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DESCENT FOR COMPACT 0-DIMENSIONAL SPACES

Dedicated to Walter Tholen on the occasion of his 60th birthday

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Abstract. Using the reflection of the category $\mathcal{C}$ of compact 0-dimensional topological spaces into the category of Stone spaces we introduce a concept of a fibration in $\mathcal{C}$. We show that: (i) effective descent morphisms in $\mathcal{C}$ are the same as the surjective fibrations; (ii) effective descent morphisms in $\mathcal{C}$ with respect to the fibrations are all surjections.

Introduction

Our original intention was to describe effective descent morphisms in the category $\mathcal{C}$ of compact 0-dimensional topological spaces by combining the following well-known facts:

- A compact 0-dimensional space is nothing but a set equipped with a surjection into a Stone space (see Theorem 2.1 for the precise formulation).
- The effective descent morphisms in the categories of sets and of Stone spaces are just surjections.

It is still the main purpose of the paper, although it turned out that:

- Not all pullbacks exist in $\mathcal{C}$. Therefore the definition of an effective descent morphism p in $\mathcal{C}$ should include the requirement: *all pullbacks along p must exist* (see Definition 3.2).
- When p is surjective, that requirement holds if and only if p is a fibration in a suitable sense (see Definition 2.2), which is very different from what is happening in the situations studied by H. Herrlich [1], and which makes the descent problem much easier. In a somewhat different situation, this is made clear in [3].
- The surjectivity requirement does not create any problem since it is independently forced by the reflection of isomorphisms by the pullback functor along an effective descent morphism.

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- Therefore the problem of describing effective descent morphisms in $\mathcal{C}$ has an easy solution: Theorem 3.3 says that they are the same as the surjective fibrations.
- However, this suggests a new question, namely, what are the effective descent morphisms with respect to fibrations? Fortunately there is a complete answer again: they are all surjections (Theorem 3.1).
- In particular, even though the spaces we consider are not necessarily Hausdorff spaces, which prevents their convergence relations to be maps, our characterization of their effective descent morphisms avoids using the Reiterman-Tholen characterization of effective descent morphisms in the category of all topological spaces [4].

Accordingly, the paper is organized as follows:

Section 1 contains preliminary categorical observations with no topology involved. The ground category $\mathcal{C}$ there is constructed as a full subcategory in the comma category $(\mathcal{S} \downarrow U)$, where $U : \mathcal{X} \to \mathcal{S}$ is a pullback preserving functor between categories with pullbacks, using also a distinguished class $\mathbb{E}$ of morphisms in $\mathcal{S}$. This class is also used to define what we call fibrations in $\mathcal{C}$. The sufficient conditions for a morphism to be an effective descent morphism (globally or with respect to the class of fibrations) given in Section 1 will become also necessary in the topological context of Sections 2 and 3.

Section 2 begins by recalling relevant topological concepts, presents the category of compact 0-dimensional spaces as a special case of $\mathcal{C}$ above, introduces fibrations of 0-dimensional spaces accordingly, and ends by proving that a surjective morphism in $\mathcal{C}$ admits all pullbacks along morphisms with the same codomain if and only if it is a fibration.

The purpose of Section 3 is to formulate and prove the two main results, namely the above mentioned Theorems 3.1 and 3.3.

1. Categorical framework

We fix the following data: categories $\mathcal{S}$ and $\mathcal{X}$ with pullbacks, a pullback preserving functor $U : \mathcal{X} \to \mathcal{S}$ and a class $\mathbb{E}$ of morphisms in $\mathcal{S}$ that has the following properties:

- contains all isomorphisms;
- is pullback stable;
- is closed under composition;
- forms a *stack* (=coincides with its *localization*), which means that if

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "w"'] & \cdot \end{tikzcd}$$

is a pullback diagram with w being an effective descent morphism, then $u \in \mathbb{E} \Rightarrow v \in \mathbb{E}$.

Let $\mathcal{C} = \mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathbb{E}]$ be the full subcategory in the comma category $(\mathcal{S} \downarrow U)$ with objects all triples $A = (A_1, e_A, A_0)$, in which $e_A : A_1 \to U(A_0)$ is in $\mathbb{E}$; accordingly, a morphism $A \to B$ in $\mathcal{C}$ is a pair $f = (f_1, f_0)$, in which $f_1 : A_1 \to B_1$ and $f_0 : A_0 \to B_0$ are morphisms in $\mathcal{S}$ and $\mathcal{X}$ respectively,

$$\begin{tikzcd} A_1 \arrow[r, "e_A"] \arrow[d, "f_1"'] & U(A_0) \arrow[d, "U(f_0)"] \\ B_1 \arrow[r, "e_B"'] & U(B_0) \end{tikzcd}$$

such that $U(f_0)e_A = e_B f_1$.

1.1. Definition. *A morphism $f : A \to B$ in $(\mathcal{S} \downarrow U)$ is said to be a fibration if the morphism*

$$< f_1, e_A >: A_1 \to B_1 \times_{U(B_0)} U(A_0)$$

is in $\mathbb{E}$.

1.2. Observation. *If $f : A \to B$ is a fibration, and B is in $\mathcal{C}$, then, since the class $\mathbb{E}$ is pullback stable, A also is in $\mathcal{C}$.*

1.3. Proposition. *Let*

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{1.1}$$

be a pullback diagram in $(\mathcal{S} \downarrow U)$ with $p : E \to B$ in $\mathcal{C}$. Then:

(a) If f is a fibration, then so is g.

(b) If g is a fibration, and p_1 is an effective descent morphism in $\mathcal{S}$, then f also is a fibration.

(c) If p is a fibration and A is in $\mathcal{C}$, then D is in $\mathcal{C}$.

(d) If $\mathbb{E}$ has the (weak left) cancellation property (e', $e \cdot e' \in \mathbb{E} \Rightarrow e \in \mathbb{E}$) and p_1 and $U(p_0)$ are in $\mathbb{E}$ and D is in $\mathcal{C}$, then A is in $\mathcal{C}$.

PROOF. Consider the diagram

$$\begin{tikzcd}
D_1 \arrow[rrrr, "q_1"] \arrow[dr, "d"] \arrow[ddrr, "e_D"] \arrow[ddd, "g_1"'] & & & & A_1 \arrow[dr, "a"] \arrow[ddrr, "e_A"] \arrow[ddd, "f_1"'] & & \\
& S \arrow[rrrr, "h"] \arrow[dr, "s'"'] \arrow[ddl, "s"] & & & & T \arrow[dr, "t'"] \arrow[ddl, "t"] & \\
& & U(D_0) \arrow[rrrr, "U(q_0)"] \arrow[dd, "U(g_0)"] & & & & U(A_0) \arrow[dd, "U(f_0)"] \\
E_1 \arrow[rrrr, "p_1"] \arrow[drr, "e_E"'] & & & & B_1 \arrow[drr, "e_B"] & & \\
& & U(E_0) \arrow[rrrr, "U(p_0)"] & & & & U(B_0)
\end{tikzcd}$$

in which:

- the enveloping cube represents the diagram (1.1);
- $e_E s = U(g_0)s'$ and $e_B t = U(f_0)t'$ are pullbacks;
- $d =< g_1, e_D >$, $a =< f_1, e_A >$, and $h = p_1 \times U(q_0)$ are the suitable induced morphisms.

Since the front square $U(p_0)U(g_0) = U(f_0)U(q_0)$ and the quadrilaterals $e_E s = U(g_0)s'$ and $e_B t = U(f_0)t'$ are pullbacks, so is the quadrilateral $p_1 s = th$. Next, since $p_1 g_1 = f_1 q_1$ and $p_1 s = th$ are pullbacks, so is $hd = aq_1$. This proves (a).

(b): Since p_1 is an effective descent morphism and $p_1 s = th$ is a pullback, h also is an effective descent morphism ([5], Theorem 3.1). Since $hd = aq_1$ is a pullback, this proves (b).

For (c) and (d), in order to use the same observations, let us "turn the diagram (1.1) around the diagonal connecting D and B", i.e. let us reformulate (c) and (d) as follows:

(c$'$) If f is a fibration and E is in $\mathcal{C}$, then D is in $\mathcal{C}$.

(d$'$) If f_1 and $U(f_0)$ are in $\mathbb{E}$ and D is in $\mathcal{C}$, then E is in $\mathcal{C}$.

Proof of (c$'$):

- Since f is a fibration, a is in $\mathbb{E}$.
- Since E is in $\mathcal{C}$ and $e_E s = U(g_0)s'$ is a pullback, s' is in $\mathbb{E}$.
- Since a and s' are in $\mathbb{E}$, so is e_D, i.e. D is in $\mathcal{C}$.

Proof of (d′):

- Since f_1 and $U(f_0)$ are in $\mathbb{E}$, so are g_1 and $U(g_0)$.
- Since $g_1, U(g_0)$ and e_D are in $\mathbb{E}$, the cancellation property of (d′) implies that e_E is in $\mathbb{E}$, as desired.

■

From Observation 1.2 and Proposition 1.3(a) we obtain:

1.4. Corollary. *The category $\mathcal{C}$ is closed in $(\mathcal{S} \downarrow U)$ under pullbacks along fibrations; that is, if (1.1) is a pullback diagram in $(\mathcal{S} \downarrow U)$ with f in $\mathcal{C}$ and p being a fibration in $\mathcal{C}$, then it is a pullback diagram in $\mathcal{C}$.*

When $\mathcal{S}$ has coequalizers of equivalence relations, all effective descent morphisms in $\mathcal{S}$ are regular epimorphisms. Using this fact it is easy to show that if $p : E \to B$ is a morphism in $(\mathcal{S} \downarrow U)$, for which p_0 and p_1 are effective descent morphisms in $\mathcal{X}$ and in $\mathcal{S}$ respectively, then p itself is an effective descent morphism. After that, using Proposition 1.3 and Corollary 1.4, we obtain:

1.5. Proposition. *If $\mathcal{S}$ has coequalizers of equivalence relations and $p : E \to B$ is a morphism in $\mathcal{C}$, for which p_0 and p_1 are effective descent morphisms in $\mathcal{X}$ and in $\mathcal{S}$ respectively, then*

(a) p is an effective $\mathbb{F}$-descent morphism in $\mathcal{C}$, where $\mathbb{F}$ is the class of all fibrations (in $\mathcal{C}$).

(b) if p is a fibration, then it is an effective descent morphism in $\mathcal{C}$.

2. The category of compact 0-dimensional spaces

For a topological space A, we shall write $Open(A)$ for the set of open subsets in A and $Clopen(A)$ for the set of those subsets in A that are *clopen*, i.e. closed and open at the same time. Let us recall the definitions of the following full subcategories of the category $\mathcal{T}op$ of topological spaces:

– $\mathcal{T}op_0$, the category of T_0-spaces; a space A is a T_0-space if, for every two distinct points a and a' in A, either there exists $U \in Open(A)$ with $a \in U$ and $a' \notin U$, or there exists $U \in Open(A)$ with $a' \in U$ and $a \notin U$. Note that $\mathcal{T}op_0$ is a reflective subcategory in $\mathcal{T}op$, with the reflection given by

$$A \mapsto A_0 = A/\sim, \text{ where } a \sim a' \Leftrightarrow \forall_{U \in Open(A)}(a \in U \Leftrightarrow a' \in U). \tag{2.1}$$

– 0-$\mathcal{D}im\mathcal{T}op$, the category of 0-dimensional spaces; a space is 0-dimensional if it has a basis of clopen subsets, i.e. if every open subset in it can be presented as a union of clopen subsets.

– The category of compact 0-dimensional spaces, which is the category of interest in this paper, will be simply denoted by $\mathcal{C}$; hence

$$\mathcal{C} = \mathcal{C}omp\mathcal{T}op \cap 0\text{-}\mathcal{D}im\mathcal{T}op$$

where $\mathcal{C}omp\mathcal{T}op$ is the category of compact spaces.

– $\mathcal{S}tone$, the category of Stone spaces = spaces that occur as spectra of Boolean algebras = spaces that occur as limits of finite discrete spaces = compact Hausdorff 0-dimensional spaces = compact spaces A, such that for every two distinct points a and a' in A, there exists $U \in Clopen(A)$ with $a \in U$ and $a' \notin U$. The T_0-reflection (2.1) of course induces a reflection

$$\mathcal{C} \mapsto \mathcal{S}tone, A \mapsto A_0. \tag{2.2}$$

The following theorem is a reformulation of well-known results (see also Example 3.3 in [2] for the same result for arbitrary topological spaces, which, together with other similar results was mentioned already in [1]):

2.1. Theorem. *The category $\mathcal{C}$ of compact 0-dimensional spaces is equivalent to the category $\mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathbb{E}]$ (see Section 1), for $\mathcal{X} = \mathcal{S}tone$, $\mathcal{S} = \mathcal{S}et$, $U : \mathcal{S}tone \to \mathcal{S}et$, U being the usual forgetful functor into the category of sets and $\mathbb{E}$ being the class of all surjective maps. Under this equivalence a space A corresponds to the triple (A_1, e_A, A_0), in which A_1 is the underlying set of A, A_0 is the T_0-reflection of A, and $e_A : A_1 \to U(A_0)$ is the canonical map (and we write again $A = (A_1, e_A, A_0)$). Conversely, the space corresponding to a triple (A_1, e_A, A_0) has A_1 as its underlying set and the inverse e_A-images of open sets in A_0 as its open subsets.*

According to this theorem and Definition 1.1, we introduce:

2.2. Definition. *A morphism $f : A \to B$ in $\mathcal{C}$ is said to be a fibration if so is the corresponding morphism in $\mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathbb{E}]$ of Theorem 2.1, i.e. if for every a in A and b in B with $f(a) \sim b$ there exists a' in A with $a' \sim a$ and $f(a') = b$.*

After that Proposition 1.3 helps to prove:

2.3. Theorem. *Let $p : E \to B$ be a morphism in $\mathcal{C}$. If p is surjective, then the following conditions are equivalent:*

(a) every morphism $f : A \to B$ in $\mathcal{C}$ admits pullback along p;

(b) p is a fibration.

Proof. (a)⇒(b): Suppose p is not a fibration. This means that there are e in E and b in B with

$$p(e) \sim b \text{ and } (x \in p^{-1}(b) \Rightarrow \exists_{U_x \in Clopen(E)}(x \in U_x \text{ and } e \notin U_x)). \tag{2.3}$$

We choose U_x as in (2.3) for each x in $p^{-1}(b)$, and consider two cases:

Case 1. There exists a finite subset Y in X, for which

$$p^{-1}(b) \subseteq \bigcup_{x \in Y} U_x.$$

Case 2. There is no such Y.

In *Case 1* we take

$$V = \bigcap_{x \in Y} (E \setminus U_x),$$

and observe that since Y is finite, V is clopen; and of course V contains e and has empty intersection with $p^{-1}(b)$. After that we take

$$A = \{n^{-1} | n = 1, 2, 3, \cdots\} \cup \{0\}$$

with the topology induced from the real line, and define $f : A \to B$ by $f(n^{-1}) = b$ and $f(0) = p(e)$. Suppose the pullback of p and f does exist, and let us write it as the diagram (1.1). Using the universal property of this pullback with respect to maps from a one-point space, we easily conclude that it is preserved by the forgetful functor into the category of sets. In particular, since V contains e and has empty intersection with $p^{-1}(b)$, we have

$$q(g^{-1}(E \setminus V)) = \{n^{-1} | n = 1, 2, 3, \cdots\}.$$

This is a contradiction since $g^{-1}(E \setminus V)$ being clopen in D must be compact in it, while $\{n^{-1} | n = 1, 2, 3, \cdots\}$ is not compact in A.

In *Case* 2 we take $A = \{a\}$ to be a one-point space, and define $f : A \to B$ by $f(a) = b$. Then, using (1.1) as above, we observe that $g(D) = p^{-1}(b)$ – which is again a contradiction because now $p^{-1}(b)$ is not compact.

That is, whenever p is not a fibration, there exists a morphism $f : A \to B$ in $\mathcal{C}$ that has no pullback along p.

(b)⇒(a) follows from Corollary 1.4 and Theorem 2.1. ∎

3. $\mathbb{F}$-Descent and global descent

Let $\mathcal{C}$ be as in Section 2.

3.1. Theorem. *The following conditions on a morphism $p : E \to B$ in $\mathcal{C}$ are equivalent:*

(a) p is an effective $\mathbb{F}$-descent morphism in $\mathcal{C}$;

(b) p is a surjective map.

Proof. (a)⇒(b): Suppose p is not surjective, and choose $b \in B \setminus p(E)$. Let A be the equivalence class of b with respect to the equivalence relation $\sim$ (see (2.1)). We take

$$A' = (A \setminus \{b\}) \cup \{b\} \times \{1, 2\}$$

equipped with the indiscrete topology, and define $\alpha : A' \to A$ by $\alpha(a) = a$, for $a \in A$, and $\alpha(b, 1) = b = \alpha(b, 2)$; then α becomes a morphism $(A', \alpha f) \to (A, f)$, where $f : A \to B$ is the inclusion map, in the category $\mathbb{F}(B)$ of fibrations over B (in $\mathcal{C}$). Since the image of this morphism under the pullback functor $p^* : F(B) \to F(E)$ is an isomorphism, p cannot be an effective $\mathbb{F}$-descent morphism in $\mathcal{C}$.

(b)⇒(a): Let $(p_1, p_0) : (E_1, e_E, E_0) \to (B_1, e_B, B_0)$ be the morphism in $\mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathbb{E}]$ corresponding to p under the category equivalence of Theorem 2.1, where $X = \mathcal{S}tone, \mathcal{S} = \mathcal{S}et$, $U : \mathcal{S}tone \to \mathcal{S}et$ being the usual forgetful functor into the category of sets, and $\mathbb{E}$ being the class of all surjective maps. Then p_1 is surjective and this makes p_0 surjective too. Since in both $\mathcal{S}tone$ and $\mathcal{S}et$ surjections are effective descent morphisms, this makes p an effective $\mathbb{F}$-descent morphism by Proposition 1.5(a). ■

Since $\mathcal{C}$ does not admit some pullbacks, we define effective (global-)descent morphisms in $\mathcal{C}$ as follows:

3.2. Definition. *A morphism $p : E \to B$ in $\mathcal{C}$ is said to be an effective descent morphism if every morphism $f : A \to B$ in $\mathcal{C}$ admits pullback along p, and the pullback functor*

$$p* : (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E)$$

is monadic.

3.3. Theorem. *The following conditions on a morphism $p : E \to B$ in $\mathcal{C}$ are equivalent:*

(a) p is an effective descent morphism;

(b) p is a surjective fibration.

Proof. (a)⇒(b): Surjectivity can be proved in the same way as in the proof of Theorem 3.1 (or even much simpler by considering the empty and one-point space instead of A' and A there). The fact that p must be a fibration follows from the implication (a)⇒(b) of Theorem 2.3.

(b)⇒(a) can be deduced from Proposition 1.5(b) and Theorem 2.1 with the same arguments as in the proof of Theorem 3.1(b)⇒(a). ■

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