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DUALITY IN NON-ABELIAN ALGEBRA I. FROM COVER RELATIONS TO GRANDIS EX2-CATEGORIES

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ABSTRACT. The aim of this series of papers is to develop a self-dual categorical approach to some topics in non-abelian algebra, which is based on replacing the framework of a category with that of a category equipped with a functor to it. The present paper gives some preliminary steps in this direction, where several known structures on a category, which arise in the categorical treatment of these topics, are viewed as such functors; as a result, we obtain some new conceptual links between these structures.

Introduction

As is well known, many homomorphism theorems of classical algebra can be obtained in the self-dual context of an abelian category, which includes the category of modules over any ring, and in particular, the category of abelian groups, as an example. By homomorphism theorems we mean various isomorphism theorems (e.g. first, second, etc.), as well as diagram lemmas needed for homological algebra. Non-abelian versions of these results are also valid for (not necessarily abelian) groups and group-like structures such as rings, loops, and many others. In modern categorical algebra, one of the well-established categorical contexts where these results can be abstractly obtained is that of a semi-abelian category [16] — a pointed Barr exact [2] Bourn protomodular [3] category having binary coproducts. Unlike the context of an abelian category, that of a semi-abelian category is not "self-dual". In fact, a semi-abelian category whose dual is also a semi-abelian category is necessarily an abelian category (see [16], and see also [17] for a more general result). Nevertheless, there is a hidden *functorial duality* as shown in [20]: among the pointed regular categories, the semi-abelian categories can be characterised via dual axioms on the bifibration of subobjects. The aim of this series of papers is to analyse this new phenomenon, and explore the applications of the idea of considering functorial duality in various investigations from non-abelian algebra.

By "non-abelian algebra" we mean, first of all, the study of classical non-abelian group-like structures, where functorial duality would, for instance, provide a self-dual approach

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to homomorphisms theorems, thus making our aim very much along the first sentence of the passage below from [23]:

> "A further development giving the first and second isomorphism theorems, and so on, can be made by introducing additional carefully chosen dual axioms. This will be done below only in the more symmetrical abelian case."

By "non-abelian algebra" we also mean axiomatic investigations in universal and categorical algebra which attempt to describe *general contexts* where certain results for different types of non-abelian algebraic structures can be conceptually clarified and unified. It was suggested in [18] that one such context might be that of a category equipped with a *cover relation*, i.e. a binary relation on the class of morphisms satisfying suitable conditions. Such a context allows, for instance, a very general formulation of the five lemma that unifies D. Bourn's five lemma [4] and the five lemma of M. Grandis [10] — see [26].

Cover relations abound. For instance, they arise naturally from Grothendieck topologies. Following the terminology used in [24], say that a morphism $f : X \to Y$ in a category $\mathbb{C}$ *covers* a morphism $g : W \to Y$ when the sieve $g^*(f)$ of all morphisms $h : V \to W$ such that gh factors through f, is a covering family of the topology. Cover relations also arise from factorisation systems. Given a factorisation system $(\mathcal{E}, \mathcal{M})$, say that f covers g when g factors through the morphism m in the $(\mathcal{E}, \mathcal{M})$-factorisation $f = me$ of f. These two types of cover relations are reflexive and transitive, but there are also examples of cover relations which are not reflexive and transitive — see [19].

It was remarked in [19] that it becomes possible to recover a factorisation system $(\mathcal{E}, \mathcal{M})$ from the cover relation that it induces, when $\mathcal{M}$ is a class of monomorphisms. Moreover, cover relations that correspond to such factorisation systems can be naturally characterised. We revisit this result in the present paper, where both cover relations and factorisation systems are viewed in a certain way as functors — in particular, in the case of a factorisation system, this functor is the opfibration of $\mathcal{M}$-subobjects. This part of our work is in some sense a triviality, but what is striking is that out of this triviality we discover that by imposing the resulting functors to be obtainable in a similar way also by a dual procedure, we arrive to the self-dual contexts used by M. Grandis in his "categorical foundations of homological algebra" [10].

The contexts used by M. Grandis consist of a category $\mathbb{C}$ equipped with a class $\mathcal{N}$ of morphisms, seen as abstractly defined null morphisms in the category, satisfying various axioms that mimic the behaviour of the class of null morphisms in an abelian category. This idea goes back to C. Ehresmann [8], R. Lavendhomme [22], as well as G. M. Kelly [21] (who considered it in the case of an additive category). One refers to the class $\mathcal{N}$ as an *ideal of null morphisms*, because of one of the axioms which it is required to satisfy, which states that if one of the morphisms in a composite nm belongs to the class $\mathcal{N}$, then so does the composite nm. Thus, an ideal of null morphisms is simply a subfunctor of the hom-functor. One may introduce the notions of kernel and cokernel, and exactness of a sequence relative to an ideal, which allows development of homological algebra in the context of a category equipped with an ideal — see [12] for the complete theory, as well as examples and applications.

In the first section of the paper we show that there is an abstract process which constructs a faithful amnestic functor $F : \mathbb{B} \to \mathbb{C}$, or, what we call a *form*, from a binary relation $\mathcal{C}$ on the class of morphisms in $\mathbb{C}$. The functors that we associate to a cover relation, a class $\mathcal{M}$ of monomorphisms, or an ideal $\mathcal{N}$ of null morphisms, are all instances of a form obtained from $\mathcal{C}$, where in each case the relation $\mathcal{C}$ must be suitably chosen (in the case of a cover relation, $\mathcal{C}$ is simply the same relation as the cover relation). We call these functors *the form of the cover relation*, *the form of $\mathcal{M}$-subobjects*, and *the form of $\mathcal{N}$-exact pairs*, respectively. In some cases these three forms are isomorphic (by an isomorphism of forms $F : \mathbb{B} \to \mathbb{C}$ and $F' : \mathbb{B}' \to \mathbb{C}$ we mean an isomorphism $I : \mathbb{B} \to \mathbb{B}'$ such that $F'I = F$); for instance, they are isomorphic when

- the cover relation is obtained from the Grothendieck topology generated by the class of epimorphisms in an abelian category $\mathbb{C}$,
- the class $\mathcal{M}$ is the class of monomorphisms in $\mathbb{C}$,
- the ideal $\mathcal{N}$ consists of null morphisms in $\mathbb{C}$.

The resulting form is simply the bifibration of subobjects in the abelian category $\mathbb{C}$. In general, every form of $\mathcal{M}$-subobjects and every form of $\mathcal{N}$-exact pairs is isomorphic to the form of some cover relation. Moreover, when the ideal $\mathcal{N}$ admits kernels and cokernels, the form of $\mathcal{N}$-exact pairs is isomorphic to the form of $\mathcal{M}$-subobjects where $\mathcal{M}$ is the class of kernels relative to the ideal.

In the second section we characterise forms of reflexive and transitive cover relations, and forms of $\mathcal{M}$-subobjects when $\mathcal{M}$ is a class of monomorphisms in a factorisation system $(\mathcal{E}, \mathcal{M})$, as well as when $\mathcal{M}$ satisfies weaker conditions introduced in [7], and independently in [25] — labeled as conditions (M_1) and (M_2) in this paper.

In the third section we show how requiring a form $F : \mathbb{B} \to \mathbb{C}$ arising from a cover relation to have, as well, the dual property, i.e. the property that the dual form $F^{\mathsf{op}} : \mathbb{B}^{\mathsf{op}} \to \mathbb{C}^{\mathsf{op}}$ arises from a cover relation on $\mathbb{C}^{\mathsf{op}}$, makes it into a form of $\mathcal{N}$-exact pairs, where $\mathcal{N}$ is an ideal of null morphisms such that every morphism $B \to C$ can be extended to an exact sequence $A \to B \to C \to D$. Moreover, up to an isomorphism of forms, this is a characterisation of forms which together with their duals arise from cover relations. Forms, which together with their duals arise from classes $\mathcal{M}$ of monomorphisms satisfying (M_1) and (M_2), can be characterised in a similar way by those ideals of null morphisms which admit kernels and cokernels. Finally, forms which together with their duals arise from classes $\mathcal{M}$ of monomorphisms which are part of a factorisation system $(\mathcal{E}, \mathcal{M})$, can be characterised by those ideals $\mathcal{N}$ of null morphisms which define Grandis ex2-categories. Put differently, a Grandis ex2-category can be seen as a category equipped with two factorisation systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$, such that the opfibration of $\mathcal{M}_1$-subobjects is isomorphic to the fibration of $\mathcal{E}_2$-quotients.

The main impetus for the present work can be described as follows. The structure of a cover relation is certainly not *categorically* self-dual. Indeed, a cover relation is defined for pairs of morphisms f,g having the same codomain, while its dual would give a relation defined for pairs of morphisms f,g having the same domain. It is then natural to ask what structure do we get if we force the structure of a cover relation to be *functorially* self-dual. This leads to the notion of an ideal of null morphisms. In the present paper, we point out and analyse this passage.

1. Preliminaries

Recall that, given a functor $F : \mathbb{S} \to \mathbb{C}$, the *fibre* $F^{-1}(Y)$ at an object Y in $\mathbb{C}$ is the subcategory of $\mathbb{S}$ consisting of those objects and morphisms which by F are mapped to Y and 1_Y, respectively. If F is faithful then each fibre is a preorder. A functor $F : \mathbb{S} \to \mathbb{C}$ is said to be *amnestic* when the only isomorphisms which are mapped by it to an identity morphism are the identity morphisms (see e.g. [1]). For a faithful functor F to be amnestic is equivalent to each fibre being an ordered set (class). By an *ordered set* we mean a *partially ordered set* — a set equipped with a reflexive, transitive and antisymmetric binary relation.

In this paper, by a *form* over a category $\mathbb{C}$, we mean an amnestic faithful functor $F : \mathbb{S} \to \mathbb{C}$. Thus, a form is the same as an amnestic forgetful functor in a *concrete category* over $\mathbb{C}$, in the sense of [1]. Forms certainly abound in mathematics. Among them, the following three major classes of forms should be distinguished:

- the forgetful functors for categories of algebras over monads on a category $\mathbb{C}$; we may refer to these as *algebraic forms* over $\mathbb{C}$;
- the so-called topological functors [1] (see also [5] and the reference there), which in some sense play a similar role for topological and relational structures as the algebraic forms do for algebraic structures; we may refer to these as *topological forms*;
- the amnestic functors obtained from the codomain functors $\mathcal{M} \to \mathbb{C}$, where $\mathcal{M}$ is a class of monomorphisms in $\mathbb{C}$ seen as the full subcategory of the category $\mathbb{C}^{\mathbf{2}}$ of morphisms in $\mathbb{C}$, whose objects are the elements of $\mathcal{M}$; we will refer to these as *subobject forms*.

Many concrete instances of subobject forms are in fact topological forms, while they are hardly ever algebraic. Subobject forms are often fibrations, or opfibrations, and in fact, usually they are bifibrations. A form F which is a Grothendieck fibration [13, 14] will be called a *right form*, and when F is an opfibration, we call it a *left form*; thus, F is a right form if and only if its *dual form* F^{op} is a left form. When F is both a left form and a right form, we will say that F is a *biform*. As shown in [29], topological forms are the same as what can be called *locally complete biforms*, i.e. biforms whose fibres are complete lattices.

Two forms $F_1 : \mathbb{S}_1 \to \mathbb{C}$ and $F_2 : \mathbb{S}_2 \to \mathbb{C}$ are said to be *isomorphic* when there is an isomorphism $\mathcal{I} : \mathbb{S}_1 \to \mathbb{S}_2$ such that $F_2\mathcal{I} = F_1$. Equivalence classes of (left/right/bi-) forms under the equivalence relation “F_1 is isomorphic to F_2” will be called “isomorphism *classes*" of (left/right/bi-) forms, although their sizes are bigger than the sizes of, say, "*classes* of morphisms" in a category.

Just as any preorder can be turned into an ordered set (class) by identifying isomorphic objects in it, any faithful functor $F : \mathbb{S} \to \mathbb{C}$ gives rise to a form, by identifying isomorphic objects in each fibre.

All forms that we consider in this paper can be obtained using the same general procedure which constructs a form over a category $\mathbb{C}$ from a binary relation $\mathcal{C}$ on the class of morphisms of $\mathbb{C}$. First, we construct a category from $\mathcal{C}$, denoted by $\mathsf{Mor}_{\mathcal{C}}(\mathbb{C})$. Its objects are pairs (B, f) where B is an object in $\mathbb{C}$ and f is a morphism $f : A \to B$ having the following property:

(m) $f\mathcal{C}f$ and in any diagram

$$\begin{tikzcd} & A \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ X \arrow[r, "u"'] & B \arrow[r, "v"'] & B' \end{tikzcd}$$

if $u\mathcal{C}f$ and $vf\mathcal{C}f'$ then $vu\mathcal{C}f'$.

A morphism $v : (B, f) \to (B', f')$ in $\mathsf{Mor}_{\mathcal{C}}(\mathbb{C})$ is a morphism $v : B \to B'$ in $\mathbb{C}$ such that $vf\mathcal{C}f'$. Identity morphisms and composition of morphisms in $\mathsf{Mor}_{\mathcal{C}}(\mathbb{C})$ are defined as in $\mathbb{C}$. We then get a faithful functor $\mathsf{Mor}_{\mathcal{C}}(\mathbb{C}) \to \mathbb{C}$, which maps each morphism $v : (B, f) \to (B', f')$ to the morphism $v : B \to B'$ in $\mathbb{C}$. The corresponding form will be called the *form of the relation* $\mathcal{C}$.

1.1. LEMMA. *Every morphism f in $\mathbb{C}$ has the property* (m)*, if and only if $\mathcal{C}$ is a reflexive and transitive relation such that*

(C_1) *$\mathcal{C}$ has the left preservation property [19], i.e. if $f\mathcal{C}g$ and the composites ef and eg are defined, then $ef\mathcal{C}eg$.*

We now describe two special classes of such forms, which we use in the paper.

Let $\mathcal{M}$ be a class of monomorphisms in $\mathbb{C}$. Define a relation $\mathcal{C}$ on the class of morphisms of $\mathbb{C}$ as follows: $f\mathcal{C}f'$ if and only if $f' \in \mathcal{M}$ and f factors through f', i.e. $f = f'g$ for some morphism g. It is easy to see that a morphism f satisfies (m) for this relation if and only if $f \in \mathcal{M}$. The form F of this relation will be called the *form of $\mathcal{M}$-subobjects*. Objects in each fibre $F^{-1}(X)$ are called *$\mathcal{M}$-subobjects* of the object X. They can be represented by monomorphisms $m \in \mathcal{M}$ with codomain X, with m and m' representing the same $\mathcal{M}$-subobject if and only if $m = m'i$ for some isomorphism i (which is unique once it exists). Thus, when $\mathcal{M}$ is the class of all monomorphisms, an *$\mathcal{M}$-subobject* is the same as a *subobject* in the usual sense.

Let $\mathcal{N}$ be any class of morphisms in $\mathbb{C}$. Following [10], a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of morphisms in $\mathbb{C}$ is said to be $\mathcal{N}$*-exact* (at B), when $gf \in \mathcal{N}$ and for any two morphisms u and v such that $vf \in \mathcal{N}$ and $gu \in \mathcal{N}$, we have $vu \in \mathcal{N}$. The classical example of this notion is exactness of a sequence in an abelian category, where $\mathcal{N}$ is the class of null morphisms there. The theory developed by M. Grandis in [10, 11, 12] extends many aspects of exact sequences from the context of an abelian category, to self-dual axiomatic contexts given by a category equipped with a distinguished class $\mathcal{N}$ of morphisms (thought of as null morphisms in the category), where exactness of a sequence is defined as above. The aim of his theory is to provide a foundation for non-abelian homological algebra, and it includes many examples of non-abelian categories relevant both for algebraic topology and general algebra (see [12]).

A pair (g, f) of morphisms as above, which forms an $\mathcal{N}$-exact sequence, will be called an $\mathcal{N}$*-exact pair*. Note that this notion is self-dual: (g, f) is an $\mathcal{N}$-exact pair in $\mathbb{C}$ if and only if (f, g) is an $\mathcal{N}$-exact pair in $\mathbb{C}^{\mathsf{op}}$.

Given a class $\mathcal{N}$ of morphisms in a category $\mathbb{C}$, define a relation $\mathcal{C}_{\mathcal{N}}$ on the class of morphisms of $\mathbb{C}$ as follows: $f\mathcal{C}_{\mathcal{N}}f'$ when f' is part of an $\mathcal{N}$-exact pair (g', f') such that $g'f \in \mathcal{N}$. The form of $\mathcal{C}_{\mathcal{N}}$ will be called the *form of* $\mathcal{N}$*-exact pairs* in $\mathbb{C}$. The name can be justified by the following lemma:

1.2. Lemma. *For any class $\mathcal{N}$ of morphisms in a category $\mathbb{C}$, a morphism f satisfies* (m) *for the relation $\mathcal{C} = \mathcal{C}_{\mathcal{N}}$, if and only if f is part of an $\mathcal{N}$-exact pair (g, f).*

Proof. Suppose f satisfies (m). Then $f\mathcal{C}_{\mathcal{N}}f$, which implies that f is part of an $\mathcal{N}$-exact pair (g, f). Conversely, suppose f is part of an $\mathcal{N}$-exact pair (g, f). Then, since in an $\mathcal{N}$-exact pair (g, f) we have $gf \in \mathcal{N}$, it follows that $f\mathcal{C}_{\mathcal{N}}f$. To check the second part of the condition (m), consider the diagram as in (m), and assume $u\mathcal{C}_{\mathcal{N}}f$ and $vf\mathcal{C}_{\mathcal{N}}f'$. We want to show $vu\mathcal{C}_{\mathcal{N}}f'$. Since $vf\mathcal{C}_{\mathcal{N}}f'$, there is an $\mathcal{N}$-exact pair (g', f') such that $g'vf \in \mathcal{N}$. Since $u\mathcal{C}_{\mathcal{N}}f$, there is an $\mathcal{N}$-exact pair (g'', f) such that $g''u \in \mathcal{N}$. Since $gf \in \mathcal{N}$, the fact that (g'', f) is an $\mathcal{N}$-exact pair implies $gu \in \mathcal{N}$. Now, since the pair (g, f) is $\mathcal{N}$-exact, we get that $g'vu \in \mathcal{N}$. This, together with the fact that the pair (g', f') is $\mathcal{N}$-exact, shows that $vu\mathcal{C}_{\mathcal{N}}f'$, as desired. ∎

The self-dual nature of the notion of an $\mathcal{N}$-exact pair has the following manifestation:

1.3. Proposition. *For any class $\mathcal{N}$ of morphisms in a category $\mathbb{C}$, the form of $\mathcal{N}$-exact pairs in $\mathbb{C}$ is isomorphic to the dual of the form of $\mathcal{N}$-exact pairs in $\mathbb{C}^{\mathsf{op}}$.*

Proof. The isomorphism can be established by mapping an object, represented by a morphism $f : A \to B$, in the fibre at B of the form of $\mathcal{N}$-exact pairs in $\mathbb{C}$, to the object in the fibre at B of the form of $\mathcal{N}$-exact pairs in $\mathbb{C}^{\mathsf{op}}$, which is represented by a morphism $g : B \to C$ such that (g, f) is an $\mathcal{N}$-exact pair. ∎

2. Left forms, cover relations, and factorizations

A reflexive and transitive *cover relation* in the sense of [19] is a reflexive and transitive binary relation on the class of morphisms of $\mathbb{C}$ which satisfies the condition (C_1) (see

Section 1 above) as well as the following two conditions:

(C_0) If $f\mathcal{C}g$ then f and g have the same codomain.

(C_2) For any morphism $f : A \to B$ we have $f\mathcal{C}1_B$.

The axioms for a general cover relation (i.e. one which is not reflexive and transitive) given in [19] comprise of (C_0), (C_1), and a condition which in the case of a reflexive and transitive relation satisfying (C_0) and (C_1) is easily seen to be equivalent to (C_2).

As introduced in [18], cover relations on a category $\mathbb{C}$ are binary relations on the class of morphisms of $\mathbb{C}$, defined only for those pairs of morphisms which have the same codomain. Some axioms were added later in [19], where it was shown that in a category $\mathbb{C}$, factorisation systems $(\mathcal{E}, \mathcal{M})$ where $\mathcal{M}$ is a class of monomorphisms, and a certain type of monoidal structures on $\mathbb{C}$, can be naturally seen as two different types of cover relations on $\mathbb{C}$, which in many special cases also have a several other properties in common. In [18], the structure of a cover relation was put forward as a minimal structure on a category allowing to state in it "closedness properties of internal relations" that correspond to linear Mal'tsev conditions in universal algebra [27]. These "closedness properties" can be expressed by formulas $\alpha \Rightarrow \beta$, which are similar to those that give axioms for a *geometric theory* [24], and the cover relation is used to interpret the meaning of implication in the formula. Among examples of cover relations that could fit this purpose are those defined by factorisation systems, as well as those defined by Grothendieck topologies, as briefly explained in [18]. It is precisely such type of cover relations that are of interest for the present paper, and not those which arise from monoidal structures, since the condition of being reflexive and transitive, which, apart from trivial cases is never satisfied by cover relations arising from monoidal structures, plays a principal role here.

Theorem 2.1 below states that there is a bijection between reflexive and transitive cover relations on $\mathbb{C}$ and isomorphism classes of left forms F over $\mathbb{C}$ (i.e. forms which are opfibrations) satisfying the following conditions:

(LF_1) F is *locally bounded above*, i.e. in each fibre there is a terminal object, that is, each ordered set $F^{-1}(X)$ has an upper bound (written as 1^X).

(LF_2) F is *conormal*, i.e. for any object Y of $\mathbb{C}$, and for any $W \in F^{-1}(Y)$, there exists a morphism $f : X \to Y$ in $\mathbb{C}$ such that we have $f_F(1^X) = W$.

Here f_F denotes the change-of-base functor $f_F : F^{-1}(X) \to F^{-1}(Y)$ — it maps each object $V \in F^{-1}(X)$ to the codomain of the cocartesian lifting of f at V.

2.1. Theorem. *For any category $\mathbb{C}$, there is a bijection:*

reflexive and transitive cover relations $\mathcal{C}$ on $\mathbb{C}$	$\approx$	*isomorphism classes of conormal locally bounded above left forms F over $\mathbb{C}$*

Under this bijection, an isomorphism class of a form F corresponds to the cover relation $\mathcal{C}$ defined as follows: for any two morphisms $f : X \to Y$ and $f' : X' \to Y$ in $\mathbb{C}$ we have $f\mathcal{C}f'$ if and only if $f_F(1^X) \leqslant f'_F(1^{X'})$. In the other direction, a cover relation $\mathcal{C}$ corresponds to the isomorphism class of the form of the relation $\mathcal{C}$.

Proof. Let F be a left form over $\mathbb{C}$. It is easy to see that if F is locally bounded above, then the relation $\mathcal{C} = F^*$ defined in the theorem is a reflexive and transitive cover relation. If F_1 and F_2 are isomorphic then $F_1^* = F_2^*$. Now, let $\mathcal{C}$ be a reflexive and transitive relation on the class of morphisms of $\mathbb{C}$ satisfying (C_1). Write $F = \mathcal{C}_*$ for the form of the relation $\mathcal{C}$. It is easy to show, thanks to Lemma 1.1, that F is a left form which is locally bounded above and is conormal. When further (C_0) holds, we have $(\mathcal{C}_*)^* = \mathcal{C}$. Finally, it is not difficult to verify that for any left form F which is locally bounded and conormal, the left form $(F^*)_*$ is isomorphic to F. ∎

Next, we characterize various properties of the form of $\mathcal{M}$-subobjects, where $\mathcal{M}$ is a class of monomorphisms in a category, via conditions on the class $\mathcal{M}$ first studied in [7, 9, 25, 28, 15]; we will freely use/recall some of the well known results from these works.

Consider any form $F : \mathbb{B} \to \mathbb{C}$ and let $W \in F^{-1}(Y)$. A *left universalizer* of W is a morphism $f : X \to Y$ in $\mathbb{C}$ with the universal property of being terminal among those morphisms $f : X \to Y$ which satisfy the following condition:

(LU_0) For any object X of $\mathbb{C}$, and for any $V \in F^{-1}(X)$, there exists a (unique) morphism $b : V \to W$ in $\mathbb{B}$ with the property $F(b) = f$.

Thus, f is a left universalizer of W when f satisfies the condition above and for any other morphism $f' : X' \to Y$ satisfying the same condition, there is a unique morphism $x : X' \to X$ such that $fx = f'$.

2.2. Lemma. *Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathbb{C}$, satisfying the following condition:*

(M_0) *for any object C in $\mathbb{C}$ there exists an isomorphism which belongs to the class $\mathcal{M}$ and whose codomain is C.*

Then for any monomorphism $m : X \to Y$ from the class $\mathcal{M}$, the morphism m is a left universalizer of the $\mathcal{M}$-subobject of Y represented by m, in the form of $\mathcal{M}$-subobjects.

Proof. Let F denote the form of $\mathcal{M}$-subobjects. In it clear that if W denotes the $\mathcal{M}$-subobject of Y represented by m, then (LU_0) holds for $f = m$. Suppose $f' : X' \to Y$ also satisfies (LU_0). Then, taking $f = f'$ in (LU_0), and taking V to be the $\mathcal{M}$-subobject of X represented by an isomorphism with codomain X, we will conclude that f' factors through m, i.e. $mx = f'$ for some morphism x. Since m is a monomorphism, such x is necessarily unique. ∎

The condition (M_0) in Lemma 2.2 is necessary. Indeed, for, if for example we take $\mathcal{M}$ to be the class of functions whose domain is the empty set, then the class of left universalizers for the form of $\mathcal{M}$-subobjects over the category **Set** will be the class of isomorphisms, and so in this case the assertion of the lemma does not hold.

In the case when the functor F above is a left form, the condition (LU_0) is equivalent to the following one:

(LU_1) For any object X of $\mathbb{C}$, and for any $V \in F^{-1}(X)$, we have $f_F(V) \leqslant W$.

When further F is a biform, (LU_0) becomes the same as to require that $f_F^{-1}(W)$ is the terminal object in the fibre $F^{-1}(X)$ at X, where f_F^{-1} stands for the right adjoint in the change of base adjunction

$$f_F \dashv f_F^{-1} : F^{-1}(X) \rightleftarrows F^{-1}(Y)$$

induced by f. Thus, when F is a biform, a left universalizer in the above sense is the same as a left universalizer in the sense of [20] (note that what we call in this paper a *biform* was called a *form* in [20]).

2.3. Lemma. *For any left form, every left universalizer is a monomorphism.*

Proof. This is a consequence of the fact that if a morphism f satisfies (LU_1), then so does any composite fg. ■

Right universalizers are defined dually. By the dual of the above lemma, for a right form, every right universalizer is an epimorphism.

Consider the following additional axiom on a cover relation:

(C_3) $\mathcal{C}$ *admits images* [19], i.e. for any morphism $w : W \to Y$ there exists a morphism $f : X \to Y$ such that $f\mathcal{C}w$ and f is terminal with this property, i.e. if $f' : X' \to Y$ is another morphism with $f'\mathcal{C}w$ then $f' = fx$ for a unique morphism $x : X' \to X$ (then f is called a $\mathcal{C}$*-image* of the morphism w).

It was shown in [19] that the process of assigning to a cover relation $\mathcal{C}$ the class $\mathcal{M}$ of $\mathcal{C}$-images defines a bijection between reflexive and transitive cover relations satisfying (C_3) and classes $\mathcal{M}$ of monomorphisms satisfying the following conditions, which are due to H. Ehrbar and O. Wyler [7] (see also [25, 28, 15]):

(M_1) $\mathcal{M}$ is closed under composition with isomorphisms.

(M_2) Every morphism f in $\mathbb{C}$ has a factorisation $f = me$ with $m \in \mathcal{M}$, such that for any commutative diagram of solid arrows below with $n \in \mathcal{M}$, there is a unique morphism g which makes the diagram commute:

$$\begin{tikzcd}
\bullet \arrow[d, "e"] \arrow[dr, "e'"] \arrow[dd, bend right=60, "f"'] & \\
\bullet \arrow[r, dotted, "g"'] \arrow[d, "m"] & \bullet \arrow[d, "n"] \\
\bullet \arrow[r, "h"'] & \bullet
\end{tikzcd}$$

We note immediately that already the first half of (M_2) implies (M_0), when $\mathcal{M}$ is a class of monomorphisms. Under the bijection referred to above, the cover relation $\mathcal{C}$ corresponding to the class $\mathcal{M}$ is defined as follows: $f\mathcal{C}f'$ when m factors through m', where m and m' are part of factorizations of f and f', respectively, given by (M_2). Further, it is easy to see that the form of $\mathcal{M}$-subobjects is isomorphic to the form of the corresponding cover relation.

The images for a reflexive and transitive cover relation $\mathcal{C}$ are in fact the same as left universalizers for the form of the relation $\mathcal{C}$. Thus, $\mathcal{C}$ satisfies (C_3) if and only if the corresponding left form satisfies the following condition:

(LF_3) *F admits left universalizers*, i.e. for any object Y of $\mathbb{C}$, each $W \in F^{-1}(Y)$ has a left universalizer (which is written as $\mathsf{lun}_F(W) : \mathsf{Lun}_F(W) \to Y$).

As (LF_3) is invariant under isomorphism of left forms, we get that the bijection of Theorem 2.1 restricts to a bijection between isomorphism classes of left forms satisfying (LF_1), (LF_2), and (LF_3), and reflexive and transitive cover relations admitting images. Combining this bijection with the one established in [19] and recalled above, we get:

2.4. Theorem. *For any category $\mathbb{C}$, there is a bijection:*

classes $\mathcal{M}$ of monomorphisms in $\mathbb{C}$ satisfying (M_1) and (M_2)	$\approx$	*isomorphism classes of conormal locally bounded above left forms F over $\mathbb{C}$ admitting left universalizers*

Under this bijection, an isomorphism class of a form F corresponds to the class of left universalizers for F. In the other direction, a class $\mathcal{M}$ of monomorphisms corresponds to the form of $\mathcal{M}$-subobjects.

It is well known that for a class $\mathcal{M}$ of morphisms which satisfies (M_0) and (M_1), the following conditions are equivalent:

- the codomain functor $\mathcal{M} \to \mathbb{C}$ is an opfibration (when $\mathcal{M}$ is a class of monomorphisms, this is the same as to say that the form of $\mathcal{M}$-subobjects is a left form).
- (M_2) holds.

Thus, the theorem above gives a characterization of *left forms* which are isomorphic to forms of $\mathcal{M}$-subobjects, where $\mathcal{M}$ is a class of monomorphisms which satisfies (M_0) and (M_1): such left forms are precisely those left forms which are locally bounded above, conormal, and admit left universalizers.

It is also well known that for a class $\mathcal{M}$ of monomorphisms satisfying (M_1) and (M_2), we have:

- the form of $\mathcal{M}$-subobjects is a right form (and hence a biform) if and only if all pullbacks of morphisms along those in $\mathcal{M}$ exist (in which case $\mathcal{M}$ is even stable under pullbacks);

- the class $\mathcal{M}$ is part of a factorization system $(\mathcal{E}, \mathcal{M})$ in the sense of P. Freyd and G. M. Kelly [9] if and only if $\mathcal{M}$ is closed under composition.

This produces two important restrictions of the bijection described in Theorem 2.4: a bijection between

- classes $\mathcal{M}$ of monomorphisms, such that $\mathbb{C}$ is *finitely $\mathcal{M}$-complete* in the sense of D. Dikranjan and W. Tholen [6],

- and isomorphisms classes of biforms over $\mathbb{C}$ which are locally bounded above, conormal and admit left universalizers;

and a bijection between

- classes $\mathcal{M}$ of monomorphisms which are part of a factorization system,

- and isomorphisms classes of left forms over $\mathbb{C}$ which are locally bounded above, conormal, admit left universalizers, and for which the class of left universalizers is closed under composition.

As follows from Proposition 1.4.20 in [19], the requirement above that the class of left universalizers is closed under composition can be replaced with the following condition:

(LF_4) For any object Y of $\mathbb{C}$, and for any $W \in F^{-1}(Y)$, the change-of-base functor $F^{-1}(\mathsf{Lun}_F(W)) \to F^{-1}(Y)$ induced by $\mathsf{lun}_F(W)$ is full.

Note that when F is a biform, the condition above becomes equivalent to requiring that left universalizers are *injective*, i.e. the left adjoint in the corresponding change-of-base adjunction is injective on objects. In the language of the corresponding cover relation, the condition above evidently translates to the following one:

(C_4) Any $\mathcal{C}$-image f is *$\mathcal{C}$-reflecting* [19], i.e. for any v and v', if $fv\mathcal{C}fv'$ then $v\mathcal{C}v'$.

Thus, we have:

2.5. Corollary. *There is a bijection*

classes $\mathcal{M}$ of monomorphisms in $\mathbb{C}$ which are part of a factorization system $(\mathcal{E}, \mathcal{M})$	$\approx$	*isomorphism classes of conormal locally bounded above left forms F over $\mathbb{C}$ admitting left universalizers which determine full change-of-base functors*

given by assigning to a class $\mathcal{M}$ of monomorphisms the isomorphism class of the form of $\mathcal{M}$-subobjects.

3. Biforms and ideals of null morphisms

A class $\mathcal{N}$ of morphisms in a category $\mathbb{C}$ satisfying the following conditions is called an *ideal of null morphisms* [8, 21, 22]:

(LN_0) If $n \in \mathcal{N}$ and the composite nt is defined, then $nt \in \mathcal{N}$.

(RN_0) If $n \in \mathcal{N}$ and the composite sn is defined, then $sn \in \mathcal{N}$.

Note that (LN_0) is dual to (RN_0), i.e. the first one holds for a class $\mathcal{N}$ in $\mathbb{C}$ if and only if the second one holds for the same class in $\mathbb{C}^{\mathsf{op}}$. Consider the following additional pair of dual conditions on an ideal $\mathcal{N}$:

(LN_1) Any morphism g is part of an $\mathcal{N}$-exact pair (g, f).

(RN_1) Any morphism f is part of an $\mathcal{N}$-exact pair (g, f).

When stated for the dual $F^{\mathsf{op}} : \mathbb{B}^{\mathsf{op}} \to \mathbb{C}^{\mathsf{op}}$ of a right form F, the conditions (LF_1) and (LF_2) become:

(RF_1) F is *locally bounded below*, i.e. in each fibre there is an initial object, that is, each ordered set $F^{-1}(X)$ has a lower bound (written as 0^X).

(RF_2) F is *normal*, i.e. for any object X in $\mathbb{C}$, and for any $V \in F^{-1}(X)$, there exists a morphism $f : X \to Y$ in $\mathbb{C}$ such that we have $f_F^{-1}(0^Y) = V$.

We will now show that a biform F over a category $\mathbb{C}$ which is *locally bounded* (i.e. satisfies both (LF_1) and (RF_1)) and *binormal* (i.e. satisfies both (LF_2) and (RF_2)) is determined uniquely (up to an isomorphism of forms) by the class of those morphisms $n : X \to Y$ in $\mathbb{C}$ for which the change-of-base adjunction is trivial (i.e. the change of base maps $n_F : F^{-1}(X) \to F^{-1}(Y)$ and $n_F^{-1} : F^{-1}(Y) \to F^{-1}(X)$ are constant maps) — we will call such morphisms *F-trivial morphisms*. Moreover, this gives a bijection between isomorphism classes of locally bounded binormal biforms over a category $\mathbb{C}$ and ideals $\mathcal{N}$ of null morphisms in $\mathbb{C}$ for which every morphism $B \to C$ is part of a sequence $A \to B \to C \to D$ which is exact relative to the ideal.

3.1. THEOREM. *For any category $\mathbb{C}$, there is a bijection:*

$$\begin{array}{ccc} \textit{ideals } \mathcal{N} \textit{ of null morphisms in } \mathbb{C} & \approx & \textit{isomorphism classes of binormal} \\ \textit{satisfying } (\mathsf{LN}_1) \textit{ and } (\mathsf{RN}_1) & & \textit{locally bounded biforms } F \textit{ over } \mathbb{C} \end{array}$$

Under this bijection, an isomorphism class of a biform F corresponds to the class of F-trivial morphisms. In the other direction, an ideal $\mathcal{N}$ corresponds to the isomorphism class of the form of $\mathcal{N}$-exact pairs.

Proof. Let F be a biform over $\mathbb{C}$. It is clear that the class $\mathcal{N} = F^*$ of F-trivial morphisms is an ideal of null morphisms. Note that when F is locally bounded, a composite fg of two morphisms $f : A \to B$ and $g : B \to C$ is F-trivial if and only if $f_F(1^A) \leqslant g_F^{-1}(0^C)$. When in addition F is normal, this gives that for two morphisms $f : A \to B$ and $f' : A' \to B$ we have $f_F(1^A) \leqslant f'_F(1^{A'})$ if and only if for any morphism $g : B \to C$ we have: $gf' \in F^*$ implies $gf \in F^*$. So the cover relation $\mathcal{C}$ corresponding to F is uniquely determined by the class of F-trivial morphisms, which after applying Theorem 2.1 implies that two locally bounded normal biforms are isomorphic provided they determine the same class of trivial morphisms. Next, we show that $\mathcal{N} = F^*$ satisfies (LN_1) when F is locally bounded and conormal. Given a morphism $g : B \to C$, consider the morphism $f : A \to B$ such that $f_F(1^A) = g_F^{-1}(0^C)$ (such f exists when the form is conormal). Then the composite gf is F-trivial. Moreover, if $u : A' \to B$ and $v : B \to C'$ are morphisms such that vf and gu are F-trivial, then $f_F(1^A) \leqslant v_F^{-1}(0^{C'})$ and $u_F(1^{A'}) \leqslant g_F^{-1}(0^C)$ which with the previous equality together give $u_F(1^{A'}) \leqslant v_F^{-1}(0^{C'})$ and hence vu is trivial. Thus the pair (g, f) is F^*-exact. This proves that the class $\mathcal{N} = F^*$ satisfies (LN_1), as desired. Dually, the class $\mathcal{N} = F^*$ satisfies (RN_1) when F is locally bounded and normal. It is not difficult to see that two isomorphic biforms will give rise to the same class of trivial morphisms. So, thus far we have shown that the assignment $F \mapsto F^*$ determines a map from isomorphism classes of locally bounded binormal biforms to ideals of null morphisms satisfying (LN_1) and (RN_1), and that this map is injective.

It is easy to see that when a class $\mathcal{N}$ of morphisms satisfies (LN_1), the form $F = \mathcal{N}_*$ of $\mathcal{N}$-exact pairs is a right form. Dually, in view of Proposition 1.3, when $\mathcal{N}$ satisfies (RN_1), the same form is a left form. Further, once each identity morphism $1_B : B \to B$ is part of an $\mathcal{N}$-exact pair $(g, 1_B)$, and (LN_0) holds, the form $\mathcal{N}_*$ is locally bounded above. Dually, once each identity morphism $1_B : B \to B$ is part of an $\mathcal{N}$-exact pair $(1_B, f)$, and (RN_0) holds, the form $\mathcal{N}_*$ is locally bounded below. So when $\mathcal{N}$ is an ideal of null morphisms satisfying (LN_1) and (RN_1), we get that $\mathcal{N}_*$ is a biform which is locally bounded. Moreover, in this case $\mathcal{N}_*$ is also binormal, which can be easily verified.

For a locally bounded form F over $\mathbb{C}$, a morphism $n : X \to Y$ in $\mathbb{C}$ is F-trivial if and only if there exists a morphism $b : 1^X \to 0^Y$ in the domain of F such that $F(b) = n$. This fact can be used to verify straightforwardly that for an ideal $\mathcal{N}$ of null morphisms satisfying (LN_1) and (RN_1), the class of $\mathcal{N}_*$-trivial morphisms coincides with the class $\mathcal{N}$. ∎

Consider now the following pair of dual conditions on a class $\mathcal{N}$ of morphisms in a category $\mathbb{C}$:

(LN_2) $\mathcal{N}$ *admits kernels*, i.e. for any morphism $g : B \to C$ in $\mathbb{C}$ there exists a morphism $k : A \to B$ such that $gk \in \mathcal{N}$ and k is terminal with this property, i.e. if $gk' \in \mathcal{N}$ for some morphism $k' : A' \to B$ then $k' = ku$ for a unique morphism $u : A' \to A$ (such k is called an $\mathcal{N}$*-kernel* of g).

(RN_2) $\mathcal{N}$ *admits cokernels*, i.e. for any morphism $f : A \to B$ in $\mathbb{C}$ there exists a morphism $c : B \to C$ such that $cf \in \mathcal{N}$ and c is initial with this property, i.e. if $c'f \in \mathcal{N}$ for some morphism $c' : B \to C'$ then $c' = uc$ for a unique morphism $u : C \to C'$ (such c is called an $\mathcal{N}$-*cokernel* of f).

It is not difficult to see if k is an $\mathcal{N}$-kernel of a morphism g, then the pair (g, k) is $\mathcal{N}$-exact, when $\mathcal{N}$ satisfies (LN_0). So under (LN_0), the condition (LN_2) implies (LN_1). Dually, under (RN_0), the condition (RN_2) implies (RN_1).

When $\mathcal{N}$ is the class of trivial morphisms in a locally bounded below normal biform F, the class of $\mathcal{N}$-kernels coincides with the class of left universalizers for F. In detail, an $\mathcal{N}$-kernel of a morphism $g : B \to C$ is the same as a left universalizer of $g_F^{-1}(0^C)$. This and its dual observation allow us to restrict the bijection described in Theorem 3.1 to a bijection:

$$\begin{array}{c} \text{ideals } \mathcal{N} \\ \text{of null morphisms} \\ \text{in } \mathbb{C} \text{ admitting} \\ \text{kernels and cokernels} \end{array} \quad \approx \quad \begin{array}{c} \text{isomorphism classes of binormal} \\ \text{locally bounded biforms } F \text{ over } \mathbb{C} \\ \text{admitting left and right universalizers} \end{array} \tag{1}$$

Now, as it follows by the remarks after Theorem 2.4 in Section 2, for an ideal of null morphisms admitting kernels and cokernels, the class of kernels is closed under composition if and only if the corresponding biform admits injective left universalizers. A dual result would state that for an ideal of null morphisms admitting kernels and cokernels, the class of cokernels is closed under composition if and only if the corresponding biform admits surjective right universalizers (i.e. the left adjoint in the change-of-base adjunction induced by any right universalizer is surjective on objects).

Ideals $\mathcal{N}$ of null morphisms in a category $\mathbb{C}$ which admit kernels and cokernels, and for which the classes of kernels and cokernels are closed under composition, are precisely those for which the pair $(\mathbb{C}, \mathcal{N})$ is an ex2-category in the sense of M. Grandis [10]. In fact, in the definition given in [10], the class $\mathcal{N}$ is also required to be a *closed ideal*, i.e. any morphism in $\mathcal{N}$ must factor through an identity morphism which belongs to $\mathcal{N}$. However, this is a consequence of the other requirements: consider a morphism $n : L \to M$ from the class $\mathcal{N}$. Let $k : K \to M$ be the kernel of 1_M. Then n factors though 1_K. To show that $1_K \in \mathcal{N}$, consider the composite kk' where k' is a kernel of 1_K. Notice that $kk' \in \mathcal{N}$. Then, 1_M is a cokernel of kk'. Since the class of $\mathcal{N}$-kernels is closed under composition, the composite kk' is a kernel of some morphism and hence of its cokernel 1_M. But k is also a kernel of 1_M. This implies that k' is an isomorphism, which forces $1_K \in \mathcal{N}$.

Thus, we have:

3.2. Corollary. *For any category $\mathbb{C}$, there is a bijection (1) under which an isomorphism class of a biform F corresponds to the class of F-trivial morphisms. In the other direction, an ideal $\mathcal{N}$ corresponds to the isomorphism class of the form F of $\mathcal{M}$-subobjects, where $\mathcal{M}$ is the class of $\mathcal{N}$-kernels. Furthermore, such F admits injective left universalizers and surjective right universalizers if and only if the pair $(\mathbb{C}, \mathcal{N})$ is an ex2-category.*

These results reveal a correspondence between a hierarchy of conditions on cover relations and a hierarchy of conditions on ideals of null morphisms encountered in the work of M. Grandis [10, 11, 12], which is given by the procedure of translating a condition on a cover relation into the language of the form F of the cover relation, imposing also the dual condition on the same form F, and then further translating the resulting condition into the language of the ideal of F-trivial morphisms. This correspondence, to the extent explored above, can be summarized by the following table:

Conditions on a cover relation $\mathcal{C}$:	Corresponding dual conditions on a biform F:	Corresponding dual conditions on an ideal $\mathcal{N}$:
$\mathcal{C}$ is reflexive and transitive	F is locally bounded and binormal	$\mathcal{N}$ is an ideal such that every morphism $B \to C$ is part of an exact sequence $A \to B \to C \to D$
+ $\mathcal{C}$ admits images	+ left and right universalizers exist	$\mathcal{N}$ is an ideal admitting kernels and cokernels
+ $\mathcal{C}$-images are $\mathcal{C}$-reflecting	+ left universalizers are injective and right universalizers are surjective	$\mathcal{N}$ makes an ex2-category

A different presentation of the same correspondence is to view the resulting classes of ideals as classes of pairs of "dual structures":

- For example, an ideal $\mathcal{N}$ which admits kernels and cokernels *is the same as* a pair $(\mathcal{E}, \mathcal{M})$, where $\mathcal{M}$ is a class of monomorphisms satisfying (M_1) and (M_2), and $\mathcal{E}$ is a class of epimorphisms satisfying dual conditions, such that the form of $\mathcal{M}$-subobjects is isomorphic to the *form of $\mathcal{E}$-quotients* (which is a form given by the dual of the construction for the form of $\mathcal{M}$-subobjects — construct the form of $\mathcal{E}$-subobjects in the dual category and then take the dual of the resulting form). Above, "is the same as" means that there is a bijection between these two types of structures; namely, for an ideal $\mathcal{N}$ the corresponding pair $(\mathcal{E}, \mathcal{M})$ consists of the class $\mathcal{E}$ of $\mathcal{N}$-cokernels and the class $\mathcal{M}$ of $\mathcal{N}$-kernels.

- Note that the "compatibility condition" on the pair $(\mathcal{E}, \mathcal{M})$ which says that the form of $\mathcal{M}$-subobjects is isomorphic to the form of $\mathcal{E}$-quotients, can be also presented in the following elementary way: there is a relation $\mathcal{R} \subseteq \mathcal{E} \times \mathcal{M}$ which induces a bijection between the class of all $\mathcal{M}$-subobjects and the class of all $\mathcal{E}$-quotients, and is such that for any diagram

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, dotted, "a"'] & B \arrow[r, "e"] \arrow[d, "s"] & C \arrow[d, dotted, "c"] \\ A' \arrow[r, "m'"'] & B' \arrow[r, "e'"'] & C' \end{tikzcd}$$

  of solid arrows where $(e, m) \in \mathcal{R}$ and $(e', m') \in \mathcal{R}$, the dotted arrow a exists making the left square commute, if and only if the dotted arrow c exists making the right square commute. The pairs (e, m) in this relation are those which are $\mathcal{N}$-exact for the corresponding ideal $\mathcal{N}$ of null morphisms — these in turn are those morphisms s for which the dotted arrows above always exist.

- As already observed in the introduction, in a similar way an ex2-category structure *is the same as* a pair of factorization systems such that the class $\mathcal{E}$ in the first factorization system is a class of epimorphisms, the class $\mathcal{M}$ in the second factorization system is a class of monomorphisms, and the form of $\mathcal{M}$-subobjects for the second factorization system is isomorphic to the form of $\mathcal{E}$-quotients for the first one.

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