MODEL CATEGORIES WITH SIMPLE HOMOTOPY CATEGORIES

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ABSTRACT. In the present article we describe constructions of model structures on general bicomplete categories. We are motivated by the following question: given a category $\mathcal{C}$ with a suitable subcategory $w\mathcal{C}$, when is there a model structure on $\mathcal{C}$ with $w\mathcal{C}$ as the subcategory of weak equivalences? We begin exploring this question in the case where $w\mathcal{C} = F^{-1}(\mathrm{iso}\,\mathcal{D})$ for some functor $F\colon\mathcal{C} \to \mathcal{D}$. We also prove properness of our constructions under minor assumptions and examine an application to the category of infinite graphs.

1. Introduction

Model categories are very useful structures for analyzing the homotopy-theoretic properties of various problems. However, constructing these structures is generally difficult; often, only the weak equivalences arise naturally, and much effort must be expended to find compatible sets of cofibrations and fibrations. (For examples of this, see [Hir03], [Hov99], chapter VII of [GJ99], [Ber07], or the discussion of various model structures of spectra in [MMSS01].) This paper is the first in a series which explores the general structure of such problems. It attempts to answer the following question:

1.1. QUESTION. *Given a bicomplete category $\mathcal{C}$, together with a subcategory $w\mathcal{C} \subseteq \mathcal{C}$ which is closed under two-of-three and retracts, when is there a model structure on $\mathcal{C}$ such that $w\mathcal{C}$ is the subcategory of weak equivalences?*

This question is very difficult, and we do not possess a complete answer to it. However, the study of some cases has yielded many interesting families of examples, and we present the first few here.

Often the subcategory $w\mathcal{C}$ is obtained through a functor $F\colon\mathcal{C} \to \mathcal{D}$ by defining $w\mathcal{C} \stackrel{\mathrm{def}}{=} F^{-1}(\mathrm{iso}\,\mathcal{D})$. In this paper we address the case when $\mathcal{D}$ is a preorder: a category where $|\mathrm{Hom}(A, B)| \leq 1$ for all objects $A, B \in \mathcal{D}$. Although it turns out that we cannot answer this question in full generality even with this simplification, we answer it in the following three cases:

    1. F has a right adjoint which is a section.

   2. $F: \mathcal{C} \to \mathcal{E}$, where $\mathcal{E}$ is the category with two objects and one noninvertible morphism between them.

   3. $F = R_{\mathcal{C}}$, where $R_{\mathcal{C}}$ is the universal functor from $\mathcal{C}$ to a preorder.

In fact, it turns out that the methods which allow us to answer these questions answer more general questions than the one asked here. For example, the construction which gives the model structure in case (2) can also be used to construct a model structure where the noninvertible weak equivalences are the preimage of only one of the objects. Whenever possible we state the results we obtain in full generality, only applying them to the case when $\mathcal{D}$ is a preorder when necessary.

   We will spend the majority of our time on the third type of model structure, as it is the one with the most interesting applications. It generalizes a model structure on the category of finite graphs constructed in [Dro12], and in this case gives an interesting homotopy-theoretic perspective on what the notion of a "core" for an infinite graph should be. (This will be discussed in Section 6.)

   The following theorem sums up the main results of the paper.

1.2. THEOREM. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor as described in cases 1-3. There exists a model structure on $\mathcal{C}$ such that the weak equivalences are $F^{-1}(\mathrm{iso}\, \mathcal{D})$. This model structure is left proper.*

   A recurring example in this paper is the category of semi-simplicial sets. This has as objects functors $\Delta_{inj}^{op} \to \mathbf{Set}$, where Δ_{inj} is the category of nonempty ordered sets and injections between them. Any simplicial set is also a semi-simplicial set, and the geometric realization of a semi-simplicial set is homotopy equivalent to the geometric realization of the original simplicial set. However, this category is not a model for the homotopy theory of topological spaces, in the sense that it does not have a model structure Quillen equivalent to the model structure on topological spaces. In this paper we will show that it does have several intriguing model structures on it, including several where the dimension of a semi-simplicial set is a homotopy invariant. For more details, see Examples 3.5, 4.9 and 5.4.

   As another application of this theorem we consider the model structure constructed in [Dro12] for the category of finite graphs. This model structure is interesting in that it gives a homotopy-theoretic expression of a combinatorial invariant: two graphs are weakly equivalent if and only if they have the same core. The theorem allows us to construct an analogous structure on the category of infinite graphs, and thus gives a possible generalization of the notion of "core" to the context of infinite graphs. The notion of core for infinite graphs is not agreed upon, although several candidates are defined; our new notion of core does not agree with any of the existing candidates for the notion of the core of an infinite graph.

   The organization of this paper is as follows. Section 2 discusses model structures and some categorical preliminaries necessary for the paper. Sections 3-5 discuss cases 1-3 in detail. Finally, section 6 analyzes the implications that the model structure from section 5 has for the notion of a core for infinite graphs.

NOTATION AND TERMINOLOGY. We will say that a category is *bicomplete* if it has all finite limits and colimits. We only use finite limits and colimits instead of the usual assumption of small limits and colimits as we do not need to use the techniques of cofibrant generation for constructing our model structures. Thus categories such as the category of finite graphs can be given model structures in our examples.

A *preorder* is a category where for all objects A and B, $|\operatorname{Hom}(A, B)| \leq 1$.

We write $\varnothing$ for the initial object in a category and $*$ for the terminal object.

2. Preliminaries

2.1. WEAK FACTORIZATION SYSTEMS.

2.2. DEFINITION. *In a category $\mathcal{C}$, we say that the morphism $f\colon A \to B$ has the left lifting property with respect to the morphism $g\colon C \to D$ if for any commutative diagram of solid arrows*

$$\begin{array}{ccc} A & \longrightarrow & C \\ f \downarrow & {\scriptstyle h} \nearrow & \downarrow g \\ B & \longrightarrow & D \end{array}$$

there is a morphism h which makes the complete diagram commutative. We will write $f \boxtimes g$ if f has the left lifting property with respect to g.

For any class of morphisms S, we define

$$S^{\boxtimes} = \{g \in \mathcal{C} \mid f \boxtimes g \text{ for all } f \in S\},$$
$$^{\boxtimes}S = \{f \in \mathcal{C} \mid f \boxtimes g \text{ for all } g \in S\}.$$

Note that for any set S, the sets $S^{\boxtimes}$ and $^{\boxtimes}S$ are closed under retracts.

2.3. DEFINITION. *A maximal lifting system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{C}$ is a pair of classes of morphisms, such that $\mathcal{L} = {}^{\boxtimes}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxtimes}$.*

The following theorem is well-known; for a proof (and a more general statement), see [MP12, 14.1.8].

2.4. THEOREM. [Folklore] *If $(\mathcal{L}, \mathcal{R})$ is a maximal lifting system in a category $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms and are closed under composition and retraction. Moreover, $\mathcal{L}$ is closed under coproducts and pushouts along morphisms in $\mathcal{C}$, and $\mathcal{R}$ is closed under products and pullbacks along morphisms in $\mathcal{C}$.*

We recall the definition of a weak factorization system. For more on weak factorization systems, see for example [AHR02] or [Rie14, Section 11].

2.5. DEFINITION. *A weak factorization system $(\mathcal{L}, \mathcal{R})$ in the category $\mathcal{C}$ is a maximal lifting system such that any morphism in $\mathcal{C}$ can be factored as $g \circ f$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.*

From this point on, we will write WFS for "weak factorization system." The following is a well-known result for recognizing WFSs; for a proof, see [MP12, 14.1.13].

2.6. LEMMA. [Folklore] *If $(\mathcal{L}, \mathcal{R})$ is a pair of classes of morphisms in a category $\mathcal{C}$ such that*

1. *$f \boxdot g$ for all $f \in \mathcal{L}$ and $g \in \mathcal{R}$,*

2. *all morphisms $f \in \mathcal{C}$ can be can be factored as $f_R \circ f_L$, where $f_R \in \mathcal{R}$ and $f_L \in \mathcal{L}$, and*

3. *$\mathcal{L}$ and $\mathcal{R}$ are closed under retracts,*

then $(\mathcal{L}, \mathcal{R})$ is a WFS.

As an example of how lifting properties can classify properties of morphisms, we present the following characterization of retractions and sections.

2.7. DEFINITION. *A morphism $r \colon A \to B$ in a category is called a retraction if it is possible to factorize the identity of B as $1_B = rs$ for some morphism s. Dually, a morphism $s \colon A \to B$ is called a section if it is possible to factorize the identity of A as $1_A = rs$ for some morphism r.*

2.8. LEMMA. *The class of retractions is exactly $\{\varnothing \to A \,|\, A \in \mathcal{C}\}^{\boxdot}$. Dually, the class of sections is exactly $^{\boxdot}\{A \to * \,|\, A \in \mathcal{C}\}$.*

2.9. MODEL CATEGORIES. We now recall the definition of a model structure on a category. Instead of using the most traditional approach [Hir03, Hov99] we use an equivalent axiomatization using WFSs. For a more thorough treatment of model categories along these lines, see for example [MP12, Section 14.2] or [Rie14, Section 11.2].

2.10. DEFINITION. *A model structure $\mathbb{C}$ on a bicomplete category $\mathcal{C}$ is a tuple of three subcategories of $\mathcal{C}$ called the weak equivalences ($\mathbb{C}_{we}$), the cofibrations ($\mathbb{C}_{cof}$) and the fibrations ($\mathbb{C}_{fib}$). Those three sets should satisfy the following axioms.*

WFS *The pairs*

$$(\mathbb{C}_{cof}, \mathbb{C}_{fib} \cap \mathbb{C}_{we}) \qquad (\mathbb{C}_{cof} \cap \mathbb{C}_{we}, \mathbb{C}_{fib})$$

are WFSs.

2OF3 *For composable morphisms f and g, if two of the morphisms f, g and gf are weak equivalences, then so is the third.*

We call a morphism which is both a cofibration (resp. fibration) and a weak equivalence an acyclic cofibration (resp. acyclic fibration).

One nontrivial consequence of these axioms is that $\mathbb{C}_{we}$ is closed under retracts. This result is due to Tierney, but we could not find it in his writings; for a proof of this lemma, see [MP12, 14.2.5] or [Rie14, 11.2.3].

2.11. LEMMA. [Tierney] *$\mathbb{C}_{we}$ is closed under retracts.*

The following two lemmas will be used below to construct model structures. We omit the proofs, as they are simple definition checks.

2.12. Lemma. *Given a bicomplete category $\mathcal{C}$, together with subcategories $\tilde{f}\mathcal{C} \subseteq w\mathcal{C} \subseteq \mathcal{C}$, where $\tilde{f}\mathcal{C}$ is closed under pullbacks and $w\mathcal{C}$ satisfies (2OF3), we define*

$$\mathbb{C}_{we} = w\mathcal{C} \qquad \mathbb{C}_{cof} = {}^{\boxtimes}\tilde{f}\mathcal{C} \qquad \mathbb{C}_{fib} = (\mathbb{C}_{cof} \cap \mathbb{C}_{we})^{\boxtimes}.$$

If $(\mathbb{C}_{cof}, \tilde{f}\mathcal{C})$ and $(\mathbb{C}_{cof} \cap \mathbb{C}_{we}, \mathbb{C}_{fib})$ are WFSs and $\mathbb{C}_{we} \cap \mathbb{C}_{fib} = \tilde{f}\mathcal{C}$, then $(\mathbb{C}_{we}, \mathbb{C}_{cof}, \mathbb{C}_{fib})$ is a model structure on $\mathcal{C}$.

2.13. Lemma. *Given a bicomplete category $\mathcal{C}$ together with a subcategory $w\mathcal{C}$ which satisfies (2OF3), we define*

$$\mathbb{C}_{we} = w\mathcal{C} \qquad \mathbb{C}_{cof} = \mathcal{C} \qquad \mathbb{C}_{fib} = \mathbb{C}_{we}^{\boxtimes}.$$

If $(\mathbb{C}_{we}, \mathbb{C}_{fib})$ is a WFS then $\mathbb{C}$ is a model structure on $\mathcal{C}$.

We conclude the discussion of model categories by recalling the definition of a proper model category.

2.14. Definition. *A model structure is left (resp. right) proper if the pushout (resp. pullback) of a weak equivalence along a cofibration (resp. fibration) is always a weak equivalence.*

2.15. Splitting and disjoint coproducts. Our last topic in this section is splitting and disjoint coproducts. We present several examples, as these notions interact in nontrivial ways. We write $A \sqcup B$ for the coproduct of A and B.

2.16. Definition. *A category is said to have splitting coproducts if for any morphism $f \colon X \to A \sqcup B$ there exist objects X_L and X_R such that $X \cong X_L \sqcup X_R$, and morphisms $f_L \colon X_L \to A$ and $f_R \colon X_R \to B$ such that $f \cong f_L \sqcup f_R$. (Although X_L and X_R depend on f, we do omit it from the notation.)*

2.17. Definition. *A category is said to have disjoint coproducts if, for any coproduct $A \sqcup B$, the natural injections $i_1 \colon A \to A \sqcup B$ and $i_2 \colon B \to A \sqcup B$ are monic, and the following three squares are pullback squares:*

$$
\begin{array}{ccc}
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle i_1} \\
A & \xrightarrow{i_1} & A \sqcup B
\end{array}
&
\begin{array}{ccc}
\varnothing & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle i_2} \\
A & \xrightarrow{i_1} & A \sqcup B
\end{array}
&
\begin{array}{ccc}
B & \xrightarrow{1_B} & B \\
{\scriptstyle 1_B}\downarrow & & \downarrow{\scriptstyle i_2} \\
B & \xrightarrow{i_2} & A \sqcup B
\end{array}
\end{array}
$$

2.18. Example. *We present examples of how these definitions interact.*

1. *The categories of sets and graphs both have splitting coproducts and disjoint coproducts.*

2. *The category of vector spaces and linear maps over $\mathbb{R}$ has disjoint coproducts but not splitting coproducts.*
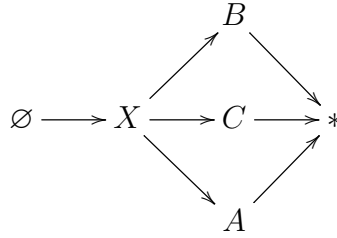
3. *The category of pointed finite sets has both disjoint and splitting coproducts.*

4. *The lattice*

$$
\begin{array}{ccc}
 & & B \\
 & \nearrow & & \searrow \\
\varnothing \longrightarrow X & & & * \\
 & \searrow & & \nearrow \\
 & & A
\end{array}
$$
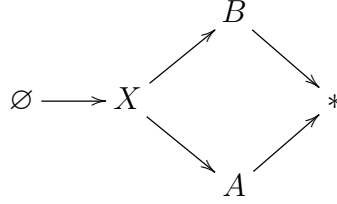
   has splitting but not disjoint coproducts, as $* = A \sqcup B$ *but the pullback of the two inclusions is* X.

5. *The lattice*

$$
\begin{array}{ccc}
 & & B \\
 & \nearrow & & \searrow \\
\varnothing \longrightarrow X \longrightarrow C & & & * \\
 & \searrow & & \nearrow \\
 & & A
\end{array}
$$

   has neither disjoint nor splitting coproducts. Just as in (4) it does not have disjoint coproducts, and it does not have splitting coproducts because $C \to * = A \sqcup B$ *cannot be written as* $(Y \to A) \sqcup (Z \to B)$ *for any* Y *or* Z *in the lattice.*

2.19. LEMMA. *Suppose that* $\mathcal{C}$ *has disjoint coproducts, and suppose that there exists a morphism* $f \colon A \to B$ *such that the diagram*

$$
\begin{array}{ccc}
 & & B \\
 & \overset{f}{\nearrow} & \downarrow {\scriptstyle i_2} \\
A & \underset{i_1}{\longrightarrow} & A \sqcup B
\end{array}
$$

commutes. Then $A \cong \varnothing$.

PROOF. We have the following diagram, where both squares are pullback squares:

$$
\begin{array}{ccc}
A & \overset{1_A}{\longrightarrow} & A \\
\downarrow & & \downarrow {\scriptstyle f} \\
\varnothing & \longrightarrow & B \\
\downarrow & & \downarrow {\scriptstyle i_2} \\
A & \underset{i_1}{\longrightarrow} & A \sqcup B
\end{array}
$$

If we consider the composite pullback, we need to take the pullback of i_1 along i_1, which (as $\mathcal{C}$ has disjoint coproducts) is A with the identity morphism. Thus the composite down the left must be the identity on A, and $A \cong \varnothing$, as desired. ∎

3. The case of a functor with an adjoint section

In many cases where a model category is required, the subcategory of weak equivalences is given as the preimage of the isomorphisms under a functor. In this section we explore the question of how much extra structure on the functor is required to show that the model structure exists directly from the existence of the functor.

Let $\mathcal{C}$ be a bicomplete category and suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor with a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ such that the counit $\epsilon\colon FG \to 1_{\mathcal{D}}$ is a natural isomorphism; in this case, we say that G is a *section* of F. We would like to define a model structure on $\mathcal{C}$ such that $\mathcal{D} = \mathrm{Ho}\,\mathcal{C}$ and F is the localization functor.

3.1. PROPOSITION. *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor with a section $G\colon \mathcal{D} \to \mathcal{C}$. We define three subcategories of $\mathcal{C}$ by*

$$\mathbb{C}^{adj}_{we} = F^{-1}(\mathrm{iso}\,\mathcal{D}) \qquad \mathbb{C}^{adj}_{cof} = \mathcal{C} \qquad \mathbb{C}^{adj}_{fib} = (\mathbb{C}^{adj}_{we})^{\boxtimes}.$$

Suppose that either

  (a) $\mathbb{C}^{adj}_{we}$ *is closed under pullbacks along $G(\mathcal{D})$, or*

  (b) *for any $f\colon A \to B$, $A \to B \times_{GF(B)} GF(A)$ is in $\mathbb{C}^{adj}_{we}$.*

Then these three subcategories form a left proper model structure. If $\mathcal{D}$ is bicomplete we can consider $\mathcal{D}$ to be a model category where the weak equivalences are the isomorphisms and all morphisms are both cofibrations and fibrations. In this case, (F,G) is a Quillen equivalence.

PROOF. First, notice that the image of G is inside $\mathbb{C}^{adj}_{fib}$. A commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & G(X) \\
{\scriptstyle f}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle G(p)} \\
B & \longrightarrow & G(Y)
\end{array}
$$

has a lift because $F(f)$ is an isomorphism in the adjoint square. Thus a lift exists in the original square and $G(p) \in (\mathbb{C}^{adj}_{we})^{\boxtimes} = \mathbb{C}^{adj}_{fib}$.

Now we check that we have a model structure on $\mathcal{C}$. As $\mathbb{C}^{adj}_{we}$ clearly satisfies (2OF3), by Lemma 2.13 we only need to check that $(\mathbb{C}^{adj}_{we}, \mathbb{C}^{adj}_{fib})$ is a WFS. We use Lemma 2.6. From the definitions we know that $\mathbb{C}^{adj}_{we}$ and $\mathbb{C}^{adj}_{fib}$ are closed under retracts, so condition (3) holds. As $\mathbb{C}^{adj}_{fib} = (\mathbb{C}^{adj}_{we})^{\boxtimes}$, condition (1) holds as well.

It remains to check condition (2): any morphism can be factored as a weak equivalence

followed by a fibration. Let $f\colon A \to B$ be any morphism, and consider the diagram

$$
\begin{array}{c}
A \xrightarrow{\quad \eta_A \quad} GF(A) \\
\quad \downarrow i \quad \nearrow \\
B \times_{GF(B)} GF(A) \longrightarrow GF(A) \\
\quad f \downarrow \quad \downarrow GF(f) \\
B \xrightarrow{\quad \eta_B \quad} GF(B)
\end{array}
$$

where $\eta\colon 1_{\mathcal{C}} \to GF$ is the unit of the adjunction. We need to show that i is a weak equivalence. If (b) holds then this is true by assumption. On the other hand, since ϵ is a natural isomorphism we know that as $F(\eta_X)$ is an isomorphism for all X, η_A and η_B are weak equivalences. If (a) holds then $B \times_{GF(B)} GF(A) \to GF(A)$ is also a weak equivalence, and by (2OF3) i is, as well. In either case we have a factorization, as desired. Thus $(\mathbb{C}^{adj}_{we}, \mathbb{C}^{adj}_{fib})$ is a WFS, as desired.

It remains to check that $\mathbb{C}^{adj}$ is left proper. Consider any diagram

$$
C \xleftarrow{\ g\ } A \xhookrightarrow{\ i\ } B
$$

where g is a weak equivalence. Since F is a left adjoint, $F(B \to B \cup_A C) = F(B) \to F(B) \cup_{F(A)} F(C)$, which is an isomorphism because $F(g)$ is an isomorphism. Thus the pushout of g along i is a weak equivalence, as desired.

We need to check that if $\mathcal{D}$ is bicomplete then (F, G) is, indeed, a Quillen equivalence. Clearly F preserves cofibrations and acyclic cofibrations, so we have a Quillen pair. It remains to show that $FA \to X$ is an isomorphism if and only if $A \to GX$ is a weak equivalence. But $FA \to X$ is an isomorphism if and only if $GFA \to GX$ is an isomorphism (as $FG \simeq 1_{\mathcal{D}}$), which is an isomorphism if and only if $A \xrightarrow{\sim} GFA \to GX$ is a weak equivalence, as desired. ∎

3.2. REMARK. *One may ask whether the model structure constructed in Proposition 3.1 is right proper. Unfortunately, we could not resolve that question. The model structures constructed in Examples 3.4 and 3.5 below are right proper, but we could not find a proof that this is generally the case.*

Conditions (a) and (b) are a bit annoying, as we do not have a conceptual explanation of why they are necessary; they are assumed simply because they are needed in the proof. Morally speaking, they should correspond to the fact that $\mathcal{D}$ is a much simpler category than $\mathcal{C}$, and thus that we don't need any more information about the problem than just the structure of $\mathcal{D}$. For very simple $\mathcal{D}$ this is the case:

3.3. COROLLARY. *If $\mathcal{D}$ is a preorder then condition (b) always holds. Thus for any functor $F\colon \mathcal{C} \to \mathcal{D}$ with a section we have a model structure $\mathbb{C}^{adj}$ with $\mathbb{C}^{adj}_{we} = F^{-1}(\mathrm{iso}\,\mathcal{D})$.*

PROOF. We need to show that $F(i\colon A \to B \times_{GF(B)} GF(A)) \in \mathrm{iso}\,\mathcal{D}$ if $\mathcal{D}$ is a preorder. Let $\pi_2\colon B \times_{GF(B)} GF(A) \to GF(A)$ be the projection morphism; then we know that $\pi_2 i = \eta_A$. Since $FG = 1_{\mathcal{D}}$, in particular we know that $F(\eta_A) = 1_{F(A)}$; thus $F(\pi_2 i) = F(\pi_2)F(i) = 1_{F(A)}$. Since $\mathcal{D}$ is a preorder, this means that $F(i)$ is an isomorphism, as desired. ■

We present a couple of examples of model structures constructed using this theorem.

3.4. EXAMPLE. Let $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ be the functor which takes a topological space to the set of its connected components. This functor has a right adjoint $-^{\delta}$ which endows a set with the discrete topology. To check that a model structure exists with weak equivalences equal to $\pi_0^{-1}(\mathrm{iso}\,\mathbf{Set})$ (the morphisms which induce bijections between connected components) we will show that condition (b) holds. Let $f\colon A \to B$ be a continuous map of spaces. Write $A = \coprod_{i \in I} A_i$ and $B = \coprod_{j \in J} B_j$, with A_i and B_j connected; by an abuse of notation, write $f\colon I \to J$ for the induced map on connected components. Thus $\pi_0(A)^{\delta} = I$ and $\pi_0(B)^{\delta} = J$, and

$$B \times_{\pi_0(B)^{\delta}} \pi_0(A)^{\delta} \cong \coprod_{i \in I} B_{f(i)}$$

with the map $A \to B \times_{GF(B)} GF(A)$ sending A_i to $B_{f(i)}$ by f. This induces a bijection on connected components, so it is a weak equivalence, as desired.

3.5. EXAMPLE. Let $s_{inj}\mathbf{Set}$ be the category of *semi-simplicial sets* as defined in the introduction. For $X \in s_{inj}\mathbf{Set}$, let $\dim X$ be the smallest integer n such that $X(0 < \cdots < k)$ is empty for all $k > n$; if such an integer does not exist then we write $\dim X = \infty$. Let $\mathbf{Z}_{\geq 0}^{+}$ be the category with objects nonnegative integers and ∞ and morphisms $n \to m$ if $n \leq m$. We have a functor $F\colon s_{inj}\mathbf{Set} \to \mathbf{Z}_{\geq 0}^{+}$ given by taking X to $\dim X$. This functor has right adjoint G which takes n to $D_n\colon \Delta_{inj}^{\overline{op}} \to \mathbf{Set}$ defined by

$$D_n(m) = \begin{cases} * & \text{if } m \leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $FG = 1$, and thus by Corollary 3.3 we have a model structure on $s_{inj}\mathbf{Set}$ where X and Y are weakly equivalent exactly when they have the same dimension.

4. Model structures from simple preorders

In the previous section, we showed that if $\mathcal{D}$ is a preorder then for any functor $F\colon \mathcal{C} \to \mathcal{D}$ with a section there exists a model structure $\mathbb{C}$ on $\mathcal{C}$ such that $\mathbb{C}_{we} = F^{-1}(\mathrm{iso}\,\mathcal{D})$. This result is not completely satisfying, however, as the condition that F has a right adjoint section is much too strong to hold in general. Thus in this section we will try to analyze this problem with the (equally strong but) different assumption that the structure of $\mathrm{Ho}\,\mathbb{C}$ is very simple. In the course of this exploration we actually construct several model structures whose homotopy categories are *not* preorders; we include them in the discussion as well, since their proofs are identical, and they give an interesting family of model structures.

The simplest that $\mathrm{Ho}\,\mathbb{C}$ could be, of course, is if it is equivalent to the trivial category. Such a model structure always exists, by setting the weak equivalences and the cofibrations to be all morphisms, and the fibrations to be the isomorphisms. This resolved, we consider the second-simplest case, when $\mathrm{Ho}\,\mathbb{C}$ has two objects and one morphism between them. Let $\mathcal{E}$ be the category with two objects, $\varnothing$ and $*$, and one non-identity morphism $\varnothing \to *$. In this case the model structure $\mathbb{C}$ divides objects of $\mathcal{C}$ into "big" objects and "small" objects, but does not distinguish between different "big" or different "small" objects.

4.1. DEFINITION. *A cut of a category $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathcal{E}$; such a cut is called trivial if $\mathcal{C} = F^{-1}(\varnothing)$ or $\mathcal{C} = F^{-1}(*)$. Given any cut F, we define $\mathfrak{I}_F = F^{-1}(\varnothing)$, $\mathcal{P}_F = F^{-1}(*)$. When F is clear from context we omit the subscript from the notation.*

We start with a more general construction which will give us three model structures associated to any cut. These three model structures will either (a) classify objects into "big" and "small", (b) distinguish between all "small" objects but have all "big" objects be equivalent, or (c) distinguish between all "big" objects but have all "small" objects be equivalent.

4.2. PROPOSITION. *Let $\mathcal{E}'$ be the total order with three objects, $\varnothing \to E \to *$. Suppose that $\mathcal{C}$ has a "double cut", a functor $F\colon \mathcal{C} \to \mathcal{E}'$. Then we have a model structure $\mathbb{C}^F$ on $\mathcal{C}$ given by*

$$\mathbb{C}^F_{cof} = \{A \to B \mid F(B) \neq \varnothing\} \cup \mathrm{iso}\,\mathcal{C} \qquad \mathbb{C}^F_{fib} = \{X \to Y \mid F(X) \neq *\} \cup \mathrm{iso}\,\mathcal{C},$$

and

$$\mathbb{C}^F_{we} = F^{-1}\{1_\varnothing, 1_*\} \cup \mathrm{iso}\,\mathcal{C}.$$

PROOF. We need to check the axioms of a model structure. $\mathbb{C}^F_{we}$ clearly satisfies (2OF3), so we focus on (WFS). We will prove that $(\mathbb{C}^F_{cof}, \mathbb{C}^F_{we} \cap \mathbb{C}^F_{fib})$ is a WFS; the other one will follow by duality. We use Lemma 2.6. As all three of the above classes are closed under retracts, condition (3) is satisfied. Note that

$$\mathbb{C}^F_{we} \cap \mathbb{C}^F_{fib} = F^{-1}(1_\varnothing) \cup \mathrm{iso}\,\mathcal{C}.$$

Thus we can say that a noninvertible morphism $f\colon X \to Y$ is an acyclic fibration when $F(Y) = \varnothing$, and we see that any morphism is either a cofibration or an acyclic fibration. In such situations factorizations trivially exist, and condition (2) is satisfied. Thus it remains to check condition (1).

Let $f\colon A \to B \in \mathbb{C}^F_{cof}$ and $g\colon X \to Y \in \mathbb{C}^F_{we} \cap \mathbb{C}^F_{fib}$, and suppose that we have a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

If either f or g is an isomorphism then this square clearly has a lift, so we assume that neither is an isomorphism. Then $F(g) = 1_\varnothing$, and in particular $F(Y) = \varnothing$. But

$F(B) \neq \varnothing$, and thus we cannot have a morphism $B \to Y$. Contradiction. Thus in any such square either f or g must be an isomorphism and $f \boxtimes g$. So condition (1) holds, and $(\mathbb{C}^F_{cof}, \mathbb{C}^F_{we} \cap \mathbb{C}^F_{fib})$ is a WFS, as desired. ∎

We now use this proposition to construct the model structures associated to a cut.

4.3. COROLLARY. *Given any cut F of a bicomplete $\mathcal{C}$ we define the model structure $\mathbb{C}^{bF}$ (the balanced model structure associated to F) on $\mathcal{C}$ by*

$$\mathbb{C}^{bF}_{we} = \{A \to B \mid B \in \mathfrak{I} \ or \ A \in \mathfrak{P}\} \cup \mathrm{iso}\,\mathcal{C} = F^{-1}(\mathrm{iso}\,\mathcal{E})$$
$$\mathbb{C}^{bF}_{cof} = \{A \to B \mid B \in \mathfrak{P}\} \cup \mathrm{iso}\,\mathcal{C} \qquad \mathbb{C}^{bF}_{fib} = \{A \to B \mid A \in \mathfrak{I}\} \cup \mathrm{iso}\,\mathcal{C}.$$

This model structure is both left and right proper.

$\mathbb{C}^{bF}$ is the model structure that can distinguish between "big" and "small" objects, but does not detect any other differences.

PROOF. Let F' be the double cut defined by composing F with the functor $\mathcal{E} \to \mathcal{E}'$ taking $\varnothing$ to $\varnothing$ and $*$ to $*$. Applying Proposition 4.2 to F' we get the desired structure.

As the model structure is self-dual, it suffices to show that it is left proper. Suppose that we have a diagram of noninvertible morphisms

$$C \longleftarrow A \overset{i}{\hookrightarrow} B.$$

As i is a cofibration, $B \in \mathfrak{P}_F$, and thus the pushout $B \to B \cup_A C \in \mathfrak{P}_F$. But then $B \to B \cup_A C$ is a weak equivalence, as desired. ∎

4.4. REMARK. *Note that we didn't use the fact that $A \to C$ is a weak equivalence, so in fact the pushout of any morphism along a noninvertible cofibration must be a weak equivalence. This reflects that the balanced model structure is not very discriminating.*

Thus for any cut $F \colon \mathcal{C} \to \mathcal{E}$ we can construct a model structure with homotopy category equivalent to $\mathcal{E}$. Note, however, that F need not be a Quillen equivalence, as it does not necessarily have an adjoint. For example, consider the category

$$
\begin{array}{ccc}
 & A \longrightarrow B & \\
\nearrow & & \searrow \\
\varnothing & & * \\
\searrow & & \nearrow \\
 & C \longrightarrow D &
\end{array}
$$
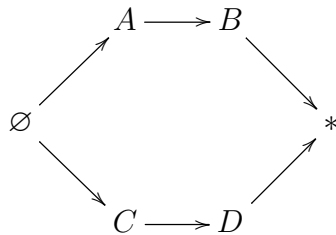
and define F to map $\varnothing$, A and B to $\varnothing$ and C, D and $*$ to $*$. Then F does not preserve either pullbacks or pushouts, so it is not a right or a left adjoint.

4.5. COROLLARY. *Given any cut F of a bicomplete $\mathcal{C}$ we have a model structure $\mathbb{C}^{rF}$ on $\mathcal{C}$ given by*

$$\mathbb{C}^{rF}_{we} = \mathcal{P} \cup \mathrm{iso}\,\mathcal{C} \qquad \mathbb{C}^{rF}_{cof} = \mathcal{C} \qquad \mathbb{C}^{rF}_{fib} = (\mathbb{C}^{rF}_{we})^{\boxtimes}.$$

This model structure is left proper. As the definition of a cut is self-dual, we also have a dual model structure $\mathbb{C}^{\ell F}$ where all morphisms are fibrations and the noninvertible weak equivalences are morphisms in $\mathcal{I}$; this model structure is right proper.

$\mathbb{C}^{rF}$ can distinguish between all objects in $\mathcal{I}$ (the "small" objects), but collapses all objects in $\mathcal{P}$ to a single one.

PROOF. We construct $\mathbb{C}^{rF}$ by composing the cut with the functor $\mathcal{E} \to \mathcal{E}'$ which takes $\varnothing$ to E and $*$ to $*$ and taking the model structure constructed in Proposition 4.2. As $(\mathbb{C}^{rF}_{we}, \mathbb{C}^{rF}_{fib})$ is a WFS, we know that $\mathbb{C}^{rF}_{we}$ is closed under pushouts. Thus $\mathbb{C}^{rF}$ is left proper.
   However, $\mathbb{C}^{rF}$ does not have to be right proper. Let $\mathcal{C}$ be the category

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & *
\end{array}
$$

and let F be the cut that takes $\varnothing$ and A to $\varnothing$ and B and $*$ to $*$. Then the only nontrivial weak equivalence is $B \to *$, and A is a fibrant object. If $\mathbb{C}^{rF}$ were proper we would have to have $\varnothing \to A$ be a weak equivalence, but in this model structure it is not. Thus in this case $\mathbb{C}^{rF}$ is not right proper, as claimed.
   The second part of the corollary follows by duality.   ∎

   In particular, the two examples constructed in this proof also prove the following:

4.6. COROLLARY. *The model structure constructed in Proposition 4.2 is not necessarily left or right proper.*

   Thus any cut in a category $\mathcal{C}$ gives at least three different (but possibly equivalent) model structures on $\mathcal{C}$. This means that any category with uncountably many cuts has uncountably many model structures, and more generally that any category with κ cuts has at least κ model structures.

4.7. EXAMPLE. Any cut of a pointed category must be trivial, which means that in a model structure associated to a cut, the weak equivalence are either all morphisms or just the isomorphisms.

4.8. EXAMPLE. The category **Set** has a single non-trivial cut, which takes the empty set to $\varnothing$ and all other sets to $*$. All model structures on **Set** where not all morphisms are weak equivalences are Quillen equivalent to either $\mathbb{C}^{bF}$ or $\mathbb{C}^{\ell F}$. (For an enumeration of the model structures on **Set**, see [Cam].)
   More generally, many **Set**-based categories (topological spaces, simplicial sets, etc.) have a single non-trivial cut, which gives rise to a similar family of model structures. However, these do not generally cover all possible model structures.

4.9. EXAMPLE. The category $s_{inj}\mathbf{Set}$ (defined in Example 3.5) has many different cuts; for example, for any n we have a cut F_n defined by $F_n(X) = \varnothing$ if $\dim X \leq n$ and $F_n(X) = *$ otherwise. Corollaries 4.3 and 4.5 give model structures which distinguish between semi-simplicial sets based on their dimensions: $\mathbb{C}^{bF_n}$ has X and Y equivalent if $\dim X, \dim Y \leq n$ or $\dim X, \dim Y > n$, $\mathbb{C}^{rF_n}$ has X and Y equivalent if $\dim X, \dim Y > n$ and $\mathbb{C}^{\ell F_n}$ has X and Y equivalent if $\dim X, \dim Y \leq n$.

It is possible for different cuts to yield equivalent model structures. For example, consider the category $\mathcal{C}$ with objects $\mathbb{R} \cup \{\pm\infty\}$, and with a morphism $a \to b$ if $a < b$. Let

$$F_a(b) = \begin{cases} \varnothing & \text{if } b < a \\ * & \text{otherwise.} \end{cases}$$

Then F_a is a cut for any finite value of a; let $\mathbb{C}_a$ be the model structure constructed by Corollary 4.5 for F_a. If we choose $a < a'$ then the functor $G: \mathbb{C}_a \to \mathbb{C}_{a'}$ given by $G(b) = b - a' + a$ preserves both cofibrations and weak equivalences and is clearly an equivalence of categories, and thus gives a Quillen equivalence between $\mathbb{C}_a$ and $\mathbb{C}_{a'}$.

However, in many cases we can show that different cuts will yield inequivalent model structures.

4.10. COROLLARY. *If a category $\mathcal{C}$ has a family of cuts $\{F_\alpha: \mathcal{C} \to \mathcal{E}\}_{\alpha \in A}$ such that if $\alpha \neq \alpha'$ then $\mathfrak{I}_\alpha$ and $\mathfrak{I}_{\alpha'}$ are not equivalent categories, then $\mathcal{C}$ has at least $|A|$ nonequivalent model structures.*

Dually, if such a family of cuts exists with $\mathcal{P}_\alpha \not\simeq \mathcal{P}_{\alpha'}$ for all distinct $\alpha, \alpha' \in A$ then $\mathcal{C}$ has at least $|A|$ nonequivalent model structures.

PROOF. Let $\alpha \neq \alpha' \in A$, and let $(\mathfrak{I}_\alpha, \mathcal{P}_\alpha)$ and $(\mathfrak{I}_{\alpha'}, \mathcal{P}_{\alpha'})$ be obtained from F_α and $F_{\alpha'}$, respectively. Let $\mathbb{C}_\alpha$ and $\mathbb{C}_{\alpha'}$ be the model structures constructed by the first part of Corollary 4.5. A zigzag of Quillen equivalences between $\mathbb{C}_\alpha$ and $\mathbb{C}_{\alpha'}$ would give an equivalence of homotopy categories. However, the homotopy category of $\mathbb{C}_\alpha$ is $(\mathfrak{I}_\alpha)_+$, the category $\mathfrak{I}_\alpha$ with a new terminal object added. As an equivalence must take terminal objects to terminal objects, an equivalence of $(\mathfrak{I}_\alpha)_+$ with $(\mathfrak{I}_{\alpha'})_+$ must give an equivalence of $\mathfrak{I}_\alpha$ with $\mathfrak{I}_{\alpha'}$; as these are inequivalent, we know that $\mathbb{C}_\alpha$ and $\mathbb{C}_{\alpha'}$ must be inequivalent, as desired.

The dual version follows from the dual version of Corollary 4.5. ∎

4.11. EXAMPLE. The model structures $\mathbb{C}^{rF_n}$ from Example 4.9 are all non-equivalent. Let $\mathcal{S}_n = F_n^{-1}(\varnothing)$; by Corollary 4.10 it suffices to check that these are nonequivalent.

We define the *monic length* of a category $\mathcal{C}$ with a terminal object to be the maximum length of a chain

$$A_0 \to A_1 \to \cdots \to A_k = * \in \mathcal{C}$$

such that each morphism is a noninvertible monomorphism and A_k is the terminal object of $\mathcal{C}$; this is an equivalence invariant. In $\mathcal{S}_n$ the terminal object is D_n, defined by

$$D_n(k) = \begin{cases} * & \text{if } k \leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

All monomorphisms in $\mathcal{S}_n$ are levelwise injections, so the monic length of $\mathcal{S}_n$ is $n + 1$, given by

$$\emptyset \to D_0 \to D_1 \to \cdots \to D_n.$$

As if $m \neq n$ then $m + 1 \neq n + 1$, we see that $\mathcal{S}_m$ and $\mathcal{S}_n$ are not equivalent, as claimed.

5. The generalized core model structure

There are two motivations for the construction of the generalized core model category. The first is a continuation of the type of analysis given in the previous section; however, in this case instead of taking $\mathcal{D}$ to be the simplest possible preorder, we take it to be the most complicated. More formally, we have the following definition:

5.1. DEFINITION. *Let $\mathcal{C}$ be a category. We define the preorder $P(\mathcal{C})$ with $\mathrm{ob}\, P(\mathcal{C}) = \mathrm{ob}\, \mathcal{C}$, and $\mathrm{Hom}_{P(\mathcal{C})}(X, Y)$ equaling the one-point set if there exists a morphism $X \to Y \in \mathcal{C}$, and the empty set otherwise. We will write $X \sim Y$ if X is isomorphic to Y in $P(\mathcal{C})$.*

There is a canonical functor $R_{\mathcal{C}}: \mathcal{C} \to P(\mathcal{C})$, such that any functor $F: \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a preorder, factors through $R_{\mathcal{C}}$. In this section, we construct a model structure on $\mathcal{C}$ such that the weak equivalences are $R_{\mathcal{C}}^{-1}(\mathrm{iso}\, P(\mathcal{C}))$. Note that P is a functor $\mathbf{Cat} \to \mathbf{PreOrd}$, which is left adjoint to the forgetful functor $U: \mathbf{PreOrd} \to \mathbf{Cat}$.[1]

The second motivation for constructing the generalized core model structure is to generalize the construction of the core model category structure in [Dro12]. The *core* of a graph is the smallest retract of the graph, and two graphs G and G' have isomorphic cores if and only if there exist morphisms $f: G \to G'$ and $g: G' \to G$ in the category of graphs. (For more on cores, see [GR01, Chapter 6].) In [Dro12], Droz constructed a model structure on the category of finite graphs where the weak equivalences are exactly the morphisms between graphs with isomorphic cores. It turns out that a similar construction will work in any category, and in particular on the category of infinite graphs. This gives rise to an application to infinite graph theory: an alternate definition of the core of an infinite graph. There is very little known about cores of infinite graphs, and it turns out that the homotopy-theoretic perspective gives an entirely new possible definition of a core. For more on this, see Section 6.

The main result of this section is the following:

5.2. THEOREM. *There is a model structure $\mathbb{C}^{core}$ with homotopy category $P(\mathcal{C})$ on any bicomplete category $\mathcal{C}$. A morphism $f: A \to B$ is a weak equivalence iff $A \sim B$. The acyclic fibrations are exactly the retractions in $\mathcal{C}$.*

If in addition $\mathcal{C}$ has splitting and disjoint coproducts then this structure is both left and right proper.

---

[1]Technically, P and U are only functors if we restrict our attention to small categories; otherwise, we need to worry about the 2-category structure of $\mathbf{Cat}$ and $\mathbf{PreOrd}$ and check that it is a 2-adjunction. However, as in the rest of this paper we are only concerned with the functor $R_{\mathcal{C}}$, which exists in any case, we blithely sweep these problems under the rug.

We call $\mathbb{C}^{core}$ the *generalized core model structure* on $\mathcal{C}$. Before we begin the proof, we present a couple of examples of such model structures.

5.3. EXAMPLE. Let **Set** be the category of sets. $P(\textbf{Set})$ is the category with two objects and one noninvertible morphism between them. The core model structure can distinguish between empty and nonempty sets, but cannot distinguish between nonempty sets. The fibrations are the surjective morphisms and the cofibrations are the injective morphisms.

More generally, for many set-based categories (such as topological spaces, simplicial sets, etc.) the core model structure has as the weak equivalences all morphisms between "nonempty" objects.

5.4. EXAMPLE. The category $s_{inj}\textbf{Set}$ (defined in Example 3.5) has a core which is more complicated than the core of simplicial sets. For example, if $\dim X > \dim Y$ then there are no morphisms $X \to Y \in s_{inj}\textbf{Set}$, and in fact dimension is a homotopy invariant in the generalized core model structure, since if there is a morphism $X \to Y$ and a morphism $Y \to X$ then $\dim X = \dim Y$. However, unlike in Example 3.5, it is not the only invariant, as there exist X and Y with $\dim X = \dim Y$ but with X and Y not isomorphic in $P(s_{inj}\textbf{Set})$.

PROOF OF THEOREM 5.2. We define $w\mathcal{C}$ to be the preimage under $R_{\mathcal{C}}$ of iso $P(\mathcal{C})$, and $\tilde{f}\mathcal{C}$ to be the subcategory of retractions in $\mathcal{C}$. These satisfy the conditions of Lemma 2.12. Let $\mathbb{C}^{core}$ be the candidate constructed as in Lemma 2.12; we will show that it satisfies the necessary conditions to be a model structure. Since $\mathbb{C}^{core}_{we}$ satisfies (2OF3) by definition, we focus on the other three conditions.

First, an observation: suppose that $f: A \to B$ is any morphism in $\mathcal{C}$. Then in $\mathbb{C}^{core}$, the canonical projection $p_1: A \times B \to A$ is an acyclic fibration, and the canonical inclusion $i_1: B \to B \sqcup A$ is a cofibration and a weak equivalence. The first follows trivially from the definition of acyclic fibration, since f and 1_A give a morphism $A \to A \times B$ which is a section of p_1. For the second, note that a canonical injection is always a cofibration as it is isomorphic to $1_B \sqcup (\varnothing \to A)$, and inclusions of the initial object are cofibrations by Lemma 2.8. It is a weak equivalence because f gives a retraction $B \sqcup A \to B$.

We now prove that $\tilde{f}\mathcal{C} = \mathbb{C}^{core}_{we} \cap \mathbb{C}^{core}_{fib}$, that is, that $\tilde{f}\mathcal{C}$ is exactly the acyclic fibrations.

We first show that $\tilde{f}\mathcal{C} \subseteq \mathbb{C}^{core}_{we} \cap \mathbb{C}^{core}_{fib}$. By definition, $\tilde{f}\mathcal{C} \subseteq w\mathcal{C} = \mathbb{C}^{core}_{we}$. We also have $\tilde{f}\mathcal{C} = (\mathbb{C}^{core}_{cof})^{\boxslash} \subseteq (\mathbb{C}^{core}_{cof} \cap \mathbb{C}^{core}_{we})^{\boxslash} = \mathbb{C}^{core}_{fib}$, as desired. Now let $f: A \to B \in \mathbb{C}^{core}_{we} \cap \mathbb{C}^{core}_{fib}$. As $f \in \mathbb{C}^{core}_{fib}$, it lifts on the right of $i_1: B \overset{\sim}{\hookrightarrow} B \sqcup B$. Let b be any morphism $B \to A$, which exists since $f \in \mathbb{C}^{core}_{we}$, so that we have a commutative diagram

$$
\begin{array}{ccc}
B & \overset{b}{\longrightarrow} & A \\
{\scriptstyle i_1}\downarrow{\scriptstyle \sim} & {\scriptstyle h}\nearrow{\scriptstyle \exists} & \downarrow{\scriptstyle f} \\
B \sqcup B & \underset{fb \sqcup 1_B}{\longrightarrow} & B
\end{array}
$$

This diagram shows that hi_2 is a section of f, so that f is a retraction and therefore $\tilde{f}\mathcal{C} \supseteq \mathbb{C}^{core}_{we} \cap \mathbb{C}^{core}_{fib}$, as desired.

Now we need to show that $(\mathbb{C}^{core}_{cof}, \tilde{f}\mathcal{C})$ and $(\mathbb{C}^{core}_{cof} \cap \mathbb{C}^{core}_{we}, \mathbb{C}^{core}_{fib})$ are WFSs. We prove this using Lemma 2.6. $\mathbb{C}^{core}_{we}$ is closed under retracts because $A \sim B$ is an equivalence relation. $\mathbb{C}^{core}_{cof}$ and $\mathbb{C}^{core}_{fib}$ are closed under retracts because they are defined by lifting properties, and $\tilde{f}\mathcal{C}$ is closed under retracts because it is equal to $\mathbb{C}^{core}_{we} \cap \mathbb{C}^{core}_{fib}$. Thus condition (3) of the lemma holds. Condition (1) holds by definition of $\mathbb{C}^{core}_{cof}$ and $\mathbb{C}^{core}_{fib}$. Thus to show that these are WFSs it suffices to check condition (2).

First we factor any morphism as a cofibration followed by an acyclic fibration. Any morphism $f \colon A \to B$ factors as

$$A \overset{i_1}{\hookrightarrow} A \sqcup B \xrightarrow[\sim]{f \sqcup 1_B} B$$

where the morphism i_1 is a canonical injection into a coproduct (and thus a cofibration) and $f \sqcup 1_B$ is a retraction. This proves condition (2), and thus $(\mathbb{C}^{core}_{cof}, \tilde{f}\mathcal{C})$ is a WFS.

Now we factor any morphism $f \colon A \to B$ as an acyclic cofibration followed by a fibration. In particular, we will show that the factorization

$$A \overset{i_1}{\underset{\sim}{\hookrightarrow}} A \sqcup (A \times B) \xrightarrow{f \sqcup p_2} B \, ,$$

where i_1 is the canonical injection and p_2 is the projection of the product on its second factor, works. By our previous analysis we know that i_1 is an acyclic cofibration, so we just need to prove that $f \sqcup p_2 \colon A \sqcup (A \times B) \to B$ is a fibration. Let $e \colon K \to L$ be any acyclic cofibration and consider any commutative diagram:

$$
\begin{array}{ccc}
K & \overset{k}{\longrightarrow} & A \sqcup (A \times B) \\
{\scriptstyle e}\big\uparrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle f \sqcup p_2} \\
L & \underset{l}{\longrightarrow} & B
\end{array}
$$

In order to show that a lift exists, it suffices to show that the lift h exists in the following diagram:

$$
\begin{array}{ccccc}
K & \overset{i_1}{\longrightarrow} & K \sqcup L & \overset{k \sqcup (kg \times l)}{\longrightarrow} & A \sqcup (A \times B) \\
{\scriptstyle e}\big\uparrow{\scriptstyle \sim} & {\scriptstyle h}\nearrow & \big\downarrow{\scriptstyle e \sqcup 1_L} & & \big\downarrow{\scriptstyle f \sqcup p_2} \\
L & \underset{1_L}{\longrightarrow} & L & \underset{l}{\longrightarrow} & B
\end{array}
$$

where g is any morphism from L to K (which exists because e is a weak equivalence). Note that the morphism $k \sqcup (kg \times l)$ is not the coproduct of two morphisms, but is rather the universal morphism induced by k and $(kg \times l)$. As $i_2 \colon L \to K \sqcup L$ is a section of $e \sqcup 1_L$, $e \sqcup 1_L \in \tilde{f}\mathcal{C}$, it lifts on the right of $e \in \mathbb{C}^{core}_{cof}$. Thus $(\mathbb{C}^{core}_{cof} \cap \mathbb{C}^{core}_{we}, \mathbb{C}^{core}_{fib})$ is a WFS, as desired.

We defer the proof of left and right properness to Proposition 5.6. ∎

Before moving on to prove properness, we need to analyze the cofibrations in this model structure. In general, the cofibrations in the core model structure are very difficult to analyze; however, in the case when $\mathcal{C}$ has splitting and disjoint coproducts it is possible:

5.5. PROPOSITION. *If $\mathcal{C}$ has splitting and disjoint coproducts, then any cofibration in the generalized core model structure, $c\colon A \to B$, is isomorphic to a canonical inclusion $i_1\colon A \to A \sqcup X$ for some object X.*

PROOF. The square

$$
\begin{array}{ccc}
A & \xrightarrow{\;i_1\;} & A \sqcup B \\
{\scriptstyle c}\downarrow & \;\;\nearrow{\scriptstyle h} & \downarrow{\scriptstyle c \sqcup 1_B} \\
B & \xrightarrow[\;1_B\;]{} & B
\end{array}
$$

commutes. As $\mathcal{C}$ has splitting coproducts, we can write $h = h_L \sqcup h_R$ with $h_L\colon B_L \to A$ and $h_R\colon B_R \to B$. Thus $c\colon A \to B_L \sqcup B_R$, so we can again use splitting to write $c = c_L \sqcup c_R$. We can then rewrite the above diagram as follows:

$$
\begin{array}{ccc}
A_L \sqcup A_R & \xrightarrow{\;i_1\;} & A \sqcup B \\
{\scriptstyle c_L \sqcup c_R}\downarrow & \;\;\nearrow{\scriptstyle h_L \sqcup h_R} & \downarrow{\scriptstyle c \sqcup 1_B} \\
B_L \sqcup B_R & \xrightarrow[\;\cong\;]{} & B
\end{array}
$$

By considering the restriction to A_R we get that the following diagram commutes:

$$
\begin{array}{ccc}
A_R & \xrightarrow{\;i_1\;} & A_R \sqcup B \\
{\scriptstyle c_R}\downarrow & & \nearrow{\scriptstyle i_2} \\
B_R & \xrightarrow[\;h_R\;]{} & B
\end{array}
$$

Thus $h_R c_R$ satisfies the conditions of Lemma 2.19 and we conclude that $A_R = \varnothing$ and $A_L \cong A$. Now consider the restriction to A_L; we get the following diagram:

$$
\begin{array}{ccc}
A_L & \xrightarrow{\;i_1\;} & A \\
{\scriptstyle c_L}\downarrow & \;\;\nearrow{\scriptstyle h_L} & \downarrow{\scriptstyle c} \\
B_L & \xrightarrow[\;i_1\;]{} & B
\end{array}
$$

As $A_L = A$ we know that c factors through $i_1\colon B_L \to B$ as $i_1 c_L$. The upper triangle says that $h_L c_L = 1_A$, and the lower triangle and the fact that c factors through i_1 says that $i_1 c_L h_L = i_1$; as i_1 is monic, $c_L h_L = 1_{B_L}$ and we see that c_L is an isomorphism. So we are done. ∎

We can now prove that the generalized core model structure is left proper and right proper.

5.6. PROPOSITION. *In a category with splitting and disjoint coproducts the generalized core model structure is left proper and right proper.*

PROOF. We first need to prove left properness: that the pushout of a weak equivalence along a cofibration is a weak equivalence. By Proposition 5.5, we can assume that the cofibration is a canonical inclusion $i_1: A \to A \sqcup C$ and that the weak equivalence is $w: A \to B$; then we have a pushout square

$$
\begin{array}{ccc}
A & \xrightarrow{\ i_1\ } & A \sqcup C \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w \sqcup 1_C} \\
B & \xrightarrow{\ i_1\ } & B \sqcup C
\end{array}
$$

We want to show that $w \sqcup 1_C$ is a weak equivalence, or in other words that there is a morphism $B \sqcup C \to A \sqcup C$. As w is a weak equivalence there exists a morphism $f: B \to A$; then $f \sqcup 1_C$ is the desired morphism, and we are done.

We now consider right properness. In any model category we can factor a weak equivalence as an acyclic cofibration followed by an acyclic fibration. We know that acyclic fibrations are preserved by pullbacks, so in order to show right properness it suffices to show that the pullback of an acyclic cofibration along a fibration is a weak equivalence.

By Proposition 5.5 we can assume that our cofibration is a canonical injection $i_1: A \to A \sqcup B$. Let $f: C \to A \sqcup B$ be the fibration along which we want to take a pullback. By splitting of coproducts, we can write $f = f_L \sqcup f_R$ with $f_L: C_L \to A$ and $f_R: C_R \to B$. Let D be the pullback of our two morphisms, so that we have a diagram

$$
\begin{array}{ccc}
D & \longrightarrow & C_L \sqcup C_R \\
\downarrow & & \downarrow{\scriptstyle f_L \sqcup f_R} \\
A & \overset{\sim}{\underset{i_1}{\hookrightarrow}} & A \sqcup B
\end{array}
$$

We want to show that there exists a morphism $g: C_L \sqcup C_R \to D$. Suppose that there exists a morphism $g': C_R \to C_L$. Then the commutative diagram

$$
\begin{array}{ccccc}
C_L \sqcup C_R & \xrightarrow{\ 1_{C_L} \sqcup f_L g'\ } & C_L & \xrightarrow{\ i_1\ } & C_L \sqcup C_R \\
\downarrow{\scriptstyle f_L \sqcup g'} & & & & \downarrow{\scriptstyle f_L \sqcup f_R} \\
A & & \overset{\sim}{\underset{i_1}{\hookrightarrow}} & & A \sqcup B
\end{array}
$$

shows that the morphism $C_L \sqcup C_R \to D$ exists, as desired. Thus all that we have left to show is that g' exists.

Since i_1 is a weak equivalence there exists a morphism $r\colon B \to A$. We consider the following diagram, where h exists because $f_L \sqcup f_R$ is a fibration and $C_R \to C_R \sqcup C_R$ is an acyclic cofibration:

$$
\begin{array}{ccc}
C_R & \xrightarrow{\ \ i_2\ \ } & C_L \sqcup C_R \\
{\scriptstyle i_2}\Big\downarrow{\scriptstyle \sim} & \overset{h}{\nearrow} & \Big\downarrow{\scriptstyle f_L \sqcup f_R} \\
C_R \sqcup C_R & \xrightarrow[(rf_R)\sqcup f_R]{} & A \sqcup B
\end{array}
$$

As we have splitting coproducts, we can write $hi_1\colon C_R \to C_L \sqcup C_R$ as a coproduct of $h_L\colon X \to C_L$ and $h_R\colon Y \to C_R$. If we can show that there exists a morphism $Y \to C_L$ we will be done, as $C_R \cong X \sqcup Y$. We have a commutative square

$$
\begin{array}{ccc}
Y & \xrightarrow{\ f_R h_R\ } & B \\
{\scriptstyle rf_R i_2}\Big\downarrow & & \Big\downarrow{\scriptstyle i_2} \\
A & \xrightarrow[\ \ i_1\ \ ]{} & A \sqcup B
\end{array}
$$

As $\mathcal{C}$ has disjoint coproducts the pullback of i_1 and i_2 is $\varnothing$; thus we have a morphism $Y \to \varnothing \to C_L$ and we are done. ∎

. At the beginning of this section we made the choice of setting the acyclic fibrations to be the retractions. Instead, we could have taken the dual definition, and constructed a model structure where the acyclic cofibrations are the sections:

5.7. Theorem. *There is a model structure $\mathbb{C}^{cocore}$ on $\mathcal{C}$ where $f\colon A \to B$ is a weak equivalence exactly when $A \sim B$ and the acyclic cofibrations are the sections. If $\mathcal{C}$ has splitting and disjoint coproducts then this model structure is right proper; if in addition binary coproducts distribute over binary products then it is left proper.*

Proof. The proof that the model structure exists follows by duality from the proof of Theorem 5.2. We defer the proof of properness to Corollary 5.9 and Proposition 5.10. ∎

We call this model structure the *generalized cocore model structure*. Morally speaking, the generalized core and the generalized cocore model structures should be Quillen equivalent, although we do not know how to prove this in full generality. In the case when $\mathcal{C}$ has splitting and disjoint coproducts, however, this does turn out to be the case:

5.8. Proposition. *If $\mathcal{C}$ has splitting and disjoint coproducts, then the identity functor is a left Quillen equivalence from the generalized core model structure to the generalized cocore model structure.*

Proof. We know that $\mathbb{C}_{we}^{core} = \mathbb{C}_{we}^{cocore}$, so it suffices to show that $\mathbb{C}_{fib}^{core} \supseteq \mathbb{C}_{fib}^{cocore}$. Equivalently, it suffices to show that $\mathbb{C}_{cof}^{core} \cap \mathbb{C}_{we}^{core} \subseteq \mathbb{C}_{cof}^{cocore} \cap \mathbb{C}_{we}^{cocore}$.

In the generalized cocore model structure the acyclic cofibrations are sections. In the generalized core model structure the acyclic cofibrations are those morphisms $f\colon A \to A \sqcup B$ for which a morphism $g\colon B \to A$ exists. If such a morphism exists then the induced

morphism $1_A \sqcup g$ is clearly a retraction for f, so all acyclic cofibrations in the generalized core model structure have retractions. Thus all acyclic cofibrations in the core model structure are also acyclic cofibrations in the cocore model structure, as desired. ∎

Right properness of the generalized cocore model structure follows directly from this proposition.

5.9. COROLLARY. *If $\mathcal{C}$ has splitting and disjoint coproducts then the generalized cocore model structure is right proper.*

PROOF. By Proposition 5.8 we know that the identity functor is a right Quillen equivalence from the generalized cocore model structure to the generalized core model structure; as the two structures have the same weak equivalences it suffices to show that the pullback is a weak equivalence in the generalized core model structure. This follows from Proposition 5.6. ∎

To finish the discussion of the cocore model structure, we would like to show that the generalized cocore model structure is left proper. However, because of the way we defined the acyclic fibrations, it turns out to be very difficult to do so in general. By introducing a further assumption we get the following result.

5.10. PROPOSITION. *If $\mathcal{C}$ has splitting and disjoint coproducts and, moreover, if binary products distribute over binary coproducts, the generalized cocore model structure is left proper.*

PROOF. Let $c\colon A \to B$ be a cofibration. Since in our model structure the acyclic cofibrations are the sections, the projections $C \times D \to C$ are fibrations. In particular, $p_2\colon (A \sqcup *) \times B \to B$ is a fibration. However, as products distribute over coproducts we know that $(A \sqcup *) \times B \cong (A \times B) \sqcup B$, so there exists a morphism $B \to (A \sqcup *) \times B$. Thus the morphism $p_2 \sqcup 1_B\colon (A \times B) \sqcup B \to B$ is an acyclic fibration.

We consider the following commutative diagram and deduce the existence of a lifting morphism h.

$$
\begin{array}{ccc}
A & \xrightarrow{\ i_1 \circ (1_A \times c)\ } & (A \times B) \sqcup B \\
{\scriptstyle c}\Big\uparrow\Big\downarrow & {\overset{h}{\nearrow}} & \Big\downarrow{\scriptstyle \sim}{\scriptstyle p_2 \sqcup 1_B} \\
B & \xrightarrow[\ 1_B\ ]{} & B
\end{array}
$$

By applying the logic used in Proposition 5.5, we see that this diagram is induced from two diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ 1_A \times c\ } & A \times B \\
{\scriptstyle c_L}\Big\uparrow\Big\downarrow & {\overset{h_L}{\nearrow}} & \Big\downarrow{\scriptstyle p_2} \\
B_L & \xrightarrow[\ i_1\ ]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
\varnothing & \xrightarrow{\ \ \ } & B \\
\Big\downarrow & {\overset{h_R}{\nearrow}} & \Big\downarrow{\scriptstyle 1_B} \\
B_R & \xrightarrow[\ i_2\ ]{} & B
\end{array}
$$

Note that $p_1 h_L c_L = p_1(1_A \times c) = 1_A$, so c_L is a section. Thus any cofibration is a composition of a section and a canonical inclusion.

Thus it suffices to show that the pushout of a weak equivalence along a canonical inclusion or a section is still a weak equivalence. The pushout of a weak equivalence $f\colon A \to C$ along a canonical inclusion $i_1\colon A \to A \sqcup B$ is just $f \sqcup 1_B\colon A \sqcup B \to C \sqcup B$, which is clearly also a weak equivalence. The pushout of a section is another section, and as all sections are weak equivalences by (2OF3) the pushout of a weak equivalence along a section is another weak equivalence, as desired. So we are done. ∎

We conclude this section with an application of this theorem to the core model structure defined in [Dro12] on the category of finite graphs. This model structure agrees with the generalized core model structure defined in Theorem 5.2.

5.11. COROLLARY. *The categories of finite graphs and of infinite graphs have splitting and disjoint coproducts and binary products distribute over binary coproducts. Thus the generalized core and generalized cocore model structures on each are both left proper and right proper.*

PROOF. We will show that the categories of graphs have the desired properties; the rest follows from the above results. First we check splitting coproducts. Suppose that we have a morphism $f\colon X \to A \sqcup B$; this is a map from the set of vertices of X to the disjoint union of the vertices of A and B. Let X_L be the complete subgraph of X on the preimage of the vertices of A and let X_R be the complete subgraph on the preimage of the vertices of B. X_L and X_R are disjoint subgraphs of X whose union is X, so we see that $X \cong X_L \sqcup X_R$ and $f = (f|_A\colon X_L \to A) \sqcup (f|_B\colon X_R \to B)$. Thus we have splitting coproducts.

To check that we have disjoint coproducts we just need to check the definition on the vertices, where it holds because it holds in the category of sets.

It remains to show that binary products distribute over binary coproducts. In particular, we want to show that for graphs A, B and C we have

$$A \times (B \sqcup C) \cong (A \times B) \sqcup (A \times C).$$

This follows from the definitions of graphs and the fact that products distribute over coproducts in the category of sets. ∎

6. Concepts of cores for infinite graphs

We called the model structure constructed in Section 5 the "generalized core model structure" because in the case when $\mathcal{C}$ is the category of finite graphs[2], homotopy types correspond exactly to cores. More precisely, in the model structure two graphs are weakly equivalent exactly when they have the same core. (For more on the core, see [GR01], section 6.2.) Inspired by this, we can consider the generalized core model structure on the category of all graphs, and ask for a classification of the homotopy types of this category. One conjecture is that there should be a notion of a "core" for a (possibly infinite) graph such that cores classify homotopy types in the generalized core model structure.

---

[2]By "finite graph" we mean an undirected graph with no repeated edges.

Diverse generalizations of the notion of core to infinite graphs have been explored by Bauslaugh in [Bau95].

1. An s-core is a graph such that all endomorphisms are surjections (on the vertices).

2. An r-core is a graph without proper retractions.

3. An a-core is a graph such that all endomorphisms are automorphisms.

4. An i-core is a graph for which all endomorphisms are injections.

5. An e-core is a graph such that all endomorphisms preserve non-adjacency.

These definitions are known to be equivalent for finite graphs, and are all proved to be different when considering infinite graphs in [Bau95].

Once a definition of core is chosen, we can define *a core of a graph G* as one of its subgraphs H, which is a core and for which a morphism $G \to H$ exists. It would also make sense to define the core as a retraction of G; however, as this definition is more restrictive than the previous one, the results of this section will also hold under this definition.

It is natural to ask if applying our generalized core construction to the category of all graphs gives a notion of core that corresponds to one of those defined above. More precisely, is it the case that two graphs are weakly equivalent if they have the same "core", for some notion of "core" defined above? The answer turns out to be "no." We prove this by exhibiting two graphs, one of which does not contain a core in the sense of (1)-(3), and one of which does not contain a core in the sense of (3)-(5). As every graph has a "homotopy type" in the generalized core model structure, this means that none of these definitions of a core classify homotopy types in the case of the generalized core model structure.

Note that while the definitions above were originally given for general "structures" (understood as combinatorial structures), and exemplified by oriented graphs, it can be shown ([PT80]) that all of the relevant examples and results can be transferred to the category of undirected graphs using a well-chosen fully faithful "edge-replacement" functor. Thus it suffices to show that there exist directed infinite graphs with no core, and it will also hold for undirected graphs.

We construct our examples by adapting methods from [Bau95].

6.1. THEOREM. *Let G be the graph with vertices $\{1, 2, \ldots\}$ and with an edge from n to $n+1$ for all n. Then G has no s-core, r-core or a-core. The zipper graph in Figure 1 has no a-core, i-core or e-core.*

PROOF. Any endomorphism φ of G is uniquely determined by $\varphi(1)$, and must have an image isomorphic to itself. Thus G has a core if and only if it is a core. However, it is clearly not an s-core, an r-core or an a-core, and thus G has none of these cores.

The zipper graph is composed of three infinite rays with a common point, two of the rays going to the common point, one ray coming out of the common point and additional decorations. We observe that the endomorphisms of the zipper graph map the outgoing
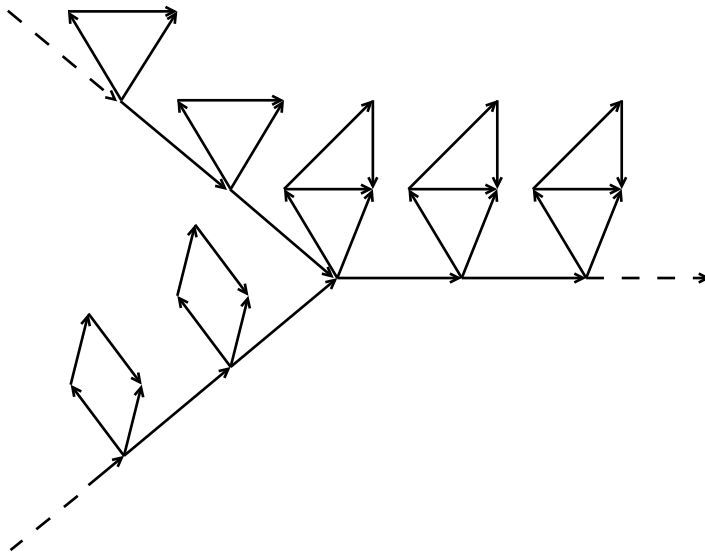
Figure 1: Prolonging this graph in three directions without end, we obtain the *zipper* graph.

ray to itself by a "shift toward the right". The decorations insure the absence of an automorphism mapping one of the incoming rays to the other. Since the non-trivial endomorphisms of the zipper graph are non-injective but surjective, the zipper graph has no i-core or a-core. Moreover, looking at non-adjacent vertices of the decorations of the lower incoming ray, we see that they can sometimes be mapped to adjacent vertices. This shows that the zipper graph has no e-core and concludes the proof of our theorem. ∎

We conclude that the generalized core model structure has a notion of homotopy type which does not correspond to any of Bauslaugh's definition of cores.

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