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FORMS AND EXTERIOR DIFFERENTIATION IN CARTESIAN DIFFERENTIAL CATEGORIES

G.S.H. CRUTTWELL

Abstract. Cartesian differential categories abstractly capture the notion of a differentiation operation. In this paper, we develop some of the theory of such categories by defining differential forms and exterior differentiation in this setting. We show that this exterior derivative, as expected, produces a cochain complex.

1. Introduction

Differential categories [Blute et. al. 2006] and Cartesian differential categories [Blute et. al. 2008] were defined so as to abstractly capture the essential properties of the derivative. Since then, much work has been done on describing and classifying different types of examples of these structures. For example, see [Cockett and Seely 2011], [Cockett et. al 2011], [Blute et al. 2012], [Manzonetto 2012], [Laird et. al. 2013], and [Cockett and Cruttwell 2013] on how these structures relate to derivatives throughout mathematics and logic. However, less work has been done on describing the theory of these structures: how, given a differential or Cartesian differential category, one can define and prove various definitions and theorems familiar from ordinary calculus in this abstract setting.

In this paper, we examine one aspect of this by defining differential forms and exterior differentiation in the abstract setting of a generalized Cartesian differential category [Cruttwell 2013]. These are a slight generalization of Cartesian differential categories that allow for additional examples. In particular, while smooth maps between Cartesian spaces are a Cartesian differential category, smooth maps between open subsets of Cartesian spaces are an example of a generalized Cartesian differential category which is not a Cartesian differential category. Since forms and exterior differentiation are much more interesting when applied to open subsets of Cartesian spaces, we would like to work in this more general setting. Thus, in the setting of a generalized Cartesian differential category, we define differential forms, we define an exterior differentiation operation for these differential forms, and we show the essential properties of exterior differentiation, namely that it is a natural operation which, when applied twice, gives the 0 map.

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One of the initial difficulties in doing this is determining how to define differential forms. In the standard setting of smooth maps between open subsets of Cartesian spaces, one way to define a differential form is as a multilinear alternating map. However, since Cartesian differential categories have no vector space or monoidal structure, it is not immediately obvious what it means to say that a map in a Cartesian differential category is multilinear. What is required is an adaptation of the notion of linear map from [Blute et. al. 2008]. There, the authors define a map to be linear if its derivative takes a particularly simple form (see definition 2.4). In this paper, we extend this definition to be able to talk of multilinear maps. The resulting definition (2.9) captures the ordinary definition of a multilinear map solely in terms of properties of its derivative.

Once we have the abstract definition of multilinearity, we then give an abstract definition of the exterior derivative of multilinear alternating maps. The main results of the paper are then showing that this definition of exterior differentiation has all the ordinary properties of the exterior derivative: (i) that it produces another differential form, (ii) that it is a natural transformation and (iii) that the result of applying the exterior derivative twice is zero. The standard way to prove these results in the setting of smooth maps between open subsets of Cartesian spaces is to approach the problem indirectly (see, for example, pages 210-213 of [Spivak 1997]). However, we cannot adapt the standard proof in our general setting, as it uses structure that is not available to us. Thus, we must prove the results directly, and this takes some work.

In the final section, we describe how this abstract approach relates to exterior differentiation for finite and infinite-dimensional smooth manifolds and diffeological spaces, and discuss possible extensions of this work to even more general settings.

2. Generalized Cartesian differential categories

Consider a smooth map f from some open subset of $U \subseteq \mathbb{R}^n$ to some open subset $V \subseteq \mathbb{R}^m$. Its Jacobian at a point $x \in U$ is then an $n \times m$ matrix, that is, a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Looking at this in another way, one can view the Jacobian of $f : U \longrightarrow V$ as being a map from

$$\mathbb{R}^n \times U \longrightarrow \mathbb{R}^m$$

which is linear in its first variable, but has other properties (such as the chain rule) as well. Describing these properties abstractly is the idea behind a generalized Cartesian differential category.

Before we give the definition, we briefly describe some notation we use throughout the paper. First of all, composites will be written in diagrammatic order, so that f, followed by g, is written fg. Secondly, if $(A, +, 0)$ is a commutative monoid in a category and there are maps $f, g : X \longrightarrow A$, we write $f + g : X \longrightarrow A$ for $\langle f, g\rangle+$ and $0 : X \longrightarrow A$ for $!0$. Note that these operation are left-additive; that is, for $h : Y \longrightarrow X$, $h(f + g) = hf + hg$ and $h0 = 0$. Finally, if B is also a monoid, then a map $h : A \longrightarrow B$ which has the property that $(f + g)h = fh + gh$ and $0h = 0$ will be called **additive**. The following definition is from [Cruttwell 2013], but is only a slight generalization of the central definition of [Blute et. al. 2008].

2.1. DEFINITION. *A* ***generalized Cartesian differential category*** *consists of a category* $\mathbb{X}$ *with chosen products, which has, for each object* X*, a commutative monoid* $L(X) = (A, +, 0)$*, with* $L(L(X)) = L(X)$ *and* $L(X \times Y) = L(X) \times L(Y)$*. In addition, for each map* $f : X \longrightarrow Y$*, there is a map*

$$D(f) : L(X) \times X \longrightarrow L(Y)$$

such that:

[CD.1] $D(+) = \pi_0+$ *and* $D(0) = \pi_0 0$*;*

[CD.2] $\langle a + b, c\rangle D(f) = \langle a, b\rangle D(f) + \langle b, c\rangle D(f)$ *and* $\langle 0, a\rangle D(f) = 0$*;*

[CD.3] $D(\pi_0) = \pi_0\pi_0$*,* $D(\pi_1) = \pi_0\pi_1$*, and* $D(1) = \pi_0$*;*

[CD.4] $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$*;*

[CD.5] $D(fg) = \langle D(f), \pi_1 f\rangle D(g)$*;*

[CD.6] $\langle\langle a, 0\rangle, \langle c, d\rangle\rangle D(D(f)) = \langle a, d\rangle D(f)$*;*

[CD.7] $\langle\langle 0, b\rangle, \langle c, d\rangle\rangle D(D(f)) = \langle\langle 0, c\rangle, \langle b, d\rangle\rangle D(D(f))$*;*

A ***Cartesian differential category*** *is a generalized Cartesian differential category in which* $L(A) = A$ *for every object* A*.*

We can get some understanding for these axioms by considering how they work in the example of smooth maps between open subsets of Cartesian spaces. In this example, for $U \subseteq \mathbb{R}^n$, we define $L(U) = \mathbb{R}^n$. For a smooth map $f : U \longrightarrow V$, $D(f)(v, x)$ is defined to be the Jacobian of f, evaluated at x, then multiplied by the vector v. **[CD.1]** describes how to differentiate addition and zero maps. **[CD.2]** says that the derivative is additive in its first variable. **[CD.3]** and **[CD.4]** describe how to differentiate projections, pairings, and identity maps.

To understand **[CD.5]**, it may be useful to look at how the above structure relates to a smooth map $\mathbb{R} \longrightarrow \mathbb{R}$. Here, if $f'(x) : \mathbb{R} \longrightarrow \mathbb{R}$ is the ordinary derivative of f, then $D(f)(v, x) = f'(x) \cdot v$. Then for another smooth map $g : \mathbb{R} \longrightarrow \mathbb{R}$, by the chain rule,

$$D(fg)(v, x) = g'(f(x)) \cdot f'(x) \cdot v$$

so that we can write

$$D(fg) = \langle Df, \pi_1 f\rangle D(g).$$

In other words, **[CD.5]** is how to express the chain rule in this formalism.

To understand **[CD.6]** and **[CD.7]**, it is useful to see how to recover partial derivatives from the operator D. If we have a map $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, then

$$\frac{\partial g}{\partial x_1}(a_1, a_2) = \langle 1, 0, a_1, a_2\rangle D(g) \text{ and } \frac{\partial g}{\partial x_2}(a_1, a_2) = \langle 0, 1, a_1, a_2\rangle D(g).$$

Now, given a map $f : \mathbb{R} \longrightarrow \mathbb{R}$, $D(f) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by $D(f)(v, x) = f'(x) \cdot v$. Then $\frac{\partial D(f)}{\partial v} = f'(x)$. In other words, given how partial derivatives relate to the D operation, we have for any a, c, d

$$\langle\langle a, 0\rangle, \langle c, d\rangle\rangle D(D(f)) = \langle a, d\rangle D(f)$$

which is **[CD.6]**. Thus **[CD.6]** represents the linearity of D in its first variable.

[CD.7] is the independence of order of partial differentiation. Indeed, if we have a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, then as above

$$\frac{\partial f}{\partial x_1}(a_1, a_2) = \langle 1, 0, a_1, a_2\rangle D(f)$$

and then using **[CD.3]**,**[CD.4]**, and **[CD.5]**,

$$\frac{\partial f}{\partial x_1 \partial x_2}(a_1, a_2) = \langle\langle 0, 0\rangle\langle 0, 1\rangle\langle 1, 0\rangle\langle a_1, a_2\rangle\rangle D^2(f).$$

The independence of order of partial differentiation says that this is equal to

$$\frac{\partial f}{\partial x_2 \partial x_1}(a_1, a_2) = \langle\langle 0, 0\rangle\langle 1, 0\rangle\langle 0, 1\rangle\langle a_1, a_2\rangle\rangle D^2(f).$$

The axiom **[CD.7]** is the generalization of this to arbitrary maps, so it asks that for any map f,

$$\langle\langle 0, b\rangle, \langle c, d\rangle\rangle D(D(f)) = \langle\langle 0, c\rangle, \langle b, d\rangle\rangle D(D(f)).$$

2.2. Examples. In addition to the standard example give above, there are many other examples of generalized Cartesian differential categories. All examples but the last are from [Cruttwell 2013]:

(i) Convenient vector spaces are certain locally convex vector spaces with a well-defined notion of smooth map (see [Kriegl and Michor 1997]). The open subsets of convenient vector spaces form a generalized Cartesian differential category, with differential as described in [Blute et al. 2012].

(ii) Any category with finite products has an associated cofree generalized Cartesian differential category. For details, see corollary 2.13 in [Cruttwell 2013], which generalizes work in [Cockett and Seely 2011].

(iii) Each model of the differential lambda calculus of [Erhard and Regnier 2003] is a Cartesian differential category, as described in [Manzonetto 2012].

(iv) The coKleisli category of any differential storage category is a Cartesian differential category ([Blute et. al. 2008], proposition 3.2.1) and hence is a generalized Cartesian differential category. This includes such categories as **rel**, the category of sets and relations.

(v) In any category with an "abstract tangent functor" [Rosický 1984], the category of "tangent spaces" forms a Cartesian differential category, by theorems 4.15 and 4.11 of [Cockett and Cruttwell 2013]. For example, this includes the tangent spaces of infinitesimally linear objects in a model of synthetic differential geometry ([Kock 2006]).

2.3. Linear objects and linear maps. Linear maps are an important subclass of maps in any Cartesian differential category, and the same is true in the generalized version. Since these categories do not assume any sort of vector space or monoidal structure, linearity is defined directly through a property of the derivative.

2.4. Definition. *In a generalized Cartesian differential category, say that an object A is* ***linear*** *if $L(A) = A$. Say that a map $f : A \longrightarrow B$ between linear objects is* ***linear*** *if $D(f) = \pi_0 f$.*

It may be useful to consider how this definition of linear corresponds to the ordinary definition of linear in the case when we are dealing with smooth maps between open subsets of Cartesian spaces. Here, the linear objects are simply the Cartesian spaces. To understand linear maps, consider first the case of a smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}$. If f is linear in the vector space sense, then $f(x) = \lambda x$ for some λ, so that $D(f)(v, x) = f'(x) \cdot v = \lambda \cdot v = f(v)$, so that $D(f) = \pi_0 f$. Thus f is linear in the differential sense above. Conversely, if f is linear in the differential sense, then in particular $f'(x) = D(f)(1, x) = f(1)$. Thus $f(x) = f(1) \cdot x + c$. But substituting $x = 1$ gives $c = 0$, so $f(x) = f(1) \cdot x$, so f is linear in the vector space sense.

A similar result holds for a smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. For simplicity we will consider the case $n = 2$. If f is linear in the vector space sense, then $f(x_1, x_2) = \lambda_1 \cdot x_1 + \lambda_2 x_2$, so that $D(f)(v_1, v_2, x_1, x_2) = (\lambda_1, \lambda_2) \cdot (v_1, v_2) = f(v_1, v_2)$, so $D(f) = \pi_0 f$. Hence f is linear in the differential sense. Conversely, if f is linear in the differential sense, then in particular

$$\frac{\partial f}{\partial x_1} = D(f)(1, 0, x_1, x_2) = f(1, 0) \text{ and } \frac{\partial f}{\partial x_2} = D(f)(0, 1, x_1, x_2) = f(0, 1)$$

so that $f(x_1, x_2) = f(1, 0) \cdot x_1 + f(0, 1) \cdot x_2 + c$, but substituting $x_1 = 1$ and $x_2 = 0$ gives $c = 0$, so $f(x_1, x_2) = f(1, 0) \cdot x_1 + f(0, 1) \cdot x_2$, so that f is linear in the vector space sense. Thus this differential definition of linear captures the ordinary notion of linearity without referring to any explicit vector space or monoidal structure.

The following are some basic properties of linear maps in a generalized Cartesian differential category; the proofs are as in lemma 2.2.2 of [Blute et. al. 2008].

2.5. Lemma. *In a generalized Cartesian differential category:*

(i) if f is linear, then f is additive;

(ii) for any linear object A, the addition map $+ : A \times A \longrightarrow A$ and the zero map $0 : 1 \longrightarrow A$ are linear;

(iii) composites of linear maps are linear, and identities are linear;

(iv) projections are linear, and pairings of linear maps are linear.

2.6. Multilinear maps. We have just seen how to define linear maps between linear objects by using the derivative. We now turn to defining multilinear maps. We first need to define the domain for such a map: the space of n tangent vectors at a single point.

2.7. Lemma. *If $\mathbb{X}$ is a generalized Cartesian differential category, then for any $n \geq 1$ there is an endofunctor $T_n : \mathbb{X} \longrightarrow \mathbb{X}$ which is defined on an object M by*

$$T_n(M) := L(M)^n \times M$$

and on a map $f : M \longrightarrow M'$ by

$$T_n(f) := \langle \langle \pi_0, \pi_n \rangle D(f), \langle \pi_1, \pi_n \rangle D(f), \ldots \langle \pi_{n-1}, \pi_n \rangle D(f), \pi_n f \rangle$$

Proof. The fact that T_n preserves composition follows from [**CD.5**]:

$$D(fg) = \langle D(f), \pi_1 f \rangle D(g),$$

and the fact that T_n preserves identities follows from [**CD.3**] ($D(1) = \pi_0$). ∎

The functor T_1, which we sometimes write as T, is the tangent bundle functor. Its properties are studied in more detail in [Cockett and Cruttwell 2013].

In the definitions below, we will often be dealing with the first and second derivatives of maps with domain and codomains of the form

$$T(T_n M) = L(M)^{n+1} \times L(M)^n \times M.$$

A map into such an object has $2n$ terms, and we will often use a $|$ to distinguish the first set of n terms from the last set of n terms.

As we shall see below, differential n-forms on M will be certain maps from $T_n M$ to a linear object A. We would like to be able to define what it means for such a map to be "linear" in one of its terms. To define this, we first need some special maps.

2.8. Definition. *For any $n \geq 1$ and $0 \leq i \leq n-1$, define the map e_i by*

$$L(M) \times T_n(M) \xrightarrow{e_i := \langle 0, \ldots, 0, \pi_0, 0, \ldots 0 | \pi_1, \pi_2, \ldots \pi_{n+1} \rangle} T(T_n(M))$$

where the π_0 is in the ith position.

We can now define when a map with domain $T_n(M)$ and codomain a linear object is "linear in each of the first n variables"

2.9. Definition. *If A is a linear object and $0 \leq i \leq n-1$, say that a map $f : T_nM \longrightarrow A$ is **linear in the ith variable** if the diagram*

$$\begin{tikzcd} L(M) \times T_n(M) \arrow[r, "e_i"] \arrow[d, "\langle \pi_1, \pi_2, \ldots \pi_i, \pi_0, \pi_{i+2}, \ldots \pi_{n+1} \rangle"'] & T(T_nM) \arrow[d, "D(f)"] \\ T_nM \arrow[r, "f"'] & A \end{tikzcd}$$

*commutes. Say that f is **multilinear** if it is linear for each such i.*

Note that the map on the left excludes the π_{i+1} term. In the case $n = 1$, only one equality ($i = 0$) must be satisfied, namely

$$\begin{tikzcd} L(M) \times L(M) \times M \arrow[r, "\langle \pi_0, 0, \pi_1, \pi_2 \rangle"] \arrow[d, "\langle \pi_0, \pi_2 \rangle"'] & L(M) \times L(M) \times L(M) \times M \arrow[d, "D(f)"] \\ L(M) \times M \arrow[r, "f"'] & A \end{tikzcd}$$

There is a canonical class of maps that satisfy this definition: for any $f : M \longrightarrow N$, the map $D(f) : L(M) \times M \longrightarrow L(N)$ is multilinear, since the required equality in this case is

$$\langle \pi_0, 0, \pi_1, \pi_2 \rangle D^2(f) = \langle \pi_0, \pi_2 \rangle D(f)$$

which is exactly [**CD.6**].

The definition above is thus a generalization of property [**CD.6**] for derivatives. Even so, however, it may seem somewhat arbitrary. In a future paper, we will consider a general theory of linear bundles in a tangent category (a tangent category is a category equipped with an abstract tangent bundle functor, see [Cockett and Cruttwell 2013] for more detail). The definition above can then be seen as a linear bundle morphism between particular linear bundles in the tangent category associated to the generalized Cartesian differential category.

The following is a generalization of lemma 2.2.1(i) from [Blute et. al. 2008] ("linear maps are additive"), and the proof is essentially the same.

2.10. Proposition. *If $\omega : T_nM \longrightarrow A$ is multilinear, then ω is additive in each of its first n variables.*

Proof. Consider

$$\begin{aligned} & \langle a_0, a_1, \ldots a_i + a_i' \ldots a_n, p \rangle \omega \\ = \; & \langle 0, 0, \ldots, a_i + a_i' \ldots 0 | a_0, a_1, \ldots a_n, p \rangle D(\omega) \text{ (by linearity of } \omega) \\ = \; & \langle 0, \ldots, a_i, \ldots 0 | a_0, a_1, \ldots a_n, p \rangle D(\omega) + \langle 0, \ldots, a_i', \ldots 0 | a_0, a_1, \ldots a_n, p \rangle D(\omega) \text{ (by } [\textbf{CD.2}]) \\ = \; & \langle a_0, a_1, \ldots a_i, \ldots a_n, p \rangle \omega + \langle a_0, a_1, \ldots a_i', \ldots a_n, p \rangle \omega \text{ (by linearity of } \omega) \end{aligned}$$

so that ω preserves addition in its ith variable. Preservation of 0 is similar. ∎

The last thing we need to define is when such maps are alternating.

2.11. Definition. *Suppose is M an object of a Cartesian differential category $\mathbb{X}$, A is a linear object in $\mathbb{X}$, and $n \geq 1$. Say that a map $f : T_n M \longrightarrow A$ is **alternating** if for any $0 \leq i, j \leq n-1$,*

$$\langle \pi_0, \ldots, \pi_i, \ldots, \pi_i, \ldots \pi_{n-1}, \pi_n \rangle \omega = 0$$

(where the second π_i is in the j position).

3. Forms and exterior differentiation

We are now in a position to define differential forms and exterior differentiation of forms, and to prove this operation's essential properties.

3.1. Definition. *For M an object of a Cartesian differential category $\mathbb{X}$, A a linear object in $\mathbb{X}$, and $n \geq 1$, a **differential n-form on M with values in A** is a map $\omega : T_n(M) \longrightarrow A$ which is multilinear and alternating. Denote the set of n-forms on M with values in A by $\Omega_n(M; A)$. Define $\Omega_0(M; A)$ as simply the hom-set $\mathbb{X}(M, A)$.*

It is worth noting that most standard definitions of differential n-form define them as maps $M \longrightarrow A^{L(M)^n}$ (see, for example, [Spivak 1997], pg. 207). That is, they curry the above maps. Since we do not assume our Cartesian differential categories are Cartesian closed, we use the uncurried format given above, which only requires products. In fact, for convenient vector spaces, the above definition of differential form is the only appropriate one. In section 33 of [Kriegl and Michor 1997], the authors consider 12 different definitions of differential form, all of which are equivalent for Cartesian spaces, but which are different for convenient vector spaces. They determine that only one definition, the one given above, has all of the necessary properties of a differential form.

As usual, the alternating property of a differential form implies skew-symmetry.

3.2. Lemma. *If $\omega : T_n(M) \longrightarrow A$ is alternating, then it is also skew-symmetric; that is, for any $0 \leq i, j \leq n-1$,*

$$\langle \pi_0, \ldots, \pi_i, \ldots, \pi_j, \ldots \pi_{n-1}, \pi_n \rangle \omega + \langle \pi_0, \ldots, \pi_j, \ldots, \pi_i, \ldots \pi_{n-1}, \pi_n \rangle \omega = 0.$$

Proof. Since ω is additive in each of its first n variables,

$$\begin{aligned}
& \langle \pi_0, \ldots, \pi_i, \ldots, \pi_j, \ldots \pi_{n-1}, \pi_n \rangle \omega + \langle \pi_0, \ldots, \pi_j, \ldots, \pi_i, \ldots \pi_{n-1}, \pi_n \rangle \omega \\
= & \langle \pi_0, \ldots, \pi_i + \pi_j, \ldots, \pi_j + \pi_i, \ldots \pi_{n-1}, \pi_n \rangle \omega \\
= & 0
\end{aligned}$$

since ω is alternating. ∎

Before proving our next result, we note a useful consequence of **[CD.1]**.

3.3. LEMMA. *If A is a linear object and we have maps $f, g : X \longrightarrow A$, then $D(f+g) = D(f) + D(g)$ and $D(0) = 0$.*

PROOF. Indeed, using **[CD.1]**, **[CD.4]**, and **[CD.5]**:

$$D(f+g) = D(\langle f, g\rangle +) = \langle D(\langle f, g\rangle), \pi_1\langle f, g\rangle\rangle D(+)$$

$$= \langle\langle Df, Dg\rangle, \pi_1\langle f, g\rangle\rangle\pi_0 + = \langle Df, Dg\rangle + = Df + Dg,$$

and similarly for the preservation of 0. ∎

We can then use this to prove:

3.4. LEMMA. *For each M, A, and n, $\Omega_n(M; A)$ is a monoid, with monoid structure inherited from the hom-set $\mathbb{X}(T_n(X), A)$.*

PROOF. It is clear that $0 \in \Omega_n(M; A)$, and that the sum of two alternating maps is alternating. Thus, the only thing we need to check is that the sum of two multilinear maps is multilinear; this follows almost immediately from the fact, proven above, that $D(f+g) = D(f) + D(g)$.

∎

We would like to view $\Omega_n(-; A)$ as a functor from $\mathbb{X}^{\text{op}}$ to the category of monoids. Note that since each T_n is a functor, we have a functor $\mathbb{X}(T_n(-), A)$: $\mathbb{X}^{\text{op}} \longrightarrow \mathbf{set}$, and we will use this as the action on arrows for $\Omega_n(-; A)$. However, we need to check that when applied to an alternating multilinear map, the result of this functorial action is still alternating multilinear.

3.5. LEMMA. *Let $f : M' \longrightarrow M$, and $\omega \in \Omega_n(M; A)$. Then the composite*

$$T_n(M') \xrightarrow{T_n(f)} T_n(M) \xrightarrow{\omega} A$$

is in $\Omega_n(M'; A)$.

PROOF. Since $T_n(f)$ works with each of the first p components equally, if ω is alternating, then so is $T_n(f)\omega$.

For multilinearity, let $0 \leq i \leq n-1$, and consider

$$e_i D(T_n(f)\omega) = e_i\langle D(T_n(f)), \pi_1 T_n(f)\rangle D(\omega)$$

by **[CD.5]**. Recall that

$$T_n(f) = \langle \ldots \langle \pi_j, \pi_n\rangle D(f) \ldots \pi_n f\rangle$$

So that, by **[CD.3]** and **[CD.4]**,

$$D(T_n(f)) = \langle \ldots \langle \pi_0\pi_j, \pi_0\pi_n, \pi_1\pi_j, \pi_1\pi_n\rangle D^2(f) \ldots \langle \pi_0\pi_n, \pi_1\pi_n\rangle\rangle$$

Thus

$$e_iD(T_n(f)\omega) = \langle \dots e_i\langle \pi_0\pi_j, \pi_0\pi_n, \pi_1\pi_j, \pi_1\pi_n\rangle D^2(f) \dots e_i\langle \pi_0\pi_n, \pi_1\pi_n\rangle D(f)|e_i\pi_1T_n(f))\rangle D(\omega)$$

We consider each of the terms inside the bracketing separately. For $i \neq j$, by the definition of e_i,

$$e_i\langle \pi_0\pi_j, \pi_0\pi_n, \pi_1\pi_j, \pi_1\pi_n\rangle D^2(f) = \langle 0, 0, \pi_{j+1}, \pi_{n+1}\rangle D^2(f) = 0$$

by [**CD.2**]. For $i = j$,

$$e_i\langle \pi_0\pi_j, \pi_0\pi_n, \pi_1\pi_j, \pi_1\pi_n\rangle D^2(f) = \langle \pi_0, 0, \pi_{j+1}, \pi_{n+1}\rangle D^2(f) = \langle \pi_0, \pi_{n+1}\rangle D(f)$$

by [**CD.6**]. Finally, the last term in the bracketing before | is

$$e_i\langle \pi_0\pi_n, \pi_1\pi_n\rangle D(f) = \langle 0, \pi_{n+1}\rangle D(f) = 0$$

by [**CD.2**]. Thus

$$e_iD(T_n(f)\omega) = \langle 0, \dots, \langle \pi_0, \pi_{n+1}\rangle D(f), \dots 0|e_i\pi_1T_n(f)\rangle D(\omega)$$

where the only non-zero term before the bracket is in the ith position. But then by the definition of e_i, we can rewrite this as

$$\langle\langle \pi_0, \pi_{n+1}\rangle D(f), \langle \pi_1, \pi_2, \dots \pi_{n+1}\rangle T_n(f)\rangle e_iD(\omega).$$

By multilinearity of ω, this equals

$$\langle\langle \pi_0, \pi_{n+1}\rangle D(f), \langle \pi_1, \pi_2, \dots \pi_{n+1}\rangle T_n(f)\rangle \langle \pi_1, \pi_2, \dots \pi_i, \pi_0, \pi_{i+2}, \dots \pi_{n+1}\rangle\omega$$

which, recalling the definition of $T_n(f)$, is equal to

$$\langle \pi_1, \pi_2, \dots \pi_i, \pi_0, \pi_{i+2}, \dots \pi_{n+1}\rangle T_n(f)\omega.$$

So $T_n(f)\omega$ is multilinear, as required. ∎

Thus, we have the following result:

3.6. PROPOSITION. *If* ***mon*** *is the category of monoids and monoid homomorphisms, then for any $n \geq 0$, the above data defines a functor $\Omega_n(-; A) : \mathbb{X}^{op} \longrightarrow$* ***mon****.*

PROOF. The only thing left to check is that for a map $f : Y \longrightarrow X$, $\Omega_n(f; A)$ is a monoid homomorphism. But this follows from left additivity: $T_n(f)(\omega_1+\omega_2) = T_n(f)\omega_1+T_n(f)\omega_2$, and $T_n(f)(0) = 0$. ∎

3.7. Exterior differentiation. As we have seen in the previous sections, we can define differential forms if the coefficient object is any linear object; that is, one with $A = L(A)$. To define the exterior derivative, however, will require more: we will need these linear objects to not just be monoids, but groups. For discussion on why negatives are necessary, see the remarks following the definition of the exterior derivative below. For now, we will briefly describe these objects and their properties.

3.8. Definition. *Say that an object A in a Cartesian differential category $\mathbb{X}$ is a **linear group** if $A = L(A)$, and, in addition to its monoid structure, $A = L(A)$ has a map $n : A \longrightarrow A$ making it into a group object.*

3.9. Lemma. *If A is a linear group, then:*

(i) for each M, $\mathbb{X}(M, A)$ is a group, with $-f := fn$,

(ii) $D(-f) = -D(f)$;

(iii) n is a linear map.

*Moreover, for any object M and $n \geq 0$, $\Omega_n(-; A)$ is a functor to **ab**.*

Proof. All results are straightforward. ∎

Before defining the exterior derivative, we need to define another important set of maps.

3.10. Definition. *For $n \geq 1$ and M an object, define*

$$L(M) \times T_n(M) \xrightarrow{z_i := \langle 0,0,\ldots,0,\pi_i | \pi_0,\pi_1,\ldots\widehat{\pi}_i\ldots\pi_{n+1}\rangle} T(T_n(M))$$

where $\widehat{\pi}_i$ indicates the exclusion of that term.

It is important to note the difference with the definition of z_i and with the earlier maps e_i. In the definition of multilinearity, we considered the maps

$$e_i = \langle 0, 0, \ldots, \pi_0, 0, \ldots, 0, 0 | \pi_1, \pi_1, \ldots \pi_{i-1}, \pi_{i+1} \ldots \pi_p, \pi_{p+1} \rangle$$

where the π_0 is in the ith position. These maps have the same domain and codomain as the z_i's. However, in the e_i's, we always assign a zero to the nth term. By comparison, in the z_i's, we assign a *non-zero* term to the nth term, namely π_i.

We can now define the exterior derivative.

3.11. Definition. *Suppose A is a linear group, and $\omega \in \Omega_n(M; A)$. For $n \geq 1$, define the **exterior derivative of** ω, denoted $\partial_n(\omega)$, to be the map*

$$T_{n+1}(M) \xrightarrow{\partial_n} A$$

given by

$$\partial_n(\omega) := \sum_{i=0}^{n}(-1)^i z_i D(\omega)$$

For a 0-form $\omega : M \longrightarrow A$, define $\partial_0(\omega) := D(\omega)$.

This corresponds to the usual definition of the exterior derivative in the standard example. However, it will be useful to see why, from a structural point of view, this particular definition of the exterior derivative is used. In particular, one may wonder why the simpler expression

$$\langle 0, 0, \ldots, 0, \pi_0 | \pi_1, \pi_2, \ldots \pi_n, \pi_{n+1}\rangle D(\omega)$$

which is not a sum and does not require negatives in A, is not used. The problem is that this definition of the exterior derivative will not be a natural map from $\Omega_n(M; A)$ to $\Omega_{n+1}(M; A)$. To see this intuitively, note that in the domain of the map

$$L(M)^n \times L(M) \times L(M)^n \times M \xrightarrow{D(\omega)} A,$$

the $(n+1)$st $L(M)$ has a different status than the $L(M)$'s before or after it; in particular, while ω alternating implies $D(\omega)$ is alternating in those $L(M)$'s before or after the second one, it is not alternating in that one. Thus, making a choice to take one of the i elements in $T_{n+1}(M)$ (in particular, the 0th one) and selecting it to go in that slot is a non-natural choice.

To see this concretely, suppose we have a differential 1-form $\omega : L(M) \times M \longrightarrow A$ and a map $f : M' \longrightarrow M$. For naturality of δ_1, we would need to verify that

$$T_2(f)(\partial_1(\omega)) = \partial_1(T(f)(\omega)).$$

Both are maps

$$L(M') \times L(M') \times M' \longrightarrow A.$$

Using the "wrong" definition of the exterior derivative ∂_1 given above, after calculations, one find that the term on the left is

$$\langle 0, \langle \pi_0, \pi_2\rangle D(f), \langle \pi_1, \pi_2\rangle D(f), \pi_2 f\rangle D(\omega)$$

while the term on the right is

$$\langle \langle 0, \pi_0, \pi_1, \pi_2\rangle D^2(f), \langle \pi_0, \pi_2\rangle D(f), \langle \pi_1, \pi_2\rangle D(f), \pi_2 f\rangle D(\omega).$$

For any non-trivial f, then, the expressions are not equal, and the difference is a $D^2(f)$ term in the second expression which is a 0 term in the first.

The actual definition of the exterior derivative avoids this problem by considering an alternating sum of all the possible choices for placing a term in the privileged $L(M)$. As we will see in the proof of naturality, one then uses **[CD.7]** and the fact that ω is alternating to cancel out the $D^2(f)$ terms that appear.

Before proving naturality, however, we first need to prove that the exterior derivative of a differential form produces another differential form.

3.12. Proposition. *For each $\omega \in \Omega_n(M;A)$, its exterior derivative $\partial_n(\omega)$ is in $\Omega_{n+1}(M;A)$.*

Proof. We will first show that $\partial_n(\omega)$ is alternating. Suppose $j < k$. Repeating the jth projection in the kth slot of the exterior derivative

$$\partial_n(\omega) = \sum_{i=0}^{n}(-1)^i\langle 0,0,\ldots,0,\pi_i|\pi_0,\pi_1,\ldots\widehat{\pi_i}\ldots\pi_{n+1}\rangle D(\omega)$$

gives the sum

$$\sum_{i=0,i\neq j,k}^{n}(-1)^i\langle 0,0,\ldots,0,\pi_i|\pi_0,\pi_1,\ldots\pi_j\ldots\pi_j\ldots\pi_n,\pi_{n+1}\rangle D(\omega)$$

as well as the $i=j$ term

$$(-1)^j\langle 0,0,\ldots 0,\pi_j|\pi_0\ldots\pi_j\ldots\pi_{n+1}\rangle D(\omega)$$

where the π_j after $|$ is in the kth position, and the $i=k$ term

$$(-1)^k\langle 0,0,\ldots 0,\pi_k|\pi_0\ldots\pi_j\ldots\pi_{n+1}\rangle D(\omega)$$

where the π_j after $|$ is in the *jth* position. Now since ω is alternating, the map

$$D(\omega): L(M)^n \times L(M) \times L(M)^n \times M \longrightarrow A$$

is alternating in both the first and second set of n variables. In particular, each term in the sum with $i \neq j,k$ is 0. For the $i=j$ term, since ω is skew-symmetric, we can transpose the π_j term to the jth slot by multiplying by $(-1)^{k-j+1}$. This term is then (-1) times the $i=k$ term, and hence the sum of these two terms is 0. Thus the entire term sums to 0, as required.

We now wish to show that $\partial_n(\omega)$ is multilinear. That is, we wish to show $\partial_n(\omega)$ is linear in each i for $0 \leq i \leq n$. Fix some $0 \leq j \leq n$. We will show that each map $z_jD(\omega)$ is linear in i, and hence $\partial_n(\omega)$, which is an alternating sum of $z_jD(\omega)$'s, is also linear in i.

Thus, we want to show that

$$e_iD(z_jD(\omega)) = e_i\langle\pi_0z_j,\pi_1z_j\rangle D^2(\omega)\ (\dagger)$$

is equal to

$$\langle\pi_1\ldots\pi_0\ldots\pi_{n+2}\rangle z_jD(\omega)$$

where the π_0 term is in the ith slot. We will consider the cases $i=j$, $i<j$, and $j<i$ separately.

For $i=j$, using the definitions of e_i and z_j, the expression $\dagger$ is equal to

$$\langle 0,0,\ldots\pi_0|0,\ldots,0|0,0,\ldots\pi_{i+1}|\pi_1,\pi_2\ldots\widehat{\pi_{i+1}}\ldots\pi_{n+2}\rangle D^2(\omega)$$

which, by [**CD.6**], equals

$$\langle 0, 0, \ldots \pi_0 | \pi_1, \pi_2 \ldots \widehat{\pi_{i+1}} \ldots \pi_{n+2} \rangle D(\omega)$$

which by the definition of z_i equals

$$\langle \pi_1 \ldots \pi_0 \ldots \pi_{n+2} \rangle z_i D(\omega)$$

as required.

For $i < j$, the expression † is equal to

$$\langle 0, 0 \ldots 0 | 0, \ldots \pi_0 \ldots 0 | 0, 0, \ldots \pi_{j+1} | \pi_1, \pi_2 \ldots \widehat{\pi_{j+1}} \ldots \pi_{n+2} \rangle D^2(\omega)$$

(where the π_0 is in the i slot). By [**CD**.**7**], this equals

$$\langle 0, 0 \ldots 0 | 0, 0, \ldots \pi_{j+1} | 0, \ldots \pi_0 \ldots 0 | \pi_1, \pi_2 \ldots \widehat{\pi_{j+1}} \ldots \pi_{n+2} \rangle D^2(\omega) \ (\dagger\dagger)$$

Since ω itself is linear in the ith term, we have

$$e_i D(\omega) = \langle \pi_1, \pi_2, \ldots \pi_0 \ldots \pi_{n+1} \rangle \omega$$

where the π_0 is in the ith slot. Applying D to both sides of this equation tells us that

$$\langle 0, \ldots \pi_0 \pi_0 \ldots 0 | \pi_0 \pi_1 | 0 \ldots \pi_1 \pi_0 \ldots 0 | \pi_1 \pi_1 \rangle D^2(\omega)$$

(where the terms $\pi_0 \pi_0$ and $\pi_1 \pi_0$ are in their respective ith slots) equals

$$\langle \pi_0 \pi_1, \pi_0 \pi_2 \ldots \pi_0 \pi_0 \ldots \pi_0 \pi_{n+1} | \pi_1 \pi_1, \pi_1 \pi_2 \ldots \pi_1 \pi_0 \ldots \pi_1 \pi_{n+1} \rangle D(\omega)$$

Applying this equality to ††, we get

$$\langle 0, 0, \ldots \pi_{j+1} | \pi_1, \pi_2 \ldots \widehat{\pi_{j+1}} \ldots \pi_{n+2} \rangle D(\omega)$$

which in turn is equal to

$$\langle \pi_1 \ldots \pi_0 \ldots \pi_{n+2} \rangle z_j D(\omega)$$

as required.

Finally, for the case $j < i$, the expression † is equal to

$$\langle 0, 0 \ldots 0 | 0, \ldots \pi_0 \ldots 0 | 0, 0, \ldots \pi_{j+1} | \pi_1, \pi_2 \ldots \widehat{\pi_{j+1}} \ldots \pi_{n+2} \rangle D^2(\omega)$$

where the π_0 is in the $i-1$ slot. We then proceed as in the case $i < j$, except we use the linearity of ω in its $i-1$ term instead of its i term.

■

3.13. Properties of exterior differentiation. The purpose of this section is to prove the two fundamental properties of exterior differentiation: (i) that for each n, ∂_n is natural; (ii) that applying ∂_n then ∂_{n+1} to a differential n-form produces 0.

3.14. Proposition. *For each $n \geq 0$ and differential group A, exterior differentiation*

$$\partial_n : \Omega_n(-, A) \longrightarrow \Omega_{n+1}(-, A)$$

is a natural transformation.

Proof. Let $f : M' \longrightarrow M$, and fix some $\omega \in \Omega_n(M, A)$. We need to show that

$$\partial_n(\Omega_n(f; A)(\omega)) = \Omega_{n+1}(f; A)(\partial_n(\omega)).$$

For $n = 0$, naturality asks that $D(fw) = T(f)D(\omega)$; this follows immediately from the chain rule, [**CD.5**].

We will first demonstrate the case $n = 1$ to get the reader familiar with some of the manipulations used in the general case. We begin by calculating the left term. We have

$$\Omega_1(f; A)(\omega) = T(f)\omega = \langle Df, \pi_1 f\rangle\omega.$$

We then apply ∂_1 to that expression, which consists of the sum of two terms. We consider the first term in the sum:

$$\begin{aligned}
& \langle 0, \pi_0, \pi_1, \pi_2\rangle D(\langle Df, \pi_1 f\rangle\omega) \\
= \;& \langle 0, \pi_0, \pi_1, \pi_2\rangle\langle D(\langle Df, \pi_1 f\rangle), \pi_1\langle Df, \pi_1 f\rangle\rangle D(\omega) \\
= \;& \langle 0, \pi_0, \pi_1, \pi_2\rangle\langle\langle D^2 f, \langle\pi_0\pi_1, \pi_1\pi_1\rangle Df, \langle\pi_0\pi_0, \pi_1\pi_1\rangle Df, \pi_1\pi_1 f\rangle D(\omega) \\
= \;& \langle\langle 0, \pi_0, \pi_1, \pi_2\rangle D^2 f, \langle\pi_0, \pi_2\rangle Df, \langle\pi_1, \pi_2\rangle Df, \pi_2 f\rangle D(\omega)
\end{aligned}$$

Let $a = \langle 0, \pi_0, \pi_1, \pi_2\rangle D^2 f$, $b = \langle\pi_0, \pi_2\rangle Df$, $c = \langle\pi_1, \pi_2\rangle Df$, and $x = \pi_2 f$. Then the above is $\langle a, b, c, x\rangle D(\omega)$. Note that by [**CD.7**], a is also equal to $\langle 0, \pi_1, \pi_0, \pi_2\rangle D^2(f)$. Thus the second term in the sum is $\langle a, c, b, x\rangle D(\omega)$, and hence

$$\partial_1(\Omega_1(f; A)(\omega)) = \langle a, b, c, x\rangle D(\omega) - \langle a, c, b, x\rangle D(\omega).$$

Now, by [**CD.2**] we can write this as

$$\langle a, 0, c, x\rangle D(\omega) + \langle 0, b, c, x\rangle D(\omega) - \langle a, 0, b, x\rangle D(\omega) - \langle 0, c, b, x\rangle D(\omega) \ (\dagger).$$

But, ω is linear, so $\langle a, 0, c, x\rangle D(\omega) = \langle a, x\rangle\omega = \langle a, 0, b, x\rangle D(\omega)$. Hence the left composite of the naturality equation is

$$\langle 0, b, c, x\rangle D(\omega) - \langle 0, c, b, x\rangle D(\omega).$$

We now calculate the right side of the naturality equation. Again it will be a sum with two terms. The first term of the sum is:

$$\begin{array}{rl} & T_2(f)\langle 0,\pi_0,\pi_1,\pi_2\rangle D(\omega) \\ = & \langle\langle\pi_0,\pi_2\rangle Df,\langle\pi_1,\pi_2\rangle Df,\pi_2 f\rangle\langle 0,\pi_0,\pi_1,\pi_2\rangle D(\omega) \\ = & \langle 0,\langle\pi_0,\pi_2\rangle Df,\langle\pi_1,\pi_2\rangle Df,\pi_2 f\rangle D(\omega) \\ = & \langle 0,b,c,x\rangle D(\omega) \end{array}$$

and similarly the second term of the sum is $\langle 0,c,b,x\rangle D(\omega)$. Thus

$$\Omega_2(f;G)(\partial_1(\omega)) = \langle 0,b,c,x\rangle D(\omega) - \langle 0,c,b,x\rangle D(\omega) = \partial_1(\Omega_1(f;A)(\omega)),$$

so that ∂_1 is natural.

We now turn to the general case. As above, we begin by calculating the left side of the equation first. As above, there will be terms with $D^2(f)$, and a key element of the proof will be to use separate those terms out using **[CD.2]** and cancel them.

The left side is a sum with $n+1$ terms. The ith term of this sum is

$$\begin{array}{rl} & (-1)^i z_i D(T_n(f)\omega) \\ = & (-1)^i z_i\langle D(T_n(f),\pi_1 T_n(f)\rangle D(\omega) \text{ (by } \mathbf{[CD.5]}) \\ = & (-1)^i z_i\langle\ldots\langle\pi_0\pi_j,\pi_0\pi_n,\pi_1\pi_j,\pi_1\pi_n\rangle D^2(f)\ldots\langle\pi_0\pi_n,\pi_1\pi_n\rangle D(f)|\ldots\langle\pi_1\pi_j,\pi_1\pi_n\rangle D(f)\ldots\rangle D(\omega) \end{array}$$

Using **[CD.2]**, we can separate out each of the first $n+1$ variables before the $D(\omega)$ expression. For example, we can write $\langle a,b,c|d,e,f\rangle D(\omega)$ as

$$\langle a,0,0|d,e,f\rangle D(\omega) + \langle 0,b,0|d,e,f\rangle D(\omega) + \langle 0,0,c|d,e,f\rangle D(\omega).$$

Doing this to the above expression gives $n+1$ separate terms, with the first n terms being of the form

$$(-1)^i z_i\langle 0,0,\ldots,\langle\pi_0\pi_j,\pi_0\pi_n,\pi_1\pi_j,\pi_1\pi_n\rangle D^2(f),0,\ldots,0|\ldots\langle\pi_1\pi_j,\pi_1\pi_n\rangle D(f),\ldots\rangle D(\omega)$$

(where $0 \le j \le n-1$) and the final term being

$$(-1)^i z_i\langle 0,0,\ldots\langle\pi_0\pi_n,\pi_1\pi_n\rangle D(f)|\ldots\langle\pi_1\pi_j,\pi_1\pi_n\rangle D(f)\ldots\rangle D(\omega)(\star)$$

As we shall see, this last term will appear in the right side of the naturality equation, so we will leave it aside for now. To simplify the first n terms, let us define $a(i) := \langle\pi_i,\pi_n\rangle D(f)$, $b(i,j) := \langle 0,\pi_i,\pi_j,\pi_n\rangle D^2(f)$, and $x = \pi_n f$. Then by the definition of z_i, for $i \le j$ the jth term equals

$$(-1)^i\langle 0,0,\ldots,b(i,j+1),0,\ldots,0|a(0)\ldots a(j),\ldots x\rangle D(\omega)$$

while for $j < i$ the jth term equals

$$(-1)^i\langle 0,0,\ldots,b(i,j),0,\ldots,0|a(0)\ldots a(j),\ldots x\rangle D(\omega)$$

Then by linearity of ω, for $i \leq j$ we have

$$(-1)^i\langle(a(0), a(1), \ldots, a(j), b(i, j+1), a(j+2), \ldots a(n), x\rangle\omega,$$

and for $i > j$ we have

$$(-1)^i\langle a(0), a(1), \ldots, a(j), b(i, j), a(j+1), \ldots a(n), x\rangle\omega,$$

where in both sequences the term $a(i)$ is not present. As a sequence of numbers, the effect in both cases is to take the sequence $\langle a(0), a(1), a(2), \ldots, a(n)\rangle$, remove the ith term, place it to the left of the $j+1$st term, and group those two terms together with a $b(i, j)$.

We now claim that for $i \leq j$, the (i, j)th term and the $(j+1, i)$th terms sum to 0. To see this, first note that the (i, j)th term contains $b(i, j)$ while the $(j+1, i)$th term contains $b(j, i)$. But since

$$b(i, j) = \langle 0, \pi_i, \pi_j, \pi_n\rangle D^2(f),$$

by **[CD.7]**, $b(i, j) = b(j, i)$. In the (i, j)th term, this is in the ith position; in the $(j+1, i)$th term it is in the $j+1$st position. However, since ω is skew-symmetric, we can transpose the $b(j, i)$ in the $(j+1, i)$st term to the ith position by multiplying by $(-1)^{j-i}$. Since the original parity of the term is $(-1)^{j+1}$, this gives parity $(-1)^{i+1}$, which is exactly the opposite parity of the ith term. Thus, the two terms sum to 0; and in particular, allowing i and j to range over all possible $i \leq j$, this cancels out all the above terms.

As a result, all that remains on the left-side composite are terms of the form $\star$:

$$(-1)^i z_i\langle 0, 0, \ldots \langle\pi_0\pi_n, \pi_1\pi_n\rangle D(f)| \ldots \langle\pi_1\pi_j, \pi_1\pi_n\rangle D(f) \ldots\rangle D(\omega)$$

which, by definition of z_i, are equal to

$$(-1)^i\langle 0, 0, \ldots 0, a(i)|a(0), a(1), \ldots \widehat{a(i)}, \ldots a(n)\rangle D(\omega).$$

So that the left side is simply

$$\sum_{i=0}^{n}(-1)^i\langle 0, 0, \ldots 0, a(i)|a(0), a(1), \ldots \widehat{a(i)}, \ldots a(n)\rangle D(\omega).$$

But this is equal to

$$T_n(f)\partial_n(\omega),$$

by the definitions of $T_n(f)$ and ∂_n. Thus ∂_n is indeed natural. ∎

We now turn to proving the "square-zero" property of exterior differentiation.

3.15. Proposition. *For any $n \geq 0$ and differential group A, the composite*

$$\Omega_n(-; A) \xrightarrow{\partial_n} \Omega_{n+1}(-; A) \xrightarrow{\partial_{n+1}} \Omega_{n+2}(-; A)$$

is the 0 map.

Proof. Fix some object M and some $\omega \in \Omega_n(M;A)$. We want to show that $\partial_{n+1}(\partial_n(\omega)) = 0$. As with the proof of naturality, looking at some of the initial cases will help build intuition for the general case.

For $n = 0$, $\partial_0(\omega) = D(\omega)$ and then

$$\partial_1(\partial(\omega)) = \langle 0, \pi_0, \pi_1, \pi_2\rangle D^2(\omega) - \langle 0, \pi_1, \pi_0, \pi_2\rangle D^2(\omega).$$

The statement that this equals 0 is precisely **[CD.7]**.

We now consider the case $n = 1$. Here

$$\partial_1(\omega) = \langle 0, \pi_0, \pi_1, \pi_2\rangle D(\omega) - \langle 0, \pi_1, \pi_0, \pi_2\rangle D(\omega)$$

Note that $\partial_2(\partial_1(\omega))$ will have six terms. We consider the first one:

$$\begin{aligned} & \langle 0, 0, \pi_0, \pi_1, \pi_2, \pi_3\rangle D(\langle 0, \pi_0, \pi_1, \pi_2\rangle D(\omega)) \\ = \; & \langle 0, 0, \pi_0, \pi_1, \pi_2, \pi_3\rangle \langle \langle 0, \pi_0\pi_0, \pi_0\pi_1, \pi_2\pi_2\rangle, \pi_1\langle 0, \pi_0, \pi_1, \pi_2\rangle\rangle D^2(\omega) \\ = \; & \langle \langle 0, 0, 0, \pi_0\rangle, \langle 0, \pi_1, \pi_2, \pi_3\rangle\rangle D^2(\omega) \end{aligned}$$

Now, for $i \in \{0, 1, 2\}$, define $a_i = \langle 0, \pi_i\rangle$ and $b_i = \langle \pi_i, \pi_3\rangle$. Then the above term equals

$$\langle 0, a_0, a_1, b_2\rangle D^2(\omega)$$

Then by similar calculations $\partial_2(\partial_1(\omega))$ equals

$$\begin{aligned} & \langle 0, a_0, a_1, b_2\rangle D^2(\omega) - \langle 0, a_0, a_2, b_1\rangle D^2(\omega) - \langle 0, a_1, a_0, b_2\rangle D^2(\omega) \\ & + \langle 0, a_1, a_2, b_0\rangle D^2(\omega) + \langle 0, a_2, a_0, b_1\rangle D^2(\omega) - \langle 0, a_2, a_1, b_0\rangle D^2(\omega) \end{aligned}$$

Recalling that **[CD.7]** lets us flip interior terms, one can see that the above sum equals 0, as required.

We now turn to the general case. By definition

$$\partial_n(\omega) = \sum_{j=0}^{n} (-1)^j z_j D(\omega)$$

and so

$$\begin{aligned} & \partial_{n+1}(\partial_n(\omega)) \\ = \; & \partial_{n+1}\left(\sum_{j=0}^{n} (-1)^j z_j D(\omega)\right) \\ = \; & \sum_{i=0}^{n+1} (-1)^i z_i D\left(\sum_{j=0}^{n} (-1)^j z_j D(\omega)\right) \\ = \; & \sum_{i=0}^{n+1} z_i \sum_{j=0}^{n} (-1)^j \langle D(z_j), \pi_1 z_j\rangle D^2(\omega) \text{ (by } \mathbf{[CD.2]} \text{ and } \mathbf{[CD.5]}) \\ = \; & \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j} z_i \langle \pi_0 z_j, \pi_1 z_j\rangle D^2(\omega) \text{ (by left additivity and } \mathbf{[CD.3,4]}) \end{aligned}$$

To simplify this further, we need to find $\langle z_i\pi_0 z_j, z_i\pi_1 z_j\rangle$. Let $a_i = \langle 0, 0, \dots 0, \pi_i\rangle$ and $b_{i,j} = \langle \pi_0, \pi_1, \dots \hat{\pi}_i \dots \hat{\pi}_j \dots \pi_n\rangle$. Then by the definition of the z_i's,

$$\langle z_i\pi_0 z_j, z_i\pi_1 z_j\rangle = \langle 0, a_i, a_k, b_{i,k}\rangle$$

where

$$k = \begin{cases} j+1 & \text{if } i \leq j; \\ j & \text{if } j < i. \end{cases}$$

Thus

$$\partial_{n+1}(\partial_n(\omega)) = \sum_{i=0}^{n+1}\sum_{j=0}^{n}(-1)^{i+j}\langle 0, a_i, a_k, b_{i,k}\rangle D^2(\omega).$$

We now claim that for $i \leq j$, the (i,j) term in the above sum cancels out the $(j+1,i)$ term. Indeed, for $i \leq j$, the (i,j) term is

$$(-1)^{i+j}\langle 0, a_i, a_{j+1}, b_{i,j+1}\rangle D^2(\omega)$$

while the $(j+1,i)$ term is

$$(-1)^{i+j+1}\langle 0, a_{j+1}, a_i, b_{j+1,i}\rangle D^2(\omega).$$

But $b_{i,j+1} = b_{j+1,i}$ (both simply exclude the projections π_i and π_{j+1}) and

$$\langle 0, a_i, a_{j+1}, b_{i,j+1}\rangle D^2(\omega) = \langle 0, a_{j+1}, a_i, b_{i,j+1}\rangle D^2(\omega)$$

by [**CD.7**]. Then since $(-1)^{i+j+1} = (-1)(-1)^{i+j}$, the sum of the two terms is 0. As i ranges over all $0 \leq i \leq n+1$ and j ranges over all $0 \leq j \leq n$, all terms cancel out, leaving a sum of 0.

∎

Thus, for any linear group A, each object $M \in \mathbb{X}$ has an associated cochain complex

$$\Omega_0(M;A) \xrightarrow{\partial_0} \Omega_1(M;A) \xrightarrow{\partial_1} \dots \Omega_n(M;A) \xrightarrow{\partial_n} \Omega_{n+1}(M;A) \xrightarrow{\partial_{n+1}} \dots$$

from which one can define its de Rham cohomology groups.

3.16. Future work. For Cartesian spaces or convenient vector spaces, the previous sections only show us how to define forms and the exterior derivative for open subsets. However, we can easily extend this definition to manifolds and sheaves if the associated differential category has a "differential site".

3.17. Definition. *A **differential coverage** on a generalized Cartesian differential category $\mathbb{X}$ is a coverage T for which:*

- *the site $(\mathbb{X}, T)$ is subcanonical,*
- *for each n, the functor $T_n : \mathbb{X} \longrightarrow \mathbb{X}$ preserves covers.*

*A **differential site** is a generalized Cartesian differential category equipped with a differential coverage.*

For example, both ordinary Cartesian spaces and convenient vector spaces have such a site, where U_i covers U if the union of the U_i's covers U.

3.18. LEMMA. *If $(\mathbb{X}, T)$ is a differential site, then for each p and differential group G, the functor $\Omega_n(-; G)$ is a sheaf on $(\mathbb{X}, T)$.*

PROOF. Since the site is subcanonical, each presheaf functor $\mathbb{X}(-; G)$ is a sheaf. Then, since T_n preserves covers, $\mathbb{X}(T_n(-); G)$ is also a sheaf, and so $\Omega_n(-; G)$ is a sheaf. ■

3.19. DEFINITION. *Suppose that $(\mathbb{X}, T)$ is a differential site with A a linear object and $F \in$ **Sh**(X, T). We define a **differential** n**-form on** F **with values in** A to be a natural transformation $F \longrightarrow \Omega_n(-; A)$.*

The natural transformations

$$\partial_n : \Omega_n(-; A) \longrightarrow \Omega_{n+1}(-; A)$$

are then maps in the sheaf category, and if $\omega : F \longrightarrow \Omega_n(-; A)$ is an n-form on F, we can then simply define $\partial_n(\omega)$ to be the composite $\omega\partial_n$. This reproduces the de Rham cochain complex of ordinary or convenient manifolds when the manifolds are considered as sheaves on the appropriate sites. Moreover, it also reproduces the more general de Rham cochain complex of diffeological spaces, since diffeological spaces are certain sheaves on the Cartesian site (see [Baez and Hoffnung 2011], proposition 24), and their de Rham cohomology is defined as above (see [Iglesias-Zemmour 2013], chapter 6).

Ideally, one would like to be able to define forms and exterior differentiation for any category which is an "abstract categorical setting for differential geometry". One approach to defining such categories is to consider a category with an abstract "tangent bundle functor" [Rosický 1984]. As described in Example 2.2(v) of this paper, such tangent categories are closely related to Cartesian differential categories, and so in future we hope to show that the definitions and results here can be extended to a general tangent category.

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Department of Mathematics and Computer Science,
Mount Allison University, Sackville, Canada.
Email: `gcruttwell@mta.ca`

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