7]. Since the category of weakly continuous functors $\mathrm{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}} \to \mathbf{Set}$ coincides with the free completion $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ of $\mathrm{Ho}(\mathcal{K}_\lambda)$ under λ-filtered colimits, the question is whether the natural functor $E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ is (essentially) surjective on objects. Our main result is that if $\mathcal{K}$ is combinatorial (in the sense of J. H. Smith) then there is always

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a regular cardinal λ such that E_λ is not only surjective on objects but also full, which means that we obtain Brown representability not only for objects but also for morphisms. In the rest of this introduction we will explain our approach in more detail.

A model category $\mathcal{K}$ is combinatorial if it is accessible and cofibrantly generated. In very general terms, the theory of accessible categories studies those categories $\mathcal{K}$ which are determined by a full subcategory $\mathcal{A}$ consisting of "small" objects (see [33] and [1]). A typical example is the free completion $\mathrm{Ind}(\mathcal{A})$ of the category $\mathcal{A}$ under filtered colimits introduced by Grothendieck [3]. Categories $\mathrm{Ind}(\mathcal{A})$ where $\mathcal{A}$ is a small category are precisely finitely accessible categories. The most of Quillen model categories are finitely accessible and we may ask in what extent is $\mathrm{Ho}(\mathrm{Ind}(\mathcal{A}))$ determined by $\mathrm{Ho}(\mathcal{A})$. Here, since $\mathcal{A}$ is not necessarily a model category, we understand $\mathrm{Ho}(\mathcal{A})$ as the full subcategory of $\mathrm{Ho}(\mathrm{Ind}(\mathcal{A}))$. The best but very rare case is that

$$\mathrm{Ho}(\mathrm{Ind}(\mathcal{A})) \cong \mathrm{Ind}(\mathrm{Ho}(\mathcal{A}))\,,$$

which means that $\mathrm{Ho}(\mathcal{K})$ is finitely accessible as well. We will show that this happens for truncated simplicial sets $\mathbf{SSet}_n = \mathbf{Set}^{\mathbf{\Delta}_n}$ where $\mathbf{\Delta}_n$ is the category of ordinals $\{1, 2, \dots, n\}$. But the homotopy category $\mathrm{Ho}(\mathbf{SSet})$ of simplicial sets is not concrete (see [17]) and thus it cannot be accessible.

However, very often, one has the comparison functor

$$E : \mathrm{Ho}(\mathrm{Ind}(\mathcal{A})) \to \mathrm{Ind}(\mathrm{Ho}(\mathcal{A}))$$

and one can ask whether this functor is at least full and (essentially) surjective on objects. Non-faithfulness of this functor corresponds to the presence of phantoms in $\mathrm{Ho}(\mathrm{Ind}(\mathcal{A}))$, i.e., of morphisms in $\mathrm{Ho}(\mathrm{Ind}(\mathcal{A}))$ which are not determined by their restrictions on objects from $\mathrm{Ho}(\mathcal{A})$.

A sufficient condition for having $E : \mathrm{Ho}(\mathrm{Ind}(\mathcal{A})) \to \mathrm{Ind}(\mathrm{Ho}(\mathcal{A}))$ is that $\mathrm{Ho}(\mathcal{A})$ has weak finite colimits. In this case, $\mathrm{Ind}(\mathrm{Ho}(\mathcal{A}))$ is the full subcategory of the functor category $\mathbf{Set}^{\mathrm{Ho}(\mathcal{A})^{\mathrm{op}}}$ consisting of functors $\mathrm{Ho}(\mathcal{A})^{\mathrm{op}} \to \mathbf{Set}$ which are weakly left exact; they correspond to cohomological functors. Hence the essential surjectivity of E on objects precisely corresponds to the fact that every cohomological functor $\mathrm{Ho}(\mathcal{A})^{\mathrm{op}} \to \mathbf{Set}$ is representable, i.e., to Brown representability of $\mathrm{Ho}(\mathrm{Ind}(\mathcal{A}))$. The classical case is when $\mathrm{Ind}(\mathcal{A})$ is the category of spectra and $\mathcal{A}$ the category of finite spectra. Following Adams [2], the functor $E : \mathrm{Ho}(\mathrm{Ind}(\mathcal{A})) \to \mathrm{Ind}(\mathrm{Ho}(\mathcal{A}))$ is full and essentially surjective on objects.

Categories $\mathrm{Ind}(\mathcal{A})$ are important for large $\mathcal{A}$ as well, for instance in the dual setting of $\mathrm{Pro}(\mathcal{A}) = (\mathrm{Ind}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$, i.e., $\mathrm{Pro}(\mathcal{A})$ is the free completion of $\mathcal{A}$ under filtered limits. Here, the functor

$$E : \mathrm{Ho}(\mathrm{Pro}(\mathcal{A})) \to \mathrm{Pro}(\mathrm{Ho}(\mathcal{A}))$$

is considered in [15]. Their rigidification question asks which objects belong to the image of E. In the special case of $\mathcal{A}$ being the category $\mathbf{Top}$ of topological spaces, the functor E is the comparison between the strong shape category and the shape category (see [37]).

For every regular cardinal λ, there is the free completion $\mathrm{Ind}_\lambda(\mathcal{A})$ of $\mathcal{A}$ under λ-filtered colimits. While $\mathcal{K} = \mathrm{Ind}(\mathcal{A})$ consists of filtered colimits of objects from $\mathcal{A}$, $\mathrm{Ind}_\lambda(\mathcal{A})$ is the full subcategory of $\mathcal{K}$ consisting of λ-filtered colimits of objects from $\mathcal{A}$. For any $\mathcal{A}$, $\mathrm{Ind}(\mathcal{A}) = \mathrm{Ind}_\lambda(\mathcal{K}_\lambda)$ where $\mathcal{K}_\lambda$ consists of filtered colimits of objects from $\mathcal{A}$ of size $< \lambda$ (see [33], 2.3.11). In the same way as above, we get the functor

$$E_\lambda : \mathrm{Ho}(\mathrm{Ind}_\lambda(\mathcal{K}_\lambda)) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$$

(then $E = E_\omega$). As we have mentioned, our main result says that, for any combinatorial model category $\mathcal{K}$, there is a regular cardinal λ such that E_λ is full and essentially surjective on objects. This means that the homotopy category $\mathrm{Ho}(\mathcal{K})$ is approximated by the category $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ of cohomological functors $\mathrm{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}} \to \mathbf{Set}$ defined on λ-small objects. This approximation does not distinguish morphisms $f, h : K \to L$ in $\mathrm{Ho}(\mathcal{K})$ which are λ-phantom equivalent, i.e., which have the same composition with each morphism $A \to K$ where A is λ-small.

For instance, if $\mathrm{Ind}(\mathcal{A})$ is the category of simplicial sets then $\lambda = \omega_1$. It seems to be unknown whether E_ω is essentially surjective in this case, i.e., whether every weakly left exact functor $H : \mathrm{Ho}(\mathcal{A})^{\mathrm{op}} \to \mathbf{Set}$ is representable. E. M. Brown proved this in the special case when H takes countable values (see [7]) and F. Adams [2] in the case that H takes values in the category of groups.

In the case when $\mathrm{Ind}(\mathcal{A})$ is the category $\mathbf{Sp}$ of spectra, our result again yields that E_λ is full and essentially surjective on objects for $\lambda = \omega_1$. A consequence is that $\mathrm{Ho}(\mathbf{Sp})$ has minimal λ-filtered colimits of objects from $\mathrm{Ho}(\mathbf{Sp}_\lambda)$ for $\lambda = \omega_1$ and, more generally, for each $\omega_1 \triangleleft \lambda$. This has been known for $\lambda = \omega$ (see [34]). We thus contribute to the still open problem whether $\mathrm{Ho}(\mathbf{Sp})$ has all minimal filtered colimits (see [34]).

There is well known that, for each model category $\mathcal{K}$, the homotopy category $\mathrm{Ho}(\mathcal{K})$ has weak (co)limits (see, e.g., [34], [24] or [9]). They are constructed from coproducts and homotopy pushouts in the same way as colimits are constructed from coproducts and pushouts. The trivial observation about colimits is that a category $\mathcal{D}$ with a terminal object d^* has each colimit $\operatorname{colim} D$, $D : \mathcal{D} \to \mathcal{K}$ isomorphic to Dd^*. We will show, which is non-trivial and seems to be new, that the same holds for our weak colimits in homotopy categories. This implies that, for any combinatorial model category $\mathcal{K}$, there is a regular cardinal λ such that homotopy λ-filtered colimits are weak colimits, which leads to our generalized Brown representability. The proved property of weak λ-filtered colimits was called $\mathrm{Ho}(\mathcal{K}_\lambda)$-priviliged in [21].

2. Basic concepts

A *model structure* on a category $\mathcal{K}$ will be understood in the sense of Hovey [22], i.e., as consisting of three classes of morphisms called weak equivalences, cofibrations and fibrations which satisfy the usual properties of Quillen [38] and, moreover, both (cofibration, trivial fibrations) and (trivial cofibrations, fibration) factorizations are functorial. Recall that trivial (co)fibrations are those (co)fibrations which are in the same time weak equivalences. The (cofibration, trivial fibration) factorization is *functorial* if there is a functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ and natural transformations $\alpha : \mathrm{dom} \to F$ and $\beta : F \to \mathrm{cod}$ such that $f = \beta_f \alpha_f$ is the (cofibration, trivial fibration) factorization of f. Here $\mathcal{K}^{\to}$ denotes the category of morphisms in $\mathcal{K}$ and dom : $\mathcal{K}^{\to} \to \mathcal{K}$ (cod : $\mathcal{K}^{\to} \to \mathcal{K}$) assign to each morphism its (co)domain. The same for (trivial cofibration, fibration) factorization (see [39]) .

A *model category* is a complete and cocomplete category together with a model structure. In a model category $\mathcal{K}$, the classes of weak equivalences, cofibrations and fibrations will be denoted by $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$, resp. Then $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{W}$ denote trivial cofibrations and trivial fibrations, resp. We have

$$\mathcal{F}_0 = \mathcal{C}^{\square}, \quad \mathcal{F} = \mathcal{C}_0^{\square}, \quad \mathcal{C} = {}^{\square}\mathcal{F}_0 \quad \text{and} \quad \mathcal{C}_0 = {}^{\square}\mathcal{F}$$

where $\mathcal{C}^{\square}$ denotes the class of all morphisms having the right lifting property w.r.t. each morphism from $\mathcal{C}$ and ${}^{\square}\mathcal{F}$ denotes the class of all morphisms having the left lifting property w.r.t. each morphism of $\mathcal{F}$. $\mathcal{K}$ is called *cofibrantly generated* if there are sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{F}_0 = \mathcal{I}^{\square}$ and $\mathcal{F} = \mathcal{J}^{\square}$. If $\mathcal{K}$ is locally presentable then $\mathcal{C}$ is the closure of $\mathcal{I}$ under pushouts, transfinite compositions and retracts in comma-categories $K \downarrow \mathcal{K}$ and, analogously, $\mathcal{C}_0$ is this closure of $\mathcal{J}$.

An object K of a model category $\mathcal{K}$ is called *cofibrant* if the unique morphism $0 \to K$ from an initial object is a cofibration and K is called *fibrant* if the unique morphism $K \to 1$ to a terminal object is a fibration. Let $\mathcal{K}_c$, $\mathcal{K}_f$ or $\mathcal{K}_{cf}$ denote the full subcategories of $\mathcal{K}$ consisting of objects which are cofibrant, fibrant or both cofibrant and fibrant resp. We get the *cofibrant replacement functor* $R_c : \mathcal{K} \to \mathcal{K}$ and the *fibrant replacement functor* $R_f : \mathcal{K} \to \mathcal{K}$. We will denote by $R = R_f R_c$ their composition and call it the *replacement functor*. The codomain restriction of the replacement functors are $R_c : \mathcal{K} \to \mathcal{K}_c$, $R_f : \mathcal{K} \to \mathcal{K}_f$ and $R : \mathcal{K} \to \mathcal{K}_{cf}$.

Let $\mathcal{K}$ be a model category and K an object of $\mathcal{K}$. Recall that a *cylinder object* $C(K)$ for K is given by a (cofibration, weak equivalence) factorization

$$\nabla : K \amalg K \xrightarrow{\gamma_K} C(K) \xrightarrow{\sigma_K} K$$

of the codiagonal ∇. Morphisms $f, g : K \to L$ are *left homotopic* if there is a morphism $h : C(K) \to L$ with

$$f = h\gamma_{1K} \quad \text{and} \quad g = h\gamma_{2K}$$

where $\gamma_{1K} = \gamma_K i_1$ and $\gamma_{2K} = \gamma_K i_2$ with $i_1, i_2 : K \to K \amalg K$ being the coproduct injections. In fact, cylinder objects form a part of the *cylinder functor* $C : \mathcal{K} \to \mathcal{K}$ and $\gamma_1, \gamma_2 : \mathrm{Id} \to C$ are natural transformations.

On $\mathcal{K}_{cf}$, left homotopy $\sim$ is an equivalence relation compatible with compositions, it does not depend on a choice of a cylinder object and we get the quotient

$$Q : \mathcal{K}_{cf} \to \mathcal{K}_{cf}/\sim .$$

The composition

$$P : \mathcal{K} \xrightarrow{R} \mathcal{K}_{cf} \xrightarrow{Q} \mathcal{K}_{cf}/\sim$$

is, up to equivalence, the projection of $\mathcal{K}$ to the homotopy category $\mathrm{Ho}(\mathcal{K}) = \mathcal{K}[\mathcal{W}^{-1}]$ (see [22]). In what follows, we will often identify $\mathcal{K}_{cf}/\sim$ with $\mathrm{Ho}(\mathcal{K})$.

A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-filtered colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered colimit of objects from $\mathcal{A}$.

Here, an object K of a category $\mathcal{K}$ is called λ-*presentable* if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves λ-filtered colimits; $\mathbf{Set}$ is the category of sets. A category is called *accessible* if it is λ-accessible for some regular cardinal λ. The theory of accessible categories was created in [33] and for its presentation one can consult [1]. We will need to know that λ-accessible categories are precisely categories $\mathrm{Ind}_\lambda(\mathcal{A})$ where $\mathcal{A}$ is a small category. If idempotents split in $\mathcal{A}$ then $\mathcal{A}$ precisely consists of λ-presentable objects in $\mathrm{Ind}(\mathcal{A})$. In what follows, we will denote by $\mathcal{K}_\lambda$ the full subcategory of $\mathcal{K}$ consisting of λ-presentable objects.

A *locally λ-presentable category* is defined as a cocomplete λ-accessible category and it is always complete. Locally λ-presentable categories are precisely categories $\mathrm{Ind}_\lambda(\mathcal{A})$ where the category $\mathcal{A}$ has λ-*small* colimits, i.e., colimits of diagrams $D : \mathcal{D} \to \mathcal{A}$ where $\mathcal{D}$ has less then λ morphisms. In general, the category $\mathrm{Ind}_\lambda(\mathcal{A})$ can be shown to be the full subcategory of the functor category $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ consisting of λ-filtered colimits H of hom-functors $\hom(A, -)$ with A in $\mathcal{A}$. In the case that $\mathcal{A}$ has λ-small colimits this is equivalent to the fact that $H : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ preserves λ-small limits. More generally, if $\mathcal{A}$ has weak λ-small colimits then $\mathrm{Ind}_\lambda(\mathcal{A})$ precisely consists of left λ-covering functors (see [26] 3.2). Let us recall that a weak colimit of a diagram $D : \mathcal{D} \to \mathcal{A}$ is a cocone from D such that any other cocone from D factorizes through it but not necessarily uniquely. If $\mathcal{X}$ is a category with weak λ-small limits then a functor $H : \mathcal{X} \to \mathbf{Set}$ is *left λ-covering* if, for each λ-small diagram $D : \mathcal{D} \to \mathcal{X}$ and its weak limit X, the canonical mapping $H(X) \to \lim HD$ is surjective (see [8] for $\lambda = \omega$). A left λ-covering functor preserves all λ-small limits which exist in $\mathcal{X}$. Moreover, a functor $H : \mathcal{X} \to \mathbf{Set}$ is left λ-covering iff it is weakly λ-continuous, i.e., iff it preserves weak λ-small limits. This immediately follows from [8], Proposition 20 and the fact that surjective mappings in $\mathbf{Set}$ split. A functor H is called *weakly continuous* if it preserves weak limits. Hence a weakly continuous functor $H : \mathcal{X} \to \mathbf{Set}$ preserves all existing limits.

A functor $F : \mathcal{K} \to \mathcal{L}$ is called λ-*accessible* if $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible categories and F preserves λ-filtered colimits. An important subclass of λ-accessible functors are those functors which also preserve λ-presentable objects. In the case that idempotents split in $\mathcal{B}$, those functors are precisely functors $\mathrm{Ind}_\lambda(G)$ where $G : \mathcal{A} \to \mathcal{B}$ is a functor. The uniformization theorem of Makkai and Paré says that for each λ-accessible functor F there are arbitrarily large regular cardinals μ such that F is μ-accessible and preserves μ-presentable objects (see [1] 2.19). In fact, one can take $\lambda \triangleleft \mu$ where $\triangleleft$ is the set theoretical relation between regular cardinals corresponding to the fact that every λ-accessible category is μ-accessible (in contrast to [1] and [33], we accept $\lambda \triangleleft \lambda$). For every λ there are arbitrarily large regular cardinals μ such that $\lambda \triangleleft \mu$. For instance, $\omega \triangleleft \mu$ for every regular cardinal μ.

3. Combinatorial model categories

We will follow J. H. Smith and call a model category $\mathcal{K}$ λ-*combinatorial* if $\mathcal{K}$ is locally λ-presentable and both cofibrations and trivial cofibrations are cofibrantly generated by sets $\mathcal{I}$ and $\mathcal{J}$ resp. of morphisms having λ-presentable domains and codomains. Then both trivial fibrations and fibrations are closed in $\mathcal{K}^{\to}$ under λ-filtered colimits. $\mathcal{K}$ will be called *combinatorial* if it is λ-combinatorial for some regular cardinal λ.

The following result is due to J. H. Smith and is presented in [12], 7.1 and 7.2. We just add a little bit more detail to the proof.

3.1. Proposition. [Smith] *Let $\mathcal{K}$ be a combinatorial model category. Then the functors $\mathcal{K}^{\to} \to \mathcal{K}$ giving (cofibration, trivial fibration) and (trivial cofibration, fibration) factorizations are accessible.*

Proof. This means that there is a regular cardinal λ such that $\mathcal{K}$ (and hence $\mathcal{K}^{\to}$ are locally λ-presentable and the (cofibration, trivial fibration) factorization $A \to C \to B$ of a morphism $A \to B$ preserves λ-filtered colimits; the same for the (trivial cofibration, fibration) factorization. There is a regular cardinal λ such that $\mathcal{K}$ is locally λ-presentable and domains and codomains of morphisms from the generating set $\mathcal{I}$ of cofibrations are λ-presentable. For every morphism $f : A \to B$ form a colimit $F_0 f$ of the diagram

$$\begin{tikzcd} A & \\ X \arrow[u, "u"] \arrow[r, "h"'] & Y \end{tikzcd}$$

consisting of all spans (u, h) with $h : X \to Y$ in $\mathcal{I}$ such that there is $v : Y \to B$ with $vh = fu$. Let $\alpha_{0f} : A \to F_0 f$ denote the component of the colimit cocone (the other components are $Y \to F_0 f$ and they make all squares

$$\begin{tikzcd} A \arrow[r, "\alpha_{0f}"] & F_0 f \\ X \arrow[u, "u"] \arrow[r, "h"'] & Y \arrow[u] \end{tikzcd}$$

to commute). Let $\beta_{0f} : F_0 f \to B$ be the morphism induced by f and v's. Then $F_0 : \mathcal{K}^{\to} \to \mathcal{K}$ is clearly λ-accessible. Let $F_i f, \alpha_{if}$ and β_{if}, $i \le \lambda$, be given by the following transfinite induction: $F_{i+1} f = F_0 \beta_{if}$, $\alpha_{i+1,f} = \alpha_{0,\beta_{if}} \alpha_{if}$, $\beta_{i+1,f} = \beta_{0,\beta_{if}}$ and the limit step is given by taking colimits. Then all functors $F_i : \mathcal{K}^{\to} \to \mathcal{K}$, $i \le \lambda$ are λ-accessible and F_λ yields the desired (cofibration, trivial fibration) factorization. ∎

3.2. Remark. Following the uniformization theorem ([1] Remark 2.19), there is a regular cardinal μ such that the functors from 3.1 are μ-accessible and preserve μ-presentable objects. This means that the factorizations $A \to C \to B$ of a morphism $A \to B$ have C μ-presentable whenever A and B are μ-presentable. This point is also well explained in [12].

3.3. Notation. Let $\mathcal{K}$ be a locally presentable model category. Consider the following conditions

(G^1_λ) the functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ giving the (cofibration, trivial fibration) factorization is λ-accessible and preserves λ-presentable objects,

(G^2_λ) the replacement functor $R : \mathcal{K} \to \mathcal{K}$ (being the composition $R = R_f R_c$ of the cofibrant and the fibrant replacement functors) is λ-accessible and preserves λ-presentable objects, and

(G^3_λ) weak equivalences are closed under λ-filtered colimits in $\mathcal{K}^{\to}$.

3.4. Remark. (1) (G^1_λ) implies that the functor R_c is λ-accessible and preserves λ-presentable objects. Thus (G^2_λ) only adds that R_f is λ-accessible and preserves λ-presentable objects. (G^1_λ) also implies

(G^4_λ) the cylinder functor $C : \mathcal{K} \to \mathcal{K}$ is λ-accessible and preserves λ-presentable objects.

(2) Following [1] 2.11 and 2.20, if $\mathcal{K}$ satisfies (G^i_λ) and $\lambda \triangleleft \mu$ then $\mathcal{K}$ satisfies (G^i_μ) for $i = 1, 2, 3, 4$. In particular, if $\mathcal{K}$ satisfies (G^i_ω) then it satisfies (G^i_λ) for any regular cardinal λ (see [1] 2.13 (1)).

3.5. Proposition. *Let $\mathcal{K}$ be a combinatorial model category. Then there is a regular cardinal λ such that $\mathcal{K}$ satisfies the conditions (G^i_λ) for $i = 1, 2, 3$.*

Proof. Let $\mathcal{K}$ be a combinatorial model category. It immediately follows from 3.4 that there are arbitrarily large regular cardinals λ such that the conditions (G^1_λ) and (G^2_λ) are satisfied. Following [12] 7.3, there is λ such that (G^3_λ) holds. This proves the theorem. ∎

Combinatorial model categories form a very broad class. We will discuss the conditions (G^i_λ) in a couple of examples.

3.6. Examples. (i) The model category **SSet** of simplicial sets is ω-combinatorial and satisfies (G^1_ω), ($\mathrm{G}^2_{\omega_1}$) and (G^3_ω). The first and the third statements are clear and the second one follows from the fact that finitely presentable simplicial sets have ω_1-presentable fibrant replacements. This observation can be found in [27], Section 5, as well.

The same is true for the model category $\mathbf{SSet}_*$ of pointed simplicial sets.

(ii) The category **Sp** of spectra with the strict model category structure (in the sense of [5]) is ω-combinatorial (see [40] A.3). We will show that it satisfies the conditions ($\mathrm{G}^1_{\omega_1}$), ($\mathrm{G}^2_{\omega_1}$) and (G^3_ω).

Let us recall that a spectrum X is a sequence $(X_n)_{n=0}^{\infty}$ of pointed simplicial sets equipped with morphisms $\sigma_n^X : \Sigma X_n \to X_{n+1}$ where Σ is the suspension functor. This means that $\Sigma X_n = S^1 \wedge X_n$ where $S^1 \wedge -$ is the smash product functor, i.e., a left adjoint to

$$-^{S^1} = \hom(S^1, -) : \mathbf{SSet}_* \to \mathbf{SSet}_*.$$

A spectrum X is ω_1-presentable iff all X_n, $n \geq 0$, are ω_1-presentable in $\mathbf{SSet}_*$. The strict model structure on $\mathbf{Sp}$ has level equivalences as weak equivalences and level fibrations as fibrations. This means that $f : X \to Y$ is a weak equivalence (fibration) iff all $f_n : X_n \to Y_n$ are weak equivalences (fibrations) in $\mathbf{SSet}_*$. A morphism $f : X \to Y$ is a (trivial) cofibration iff $f_0 : X_0 \to Y_0$ is a (trivial) cofibration and all induced morphisms $t_n : Z_n \to Y_n$, $n \geq 1$, from pushouts are (trivial) cofibrations

$$\begin{tikzcd}
\Sigma X_{n-1} \arrow[r, "\sigma_{n-1}^X"] \arrow[d, "\Sigma f_{n-1}"'] & X_n \arrow[d] \arrow[ddr, "f_n"] & \\
\Sigma Y_{n-1} \arrow[r] \arrow[drr, "\sigma_{n-1}^Y"'] & Z_n \arrow[dr, "t_n"] & \\
& & Y_n
\end{tikzcd}$$

(see [5], [25] or [23]). Then a (cofibration, trivial fibration) factorization $X \xrightarrow{g} Z \xrightarrow{h} Y$ of a morphism $f : X \to Y$ is made as follows.

One starts with a (cofibration, trivial fibration) factorization

$$f_0 : X_0 \xrightarrow{g_0} Z_0 \xrightarrow{h_0} Y_0$$

in $\mathbf{SSet}_*$. Then one takes a (cofibration, trivial fibration) factorization

$$t : Z_1' \xrightarrow{u} Z_1 \xrightarrow{h_1} Y_1$$

of the induced morphism from a pushout

$$\begin{tikzcd}
\Sigma X_0 \arrow[r, "\sigma_0^X"] \arrow[d, "\Sigma g_0"'] & X_1 \arrow[d, "q"'] \arrow[ddr, "f_1"] & \\
\Sigma Z_0 \arrow[r, "p"] \arrow[drr, "\sigma_0^Y \cdot \Sigma h_0"'] & Z_1' \arrow[dr, "t"] & \\
& & Y_1
\end{tikzcd}$$

and puts $\sigma_1^Z = up$ and $g_1 = uq$. This yields

$$f_1 : X_1 \xrightarrow{g_1} Z_1 \xrightarrow{h_1} Y_1$$

and one continues the procedure. Analogously, one constructs a (trivial cofibration, fibration) factorization. It is now easy to see that the strict model structure on **Sp** satisfies ($\mathrm{G}^1_{\omega_1}$) and ($\mathrm{G}^2_{\omega_1}$). (G^3_{ω}) is follows from (i).

(iii) The model category **Sp** of spectra with the stable Bousfield-Friedlander model category structure (see [5]) is ω-combinatorial (see [40] A.3). We will show that it satisfies the conditions ($\mathrm{G}^1_{\omega_1}$), ($\mathrm{G}^2_{\omega_1}$) and (G^3_{ω}) too.

The stable model structure is defined as a Bousfield localization of the strict model structure, i.e., by adding a set of new weak equivalences. Cofibrations and trivial fibrations remain unchanged, which means that the condition ($\mathrm{G}^1_{\omega_1}$) is satisfied following (ii). Weak equivalences are closed under filtered colimits in $\mathbf{Sp}^{\rightarrow}$, which follows from their characterization (see, e.g., [25] 4.2.2). Stably fibrant spectra are those strictly fibrant spectra X for which the adjoint transposes $\widetilde{\sigma}^X_h : X_n \to X^{S^1}_{n+1}$ of structure morphisms are weak equivalences. For checking the condition ($\mathrm{G}^2_{\omega_1}$), it suffices to show that the stable fibrant replacement functor R_f preserves ω_1-presentable objects. But this follows from [5] or [23]: consider the functor

$$\Theta : \mathbf{Sp} \to \mathbf{Sp}$$

such that

$$(\Theta X)_n = X^{S^1}_{n+1}$$

and

$$\sigma^{\Theta X}_n = (\widetilde{\sigma}^X_n)^{S^1}\,.$$

Let $\Theta^\infty X$ be a colimit of the chain

$$X \xrightarrow{\iota_X} \Theta X \xrightarrow{\Theta\iota_X} \Theta^2 X \to \dots$$

where $\iota_{X_n} = \widetilde{\sigma}^X_n : X_n \to X^{S^1}_{n+1}$. Then Θ^∞ is stably fibrant (see [23] 4.6). Then a stable fibrant replacement of X is defined by a (cofibrant, trivial fibrant) factorization of ι^∞_X

$$X \longrightarrow R_f X \longrightarrow \Theta^\infty X\,.$$

It is easy to see that $R_f X$ is ω_1-presentable whenever X is ω_1-presentable.

4. Weak colimits

There is well known that the homotopy category of any model category $\mathcal{K}$ has products, coproducts, weak limits and weak colimits. We will recall their constructions.

4.1. Remark. (i) Let K_i, $i \in I$ be a set of objects of $\mathcal{K}$. Without any loss of generality, we may assume that they are in $\mathcal{K}_{cf}$. Then their product in $\mathcal{K}$

$$p_i : K \to K_i$$

is fibrant and let

$$q_K : R_c K \to K$$

be its cofibrant replacement. Then $R_cK \in \mathcal{K}_{cf}$ and

$$Q(p_i q_K) : QR_cK \to QK_i$$

is a product in $\mathrm{Ho}(\mathcal{K})$. Recall that $Q : \mathcal{K}_{cf} \to \mathrm{Ho}(\mathcal{K}) = \mathcal{K}_{cf}/\sim$ is the quotient functor.

In fact, consider morphisms

$$Qf_i : QL \to QK_i\,, \qquad i \in I$$

in $\mathcal{K}_{cf}/\sim$. Let $f : L \to K$ be the induced morphism and $g : L \to R_cK$ be given by the lifting property:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & R_cK \arrow[d, "q_K"] \\ L \arrow[r, "f"'] \arrow[ur, dashed, "g"] & K \end{tikzcd}$$

We have $Q(p_i q_K g) = Qf_i$ for each $i \in I$. The unicity of g follows from the facts that Qq_K is an isomorphism and that left homotopies h_i from $p_i f$ to $p_i f'$, $i \in I$, lift to the left homotopy from f to f'.

Since $\mathcal{K}^{\mathrm{op}}$ is a model category and

$$\mathrm{Ho}(\mathcal{K}^{\mathrm{op}}) = (\mathrm{Ho}(\mathcal{K}))^{\mathrm{op}}\,,$$

$\mathrm{Ho}(\mathcal{K})$ has coproducts.

(ii) In order to show that $\mathrm{Ho}(\mathcal{K})$ has weak colimits, it suffices to prove that it has weak pushouts. In fact, a weak coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & D \end{tikzcd}$$

is given by a weak pushout

$$\begin{tikzcd} B \arrow[r, "h"] & D \\ A \amalg B \arrow[u, "{(f,\mathrm{id}_B)}"] \arrow[r, "{(g,\mathrm{id}_B)}"'] & B \arrow[u, "h"'] \end{tikzcd}$$

and weak colimits are constructed using coproducts and weak coequalizers in the same way as colimits are constructed by coproducts and coequalizers.

Let

$$\begin{tikzcd} B & \\ A \arrow[u, "f"] \arrow[r, "g"'] & D \end{tikzcd}$$

be a diagram in $\mathcal{K}$. Consider a pushout

$$\begin{tikzcd} B_1 \arrow[r, "\overline{g}"] & E \\ A \arrow[u, "f_1"] \arrow[r, "g_1"'] & D_1 \arrow[u, "\overline{f}"'] \end{tikzcd}$$

in $\mathcal{K}$ where $f = f_2 f_1$ and $g = g_2 g_1$ are (cofibration, trivial fibration) factorizations. Then

$$\begin{tikzcd} PB_1 \arrow[r, "P\overline{g}"] & PE \\ PA \arrow[u, "Pf_1"] \arrow[r, "Pg_1"'] & PD \arrow[u, "P\overline{f}"'] \end{tikzcd}$$

is a weak pushout in $\mathrm{Ho}(\mathcal{K})$ which is called the *homotopy pushout* of the starting diagram. Recall that $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ is the canonical functor.

Following [9], we will call the resulting weak colimits in $\mathrm{Ho}(\mathcal{K})$ *standard*. By duality, $\mathrm{Ho}(\mathcal{K})$ has weak limits.

(iii) Consider a diagram $D : \mathcal{D} \to \mathcal{K}$, its colimit $(\bar{\delta}_d : Dd \to \overline{K})$ and $(\delta_d : Dd \to K)$ such that $(P\delta_d : PDd \to PK)$ is a standard weak colimit of PD. There is the comparison morphism $p : K \to \overline{K}$ such that $P(k)\delta_d = P(\bar{\delta}_d)$ for each $d \in \mathcal{D}$. It suffices to find this morphism for a pushout diagram

$$\begin{tikzcd} B \arrow[r, "g'"] & \overline{E} \\ A \arrow[u, "f"] \arrow[r, "g"'] & D \arrow[u, "f'"'] \end{tikzcd}$$

But it is given by $p\overline{g} = g'f_2$ and $p\overline{f} = f'g_2$; we use the notation from (ii).

(iv) There is another construction of weak pushouts in $\mathrm{Ho}(\mathcal{K})$. Consider

$$\begin{tikzcd} B & \\ A \arrow[u, "f"] \arrow[r, "g"'] & D \end{tikzcd}$$

in $\mathcal{K}_{cf}$. Form the *double mapping cylinder* of f, g, i.e., the colimit

$$\begin{tikzcd}
B \arrow[rr, "\overline{g}"] & & E \\
A \arrow[u, "f"] \arrow[r, "\gamma_{1A}"] & C(A) \arrow[ur, "t"] & \\
& A \arrow[u, "\gamma_{2A}"] \arrow[r, "g"'] & D \arrow[uu, "\overline{f}"']
\end{tikzcd}$$

of the diagram

$$\begin{tikzcd}
B & & \\
A \arrow[u, "f"] \arrow[r, "\gamma_{1A}"] & C(A) & \\
& A \arrow[u, "\gamma_{2A}"] \arrow[r, "g"'] & D
\end{tikzcd}$$

where $C(A)$ is the cylinder object. Then

$$\begin{tikzcd}
QB \arrow[r, "P\overline{g}"] & PE \\
QA \arrow[u, "Qf"] \arrow[r, "Qg"'] & QD \arrow[u, "P\overline{f}"']
\end{tikzcd}$$

is a weak pushout in $\mathrm{Ho}(\mathcal{K})$ (cf. [28]).

We will show that homotopy pushouts and double mapping cylinders are naturally weakly equivalent in **SSet**. The double mapping cylinder is given by the pushouts

$$\begin{tikzcd}
B \arrow[r, "j_f"] & E_1 \arrow[r, "\overline{g}_2"] & E \\
A \arrow[u, "f"] \arrow[r, "\gamma_{1A}"'] & C(A) \arrow[u, "f'"'] & \\
& A \arrow[u, "\gamma_{2A}"'] \arrow[r, "g"'] & D \arrow[uu, "\overline{f}"']
\end{tikzcd}$$

The left (square) pushout is called the *mapping cylinder* of f. Since γ_{1A} is a trivial cofibration, j_f is a trivial cofibration too. Since $f\sigma_A\gamma_{1A} = f$, there is a unique morphism

$q_f : E_1 \to B$ with $q_f j_f = \mathrm{id}_B$ and $q_f f' = f\sigma_A$. Thus q_f is a weak equivalence. Since, in **SSet**, $f'\gamma_{2A}$ is a cofibration,

$$q_f(f'\gamma_{2A}) = f\sigma_A\gamma_{2A} = f$$

is a (cofibration, weak equivalence) factorization of f. Hence, following [20] 13.3.4 and 13.3.8, the right (rectangle) pushout is naturally weakly equivalent to the homotopy pushout of f and g.

(v) Another, and very important, colimit construction in model categories are *homotopy colimits* (see, e.g., [6], [13], [20]). Both coproducts and homotopy pushouts described above are instances of this concept. While weak colimits correspond to homotopy commutative diagrams, homotopy colimits corrspond to homotopy coherent ones. So, one cannot expect that homotopy colimits are weak colimits. There is a construction of homotopy colimits using coproducts and homotopy pushouts which is presented in [32] 5.1.1 in the context of Segal categories. It extends the usual construction to higher homotopies and witnesses the fact that homotopy colimits are weak colimits. There is always a morphism $\operatorname{wcolim} D \to \operatorname{hocolim} D$ from the standard weak colimit to the homotopy colimit for each diagram $D : \mathcal{D} \to \mathcal{K}$.

Homotopy colimits tend to have homotopy cofinality properties. For example, if $\mathcal{D}$ has a terminal object d^* then the natural morphism $Dd^* \to \operatorname{hocolim} D$ is a weak equivalence (see [20] 19.6.8(1) or [6] 3.1(iii)). We will show that standard weak colimits have this property in each model category $\mathcal{K}$. As the author knows, this is a new result which will be used to prove the Brown representability property for morphisms (see Theorem 5.7). It also has an impact to a relation between homotopy filtered colimits and homotopy colimits (see Remark 4.4).

4.2. Proposition. *Let $\mathcal{K}$ be a model category, $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ the canonical functor, $\mathcal{D}$ a small category having a terminal object d^*, $D : \mathcal{D} \to \mathcal{K}_c$ a functor and $(P\delta_d : PDd \to PK)$ the standard weak colimit of PD. Then δ_{d^*} is a weak equivalence*[1].

Proof. Consider the construction of K using coproducts and weak coequalizers

$$\begin{tikzcd}
Dd \arrow[d, "u_e"'] \arrow[dr, "v_d"] & & \\
\coprod\limits_{e:d\to d'} Dd \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod\limits_{d} Dd \arrow[r, "h"] & K \\
Dd \arrow[u, "u_e"] \arrow[r, "De"'] & Dd' \arrow[u, "v_{d'}"'] &
\end{tikzcd}$$

where $u_e : Dd \to C$, $e : d \to d'$ in $\mathcal{D}$ and $v_d : Dd \to \coprod\limits_d Dd$, d in $\mathcal{D}$ are coproduct injections and h is a standard weak coequalizer of f and g.

[1]See erratum: *TAC* Vol. 20, No. 2.

I. Assume at first that $\mathcal{K} = \mathbf{SSet}$. This means that h is given by the double mapping cylinder (see 4.1 (iv))

$$\begin{tikzcd}
B \arrow[rr, "h"] & & K \\
A \amalg B \arrow[u, "{(f,\mathrm{id}_B)}"] \arrow[r, "\gamma_{1,A\amalg B}"] & C(A \amalg B) \arrow[ur, "t"] & \\
& A \amalg B \arrow[u, "\gamma_{2,A\amalg B}"'] \arrow[r, "{(g,\mathrm{id}_B)}"'] & B \arrow[uu, "h'"']
\end{tikzcd}$$

where $A = \coprod\limits_e Dd$ and $B = \coprod\limits_d Dd$. We have $\delta_d = hv_d$ for d in $\mathcal{D}$. Since the cylinder functor $C(-) = \Delta_1 \times -$ preserves colimits, we have $C(A \amalg B) = C(A) \amalg C(B)$ and $\gamma_{i,A\amalg B} = \gamma_{iA} \amalg \gamma_{iB}$ for $i = 1, 2$. It is easy to see that $tC(j_2) : C(B) \to K$ is a monomorphism and thus a cofibration; here

$$A \xrightarrow[j_1]{} A \amalg B \xleftarrow[j_2]{} B$$

are coproduct injections. Since σ_B is a weak equivalence, the morphism w in the pushout below is a weak equivalence (see [20] 18.1.2):

$$\begin{tikzcd}
B \arrow[r, "\bar{t}"] & L \\
C(B) \arrow[u, "\sigma_B"] \arrow[r, "tC(j_2)"'] & K \arrow[u, "w"']
\end{tikzcd}$$

Clearly, L appears as a colimit

$$\begin{tikzcd}
B \arrow[rr, "wh"] & & L \\
A \amalg B \arrow[u, "{(f,\mathrm{id}_B)}"] \arrow[r, "\gamma_{1,A}\amalg \mathrm{id}_B"] & C(A) \amalg B \arrow[ur, "{(wt_1,wh)}"'] & \\
& A \amalg B \arrow[u, "\gamma_{2,A}\amalg \mathrm{id}_B"'] \arrow[r, "{(g,\mathrm{id}_B)}"'] & B \arrow[uu, "wh"']
\end{tikzcd}$$

where $t = (t_1, t_2) : C(A) \amalg C(B) \to K$. Let $e_d : d \to d^*$ denote the unique morphism from d to the terminal object d^* in $\mathcal{D}$. Since

$$(De_d)_e \sigma_A \gamma_{1A} = (De_d)_e = (De_d)_d f$$

and

$$(De_d)_e \sigma_A \gamma_{2A} = (De_d)_e = \big(D(e_{d'}e)\big)_e = (De_d)_d g\,,$$

we get a unique morphisms $p : L \to Dd^*$ such that $pwh = (De_d)_d$ and $pwt_1 = (De_d)_{e:d\to d'} \cdot \sigma_A$. For the morphism $whv_{d^*} : Dd^* \to L$ we have $p(whv_{d^*}) = \mathrm{id}_{Dd^*}$. It suffices to find a left homotopy $q : C(L) \to L$ such that $q\gamma_{1L} = \mathrm{id}_L$ and $q\gamma_{2L} = (whv_{d^*})p$. In this case, $P(whv_{d^*}p)$ is an isomorphism and thus $w\delta_{d^*} = whv_{d^*}$ is a weak equivalence. Therefore δ_{d^*} is a weak equivalence.

We have a colimit

$$\begin{tikzcd}
C(B) \arrow[rr, "C(wh)"] & & C(L) \\
C(A) \amalg C(B) \arrow[u, "{(C(f),\mathrm{id}_{CB})}"] \arrow[r, "C(\gamma_{1,A}\amalg \mathrm{id})"] & CC(A) \amalg C(B) \arrow[ur, "{(C(wt_1),C(wh))}"'] & \\
& C(A) \amalg C(B) \arrow[u, "C(\gamma_{2,A}\amalg \mathrm{id})"'] \arrow[r, "{(C(g),\mathrm{id}_B)}"'] & C(B) \arrow[uu, "C(wh)"']
\end{tikzcd}$$

The desired morphism $q : C(L) \to L$ will be uniquely determined by the composition $\overline{q} = qC(wt_1) : CC(A) \to L$, i.e., by the morphisms

$$\overline{q}_e = \overline{q}CC(u_e) : CC(Dd) \to L$$

where $e : d \to d'$ is in $\mathcal{D}$. The simplicial set $CC(Dd) = \Delta_1 \times \Delta_1 \times Dd$ has points (ik, x) where $i, k = 0, 1$ and x is a point of Dd. Then $\overline{q}_e(0k, x) = wh(x)$ and $\overline{q}_e(1k, x) = whv_{d^*}D(e_d)(x)$. Edges (= 1-simplices) of $\Delta_1 \times \Delta_1 \times Dd$ are (s_1, s) where s_1 is an edge of $\Delta_1 \times \Delta_1$ and s is an edge of Dd. Now,

$$\overline{q}_e(s_1, s) = \begin{cases} wh(s) & \text{for} \quad s_1 = (00, 01) \\ whv_{d^*}D(e_d)(s) & \text{for} \quad s_1 = (10, 11) \\ wt_1 C(u_{e_d})(s) & \text{otherwise.} \end{cases}$$

Recall that $\Delta_1 \times \Delta_1$ has the following edges

$$\begin{tikzcd}
10 \arrow[r] & 11 \\
00 \arrow[u] \arrow[ur] \arrow[r] & 01 \arrow[u]
\end{tikzcd}$$

Analogously, we define $\overline{q}_e$ on n-simplices for $n > 1$.

II. Assume now that $\mathcal{K}$ is the functor category $\mathbf{SSet}^{\mathcal{X}}$ (where $\mathcal{X}$ is a small category) with the Bousfield-Kan model category structure. This means that (trivial) cofibrations in $\mathcal{K}$ are generated by $F_X(\mathcal{C})$ $\big(F_X(\mathcal{C}_0)\big)$ where

$$F_X : \mathbf{SSet} \to \mathcal{K}$$

is left adjoint to the evaluation $ev_X : \mathcal{K} \to \mathbf{SSet}$ at $X \in \mathcal{X}$ (i.e., $ev_X(A) = A(X)$). Hence any generating cofibration $F_X(f)$, $f : K \to L$ is a pointwise cofibration because

$$\big(F_X(f)\big)_Y = \coprod_{\mathcal{X}(X,Y)} f : \coprod_{\mathcal{X}(X,Y)} K \to \coprod_{\mathcal{X}(X,Y)} L\,.$$

Since the evaluation functors ev_X preserves colimits, every cofibration φ in $\mathcal{K}$ is a pointwise cofibration (i.e., φ_X, $X \in \mathcal{X}$ are cofibrations). Since homotopy pushouts are homotopy invariant in **SSet** (see [20] 13.3.4), they do not depend on a choice of a (cofibration, trivial fibration) factorization. Consequently, the homotopy pushouts, and thus the standard weak colimits, in $\mathcal{K}$ are pointwise in the sense that $\big(\delta_X : (Dd)(X) \to K(X)\big)$ is a standard weak colimit for each $X \in \mathcal{X}$ provided that $(\delta : Dd \to K)$ is a standard weak colimit in $\mathcal{K}$. Since weak equivalences in $\mathcal{K}$ are precisely pointwise weak equivalences, the claim follows from I.
III. Let $\mathcal{K}$ be an arbitrary model category. Following [11], there is a left Quillen functor $H : \mathbf{SSet}^{\mathcal{D}^{\mathrm{op}}} \to \mathcal{K}$ such that $HY = D$ where $Y : \mathcal{D} \to \mathbf{SSet}^{\mathcal{D}^{\mathrm{op}}}$ is given by taking the discrete simplicial presheaves. Since H is left Quillen, it preserves colimits, (trivial) cofibrations and weak equivalences between cofibrant objects (see [22]). Since discrete simplicial presheaves are cofibrant, H preserves the standard weak colimit $(\delta_d : YDd \to K)_d$ of $YD : \mathcal{D} \to \mathbf{SSet}^{\mathcal{D}^{\mathrm{op}}}$. Since cofibrant objects are closed under coproducts and homotopy pushouts, K is cofibrant. Hence $H(\delta_{d^*})$ is a weak equivalence, which proves the claim. ∎

4.3. Proposition. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ) and (G^3_λ). Let $D : \mathcal{D} \to \mathcal{K}_c$ a λ-filtered diagram where* $\operatorname{card}\mathcal{D} \geq \lambda$. *Then the comparison morphism from 4.1(iii) is a weak equivalence*[2].

Proof. Since $\mathcal{D}$ is λ-filtered and $\operatorname{card}\mathcal{D} \geq \lambda$, $\mathcal{D}$ is a λ-directed union of subcategories $\mathcal{E}$ such that $\operatorname{card}\mathcal{E} < \lambda$ and $\mathcal{E}$ has a terminal object $d_\mathcal{E}$. Let $D_\mathcal{E} : \mathcal{E} \to \mathcal{K}_c$ be the domain restriction of D. Following 4.2 the comparison morphism $p_\mathcal{E} : K_\mathcal{E} \to \overline{K}_\mathcal{E}$ for $D_\mathcal{E}$ is a weak equivalence for each $\mathcal{E}$ (because $p_\mathcal{E} = \delta_{d_\mathcal{E}}$ in this case). Following (G^1_λ), the formation of homotopy pushouts preserves λ-filtered colimits. Therefore $K = \operatorname{colim}_{\mathcal{E}} K_\mathcal{E}$ and, of course, $\overline{K} = \operatorname{colim}_{\mathcal{E}} \overline{K}_\mathcal{E}$. Consequently, $p = \operatorname{colim}_{\mathcal{E}} p_\mathcal{E}$ and, following (G^3_λ), p is a weak equivalence. ∎

4.4. Remark. (1) We have not needed the full strength of (G^1_λ) – it suffices to assume that F is λ-accessible.

(2) Under (G^3_λ), homotopy λ-filtered colimits are the same as λ-filtered colimits (in the sense that the comparison morphism from 4.1 (v) is a weak equivalence), see, e.g., [12], the proof of 4.7. The consequence is that, in locally λ-presentable model categories satisfying (G^1_λ) and (G^3_λ), standard weak λ-filtered colimits and homotopy λ-filtered colimits of diagrams $D : \mathcal{D} \to \mathcal{K}$ with $\operatorname{card}\mathcal{D} \geq \lambda$ coincide.

[2] See erratum: *TAC* Vol. 20, No. 2.

5. Brown model categories

Given a small, full subcategory $\mathcal{A}$ of a category $\mathcal{K}$, the *canonical functor*

$$E_{\mathcal{A}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

assigns to each object K the restriction

$$E_{\mathcal{A}}K = \hom(-, K)/\mathcal{A}^{\mathrm{op}}$$

of its hom-functor $\hom(-, K) : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ to $\mathcal{A}^{\mathrm{op}}$ (see [1] 1.25). This functor is (a) $\mathcal{A}$-*full* and (b) $\mathcal{A}$-*faithful* in the sense that

(a) for every $f : E_{\mathcal{A}}A \to E_{\mathcal{A}}K$ with A in $\mathcal{A}$ there is $f' : A \to K$ such that $E_{\mathcal{A}}f' = f$ and

(b) $E_{\mathcal{A}}f = E_{\mathcal{A}}g$ for $f, g : A \to K$ with A in $\mathcal{A}$ implies $f = g$.

Let $\mathcal{K}$ be a locally λ-presentable model category and denote by $\mathrm{Ho}(\mathcal{K}_\lambda)$ the full subcategory $P(\mathcal{K}_\lambda)$ of $\mathrm{Ho}(\mathcal{K})$ consisting of P-images of λ-presentable objects in $\mathcal{K}$ in the canonical functor $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$. Let E_λ denote the canonical functor $E_{\mathrm{Ho}(\mathcal{K}_\lambda)}$.

5.1. Proposition. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^2_λ) and (G^4_λ). Then the composition*

$$E_\lambda P : \mathcal{K} \to \mathbf{Set}^{\mathrm{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}}}$$

preserves λ-filtered colimits.

Proof. Consider a λ-filtered diagram $D : \mathcal{D} \to \mathcal{K}$ and its colimit $(k_d : Dd \to K)$ in $\mathcal{K}$. Since R preserves λ-filtered colimits, $(Rk_d : RDd \to RK)$ is a λ-filtered colimit. Let $X \in \mathcal{K}_\lambda$ and $f : PX \to PK$ be a morphism in $\mathrm{Ho}(\mathcal{K})$. Then $f = Q\overline{f}$ for $\overline{f} : RX \to RK$ and, since R preserves λ-presentable objects, RX is λ-presentable in $\mathcal{K}$. Thus $\overline{f} = R(k_d)\overline{g}$ for some $\overline{g} : RX \to RDd$, $d \in \mathcal{D}$.

Assume that $Q\big(R(k_d)\overline{g}_1\big) = Q\big(R(k_d)\overline{g}_2\big)$ for $\overline{g}_1, \overline{g}_2 : RX \to RDd$. Then $R(k_d)\overline{g}_1$ and $R(k_d)\overline{g}_2$ are homotopy equivalent and thus $R(k_d)\overline{g}_i = h\gamma_{iR(X)}$ for $i = 1, 2$. Since CRX is λ-presentable as well, there is $e : d \to d'$ in $\mathcal{D}$ and $\overline{h} : CRX \to RDd'$ such that $R(k_{d'})\overline{h} = h$ and $\overline{h}\gamma_{iR(X)} = RD(e)\overline{g}_i$. Therefore $Q\big(RD(e)\overline{g}_1\big) = Q\big(RD(e)\overline{g}_2\big)$. We have proved that

$$\big(\hom(PX, P(k_d)) : \hom(PX, PDd) \to \hom(PX, PK)\big)_{d \in \mathcal{D}}$$

is a λ-filtered colimit in $\mathbf{Set}$. Consequently,

$$\big(E_\lambda P(k_d) : E_\lambda PDd \to E_\lambda PK\big)$$

is a λ-filtered colimit in $\mathbf{Set}^{\mathrm{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}}}$. ∎

Let $P_\lambda : \mathcal{K}_\lambda \to \operatorname{Ho}(\mathcal{K}_\lambda)$ denote the domain and codomain restriction of the canonical functor $P : \mathcal{K} \to \operatorname{Ho}(\mathcal{K})$. We get the induced functor

$$\operatorname{Ind}_\lambda P_\lambda : \mathcal{K} = \operatorname{Ind}_\lambda \mathcal{K}_\lambda \to \operatorname{Ind}_\lambda \operatorname{Ho}(\mathcal{K}_\lambda)\,.$$

5.2. Corollary. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^2_λ) and (G^4_λ). Then $E_\lambda P \cong \operatorname{Ind}_\lambda P_\lambda$.*

Remark.This means that E_λ factorizes through the inclusion

$$\operatorname{Ind}_\lambda \operatorname{Ho}(\mathcal{K}_\lambda) \subseteq \mathbf{Set}^{\operatorname{Ho}(\mathcal{K}_\lambda)^{\mathrm{op}}}$$

and that the codomain restriction of E_λ, which we denote E_λ as well, makes the composition $E_\lambda P$ isomorphic to $\operatorname{Ind}_\lambda P_\lambda$.

Proof. Since both $E_\lambda P$ and $\operatorname{Ind}_\lambda P_\lambda$ have the same domain restriction on $\mathcal{K}_\lambda$, the result follows from 5.1. ∎

For $\lambda < \mu$ we get a unique functor

$$F_{\lambda\mu} : \operatorname{Ind}_\mu(\operatorname{Ho}(\mathcal{K}_\mu)) \to \operatorname{Ind}_\lambda(\operatorname{Ho}(\mathcal{K}_\lambda))$$

which preserves μ-filtered colimits and whose domain restriction on $\operatorname{Ho}(\mathcal{K}_\mu)$ coincides with that of E_λ.

5.3. Corollary. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^2_μ) and (G^4_μ) for a regular cardinal $\lambda < \mu$. Then $F_{\lambda\mu} E_\mu \cong E_\lambda$.*

Proof. Following 5.2, we have $E_\mu \cong \operatorname{Ind}_\mu(P_\mu)$ and thus the functors $F_{\lambda\mu}E_\mu P \cong F_{\lambda\mu}\operatorname{Ind}_\mu(P_\mu)$ and $E_\lambda P$ have the isomorphic domain restrictions on $\mathcal{K}_\mu$. We will show that the functor $E_\lambda P$ preserves μ-filtered colimits. Since $F_{\lambda\mu}\operatorname{Ind}_\mu(P_\mu)$ has the same property, we will obtain that $F_{\lambda\mu}E_\mu P \cong E_\lambda P$ and thus $F_{\lambda\mu}E_\mu \cong E_\lambda$.

The functor $E_\lambda P$ preserves μ-filtered colimits iff for every object A in $\mathcal{K}_\lambda$ the functor

$$hom(PA, P-) : \mathcal{K} \to \mathbf{Set}$$

preserves μ-filtered colimits. Since $\mathcal{K}_\lambda \subseteq \mathcal{K}_\mu$, this follows from 5.1. ∎

5.4. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ), (G^2_λ) and (G^4_λ). Then the functor*

$$E_\lambda : \operatorname{Ho}(\mathcal{K}) \to \operatorname{Ind}_\lambda \operatorname{Ho}(\mathcal{K}_\lambda)$$

is essentially surjective on objects.

PROOF. Consider X in $\mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda)$ and express it as a canonical λ-filtered colimit $(\delta_d : Dd \to X)$ of objects from $\mathrm{Ho}(\mathcal{K}_\lambda)$. This means that $\mathcal{D} = \mathrm{Ho}(\mathcal{K}_\lambda) \downarrow X$ and $D : \mathrm{Ho}(\mathcal{K}_\lambda) \downarrow X \to \mathrm{Ho}(\mathcal{K}_\lambda)$ is the projection. Since E_λ is $\mathrm{Ho}(\mathcal{K}_\lambda)$-full and faithful, D lifts along E_λ, i.e., $D = E_\lambda \overline{D}$ for $\overline{D} : \mathcal{D} \to \mathrm{Ho}(\mathcal{K})$. Let $(\overline{\delta}_d : \overline{D}d \to K)$ be a standard weak colimit of $\overline{D}$. Moreover, $\overline{D}$ is the canonical diagram of K w.r.t. $\mathrm{Ho}(\mathcal{K}_\lambda)$. There is $\overline{K}$ in $\mathcal{K}_{cf}$ such that $P\overline{K} = K$. Let $\overline{\overline{D}} : \overline{\mathcal{D}} = \mathcal{K}_\lambda \downarrow \overline{K} \to \mathcal{K}_\lambda$ be the canonical diagrams of $\overline{K}$ w.r.t. $\mathcal{K}_\lambda$ and $\overline{P} : \overline{\mathcal{D}} \to \mathcal{D}$ the functor induced by $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$, i.e., $\overline{P}(f) = P(f)$ for each $f : A \to \overline{K}$, $A \in \mathcal{K}_\lambda$. Since R preserves λ-presentable objects, $\overline{P}$ is surjective on objects.

We will show that $\overline{P}$ is essentially full in the sense that for every $h : f \to g$ in $\mathcal{D}$ there is $\overline{h} : \overline{f} \to \overline{g}$ in $\overline{\mathcal{D}}$ such that there is a commutative triangle

$$\begin{tikzcd} f = \overline{P}\,\overline{f} \arrow[r, "h"] \arrow[dr, "\overline{P}\,\overline{h}"'] & g \\ & \overline{P}\,\overline{g} \arrow[u, "v"'] \end{tikzcd}$$

for some isomorphism v in $\mathcal{D}$. Indeed, we have a homotopy commutative triangle

$$\begin{tikzcd} A \arrow[r, "\overline{f}"] \arrow[d, "h'"'] \arrow[dr, phantom, "\sim"] & \overline{K} \\ B \arrow[ur, "g'"'] & \end{tikzcd}$$

where $f = \overline{P}(\overline{f})$, $g = \overline{P}(g')$, $h = P(h')$ and $A, B \in \mathcal{K}_\lambda \cap \mathcal{K}_{cf}$. Let

$$A \xrightarrow{\overline{h}} Z \xrightarrow{\overline{v}} B$$

be the (cofibration, trivial fibration) factorization of h'. Following (G^1_λ), we have $Z \in \mathcal{K}_\lambda \cap \mathcal{K}_{cf}$. Since $g'\overline{v}\overline{h} \sim \overline{f}$ and $A, B, \overline{K} \in \mathcal{K}_{cf}$, there is $\overline{g} : Z \to \overline{K}$ such that $\overline{g} \sim g'\overline{v}$ and $\overline{g}\overline{h} = \overline{f}$. Here, we use the homotopy extension property of cofibrations ([23] 7.3.12 (1)) and the fact that right and left homotopy coincide on $\mathcal{K}_{cf}$ ([22]). Thus $\overline{h} : \overline{f} \to \overline{g}$ is in $\overline{\mathcal{D}}$, $v = P(\overline{v})$ is an isomorphism and

$$v\overline{P}(\overline{h}) = P(\overline{v}\,\overline{h}) = P(h') = h\,.$$

Since $\overline{P}$ is essentially full and surjective on objects, it is essentially cofinal in the sense that, given $h_1 : f \to \overline{P}g_1$ and $h_2 : f \to \overline{P}g_2$, there is a commutative diagram

$$\begin{tikzcd}
f = \overline{P}\,\overline{f} \arrow[rr, "h_1"] \arrow[drr, "\overline{P}(\overline{h}_1)"] \arrow[dd, "h_2"'] \arrow[ddr, "\overline{P}(\overline{h}_2)"] & & \overline{P}g_1 \arrow[d, "v_1^{-1}"] \\
& & \overline{P}(\overline{g}_1) \arrow[d, "\overline{P}(u_1)"] \\
\overline{P}(g_2) \arrow[r, "v_2^{-1}"'] & \overline{P}(\overline{g}_2) \arrow[r, "\overline{P}(u_2)"'] & \overline{P}(\overline{g})
\end{tikzcd}$$

The existence of morphisms u_1 and u_2 follows from $\overline{\mathcal{D}}$ being λ-filtered. Consequently, following 5.2,

$$\begin{aligned} E_\lambda P\overline{K} &\cong (\operatorname{Ind}_\lambda P_\lambda)(\overline{K}) = \operatorname{colim} YP\overline{\overline{D}} = \operatorname{colim} Y\overline{D}\,\overline{P} \\ &\cong \operatorname{colim} Y\overline{D} = \operatorname{colim} E_\lambda\overline{D} = \operatorname{colim} D = X\,. \end{aligned}$$

■

5.5. Definition. A locally λ-presentable model category $\mathcal{K}$ will be called λ-*Brown on objects*, where λ is a regular cardinal, provided that the codomain restriction $E_\lambda : \operatorname{Ho}(\mathcal{K}) \to \operatorname{Ind}_\lambda(\operatorname{Ho}(\mathcal{K}_\lambda))$ of E_λ is essentially surjective on objects.

$\mathcal{K}$ will be called λ-*Brown on morphisms* provided that $E_\lambda : \operatorname{Ho}(\mathcal{K}) \to \operatorname{Ind}_\lambda(Ho(\mathcal{K}_\lambda))$ is full.

$\mathcal{K}$ will be called λ-*Brown* when it is both λ-Brown on objects and on morphisms. It will be called *Brown* if it is λ-Brown for some regular cardinal λ.

5.6. Remark. (i) Theorem 5.4 says that, under (G^1_λ), (G^2_λ) and (G^4_λ), a locally λ-presentable model category $\mathcal{K}$ is λ-Brown on objects. To get an analogous result for $\mathcal{K}$ being λ-Brown on morphisms, we will use the Proposition 4.3. Probably, there is an alternative way of using [12] to get that each object of $\operatorname{Ho}(\mathcal{K})$ is a canonical homotopy λ-filtered colimit of objects from $\operatorname{Ho}(\mathcal{K}_\lambda)$. Moreover, since λ is large enough to make the canonical diagram to contain all information about higher homotopies, the canonical λ-filtered colimits are weak colimits (cf. [12]).

(ii) Whenever a compactly generated triangulated category is ω-Brown on morphisms then it is ω-Brown on objects (see [4], 11.8). In our setting, we have such a result provided that $\mathcal{K}$ is a locally finitely presentable model category satisfying the conditions (G^2_ω) and (G^4_ω). Then, by 5.2, we have $E_\omega P \cong \operatorname{Ind}_\omega P_\omega$. Now, if $\mathcal{K}$ is ω-Brown on morphisms then the functor $\operatorname{Ind}_\omega P_\omega$ is full. Since each object of $\operatorname{Ind}_\omega(\mathcal{K}_\omega)$ can be obtained by iterative taking of colimits of smooth chains (see [1]) and P_ω is essentially surjective on objects, $\operatorname{Ind}_\omega P_\omega$ is essentially surjective on objects as well. Hence $\mathcal{K}$ is ω-Brown on objects. This argument does not work for $\lambda > \omega$ because, in the proof, we need colimits of chains of cofinality ω. Thus, due to the condition (G^2_ω), this result is of a limited importance.

5.7. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ), (G^2_λ) and (G^3_λ). Then the functor*

$$E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$$

is full, i.e., $\mathcal{K}$ is λ-Brown on morphisms.

Proof. Consider a morphism $f : E_\lambda P\overline{K} \to E_\lambda PL$ and express $\overline{K}$ as a λ-filtered colimit $(\overline{\delta}_d : Dd \to \overline{K})$, $D : \mathcal{D} \to \mathcal{K}_\lambda$ of λ-presentable objects. Since E_λ is $\mathrm{Ho}(\mathcal{K}_\lambda)$-full, we may assume that $\overline{K}$ is not λ-presentable, i.e., that $\mathrm{card}\,\mathcal{D} \geq \lambda$. Let $(\delta_d : PDd \to PK)$ be a standard weak colimit of PD. Following 4.3, we may assume that $P\overline{K} = PK$ and $P(\overline{\delta}_d) = \delta_d$ for each d in $\mathcal{D}$. Following 5.1, $(E_\lambda\delta_d : E_\lambda PDd \to E_\lambda P\overline{K})$ is a λ-filtered colimit. Since E_λ is $\mathrm{Ho}(\mathcal{K}_\lambda)$-full and faithful, there is a compatible cocone $(f_d : PDd \to PL)_{d\in\mathcal{D}}$ such that $E_\lambda(f_d) = fE_\lambda(\delta_d)$ for each $d \in \mathcal{D}$. Since $(\delta_d : PDd \to P\overline{K})$ is a weak colimit, there is $g : P\overline{K} \to PL$ with $g\delta_d = f_d$ for each $d \in \mathcal{D}$. Hence

$$E_\lambda(g)E_\lambda(\delta_d) = E_\lambda(f_d) = fE_\lambda(\delta_d)$$

for each $d \in \mathcal{D}$ and thus $E_\lambda(g) = f$. ∎

5.8. Corollary. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ), (G^2_λ) and (G^3_λ). Then $\mathcal{K}$ is λ-Brown.*

5.9. Corollary. *Every combinatorial model category is Brown.*

5.10. Corollary. *The model categories* **SSet** *and* **Sp** (*with the stable model structure*) *are ω_1-Brown.*

Proof. It follows from 5.8 and 3.6. ∎

5.11. Remark. (1) In fact, both **SSet** and **Sp** are λ-Brown for every $\omega_1 \triangleleft \lambda$. We do not know whether **SSet** is ω-Brown (see the Introduction). Following [2], **Sp** is also ω-Brown.

(2) If $\mathcal{K}$ is a locally finitely presentable model category such that $\mathrm{Ho}(\mathcal{K})$ is a stable homotopy category in the sense of [24] then $\mathcal{K}$ is ω-Brown in our sense iff $\mathrm{Ho}(\mathcal{K})$ is Brown in the sense of [24].

(3) Let $\mathcal{K}$ be a λ-Brown model category and $D : \mathcal{D} \to \mathrm{Ho}(\mathcal{K}_\lambda)$ be a λ-filtered diagram. Let $(k_d : E_\lambda Dd \to K)_{d\in\mathcal{D}}$ be a colimit of $E_\lambda D$ in $\mathrm{Ind}_\lambda \mathrm{Ho}(\mathcal{K}_\lambda))$. Since $\mathcal{K}$ is λ-Brown, we have $K = E_\lambda\overline{K}$ and $k_d = E_\lambda\overline{k}_d$ for each $d \in \mathcal{D}$. Since E_λ is $\mathrm{Ho}(\mathcal{K}_\lambda)$-faithful, $(\overline{k}_d : Dd \to \overline{K})$ is a cone in $\mathrm{Ho}(\mathcal{K})$. Let $l_d : Dd \to L$ be another cone. There is a unique $t : K \to E_\lambda L$ such that $tk_d = E_\lambda l_d$ for each $d \in \mathcal{D}$. Since $\mathcal{K}$ is λ-Brown, we have $t = E_\lambda\overline{t}$ where $\overline{t} : \overline{K} \to L$. Since E_λ is $\mathrm{Ho}(\mathcal{K}_\lambda)$-faithful, $t\overline{k}_d = l_d$ for each $d \in \mathcal{D}$. Hence $\overline{k}_d : Dd \to \overline{K}$ is a weak colimit. We will call this weak colimit *minimal*.

Every object of $\mathrm{Ho}(\mathcal{K})$ is a minimal λ-filtered colimit of objects from $\mathrm{Ho}(\mathcal{K}_\lambda)$.

(4) $\mathcal{K}$ being λ-Brown can be viewed as a weak λ-accessibility of $\mathrm{Ho}(\mathcal{K})$ because $\mathrm{Ho}(\mathcal{K})$ is λ-accessible with $\mathrm{Ho}(\mathcal{K})_\lambda = \mathrm{Ho}(\mathcal{K}_\lambda)$ iff $E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ is an equivalence.

5.12. Definition. Let $\mathcal{K}$ be a model category. Morphisms $f, g : K \to L$ in $\mathrm{Ho}(\mathcal{K})$ will be called *λ-phantom equivalent* if $E_\lambda f = E_\lambda g$.

This means that $f, g : K \to L$ are λ-phantom equivalent iff $fh = gh$ for each morphism $h : A \to K$ with $A \in \mathrm{Ho}(\mathcal{K}_\lambda)$.

5.13. Proposition. *Let $\mathcal{K}$ be a λ-Brown model category. Then for each object K in $\mathrm{Ho}(\mathcal{K})$ there exists a weakly initial λ-phantom equivalent pair $f, g : K \to L$.*

Proof. Express K as a minimal weak λ-filtered colimit $(k_d : Dd \to K)_{d\in\mathcal{D}}$ of objects from $\mathrm{Ho}(\mathcal{K}_\lambda)$, take the induced morphism $p : \coprod\limits_{d\in\mathcal{D}} Dd \to K$ and its weak cokernel pair f, g

$$\coprod Dd \xrightarrow{p} K \underset{g}{\overset{f}{\rightrightarrows}} L\,.$$

Since the starting weak colimit is minimal, $E_\lambda p$ is an epimorphism in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$. Thus f and g are λ-phantom equivalent.

Let $f', g' : K \to L'$ be a λ-phantom equivalent. Then $f'p = g'p$ and thus the pair f', g' factorizes through f, g. Thus f, g is a weakly initial λ-phantom equivalent pair. ∎

5.14. Examples. We will show that the homotopy categories

$$\mathrm{Ho}(\mathbf{SSet}_n)$$

are finitely accessible for each $n = 1, 2, \dots$, i.e., that E_ω is an equivalence in this case. Recall that $\mathbf{SSet}_n = \mathbf{Set}^{\mathbf{\Delta}_n}$ where $\mathbf{\Delta}_n$ is the category of ordinals $\{1, 2, \dots, n\}$. The model category structure is the truncation of that on simplicial sets, i.e., cofibrations are monomorphisms and trivial cofibrations are generated by the horn inclusions

$$j_m : \Delta_m^k \to \Delta_m \qquad 0 < k \le m \le n.$$

Here, $\Delta_m = Y_n(m+1)$ where $Y_n : \mathbf{\Delta}_n \to \mathbf{SSet}_n$ is the Yoneda embedding for $m < n$ and Δ_n is $Y_n(n+1)$ without the $(n+1)$-dimensional simplex $\{0, 1, \dots, n\}$.

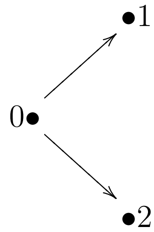
For example $\mathbf{SSet}_1 = \mathbf{Set}$ and trivial cofibrations are generated by $j_1 : 1 \to 2$. Then weak equivalences are precisely mappings between non-empty sets and $\mathrm{Ho}(\mathbf{SSet}_1)$ is the category $\mathbf{2}$; all non-empty sets are weakly equivalent. $\mathbf{SSet}_2$ is the category of oriented multigraphs with loops. Trivial cofibrations are generated by the embedding j_1 of

$$\bullet 0$$

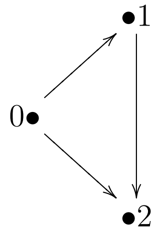
to

$$0\bullet \longrightarrow \bullet\, 1$$

(degenerated loops are not depicted), by the embedding j_2 of



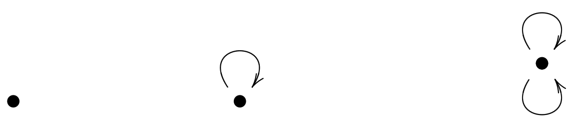
to



and their orientation variants. This makes all connected multigraphs weakly equivalent and $\mathrm{Ho}(\mathbf{SSet}_2)$ is equivalent to **Set**; the cardinality of a set corresponds to the number of connected components.

In the case of $\mathbf{SSet}_3$, 1-connected objects cease to be weakly equivalent and their contribution to $\mathrm{Ho}(\mathbf{SSet}_3)$ are trees (with a single root) of height ≤ 2. For example,

correspond to



(degenerated loops are not depicted). Therefore $\mathrm{Ho}(\mathbf{SSet}_3)$ is equivalent to the category of forests of height ≤ 2. Analogously $\mathrm{Ho}(\mathbf{SSet}_n)$ is equivalent to the category of forests of height $\leq n$. Hence it is finitely accessible.

Let us add that $\mathbf{SSet}_2$ is a natural model category of oriented multigraphs with loops (cf. [31]) and that the symmetric variants $\mathbf{Set}^{\mathbf{F}_n^{\mathrm{op}}}$, where $\mathbf{F}_n$ is the category of cardinals $\{1, \ldots, n\}$, are Quillen equivalent to $\mathbf{SSet}_n$ and left-determined by monomorphisms in the sense of [39].

6. Strongly Brown model categories

6.1. Proposition. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^2_μ) and (G^4_μ) for a regular cardinal $\lambda < \mu$. Then E_μ reflects isomorphisms provided that E_λ reflects isomorphisms.*

Proof. It follows from 5.3. ∎

6.2. Definition. A λ-Brown model category $\mathcal{K}$ will be called *strongly λ-Brown* if, in addition, E_λ also reflects isomorphisms.

$\mathcal{K}$ will be called *strongly Brown* if it is strongly λ-Brown for some regular cardinal λ.

6.3. Remark. (1) For strongly λ-Brown model categories, any minimal λ-filtered colimit $k_d : Dd \to K$ has the property that each endomorphism $t : K \to K$ satisfying $tk_d = k_d$ for each $d \in \mathcal{D}$ is an isomorphism. Thus $k_d : Dd \to K$ is a minimal colimit in the sense of [24]. Minimal colimits are determined uniquely up to an isomorphism. Another possible terminology, going back to [19], is a *stable weak colimit*.

(2) Following [24], 5.8 and 6.1, $\mathbf{Sp}$ is strongly λ-Brown for each $\omega_1 \triangleleft \lambda$. Therefore, following (3), $\mathrm{Ho}(\mathbf{Sp})$ has minimal λ-filtered colimits of objects from $\mathrm{Ho}(\mathbf{Sp}_\lambda)$ for each $\omega_1 \triangleleft \lambda$. This is known for $\lambda = \omega$ (see [34] and [24]) and there is still an open problem whether $\mathrm{Ho}(\mathbf{Sp})$ has all minimal filtered colimits.

M. Hovey [22] introduced the concept of a pre-triangulated category (distinct from that used in [35]) and showed that the homotopy category of every pointed model category is pre-triangulated in his sense. He calls a pointed model category $\mathcal{K}$ *stable* if $\mathrm{Ho}(\mathcal{K})$ is triangulated. In particular, $\mathcal{K}$ is stable provided that $\mathrm{Ho}(\mathcal{K})$ is a stable homotopy category in the sense of [24].

6.4. Proposition. *Every combinatorial stable model category is strongly Brown.*

Proof. We know that $\mathcal{K}$ is Brown (see 5.9) Following [22] 7.3.1, every combinatorial pointed model category $\mathcal{K}$ has a set $\mathcal{G}$ of weak generators. Let $\Sigma^* = \{\Sigma^n Z | Z \in \mathcal{G}, n \in \mathbf{Z}\}$. Following [35] 6.2.9, there is a regular cardinal λ such that E_λ reflects isomorphisms. ∎

6.5. Proposition. *Let $\mathcal{K}$ be a strongly λ-Brown model category. Then the functor E_λ : $\mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ preserves (existing) λ-filtered colimits of objects from $\mathrm{Ho}(\mathcal{K}_\lambda)$.*

Proof. Let $D : \mathcal{D} \to \mathrm{Ho}(\mathcal{K}_\lambda)$ be a λ-filtered diagram, $(k_d : Dd \to K)$ its colimit and $(\overline{k}_d : Dd \to \overline{K})$ its minimal colimit. We get morphisms $f : K \to \overline{K}$ and $g : \overline{K} \to K$ such that $fk_d = \overline{k}_d$ and $g\overline{k}_d = k_d$ for each d in $\mathcal{D}$. We have $gf = \mathrm{id}_K$ and $E_\lambda(fg) = \mathrm{id}_{E_\lambda \overline{K}}$. Therefore $E_\lambda f = (E_\lambda g)^{-1}$ and thus f is an isomorphism. Therefore λ-filtered colimits and minimal λ-filtered colimits of objects from $\mathrm{Ho}(\mathcal{K}_\lambda)$ coincide and the latter are sent by E_λ to λ-filtered colimits. ∎

6.6. Corollary. *Let $\mathcal{K}$ be a locally λ-presentable model category. Then* $\mathrm{Ho}(\mathcal{K})$ *is λ-accessible iff $\mathcal{K}$ is strongly λ-Brown and* $\mathrm{Ho}(\mathcal{K})$ *has λ-filtered colimits.*

6.7. Remark. If $\mathcal{K}$ is a strongly λ-Brown category then the functor $E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ is *essentially injective on objects* in the sense that $E_\lambda K \cong E_\lambda L$ implies that $K \cong L$. This means that the isomorphism classification is the same in $\mathrm{Ho}(\mathcal{K})$ and in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$.

6.8. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable model category such that E_λ is essentially injective on objects and (G^1_μ), (G^2_μ) and (G^4_μ) hold for some $\lambda \lhd \mu$. Then every weakly continuous functor $H : \mathrm{Ho}(\mathcal{K})^{\mathrm{op}} \to \mathbf{Set}$ is representable.*

Proof. Let $H : \mathrm{Ho}(\mathcal{K})^{\mathrm{op}} \to \mathbf{Set}$ be weakly continuous. Then each domain restriction $H_\alpha : \mathrm{Ho}(\mathcal{K}_\alpha)^{\mathrm{op}} \to \mathbf{Set}$ is left α-covering. Following 3.4 and 5.4, E_α is essentially surjective on objects for each $\mu \lhd \alpha$. Thus, for each $\mu \lhd \alpha$, we have A_α in $\mathrm{Ho}(\mathcal{K})$ such that $H_\alpha \cong \hom(-, A_\alpha)$. Therefore, following 5.3, we have

$$E_\mu(A_\alpha) \cong F_{\mu\alpha} E_\alpha(A_\alpha) \cong F_{\mu\alpha}(H_\alpha) \cong H_\mu \cong E_\mu(A_\mu).$$

Since E_λ is essentially injective on objects, E_μ has the same property (following 5.3) and thus $A_\alpha \cong A_\mu$ for each $\mu \lhd \alpha$. This implies that $H \cong \hom(-, A_\mu)$. ∎

6.9. Remark. The property that every weakly continuous functor $H : \mathrm{Ho}(\mathcal{K})^{\mathrm{op}} \to \mathbf{Set}$ is representable is called *the Brown representability for cohomology*, while the property that every left covering functor $H : \mathrm{Ho}(\mathcal{K}_\omega)^{\mathrm{op}} \to \mathbf{Set}$ is representable is called *the Brown representability for homology*, see [10]. The consequence of 6.8 is that, for every strongly Brown model category $\mathcal{K}$, $\mathrm{Ho}(\mathcal{K})$ satisfies the Brown representability for cohomology. This also follows from [21] 1.3.

Brown representability theorems for cohomology for triangulated categories are considered in [16], [35] and [29]. A. Neeman [35] introduced the concept of a well generated triangulated category. These categories naturally generalize compactly generated ones and they still satisfy the Brown representability for cohomology (see [35] 8.4.2). In [36] he shows that, for any Grothendieck abelian category $\mathcal{K}$, the derived category $D(\mathcal{K})$ is well generated. Since $D(\mathcal{K})$ is the homotopy category of the model category of (unbounded) chain complexes $\mathbf{Ch}(\mathcal{K})$ on $\mathcal{K}$ and this model category is combinatorial, his result follows from our 6.10. Neeman's result was generalized by H. Krause [29] to perfectly generated triangulated categories; in [30] he compares his perfect generation with the well generation of Neeman.

6.10. Proposition. *Let $\mathcal{K}$ be a combinatorial stable model category. Then* $\mathrm{Ho}(\mathcal{K})$ *is well generated.*

Proof. Following [22], $\mathrm{Ho}(\mathcal{K})$ has a set $\mathcal{A}$ of weak generators. This means that $\hom(\Sigma^n A, X) = 0$ for all $A \in \mathcal{A}$ and all $n \in \mathbb{Z}$ implies that $X \cong 0$. Following 3.5 and 3.4 (2), there is a regular cardinal λ such that $\mathcal{K}$ is locally λ-presentable and satisfies the conditions (G^i_λ) for $i = 1, 2, 3$ and $\mathcal{A} \subseteq \mathrm{Ho}(\mathcal{K}_\lambda)$.

We will find a regular cardinal $\lambda \triangleleft \mu$ such that $\mathrm{Ho}(\mathcal{K}_\mu)$ generates $\mathrm{Ho}(\mathcal{K})$ in the sense of [35] 8.1.1. Since $\mathrm{Ho}(\mathcal{K}_\mu)$ weakly generates $\mathrm{Ho}(\mathcal{K})$ it remains to show that $\mathrm{Ho}(\mathcal{K}_\mu)$ is closed under suspension and desuspension. This is the same as $\Sigma X, \Omega X \in \mathcal{K}_\mu$ for each $X \in \mathcal{K}_\mu$. But, since Σ is left adjoint to Ω, this follows from [1] 2.23 and 2.19.

It remains to show that $\mathrm{Ho}(\mathcal{K}_\mu)$ is μ-perfect in the sense of [35] 3.3.1. Recall that, following 3.4 (2), $\mathcal{K}$ satisfies the conditions (G^i_μ) for $i = 1, 2, 3$, as well. Consider a morphism $f : A \to \coprod_{i\in I} K_i$ where $\operatorname{card} I < \mu$. Without a loss of generality, we may assume that $f = P\overline{f}$ for $\overline{f} : \overline{A} \to \coprod_{i\in I} \overline{K}_i$; of course, $A = P\overline{A}$, $K_i = P\overline{K}_i$ for $i \in I$ and $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ is the canonical functor. Since $\mathcal{K}$ is locally μ-presentable, each $\overline{K}_i$, $i \in I$, is a μ-filtered colimit

$$(k^i_j : \overline{A}_{ij} \to \overline{K}_i)_{j\in J_i}$$

of μ-presentable objects $\overline{A}_{ij}$. Hence all μ-small subcoproducts $X = \coprod_{i\in I} \overline{A}_{ij_i}$ are μ-presentable and $\coprod_{i\in I} \overline{K}_i$ is their μ-filtered colimit. Thus $\overline{f}$ factorizes through some subcoproduct X and therefore f factorizes through $PX = \coprod_{i\in I} P\overline{A}_{ij_i}$. This yields [35] 3.3.1.2.

If $f = 0$ then we can assume that $\overline{f} = 0$ and, for a factorization

$$\overline{f} : \overline{A} \xrightarrow{g} X \xrightarrow{h} \coprod_{i\in I} \overline{K}_i,$$

there is a subcoproduct morphism $u : X \to X'$ such that $ug = 0$ and $h'u = h$ where $h' : X' \to \coprod_{i\in I} \overline{K}_i$. But this is the condition [35] 3.3.1.3. Hence $\mathrm{Ho}(\mathcal{K}_\mu)$ is μ-perfect. ∎

6.11. Proposition. *Let $\mathcal{K}$ be a locally λ-presentable model category such that E_λ is essentially injective on objects and (G^1_μ), (G^2_μ) and (G^4_μ) hold for some $\lambda \triangleleft \mu$. Then idempotents split in* $\mathrm{Ho}(\mathcal{K})$.

Proof. Let $f : K \to K$ be an idempotent in $\mathrm{Ho}(\mathcal{K})$. Then $\hom(-, K)$ is an idempotent in the category of all small functors (i.e., small colimits of representable functors) $\mathrm{Ho}(\mathcal{K})^{\mathrm{op}} \to$ **Set**. Let $H : \mathrm{Ho}(\mathcal{K})^{\mathrm{op}} \to \mathbf{Set}$ be its splitting. Then H is weakly continuous and thus it is representable following 6.8. Its representing object splits f. ∎

6.12. Remark. Since idempotents do not split in $\mathrm{Ho}(\mathbf{SSet})$ (see [18], [14]), no $E_\lambda : \mathrm{Ho}(\mathbf{SSet}) \to \mathrm{Ind}_\lambda(\mathcal{K}_\lambda))$ is essentially injective on objects.

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