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FINITE PRODUCTS IN PARTIAL MORPHISM CATEGORIES

Dedicated to Professor M.V. Mielke on the occasion of his seventy fifth birthday

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Abstract. In this article we give necessary and sufficient conditions for a binary product to exist in a partial morphism category. We also give necessary and sufficient conditions for the existence of a productive terminal in such categories.

1. Introduction and Preliminaries

Even though the category of partial morphisms is central to so many issues in mathematics, such as representability, [Herrlich, 1988]; theory of computation, [Asperti, Longo, 1991]; fibred mapping spaces, [Booth, Brown, 1978]; graph transformation, [Corradini, Heindel, Hermann, König, 2006]; embedding CONV and MET in a quasitopos, [Lowen, Lowen, 1988]; petri nets, [Menezes, 1998]; fuzzy graphs, [Mori, Kawahara, 1997]; and logic, [Palmgren, Vickers, 2005], to mention a few, only little has been said about limits in such categories. Dealing with limits is more delicate than dealing with colimits.

In [Cockett, Lack, 2007], colimits and limits in the restriction context have been considered in restriction categories (an abstract formulation of partial morphism categories), but only the product in the partial function category (where the base category is the category of sets and functions) is given. In [Hosseini, Mielke, 2009], the universality of monos in partial morphism categories is investigated, while in the present article, the existence of finite products is characterized, i.e., necessary and sufficient conditions on a category are given for the corresponding partial morphism category to have finite products. To this end, we recall:

[Dyckhoff, Tholen, 1987], A *pullback complement* for the composable pair (f, s) is a composable pair $(\bar{s}, \bar{f})$ such that in the right diagram, the square $sf = \bar{f}\bar{s}$ is a pullback and for every pullback square $sg = \bar{g}t$ and any morphism $h : V \to Q$ with $fh = g$, there is a unique $\bar{h} : Z \to P$ with $\bar{f}\bar{h} = \bar{g}$ and $\bar{s}h = \bar{h}t$. We then say the square $sf = \bar{f}\bar{s}$ is a pullback complement square.

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[rr, bend left, "g"] \arrow[d, "t"'] & Q \arrow[r, "f"] \arrow[d, "\bar{s}"'] \arrow[dr, phantom, "p.b.c."] & U \arrow[d, "s"] \\ Z \arrow[r, "\bar{h}"'] \arrow[rr, bend right, "\bar{g}"'] & P \arrow[r, "\bar{f}"'] & X \end{tikzcd}$$

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If $(\bar{s}, \bar{f})$ is a pullback complement for (f, s) and (f, s) is a pullback complement for $(\bar{s}, \bar{f})$, then we say the square $sf = \bar{f}\bar{s}$ is a *double pullback complement* square.

1.1. Lemma.

1. *Pullback complements are unique up to isomorphism.*

2. *If $(\bar{s}, \bar{f})$ is a pullback complement for (f, s) and f is mono, then so is $\bar{f}$.*

Proof. Obvious. ■

Recalling that a *partial morphism classifier* in $\mathcal{C}$ is a morphism $\eta : A \to \tilde{A}$ that represents partial morphisms into A, [Adamek, Herrlich, Strecker, 1990], we have:

1.2. Lemma.

1. *$\lambda : p \to \pi$, where $p : U \to W$ and $\pi : V \to W$, is a partial morphism classifier in $\mathcal{C}/W$ if and only if for any triple (i, f, g) with $i : D \rightarrowtail C, f : D \to U, g : C \to W$ such that $pf = gi$, there exists a unique morphism $\hat{f}$ such that $\pi\hat{f} = g$ and the square on the right is a pullback.*

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "p.b."] & U \arrow[d, "\lambda"] \arrow[dr, "p"] & \\ C \arrow[r, "\hat{f}"'] \arrow[rr, bend right, "///", "g"'] & V \arrow[r, "///", "\pi"'] & W \end{tikzcd}$$

In this case λ is mono.

2. *For $\kappa : V \to U$, the pullback functor $\kappa^* : \mathcal{C}/U \to \mathcal{C}/V$ preserves partial morphism classifiers.*

Proof. Obvious. ■

Using 1.2 we get:

1.3. Examples.

1. Let $\eta_A : A \to \tilde{A}$ be a partial morphism classifier and suppose the diagrams:

$$A \xleftarrow{pr_1} A \times B \xrightarrow{pr_2} B \quad \text{and} \quad \tilde{A} \xleftarrow{\pi_1} \tilde{A} \times B \xrightarrow{\pi_2} B$$

are products. Then $\eta_A \times 1_B : pr_2 \to \pi_2$ is a partial morphism classifier.

Also with $\eta_B : B \to \tilde{B}$ a partial morphism classifier and $A \xleftarrow{\pi'_1} A \times \tilde{B} \xrightarrow{\pi'_2} \tilde{B}$ a product diagram, $1_A \times \eta_B : pr_1 \to \pi'_1$ is a partial morphism classifier.

2. Let $\mathcal{C}$ be the 2-chain category $\{1_A : A \to A, 1_B : B \to B, ! : A \to B\}$. A is the initial and B is the terminal object. Now $! : pr_2 \to 1_B$, where $pr_2 = ! : A \times B = A \to B$, is a partial morphism classifier; also $1_A : pr_1 \to 1_A$, where $pr_1 = 1_A : A \times B = A \to A$ is a partial morphism classifier. However there is no partial morphism classifier with domain B in $\mathcal{C}$.

Calling a pushout square that is preserved by pullbacks a *stable pushout*, and a stable pushout that is also a pullback a *stable pulation*, we have:

1.4. Lemma.

1. *Every stable pulation square in which all the morphisms are monos is a double pullback complement.*

2. *If in the commutative diagram shown m, n and l are mons, the left square is a stable pushout and the outer square is a pullback, then the right square is a pullback.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "m"'] & C \arrow[r, "p"] \arrow[d, tail, "n"] & E \arrow[d, tail, "l"] \\ B \arrow[r, "g"'] & D \arrow[r, "q"'] & F \end{tikzcd}$$

Proof.

1. Given monos j_1, j_2, λ_1 and λ_2 with the square $j_1\lambda_1 = j_2\lambda_2$ a stable pulation, let in the right diagram the outer square be a pullback and the triangle be commutative.

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d] \arrow[rr, bend left] & A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, phantom, "s.p.l."] & B \arrow[d, "j_1"] \\ X \arrow[rr, bend right, "f"'] & C \arrow[r, "j_2"'] & P \end{tikzcd}$$

Since $j_1^{-1}(f) = \lambda_1 k$, pulling back the stable pushout square $j_1\lambda_1 = j_2\lambda_2$ along f we get the square shown, which is a pushout by stability.

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "1"'] & \cdot \arrow[d, "f^{-1}(j_2)"] \\ \cdot \arrow[r, "f^{-1}(j_1)"'] & \cdot \end{tikzcd}$$

So $f^{-1}(j_2) = 1$ and $\alpha = f^{-1}(j_1)$. Now $j_2{j_2}^{-1}(f) = ff^{-1}(j_2) = f$. Also $j_2j_2^{-1}(f)f^{-1}(j_1) = ff^{-1}(j_1) = j_1j_1^{-1}(f) = j_1\lambda_1 k = j_2\lambda_2 k$, implying $j_2^{-1}(f)f^{-1}(j_1) = \lambda_2 k$. Hence in the right diagram, the squares are pullbacks and the triangles are commutative.

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d] \arrow[rr, bend left] & A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, phantom, "p.b.c."] & B \arrow[d, "j_1"] \\ X \arrow[r, "j_2^{-1}(f)"'] \arrow[rr, bend right, "f"'] & C \arrow[r, "j_2"'] & P \end{tikzcd}$$

Uniqueness of ${j_2}^{-1}(f)$ follows from the fact that j_2 is mono. So (λ_2, j_2) is a pullback complement for (λ_1, j_1). Similarly (λ_1, j_1) is a pullback complement for (λ_2, j_2).

2. The proof in [Lack, Sobocinski, 2006] for adhesive categories works under our hypothesis too.

∎

1.5. Lemma. *If in the right diagram i_1 and i_2 are monos, the inner square is a pullback and the outer square is a stable pushout, then:*

1. *the outer square is a double pullback complement; and*

2. *the morphism φ is mono, $i_1^{-1}(\varphi) = 1, \varphi^{-1}(i_1) = j_1$, $i_2^{-1}(\varphi) = 1$ and $\varphi^{-1}(i_2) = j_2$.*

$$\begin{tikzcd} A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & B \arrow[d, "i_1"] \arrow[ddr, bend left, "j_1"] & \\ C \arrow[r, "i_2"] \arrow[drr, bend right, "j_2"'] & D & \\ & & P \arrow[ul, "\varphi"] \end{tikzcd}$$

Proof. First notice that because i_1 and i_2 are monos, so are j_1, j_2, λ_1 and λ_2.

1. Obviously the pushout square is a pullback and so a stable pulation. The result follows from 1.4.

2. In the commutative diagram on the right the left square is a stable pushout and the outer one is a pullback. 1.4, implies the right square is also a pullback. So we have $i_1^{-1}(\varphi) = 1$ and $\varphi^{-1}(i_1) = j_1$. Similarly $i_2^{-1}(\varphi) = 1$ and $\varphi^{-1}(i_2) = j_2$.

$$\begin{tikzcd}
A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & B \arrow[r, "1"] \arrow[d, "j_1"] & B \arrow[d, "i_1"] \\
C \arrow[r, "j_2"'] \arrow[rr, bend right, "i_2"'] & P \arrow[r, "\varphi"] & D
\end{tikzcd}$$

Now suppose for morphisms α and β, $\varphi\alpha = \varphi\beta$. Pullback the outer square of the above diagram along $\varphi\alpha = \varphi\beta$ to get the diagram on the right.
Since the middle square is a stable pushout, so is the left square. So $\alpha = \beta$ and therefore φ is mono.

$$\begin{tikzcd}
& Y \arrow[rr] \arrow[dl] \arrow[ddr] & & A \arrow[rr, "1"] \arrow[dl, "\lambda_1"'] \arrow[ddr, "\lambda_2"] & & A \arrow[dl, "\lambda_1"'] \arrow[ddr, "\lambda_2"] & \\
Y_1 \arrow[rr] \arrow[ddr, "\gamma_1"] & & B \arrow[rr, "1"] \arrow[ddr, "j_1"] & & B \arrow[ddr, "i_1"] & & \\
& & Y_2 \arrow[rr] \arrow[dl, "\gamma_2"'] & & C \arrow[rr, "1"] \arrow[dl, "j_2"'] & & C \arrow[dl, "i_2"'] \\
& X \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] & & P \arrow[rr, "\varphi"] & & D &
\end{tikzcd}$$

■

Calling a pushout square that is also a double pullback complement, a *double pulation complement*, we have:

1.6. Lemma. *If in the diagram on the right, i_1 and i_2 are monos, the bottom and the back faces are pullbacks and the top face is a double pulation complement, then there is a unique morphism $\varphi : \acute{D} \to D$ rendering the front and right faces as pullbacks.*

$$\begin{tikzcd}
& \acute{A} \arrow[rr, "\acute{j_1}"] \arrow[dl, "\acute{j_2}"'] \arrow[dd, "g"] & & \acute{B} \arrow[dl, "\acute{i_1}"'] \arrow[dd, "l"] \\
\acute{C} \arrow[rr, "\acute{i_2}"] \arrow[dd, "k"'] & & \acute{D} \arrow[dd, dotted, "\varphi"] & \\
& A \arrow[rr, "j_1"] \arrow[dl, "j_2"'] & & B \arrow[dl, "i_1"'] \\
C \arrow[rr, "i_2"] & & D &
\end{tikzcd}$$

Proof. Since i_1 and i_2 are monos, so are j_1, j_2, $\acute{j_1}$ and $\acute{j_2}$. Now 1.1 implies $\acute{i_1}$ and $\acute{i_2}$ are monos as well. Because the top face of the above cube is a pushout and $i_1 l \acute{j_1} = i_2 k \acute{j_2}$, there exists a unique morphism $\varphi : \acute{D} \to D$ making the front faces of the cube commutative.

Now transform the bottom square under pullback along $\varphi : \acute{D} \to D$ to get the cube on the right. Since the front faces of the above cube are commutative and those in the right cube are pullbacks, there are γ_1 and γ_2 such that $l = i_1^{-1}(\varphi)\gamma_1$, $\acute{i}_1 = \hat{i}_1\gamma_1$, $k = i_2^{-1}(\varphi)\gamma_2$ and $\acute{i}_2 = \hat{i}_2\gamma_2$. Now the equalities $l = i_1^{-1}(\varphi)\gamma_1$, $k = i_2^{-1}(\varphi)\gamma_2$, $l^{-1}(j_1) = \acute{j}_1$ and $k^{-1}(j_2) = \acute{j}_2$ imply $\gamma_1^{-1}(\hat{j}_1) = \acute{j}_1$ and $\gamma_2^{-1}(\hat{j}_2) = \acute{j}_2$.

$$\begin{tikzcd}
& \hat{A} \arrow[rr, "\hat{j}_1"] \arrow[dl, "\hat{j}_2"'] \arrow[dd] & & \hat{B} \arrow[dl, "\hat{i}_1"] \arrow[dd, "i_1^{-1}(\varphi)"] \\
\hat{C} \arrow[rr, "\hat{i}_2", near start] \arrow[dd, "i_2^{-1}(\varphi)"'] & & \acute{D} \arrow[dd, "\varphi", near start] & \\
& A \arrow[rr, "j_1", near start] \arrow[dl, "j_2"'] & & B \arrow[dl, "i_1"] \\
C \arrow[rr, "i_2"] & & D &
\end{tikzcd}$$

So we have the following diagram in which the right squares are pullbacks and the left ones are formed to be pullbacks.

$$\begin{tikzcd}
\acute{A} \arrow[r, "\delta_1"] \arrow[d, "\acute{j}_1"'] \arrow[dr, phantom, "p.b."] & \hat{A} \arrow[r, "\hat{j}_2"] \arrow[d, "\hat{j}_1"] \arrow[dr, phantom, "p.b."] & \hat{C} \arrow[d, "\hat{i}_2"] \\
\acute{B} \arrow[r, "\gamma_1"'] & \hat{B} \arrow[r, "\hat{i}_1"'] & \acute{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\acute{A} \arrow[r, "\delta_2"] \arrow[d, "\acute{j}_2"'] \arrow[dr, phantom, "p.b."] & \hat{A} \arrow[r, "\hat{j}_2"] \arrow[d, "\hat{j}_2"] \arrow[dr, phantom, "p.b."] & \hat{B} \arrow[d, "\hat{i}_1"] \\
\acute{C} \arrow[r, "\gamma_2"'] & \hat{C} \arrow[r, "\hat{i}_2"'] & \acute{D}
\end{tikzcd}$$

Because the top face of the original cube is a double pullback complement, there exist unique morphisms λ_1 and λ_2 such that the triangles in the diagram on the right are commutative and the squares are pullbacks.

$$\begin{tikzcd}
\acute{A} \arrow[r, "1"] \arrow[rr, bend left, "\acute{j}_2"] \arrow[d] & \acute{A} \arrow[r, "\acute{j}_2"] \arrow[d, "\acute{j}_1"'] \arrow[dr, phantom, "p.b.c."] & \acute{C} \arrow[d, "\acute{i}_2"] \\
\hat{B} \arrow[r, "\lambda_1"'] \arrow[rr, bend right, "\hat{i}_1"'] & \acute{B} \arrow[r, "\acute{i}_1"'] & \acute{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\acute{A} \arrow[r, "1"] \arrow[rr, bend left, "\acute{j}_1"] \arrow[d] & \acute{A} \arrow[r, "\acute{j}_1"] \arrow[d, "\acute{j}_2"'] \arrow[dr, phantom, "p.b.c."] & \acute{B} \arrow[d, "\acute{i}_1"] \\
\hat{C} \arrow[r, "\lambda_2"'] \arrow[rr, bend right, "\hat{i}_2"'] & \acute{C} \arrow[r, "\acute{i}_2"'] & \acute{D}
\end{tikzcd}$$

We have $\acute{i}_1\lambda_1\gamma_1 = \hat{i}_1\gamma_1 = \acute{i}_1$ and $\acute{i}_2\lambda_2\gamma_2 = \hat{i}_2\gamma_2 = \acute{i}_2$, yielding $\lambda_1\gamma_1 = 1$ and $\lambda_2\gamma_2 = 1$. Since λ_1 and λ_2 are monos, they are isomorphisms. It follows that $\hat{i}_1$, is isomorphic to $\acute{i}_1$ and $\hat{i}_2$ is isomorphic to $\acute{i}_2$. ∎

2. Binary Product in Partial Morphism Categories

By the *partial morphism category* $\overrightarrow{\mathcal{C}}$ is meant the category having the same objects as $\mathcal{C}$ and with morphisms $\overrightarrow{f} = [(i_f, f)] : A \to B$ equivalence classes of pairs $(i_f : D_f \to A, f : D_f \to B)$ with i_f a *universal mono* (i.e., a mono whose pullback along every morphism exists) and where equivalence of (i_f, f) and (i_g, g) means that there is an isomorphism k for which $i_f = i_g k$ and $f = gk$. The composition of morphisms $A \xrightarrow{\overrightarrow{f}} B \xrightarrow{\overrightarrow{g}} C$ is defined by $\overrightarrow{g}\,\overrightarrow{f} = [(i_f(f^{-1}(i_g)), g(i_g^{-1}(f)))]$.

Calling a category *weakly adhesive* if monos of the category are universal and pushouts of monos along monos exist and are stable, we have:

2.1. Proposition. *Suppose* $A \xleftarrow{\overrightarrow{\pi_1} = [(i_1, \pi_1)]} A \overrightarrow{\times} B \xrightarrow{\overrightarrow{\pi_2} = [(i_2, \pi_2)]} B$ *is a product in* $\overrightarrow{\mathcal{C}}$. *Then:*

1. *$A \xleftarrow{\pi_1\lambda_1} P \xrightarrow{\pi_2\lambda_2} B$ is a product diagram in $\mathcal{C}$, where λ_1 and λ_2 are obtained by the pullback square given on the right;*

$$\begin{tikzcd} P \arrow[r, tail, "\lambda_1"] \arrow[d, tail, "\lambda_2"'] \arrow[dr, phantom, "p.b."] & D_1 \arrow[d, tail, "i_1"] \\ D_2 \arrow[r, tail, "i_2"'] & A \overrightarrow{\times} B \end{tikzcd}$$

2. *the above square is a double pullback complement;*

3. *assuming $\mathcal{C}$ is weakly adhesive, the above square is a pushout; and*

4. *with λ_n, $n = 1, 2$, as in part (1) and $pr_n = \pi_n\lambda_n$, $\lambda_n : pr_n \to \pi_n$ is a partial morphism classifier (in the appropriate slice category).*

PROOF.

1. If $A \xleftarrow{f} C \xrightarrow{g} B$ is a diagram in $\mathcal{C}$, then there exists a unique morphism $\vec{h}= [(i_h, h)] : C \to A \overrightarrow{\times} B$ such that $\vec{\pi}_1\vec{h}= f$ and $\vec{\pi}_2\vec{h}= g$. This implies $i_h h^{-1}(i_1) = i_h h^{-1}(i_2) = 1$, $\pi_1 i_1^{-1}(h) = f$ and $\pi_2 i_2^{-1}(h) = g$. Since i_h is mono, $i_h = h^{-1}(i_1) = h^{-1}(i_2) = 1$ (up to isomorphism). Then $i_2 i_2^{-1}(h) = h = i_1 i_1^{-1}(h)$, and so the above pullback yields a morphism $\gamma : C \to P$ such that $\lambda_1\gamma = i_1^{-1}(h)$ and $\lambda_2\gamma = i_2^{-1}(h)$. Therefore $\pi_1\lambda_1\gamma = \pi_1 i_1^{-1}(h) = f$ and $\pi_2\lambda_2\gamma = \pi_2 i_2^{-1}(h) = g$. To show uniqueness, suppose $\theta : C \to P$ is a morphism such that $\pi_1\lambda_1\theta = f$ and $\pi_2\lambda_2\theta = g$. Then $\vec{\pi}_1\ i_1\lambda_1\theta = f =\vec{\pi}_1\ i_1\lambda_1\gamma$, $\vec{\pi}_2\ i_1\lambda_1\theta = g =\vec{\pi}_2\ i_1\lambda_1\gamma$, implying $i_1\lambda_1\theta = i_1\lambda_1\gamma$ and so $\theta = \gamma$.

2. Consider the diagram on the right in which both squares are pullbacks and the triangle commutes.

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "l^{-1}(i_1)"'] \arrow[rr, bend left] & A \times B \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & D_1 \arrow[d, "i_1"] \\ X \arrow[rr, bend right, "l"'] & D_2 \arrow[r, "i_2"'] & A \overrightarrow{\times} B \end{tikzcd}$$

By part 1, $A \xleftarrow{pr_1=\pi_1\lambda_1} A \times B \xrightarrow{pr_2=\pi_2\lambda_2} B$ is a product in $\mathcal{C}$. We have $\vec{\pi}_1\ l = [(l^{-1}(i_1), pr_1 k)]$ and $\vec{\pi}_2\ l = [(l^{-1}(i_2), \pi_2 i_2^{-1}(l))]$. Let $\alpha = i_2^{-1}(l)$. Since $i_1^{-1}(l) = \lambda_1 k$, we get $l^{-1}(i_1) = l^{-1}(i_1\lambda_1) = l^{-1}(i_2\lambda_2) = l^{-1}(i_2)\alpha^{-1}(\lambda_2)$. This in turn yields $\vec{\pi}_1\ [(l^{-1}(i_2), ll^{-1}(i_2))] = [(l^{-1}(i_1), \pi_1 i_1^{-1}(l))] =\vec{\pi}_1\ l$ and $\vec{\pi}_2\ [(l^{-1}(i_2), ll^{-1}(i_2))] = [(l^{-1}(i_2), \pi_2 i_2^{-1}(l))] =\vec{\pi}_2\ l$. Uniqueness gives, $l = [(l^{-1}(i_2), ll^{-1}(i_2))]$. It follows that $l^{-1}(i_2) = 1$ and $l = ll^{-1}(i_2) = i_2 i_2^{-1}(l)$. We have $i_2\lambda_2 k= i_1\lambda_1 k = i_1 i_1^{-1}(l) = ll^{-1}(i_1) = i_2 i_2^{-1}(l)l^{-1}(i_1)$. i_2 mono yields $\lambda_2 k = i_2^{-1}(l)l^{-1}(i_1)$. So $i_2^{-1}(l)$ is the required morphism. Uniqueness follows from the fact that i_2 is mono. Hence (λ_2, i_2) is a pullback complement for (λ_1, i_1). Similarly (λ_1, i_1) is a pullback complement for (λ_2, i_2).

3. Let $D_1 \xrightarrow{j_1} P \xleftarrow{j_2} D_2$ be a pushout of $D_1 \xleftarrow{\lambda_1} A \times B \xrightarrow{\lambda_2} D_2$. Then there exists a unique morphism $\varphi : P \longrightarrow A \overrightarrow{\times} B$ such that $i_1 = \varphi j_1$ and $i_2 = \varphi j_2$. 1.5

implies φ is mono and $i_1^{-1}(\varphi) = 1, \varphi^{-1}(i_1) = j_1$, $i_2^{-1}(\varphi) = 1$ and $\varphi^{-1}(i_2) = j_2$. Now $\overrightarrow{\pi_1} [(\varphi, \varphi)] = \overrightarrow{\pi_1}$ and $\overrightarrow{\pi_2} [(\varphi, \varphi)] = \overrightarrow{\pi_2}$ implying that φ is an isomorphism.

4. We show that $\lambda_2 : pr_2 \to \pi_2$ is a partial morphism classifier. Given (i, f, g) with $i : D \rightarrowtail C$, $f : D \longrightarrow A \times B$ and $g : C \longrightarrow B$ such that $gi = pr_2 f$, we have partial morphisms $[(i, pr_1 f)] : C \to A$ and $[(1, g)] : C \to B$. Then there exists a unique morphism $\overrightarrow{k} = [(i_k, k)]$ such that $\overrightarrow{\pi_1}\overrightarrow{k} = [(i, pr_1 f)]$ and $\overrightarrow{\pi_2}\overrightarrow{k} = g$. This implies $i_k = 1$, $k^{-1}(i_1) = i$, $\pi_1 i_1^{-1}(k) = pr_1 f$, $k^{-1}(i_2) = 1$ and $\pi_2 i_2^{-1}(k) = g$.

Consider the diagram on the right, in which the right squares are pullbacks and the left squares are formed by taking pullbacks.

$$\begin{tikzcd}
X \arrow[r, tail, "\beta"] \arrow[d, "\alpha"'] & C \arrow[r, "1"] \arrow[d] & C \arrow[d, "k"] \\
A \times B \arrow[r, tail, "\lambda_2"'] & D_2 \arrow[r, tail, "i_2"'] & A \overrightarrow{\times} B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, tail, "\acute{\beta}"] \arrow[d, "\alpha"'] & D \arrow[r, tail, "i"] \arrow[d] & C \arrow[d, "k"] \\
A \times B \arrow[r, tail, "\lambda_1"'] & D_1 \arrow[r, tail, "i_1"'] & A \overrightarrow{\times} B
\end{tikzcd}$$

Since the square $i_1\lambda_1 = i_2\lambda_2$ is a pullback, so is the square on the right. Then $X = D, \acute{\beta} = 1, \beta = i$. These and the previous equations imply:

$$\begin{tikzcd}
X \arrow[r, tail, "\acute{\beta}"] \arrow[d, tail, "\beta"'] & D \arrow[d, tail, "i"] \\
C \arrow[r, "1"'] & C
\end{tikzcd}$$

$$pr_1\alpha = \pi_1\lambda_1\alpha = \pi_1 i_1^{-1}(k)\acute{\beta} = \pi_1 i_1^{-1}(k) = pr_1 f$$
$$pr_2\alpha = \pi_2\lambda_2\alpha = \pi_2 i_2^{-1}(k)\beta = g\beta = gi = pr_2 f$$

It follows that $\alpha = f$. So we have the diagram on the right, in which the square is a pullback and $\pi_2 i_2^{-1}(k) = g$.

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "p.b."] & A \times B \arrow[d, tail, "\lambda_2"] \arrow[dr, "pr_2"] & \\
C \arrow[r, "i_2^{-1}(k)"'] & D_2 \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

To show uniqueness, suppose there exists a morphism h such that in the above diagram when $i_2^{-1}(k)$ is replaced by h, the square is a pullback and $\pi_2 h = g$. One can easily verify that $\overrightarrow{\pi_1}\ i_2 h = [(i, pr_1 f)] = \overrightarrow{\pi_1}\overrightarrow{k}$ and $\overrightarrow{\pi_2}\ i_2 h = g = \overrightarrow{\pi_2}\overrightarrow{k}$. So $i_2 h = k$. Because $k^{-1}(i_2) = 1$, we have $i_2 h = k = i_2 i_2^{-1}(k)$ and so $h = i_2^{-1}(k)$. By 1.2, we are done. Similarly $\lambda_1 : pr_1 \to \pi_1$ is a partial morphism classifier.

■

2.2. Proposition. *Suppose $\mathcal{C}$ is weakly adhesive. If $A \xleftarrow{pr_1} A \times B \xrightarrow{pr_2} B$ is a product in $\mathcal{C}$, $\lambda_n : pr_n \to \pi_n$ is a partial morphism classifier for $n = 1, 2$, where $\pi_1 : B_A \to A$, $\pi_2 : A_B \to B$ and there exist morphisms i_1 and i_2 such that the square on the right is a double pulation complement, then $A \xleftarrow{\vec{\pi_1}=[(i_1,\pi_1)]} P \xrightarrow{\vec{\pi_2}=[(i_2,\pi_2)]} B$ is a product diagram in $\vec{\mathcal{C}}$*

$$\begin{tikzcd}
A \times B \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, phantom, "d.p.l.c."] & B_A \arrow[d, "i_1"] \\
A_B \arrow[r, "i_2"'] & P
\end{tikzcd}$$

Proof. Because by 1.2, λ_1, λ_2 are monos, (λ_1, i_1) is a pullback complement for (λ_2, i_2) and (λ_2, i_2) is a pullback complement for (λ_1, i_1), by 1.1, i_1 and i_2 are monos.

To show $A \xleftarrow{\vec{\pi_1}} P \xrightarrow{\vec{\pi_2}} B$ is a product in $\vec{\mathcal{C}}$, given $A \xleftarrow{\vec{f}=[(i_f,f)]} C \xrightarrow{\vec{g}=[(i_g,g)]} B$, let $D_f \longleftarrow D_f \cap D_g \rightarrowtail D_g$ be a pullback of $D_f \overset{i_f}{\rightarrowtail} C \overset{i_g}{\leftarrowtail} D_g$. Then we have morphisms $A \xleftarrow{f i_f^{-1}(i_g)} D_f \cap D_g \xrightarrow{g i_g^{-1}(i_f)} B$, and so a unique morphism l exists such that $pr_1 l = f i_f^{-1}(i_g)$ and $pr_2 l = g i_g^{-1}(i_f)$.

Also we have unique morphisms $\tilde{f}, \tilde{g}$ such that $\pi_1 \tilde{f} = f$, $\pi_2 \tilde{g} = g$ and the below squares are pullbacks.

$$\begin{tikzcd}
D_f \cap D_g \arrow[r, "l"] \arrow[d, tail] \arrow[dr, phantom, "p.b."] & A \times B \arrow[d, tail, "\lambda_1"'] \arrow[dr, "pr_1"] & \\
D_f \arrow[r, "\tilde{f}"'] & B_A \arrow[r, "\pi_1"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
D_f \cap D_g \arrow[r, "l"] \arrow[d, tail] \arrow[dr, phantom, "p.b."] & A \times B \arrow[d, tail, "\lambda_2"'] \arrow[dr, "pr_2"] & \\
D_g \arrow[r, "\tilde{g}"'] & A_B \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

Let $D_f \overset{j_f}{\rightarrowtail} \acute{P} \overset{j_g}{\leftarrowtail} D_g$ be a pushout of $D_f \longleftarrow D_f \cap D_g \rightarrowtail D_g$.

There exists a unique morphism $k : \acute{P} \longrightarrow C$ such that $i_g = k j_g$ and $i_f = k j_f$. By 1.5, k is mono and the pushout square is a double pullback complement. Also 1.6, implies the existence of a unique morphism $h : \acute{P} \longrightarrow P$ such that the front faces in the right diagram are pullbacks.

$$\begin{tikzcd}[row sep=small, column sep=small]
 & D_f \cap D_g \arrow[rr] \arrow[dl] \arrow[dd, "l" near start] & & D_g \arrow[dl, "j_g"'] \arrow[dd, "\tilde{g}"] \\
D_f \arrow[rr, crossing over, "j_f" near end] \arrow[dd, "\tilde{f}"'] & & \acute{P} & \\
 & A \times B \arrow[rr, "\lambda_2" near start] \arrow[dl, "\lambda_1"'] & & A_B \arrow[dl, "i_2"] \\
B_A \arrow[rr, "i_1"'] & & P \arrow[from=uu, crossing over, "h" near end] &
\end{tikzcd}$$

One easily verifies that $\vec{\pi_1}\,[(k,h)] = \vec{f}$ and $\vec{\pi_2}\,[(k,h)] = \vec{g}$.

To show uniqueness, suppose there exists a morphism $\overrightarrow{\psi}$ such that $\overrightarrow{\pi_1}\overrightarrow{\psi}=\overrightarrow{f}$ and $\overrightarrow{\pi_2}\overrightarrow{\psi}=\overrightarrow{g}$. Then $i_\psi\psi^{-1}(i_1) = i_f$, $\pi_1 i_1^{-1}(\psi) = f$, $i_\psi\psi^{-1}(i_2) = i_g$ and $\pi_2 i_2^{-1}(\psi) = g$. Form the cube on the right, where all the squares are pullbacks.

$$\begin{tikzcd}
& X \arrow[rr, "m"] \arrow[dl, tail] \arrow[dd] & & A\times B \arrow[dl, tail, "\lambda_1"'] \arrow[dd, tail, "\lambda_2"] \\
D_f \arrow[rr, "i_1^{-1}(\psi)"] \arrow[dd, tail, "\psi^{-1}(i_1)"'] & & B_A \arrow[dd, "i_1"] & \\
& D_g \arrow[rr, "i_2^{-1}(\psi)"] \arrow[dl, tail, "\psi^{-1}(i_2)"'] & & A_B \arrow[dl, tail, "i_2"] \\
D_\psi \arrow[rr, "\psi"] & & P &
\end{tikzcd}$$

One can see $X = D_f \cap D_g$ and that the pair of morphisms $D_f \leftarrowtail X \rightarrowtail D_g$ equals $D_f \xleftarrow{i_f^{-1}(i_g)} D_f \cap D_g \xrightarrow{i_g^{-1}(i_f)} D_g$. Pushout stability now renders the left square in the above cube a pushout.

So $D_\psi = \acute{P}$, $\psi^{-1}(i_1) = j_f$ and $\psi^{-1}(i_2) = j_g$ and therefore we have the pullback squares on the right.

$$\begin{tikzcd}
D_f\cap D_g \arrow[r, tail, "i_f^{-1}(i_g)"] \arrow[d, "m"'] \arrow[dr, phantom, "pb"] & D_f \arrow[r, tail, "j_f"] \arrow[d, "i_1^{-1}(\psi)"] \arrow[dr, phantom, "pb"] & \acute{P} \arrow[d, "\psi"] \\
A\times B \arrow[r, tail, "\lambda_1"'] & B_A \arrow[r, tail, "i_1"'] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
D_f\cap D_g \arrow[r, tail, "i_g^{-1}(i_f)"] \arrow[d, "m"'] \arrow[dr, phantom, "pb"] & D_g \arrow[r, tail, "j_g"] \arrow[d, "i_2^{-1}(\psi)"] \arrow[dr, phantom, "pb"] & \acute{P} \arrow[d, "\psi"] \\
A\times B \arrow[r, tail, "\lambda_2"'] & A_B \arrow[r, tail, "i_2"'] & P
\end{tikzcd}$$

It follows that $pr_1 m = \pi_1\lambda_1 m = \pi_1 i_1^{-1}(\psi) i_f^{-1}(i_g) = f i_f^{-1}(i_g) = pr_1 l$. Similarly we get $pr_2 m = pr_2 l$. Therefore $m = l$.

Now we have the diagram on the right, in which the squares are pullbacks, $\pi_1 i_1^{-1}(\psi) = f$ and $\pi_2 i_2^{-1}(\psi) = g$.

$$\begin{tikzcd}
D_f\cap D_g \arrow[r, "l=m"] \arrow[d, tail, "i_f^{-1}(i_g)"'] \arrow[dr, phantom, "pb"] & A\times B \arrow[d, tail, "\lambda_1"] \arrow[dr, "pr_1"] & \\
D_f \arrow[r, "i_1^{-1}(\psi)"'] & B_A \arrow[r, "\pi_1"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
D_f\cap D_g \arrow[r, "l=m"] \arrow[d, tail, "i_g^{-1}(i_f)"'] \arrow[dr, phantom, "pb"] & A\times B \arrow[d, tail, "\lambda_2"] \arrow[dr, "pr_2"] & \\
D_g \arrow[r, "i_2^{-1}(\psi)"'] & A_B \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

It follows that $i_1^{-1}(\psi) = \tilde{f}, i_2^{-1}(\psi) = \tilde{g}$. Now $\psi j_f = \psi\psi^{-1}(i_1) = i_1 i_1^{-1}(\psi) = i_1\tilde{f}$ and similarly $\psi j_g = i_2\tilde{g}$. The uniqueness of h yields $h = \psi$. Now we have $i_f = i_\psi\psi^{-1}(i_1) = i_\psi j_f$ and $i_g = i_\psi\psi^{-1}(i_2) = i_\psi j_g$, and uniqueness of k yields $i_\psi = k$. Hence $\overrightarrow{\psi}= [(k, h)]$. ∎

2.3. Theorem. *Suppose $\mathcal{C}$ is weakly adhesive. A product $A \xleftarrow{\overrightarrow{\pi_1}} A \overrightarrow{\times} B \xrightarrow{\overrightarrow{\pi_2}} B$ exists in $\overrightarrow{\mathcal{C}}$ if and only if a product $A \xleftarrow{pr_1} A\times B \xrightarrow{pr_2} B$, along with morphisms π_1, π_2 and monos λ_1, λ_2, i_1 and i_2 exist in $\mathcal{C}$ such that $\lambda_n : pr_n \to \pi_n$ is a partial morphism classifier and the square on the right is a double pulation complement. In this case, $\overrightarrow{\pi_n}= [(i_n, \pi_n)]$.*

$$\begin{tikzcd}
A\times B \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, phantom, "d.p.l.c."] & B_A \arrow[d, "i_1"] \\
A_B \arrow[r, "i_2"'] & A \overrightarrow{\times} B
\end{tikzcd}$$

Proof. Follows from 2.1 and 2.2. ∎

2.4. Examples.

1. Let $\mathcal{C}$ be an adhesive category [Lack Sobocinski, 2004]. Then by 1.4, pushout of monos along monos are double pullback complements. So if $A \xleftarrow{pr_1} A \times B \xrightarrow{pr_2} B$ is a product in $\mathcal{C}$ and $\lambda_n : pr_n \to \pi_n$ is a partial morphism classifier, then with $\vec{\pi_1} = [(i_1, \pi_1)]$ and $\vec{\pi_2} = [(i_2, \pi_2)]$, $A \xleftarrow{\vec{\pi_1}} P \xrightarrow{\vec{\pi_2}} B$ is a product diagram in $\vec{\mathcal{C}}$, where $B_A \xrightarrow{i_1} P \xleftarrow{i_2} A_B$ is a pushout of $B_A \xleftarrow{\lambda_1} A \times B \xrightarrow{\lambda_2} A_B$.

2. Let $\mathcal{C}$ be an adhesive category in which the partial morphism classifiers $\eta_A : A \to \tilde{A}$ and $\eta_B : B \to \tilde{B}$ and the products $A \xleftarrow{pr_1} A \times B \xrightarrow{pr_2} B$, $A \xleftarrow{\pi_A} A \times \tilde{B} \xrightarrow{\pi_{\tilde{B}}} \tilde{B}$ and $\tilde{A} \xleftarrow{\pi_{\tilde{A}}} \tilde{A} \times B \xrightarrow{\pi_B} B$ exist. By 1.3, $1 \times \eta_B : pr_1 \to \pi_A$ and $\eta_A \times 1 : pr_2 \to \pi_B$ are partial morphism classifiers. So $A \xleftarrow{\vec{\pi_1} = [(i_A, \pi_A)]} P \xrightarrow{\vec{\pi_2} = [(i_B, \pi_B)]} B$ is a product in $\vec{\mathcal{C}}$, where $A \times \tilde{B} \xrightarrow{i_A} P \xleftarrow{i_B} \tilde{A} \times B$ is a pushout of $A \times \tilde{B} \xleftarrow{1 \times \eta_B} A \times B \xrightarrow{\eta_A \times 1} \tilde{A} \times B$.

3. Let $\mathcal{C}$ be a topos. Because topoi are adhesive, [Lack, Sobocinski, 2006], and all partial maps are representable, [Johnstone, 1977], by (2), $\vec{\mathcal{C}}$ has all binary products. In this case for $A, B \in \mathcal{C}$, $A \vec{\times} B = \tilde{A} \times B \cup A \times \tilde{B}$, where $\eta_A : A \to \tilde{A}$ is a partial morphism classifier and "$\cup$" is the union taken as subobjects of $\tilde{A} \times \tilde{B}$.

4. Let $\mathcal{C}$ be the category corresponding to a distributive prelattice (i.e., a preordered class with binary meets and joins, in which meet is distributive over join). One can easily verify that $\mathcal{C}$ is weakly adhesive, but it is not generally adhesive as the diagram on the right, in which $A \neq B$, suggests. Also it is not hard to see $\lambda : p \to \pi$, where $p : A \to C$ and $\pi : B \to C$, is a partial morphism classifier if and only if π is an isomorphism and the only morphism to $A \wedge B$ is the identity morphism (or equivalently, in the preordered class, $B \leq C$, $C \leq B$ and $A \wedge B$ is minimal). So $A \vec{\times} B$ exists if and only if $A \wedge B$ is minimal. In this case, since by 1.4, the square on the right is a double pulation complement, we get $A \vec{\times} B = A \vee B$.

$$\begin{tikzcd}
& A \arrow[rr] \arrow[dl] \arrow[dd] & & B \arrow[dl] \arrow[dd] \\
A \arrow[rr, crossing over] \arrow[dd] & & B & \\
& A \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
B \arrow[rr] & & B \arrow[from=uu, crossing over] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \wedge B \arrow[r] \arrow[d] \arrow[dr, phantom, "d.p.l.c."] & A \arrow[d] \\
B \arrow[r] & A \vee B
\end{tikzcd}$$

5. Let $\mathcal{C}$ be a category in which all monos are isomorphisms and not all pullbacks exist (e.g, the category generated by parallel isomorphisms f, g and a morphism h such that $hf = hg$), so that $\mathcal{C}$ is not adhesive. It can be easily verified that $\mathcal{C}$ is weakly

adhesive. Also for $A, B \in \mathcal{C}$, the existence of $A \overrightarrow{\times} B$ is equivalent to that of $A \times B$. In fact $A \overrightarrow{\times} B = A \times B$. But this is also the case because $\overrightarrow{\mathcal{C}}$ is isomorphic to $\mathcal{C}$.

6. Let $\mathcal{C}$ be a weakly adhesive category that is not adhesive and $\mathcal{D}$ be any weakly adhesive category. Then $\mathcal{C} \times \mathcal{D}$ is weakly adhesive but not adhesive. The product $(A, A') \overrightarrow{\times} (B, B')$ exists if and only if the pairs A, B and A', B' satisfy the conditions of 2.3 in $\mathcal{C}$ and $\mathcal{D}$, respectively.

Calling an object *productive*, if its product with every object exists, we have:

2.5. Lemma. *Let $\mathcal{C}$ be a category with a terminal 1 and T be a productive object. Then T is a terminal object in $\overrightarrow{\mathcal{C}}$ if and only if $T \to 1$ is a partial morphism classifier in $\mathcal{C}$.*

Proof.

$$\begin{tikzcd} A \times T \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "p.b."] & T \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd}$$

Assume T is a terminal object in $\overrightarrow{\mathcal{C}}$. Let $[(i_u, u)] : A \longrightarrow T$ be a partial morphism. Since T is productive, the right square is a pullback.

Now since T is a terminal in $\overrightarrow{\mathcal{C}}$, $T \to 1$ is mono in $\overrightarrow{\mathcal{C}}$, and so mono in $\mathcal{C}$, [Hosseini, Mielke, 2009].

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "i_u"'] & T \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd}$$

So pr_1 is a universal mono, putting $[(pr_1, pr_2)]$ in $\overrightarrow{\mathcal{C}}$. It now follows that $[(i_u, u)] = [(pr_1, pr_2)]$ and so the square on the right is a pullback.

Conversely assume $T \to 1$ is a partial morphism classifier in $\mathcal{C}$. Given an object A, we have a pullback square as shown in the first square above and so a partial morphism $[(pr_1, pr_2)] : A \longrightarrow T$. Uniqueness follows from the fact that any partial morphism from A to T can be represented by $T \longrightarrow 1$.

■

2.6. Examples.

1. Let $\mathcal{C}$ be a category with binary products. If $\mathcal{C}$ has a strict initial object 0 and a terminal object 1, then $0 \to 1$ is a partial morphism classifier and therefore 0 is a terminal object in $\overrightarrow{\mathcal{C}}$. Also 0 is an initial object in $\overrightarrow{\mathcal{C}}$. Hence $\overrightarrow{\mathcal{C}}$ has a zero object.

2. Let $\mathcal{C}$ be a topos. Then $\mathcal{C}$ has a strict initial object [Mac Lane, Moerdijk, 1992]. So $\overrightarrow{\mathcal{C}}$ has a zero object.

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