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INTERNAL CHOICE HOLDS IN THE DISCRETE PART OF ANY COHESIVE TOPOS SATISFYING STABLE CONNECTED CODISCRETENESS

We dedicate this paper to the memory of our friend R.F.C.Walters.

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Abstract. We introduce an apparent strengthening of Sufficient Cohesion that we call *stable Connected Codiscreteness* (SCC) and show that if $p : \mathcal{E} \to \mathcal{S}$ is cohesive and satisfies SCC then the internal axiom of choice holds in $\mathcal{S}$. Moreover, in this case, $p^! : \mathcal{S} \to \mathcal{E}$ is equivalent to the inclusion $\mathcal{E}_{\neg\neg} \to \mathcal{E}$.

1. Introduction

The proposal in [8] to analyze cohesion is to axiomatize the contrast between cohesion and non-cohesion. This contrast may be expressed as a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ with certain special properties that allow to effectively use the intuition that the objects of $\mathcal{E}$ are 'spaces' and those of $\mathcal{S}$ are 'sets'; that p_*X in $\mathcal{S}$ is the set of points of the space X and that p^*S in $\mathcal{E}$ is the discrete space with S as underlying set of points. A distinctive feature of this axiomatization is that it does not require any special properties on the toposes $\mathcal{E}$ and $\mathcal{S}$, so we are naturally led to the question of how do typical features of categories of non-cohesion relate to the axiomatization of the contrast between cohesion and non-cohesion. This question is further justified by the study of Cantor exposed in [5] and [6]. More concretely, one is lead to conjecture that the positive geometric conditions axiomatizing the contrast actually imply non-cohesive features on the base $\mathcal{S}$.

In this paper we show that mild conditions on the contrast $p : \mathcal{E} \to \mathcal{S}$ imply that $\mathcal{S}$ satisfies the internal axiom of choice. We also show that, in this case, $\mathcal{S}$ may be identified with the topos $\mathcal{E}_{\neg\neg}$ of sheaves for the double negation topology on $\mathcal{E}$.

We only address the feature of internal choice so, in a sense, this paper is only the first step in carrying out the program outlined in [6]. As suggested there, a next step would be to address the Generalized Continuum Hypothesis.

Although we introduce most of the necessary terminology, the reader is assumed to be familiar with [8]. The main results in the paper are elementary, so in principle, it is possible to understand the proofs just with a knowledge of elementary topos theory but,

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of course, it is better to be acquainted with examples and for these we refer also to [7], [4], [10] and [9].

2. Categories of cohesion

In this section we recall the necessary material from [8]. Fix a string of adjoints

$$\begin{tikzcd}[column sep=small] \mathcal{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift left=2, "p_*"] \\ \mathcal{S} \arrow[u, shift right=2, "p^*"'] \arrow[u, shift left=6, "p^!"] \end{tikzcd}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$. It is well known that p^* is full and faithful if and only if $p^!$ is. Let us assume that $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are indeed full and faithful. For certain detailed calculations that we need to do it will be convenient to express this in terms of the associated units and counits; so let σ and τ be the unit and counit of $p_! \dashv p^*$, α and β those of $p^* \dashv p_*$, and η and ϵ those of $p_* \dashv p^!$. Our assumption that p^* and $p^!$ are full and faithful is equivalent to $\tau : p_! p^* \to 1$, $\alpha : 1 \to p_* p^*$ and $\epsilon : p_* p^! \to 1$ being iso.

The diagrams below commute

$$\begin{tikzcd} p^* \arrow[r, "\eta_{p^*}"] \arrow[d, "p^*\epsilon^{-1}"'] & p^! p_* p^* \arrow[d, "p^!\alpha^{-1}"] \\ p^* p_* p^! \arrow[r, "\beta_{p^!}"'] & p^! \end{tikzcd} \qquad \begin{tikzcd} p_* \arrow[r, "p_*\sigma"] \arrow[d, "\tau^{-1}_{p_*}"'] & p_* p^* p_! \arrow[d, "\alpha^{-1}_{p_!}"] \\ p_! p^* p_* \arrow[r, "p_!\beta"'] & p_! \end{tikzcd}$$

and we denote the resulting natural transformations by $\phi : p^* \to p^!$ and $\theta : p_* \to p_!$.

2.1. Lemma. *The following triangles and squares*

$$\begin{tikzcd} p_*(p^*B) \arrow[r, "p_*\phi"] \arrow[d, "\theta"'] \arrow[dr, "\alpha^{-1}" description] & p_*(p^!B) \arrow[d, "\epsilon"] \\ p_!(p^*B) \arrow[r, "\tau"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\sigma"'] & p^!(p_*X) \arrow[d, "p^!\theta"] \\ p^*(p_!X) \arrow[r, "\phi"'] & p^!(p_!X) \end{tikzcd}$$

commute. In particular, $p_*\phi$ is an iso.

Proof. We first consider the diagram on the left above. Replacing ϕ and θ with their definitions we obtain

$$\begin{tikzcd} p_*(p^*B) \arrow[r, "p_*\eta"] \arrow[d, "p_*\sigma"'] \arrow[dr, "id" description] & p_*(p^!(p_*(p^*B))) \arrow[r, "p_*(p^!\alpha^{-1})"] \arrow[d, "\epsilon"] & p_*(p^!B) \arrow[dd, "\epsilon"] \\ p_*(p^*(p_!(p^*B))) \arrow[r, "p_*(p^*\tau)"'] \arrow[d, "\alpha^{-1}"'] & p_*(p^*B) \arrow[dr, "\alpha^{-1}" description] & \\ p_!(p^*B) \arrow[rr, "\tau"'] & & B \end{tikzcd}$$

To prove that the diagram on the right of the statement commutes just replace ϕ with its definition in terms of η and α, and confirm that the following

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "\sigma"'] & p^!(p_* X) \arrow[d, "p^!(p_*\sigma)"] \arrow[dr, "p^!\theta"] & \\
p^*(p_! X) \arrow[r, "\eta_{p^* p_!}"'] & p^!(p_*(p^*(p_! X))) \arrow[r, "p^!\alpha_{p_!}^{-1}"'] & p^!(p_! X)
\end{tikzcd}$$

commutes. ∎

The next fact plays a fundamental role.

2.2. Proposition. *The following are equivalent:*

1. $\theta : p_* \to p_!$ *is epi,*
2. $\phi : p^* \to p^!$ *is mono,*
3. *the unit* $\sigma : 1 \to p^* p_!$ *is epi,*
4. *the counit* $\beta : p^* p_* \to 1$ *is mono.*

Proof. The equivalence between the first two items is stated in Definition 2(c) in [8]. Details may be found in Lemma 2.3 in [4]. ∎

If the equivalent conditions of Proposition 2.2 hold then the string $p_! \dashv p^* \dashv p_* \dashv p^!$ of adjoints is said to satisfy the *Nullstellensatz*.

2.3. Lemma. *If the Nullstellensatz holds then* $p_!\phi : p_! p^* \to p_! p^!$ *is epi.*

Proof. In the naturality square

$$\begin{tikzcd}
p_*(p^* S) \arrow[r, "p_*\phi"] \arrow[d, "\theta"'] & p_*(p^! S) \arrow[d, "\theta"] \\
p_!(p^* S) \arrow[r, "p_!\phi"'] & p_!(p^! S)
\end{tikzcd}$$

both vertical maps are epi by the Nullstellensatz and the top map is an iso by Lemma 2.1. ∎

In terms of the intended intuition, the result above may be described as saying that for any A in $\mathcal{S}$, $p^! A$ has, at most, as many connected components as $p^* A$.

The following concept is fundamental in the present paper.

2.4. Definition. If $\mathcal{E}$ and $\mathcal{S}$ are cartesian closed extensive categories then $\mathcal{E}$ is called *pre-cohesive (relative to $\mathcal{S}$)* if

1. $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are full and faithful,
2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products,
3. and the Nullstellensatz holds.

We may say that $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive.

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive. An object X in $\mathcal{E}$ is called *connected* if $p_! X = 1$ and it is called *contractible* if X^A is connected for every A in $\mathcal{E}$.

2.5. Definition. The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is *sufficiently cohesive* if for every X in $\mathcal{E}$ there is a mono $X \to Y$ with Y contractible.

Since $p_!$ preserves finite products and p^* is fully faithful there is a canonical natural transformation $p_!(X^{p^* S}) \to (p_! X)^S$.

2.6. Definition. The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is said to satisfy *continuity* if the transformation $p_!(X^{p^* S}) \to (p_! X)^S$ is an iso for every X in $\mathcal{E}$ and S in $\mathcal{S}$.

This condition is the 'continuity' property required in Definition 2(b) in [8].

2.7. Definition. [Definition 2 in [8]] A *category of cohesion* $\mathcal{E}$ (relative to $\mathcal{S}$) is a pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ satisfying continuity.

We will say that $p : \mathcal{E} \to \mathcal{S}$ is cohesive.

Perhaps curiously we will not need the full strength of continuity. In fact, we will be mostly concerned with the next weaker condition.

2.8. Definition. The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is said to satisfy *epi-continuity* if the transformation $p_!(X^{p^* S}) \to (p_! X)^S$ is epi for every X in $\mathcal{E}$ and S in $\mathcal{S}$.

3. Cohesive toposes

The definition of (pre-)cohesive category does not require the existence of equalizers. There is a good reason for this as already hinted at in Theorem 1 in [8]. Having said this, we are mainly interested in the case where $p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ is a geometric morphism between toposes. In this context some of the conditions defining pre-cohesive categories have already been considered for geometric morphisms. For example, recall that a geometric morphism $f : \mathcal{E} \to \mathcal{S}$ is called *essential* if f^* has a left adjoint (denoted by $f_!$); that f is *connected* if $f^* : \mathcal{S} \to \mathcal{E}$ is fully faithful; and that f is *local* if f_* has a fully faithful right adjoint (denoted by $f^!$). Notice that local implies connected. See [3] for details.

We stress that the geometric morphisms we are interested in are localic only in extreme cases, so the localic intuition supporting the standard terminology of geometric morphisms is not very useful. For instance, many of the geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ we consider are local, but it seems useless to try to picture $\mathcal{E}$ as a space with a focal point. Hence, we will try to balance efficient reference to standard terminology on the one hand, and emphasis on 'geometric' conditions and intuition on the other. The following discussion between hyperconnectedness and the Nullstellensatz is a good example of the balance we try to achieve. (Recall that a geometric morphism $f : \mathcal{E} \to \mathcal{S}$ is *hyperconnected* if it restricts to an equivalence between $\mathcal{S}$ and the full subcategory of $\mathcal{E}$ whose objects are all subquotients of objects of the form f^*A.)

Let $p : \mathcal{E} \to \mathcal{S}$ be a geometric morphism.

3.1. LEMMA. [Lemma 3.1(i) in [4]] *If $p : \mathcal{E} \to \mathcal{S}$ is local then the following are equivalent:*

1. *p is hyperconnected,*
2. *the image of p^* is closed under subobjects in $\mathcal{E}$,*
3. *(Nullstellensatz) the canonical $\phi : p^* \to p^!$ is mono.*

PROOF. Proposition A4.6.6 in [3] gives several characterizations of hyperconnectedness. It is clear from this result that, under the assumption that $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful, the first two conditions are equivalent; and that these are also equivalent to the counit β of $p^* \dashv p_*$ being mono. This last condition is equivalent to the third item by Proposition 2.2. ∎

We insist: it does not seem useful to picture $\mathcal{E}$ as a very connected locale; on the other hand, if we picture $p^* : \mathcal{S} \to \mathcal{E}$ as the embedding of the category of discrete spaces into that of all spaces then the result above says that the Nullstellensatz is equivalent to the condition that subspaces of discrete spaces are discrete.

The next result summarizes the relation between the intuitively geometric conditions defining pre-cohesive categories and the standard localic terminology for geometric morphisms.

3.2. LEMMA. *The adjunction $p : \mathcal{E} \to \mathcal{S}$ extends to a string $p_! \dashv p^* \dashv p_* \dashv p^!$ making p pre-cohesive if and only if $p : \mathcal{E} \to \mathcal{S}$ is (as a geometric morphism) local, hyperconnected, essential and, moreover, $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.*

If the equivalent conditions of Lemma 3.2 hold then we will say that the geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *pre-cohesive*. It is now consistent with the terminology of [8] to say that the geometric morphism p is *cohesive* if it is pre-cohesive and satisfies continuity.

3.3. PROPOSITION. *If $\mathcal{E}$ and $\mathcal{S}$ are toposes and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then the following are equivalent:*

1. *p is sufficiently cohesive,*
2. *the subobject classifier of $\mathcal{E}$ is connected (i.e. $p_!\Omega = 1$),*
3. *for every X in $\mathcal{E}$ there is a mono $X \to Y$ with Y connected.*

PROOF. The equivalence between the first two items appears in Proposition 4 in [8]. The equivalence with the third item is not stated there but it is clear from the proof. Certainly, sufficient cohesion implies the third item. On the other hand, if we assume the third item then Ω embeds in a connected object but "if an injective object is embedded in a connected object, then it is a retract of that object and hence connected itself". ■

It is also relevant to recall the next restricted characterization of Sufficient Cohesion.

3.4. PROPOSITION. *If $\mathcal{S}$ is a De Morgan topos and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then, p is sufficiently cohesive if and only if $p^!2$ is connected.*

PROOF. This is the Corollary in Section VI of [8]. Just notice that the Continuity condition is not needed in the proof there. ■

Examples of cohesive toposes satisfying Sufficient Cohesion may be found in [8] and [9]. Pre-cohesive examples may also be found in [7], [4] and [10].

(The referee asked if the incompatibility of sufficient cohesion and continuity result (for presheaf toposes) in [9] may be improved be relaxing the continuity condition. Indeed, it may be strengthened by replacing continuity by *mono-continuity* in the sense that the canonical $p_!(X^{p^*A}) \to (p_!X)^A$ is mono for all X in $\mathcal{E}$ and A in $\mathcal{S}$. In more detail, consider Lemma 6.2 in [9] and the calculation there

$$p_!(R^{\mathbb{N}}) = p_!(R^{p^*\mathbb{N}}) \to (p_!R)^{\mathbb{N}} = 1^{\mathbb{N}} = 1$$

showing that $R^{\mathbb{N}}$ is connected. Continuity provides the iso $p_!(R^{p^*\mathbb{N}}) \to (p_!R)^{\mathbb{N}}$. If one weakens the hypothesis of the lemma to epi-continuity then the calculation above only shows that $p_!(R^{\mathbb{N}})$ is well-supported; which is trivial because $p_!(R^{\mathbb{N}})$ has a point. On the other hand, if we weaken the hypothesis to mono-continuity then the map $p_!(R^{p^*\mathbb{N}}) \to (p_!R)^{\mathbb{N}}$ is mono and we can still conclude that $p_!(R^{\mathbb{N}}) = 1$. The hypotheses of Proposition 7.3 and Theorem 7.4 loc. cit. can then be weakened to mono-continuity without changing the proofs.)

4. Density, Booleaness and the Nullstellensatz

Let $\mathcal{E}$ and $\mathcal{S}$ be toposes. (This is not strictly necessary but, at present, it seems distracting to try to work with maximum generality.) Let $p : \mathcal{E} \to \mathcal{S}$ be a geometric morphism such that the adjunction $p^* \dashv p_*$ extends to a string of adjoints

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift left=2, "p_*"] \\ \mathcal{S} \arrow[u, shift right=2, "p^*"'] \arrow[u, shift left=6, "p^!"] \end{tikzcd}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$. Assume that $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are full and faithful, and denote the canonical transformations by $\theta : p_* \to p_!$ and $\phi : p^* \to p^!$. (We stress that we are not assuming that the Nullstellensatz holds nor that $p_!$ preserves finite products. In standard terminology p is just assumed to be local and essential.)

4.1. Lemma. *If the Nullstellensatz holds then the rightmost adjoint functor* $p^! : \mathcal{S} \to \mathcal{E}$ *preserves* 0.

Proof. Recall that ϵ denotes the counit of $p_* \dashv p^!$ and σ the unit of $p_! \dashv p^*$. Since initial objects are strict in extensive categories the span

$$0 \xleftarrow{\epsilon} p_*(p^!0) \xrightarrow{\theta} p_!(p^!0)$$

shows that $p_!(p^!0)$ is initial. So $p^*(p_!(p^!0))$ is initial and $\sigma : p^!0 \to p^*(p_!(p^!0))$ implies that $p^!0$ is also initial. ∎

Loosely speaking, Lemma 4.1 says that the Nullstellensatz implies that the codiscrete space determined by the empty set is initial in $\mathcal{E}$. There is a partial converse.

4.2. Lemma. *If* $\mathcal{S}$ *is Boolean and* $p^!0 = 0$ *then the Nullstellensatz holds.*

Proof. By Lemma 3.1 it is enough to prove that the image of $p^* : \mathcal{S} \to \mathcal{E}$ is closed under subobjects. As before denote by α the unit of $p^* \dashv p_*$. Let $m : X \to p^*S$ be mono with S in $\mathcal{S}$. The composite

$$p_*X \xrightarrow{p_*m} p_*(p^*S) \xrightarrow{\alpha^{-1}} S$$

is mono and we let $c : C \to S$ be its complement in $\mathcal{S}$, so that the cospan

$$p^*(p_*X) \xrightarrow{p^*(p_*m)} p^*(p_*(p^*S)) \xrightarrow{p^*\alpha^{-1}} p^*S \xleftarrow{p^*c} p^*C$$

is a coproduct in $\mathcal{E}$. Since $p^*\alpha^{-1} = \beta_{p^*} : p^*(p_*(p^*S)) \to p^*S$, the left leg of the span above equals $\beta_{p^*}(p^*(p_*m)) = m\beta$. We can now take the pullback along $m : X \to p^*S$ as below

$$\begin{tikzcd}
p^*(p_*X) \arrow[r, "\beta"] \arrow[d, "id"'] & X \arrow[r, "id"] \arrow[d, "id"] & X \arrow[d, "m"] & P \arrow[l, "\pi_1"'] \arrow[d, "\pi_0"] \\
p^*(p_*X) \arrow[r, "\beta"'] & X \arrow[r, "m"'] & p^*S & p^*C \arrow[l, "p^*c"]
\end{tikzcd}$$

and hence: the map $\beta : p^*(p_*X) \to X$ is an iso if and only if P is initial. We claim that P is indeed initial. To prove this apply p_* to obtain the pullback

$$\begin{tikzcd}
p_*P \arrow[r, "p_*\pi_1"] \arrow[d, "p_*\pi_0"'] & p_*X \arrow[d, "p_*m"] \\
p_*(p^*C) \arrow[r, "p_*(p^*c)"'] & p_*(p^*S)
\end{tikzcd}$$

in $\mathcal{S}$, showing that p_*P is initial and, by adjointness, that there is a map $P \to p^!0$. By hypothesis, $p^!0 = 0$ so P is initial. ∎

The hypotheses of the last result do not allow much freedom. Recall that a subtopos is called *dense* if the direct image of the inclusion preserves 0. Let Ω be the subobject classifier of $\mathcal{E}$ and $\neg\neg : \Omega \to \Omega$ the double negation Lawvere-Tierney topology. The sheaf subtopos $\mathcal{E}_{\neg\neg} \to \mathcal{E}$ is well-known to be the smallest dense subtopos of $\mathcal{E}$ so Lemma 4.1 implies the existence of a subtopos inclusion $\mathcal{E}_{\neg\neg} \to \mathcal{S}$ factoring the inclusion $\mathcal{E}_{\neg\neg} \to \mathcal{E}$ through $p^! : \mathcal{S} \to \mathcal{E}$.

4.3. Lemma. *For any topos $\mathcal{F}$, if $\mathcal{F}_j \to \mathcal{F}$ is a Boolean dense subtopos then $j = \neg\neg$.*

Proof. For general considerations about dense subtoposes, $(\mathcal{F}_j)_{\neg\neg} = \mathcal{F}_{\neg\neg}$ (see the proof of Lemma A4.5.21 in [3]). If $\mathcal{F}_j$ is Boolean then $\neg\neg = id$ so, in this case, $\mathcal{F}_j = (\mathcal{F}_j)_{\neg\neg} = \mathcal{F}_{\neg\neg}$ and hence, $j = \neg\neg$. ∎

We summarize the above using standard terminology.

4.4. Proposition. *Let $\mathcal{S}$ be Boolean and $p : \mathcal{E} \to \mathcal{S}$ be local and essential. Then the following are equivalent:*

1. *p is hyperconnected,*
2. *the Nullstellensatz holds,*
3. *the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ is dense,*
4. *the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ coincides with $\mathcal{E}_{\neg\neg} \to \mathcal{E}$.*

Proof. The first two items are equivalent by Lemma 3.1. The second and third items are equivalent by Lemmas 4.1 and 4.2. The last two items are equivalent by Lemma 4.3. ∎

We will use the next consequence.

4.5. Corollary. *If $\mathcal{S}$ is Boolean and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ coincides with $\mathcal{E}_{\neg\neg} \to \mathcal{E}$.*

5. Stably pre-cohesive geometric morphisms

Let $p : \mathcal{E} \to \mathcal{S}$ be a geometric morphism. Every object B in $\mathcal{S}$ determines a geometric morphism $p/B : \mathcal{E}/p^*B \to \mathcal{S}/B$. The inverse image $(p/B)^* : \mathcal{S}/B \to \mathcal{E}/p^*B$ is just p^* applied in the obvious way. The right adjoint $(p/B)_* : \mathcal{E}/p^*B \to \mathcal{S}/B$ sends $x : X \to p^*B$ to the left vertical map in the pullback

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & p_*X \arrow[d, "p_*x"] \\ B \arrow[r, "\alpha"'] & p_*(p^*B) \end{tikzcd}$$

where α is the unit of the adjunction $p^* \dashv p_*$.

5.1. Lemma. *If $p^*: \mathcal{S} \to \mathcal{E}$ is full and faithful then so is $(p/B)^*: \mathcal{S}/B \to \mathcal{E}/p^*B$ for every B in $\mathcal{S}$. Moreover, in this case the right adjoint $(p/B)_*: \mathcal{E}/p^*B \to \mathcal{S}/B$ sends $x: X \to p^*B$ to the composite*

$$p_*X \xrightarrow{p_*x} p_*(p^*B) \xrightarrow{\alpha^{-1}} B$$

as an object in $\mathcal{S}/B$.

Proof. It is easy to check that $(p/B)^*$ is full and faithful if p^* is. To confirm the expression of $(p/B)_*: \mathcal{E}/p^*B \to \mathcal{S}/B$ use the fact that α is iso because p^* is assumed full and faithful. ∎

In other words, if p is connected then so is p/B.

5.2. Lemma. *If $p^*: \mathcal{S} \to \mathcal{E}$ has a left adjoint then so does $(p/B)^*: \mathcal{S}/B \to \mathcal{E}/p^*B$ for each B in $\mathcal{S}$.*

Proof. Let τ be the counit of $p_! \dashv p^*$. The functor $(p/B)_!: \mathcal{E}/p^*B \to \mathcal{S}/B$ that sends the object $x: X \to p^*B$ in $\mathcal{E}/p^*B$ to the composite

$$p_!X \xrightarrow{p_!x} p_!(p^*B) \xrightarrow{\tau} B$$

is easily proved to be left adjoint to $(p/B)^*$. ∎

That is, if p is essential then so is p/B.

Assume for the moment that $p_*: \mathcal{E} \to \mathcal{S}$ has a fully faithful right adjoint, denoted by $p^!: \mathcal{S} \to \mathcal{E}$. Then we have the canonical natural transformation by $\phi: p^* \to p^!$.

5.3. Definition. The *codiscrete fibering* of a morphism $f: A \to B$ in $\mathcal{S}$ is the map $f_1: \Phi f \to p^*B$ in $\mathcal{E}$ that appears in the pullback below

$$\begin{tikzcd}
\Phi f \arrow[r, "f_1"] \arrow[d, "f_0"'] & p^*B \arrow[d, "\phi"] \\
p^!A \arrow[r, "p^!f"'] & p^!B
\end{tikzcd}$$

Notice that if the Nullstellensatz holds for p then $f_0: \Phi f \to p^!A$ is mono.

The subobject $\Phi f \to p^!A$ may be pictured as the result of removing from the codiscrete $p^!A$ all the cohesion except that binding each of the fibers of f.

The assignment that sends $f: A \to B$ in $\mathcal{S}/B$ to $f_1: \Phi f \to p^*B$ in $\mathcal{E}/p^*B$ extends to a functor $(p/B)^!: \mathcal{S}/B \to \mathcal{E}/p^*B$.

5.4. Lemma. *If $p_*: \mathcal{E} \to \mathcal{S}$ has a fully faithful right adjoint $p^!: \mathcal{S} \to \mathcal{E}$ then the functor $(p/B)^!: \mathcal{S}/B \to \mathcal{E}/p^*B$ is a fully faithful right adjoint to $(p/B)_*: \mathcal{E}/p^*B \to \mathcal{S}/B$ for each B in $\mathcal{S}$.*

Proof. We first prove that the functor $(p/B)^! : \mathcal{S}/B \to \mathcal{E}/p^*B$ defined above is right adjoint to $(p/B)_* : \mathcal{E}/p^*B \to \mathcal{S}/B$. It is enough to check that for every $x : X \to p^*B$ in $\mathcal{E}$, $f : A \to B$ in $\mathcal{S}$ and $a : p_*X \to A$ in $\mathcal{S}$, the square on the left below commutes (in $\mathcal{S}$)

$$\begin{tikzcd} p_*X \arrow[r, "a"] \arrow[d, "p_*x"'] & A \arrow[d, "f"] \\ p_*(p^*B) \arrow[r, "\alpha^{-1}"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "x"'] & p^!(p_*X) \arrow[r, "p^!a"] & p^!A \arrow[d, "p^!f"] \\ p^*B \arrow[rr, "\phi"'] & & p^!B \end{tikzcd}$$

if and only if the rectangle on the right above commutes (in $\mathcal{E}$). Assume first that the square on the left commutes. Apply $p^!$ to it and put a naturality square on the left as below

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "x"'] & p^!(p_*X) \arrow[r, "p^!a"] \arrow[d, "p^!(p_*x)"] & p^!A \arrow[d, "p^!f"] \\ p^*B \arrow[r, "\eta"'] & p^!(p_*(p^*B)) \arrow[r, "p^!\alpha^{-1}"'] & p^!B \end{tikzcd}$$

to obtain the rectangle. On the other hand, if we assume that the rectangle on the right above commutes then we can apply p_* to it, put a naturality square on its right as below

$$\begin{tikzcd} p_*X \arrow[r, "p_*\eta"] \arrow[d, "p_*x"'] & p_*(p^!(p_*X)) \arrow[r, "p_*(p^!a)"] & p_*(p^!A) \arrow[r, "\epsilon"] \arrow[d, "p_*(p^!f)"] & A \arrow[d, "f"] \\ p_*(p^*B) \arrow[rr, "p_*\phi"'] & & p_*(p^!B) \arrow[r, "\epsilon"'] & B \end{tikzcd}$$

and use Lemma 2.1 to conclude that the bottom composite equals α^{-1}. The top map composite clearly equals a.

It remains to show that $(p/A)^!$ is fully faithful, but this follows from Lemma 5.1. ∎

The fact that local geometric morphisms are stable under slicing surely has a slicker proof using the fact that local morphisms are those whose direct image has an indexed right adjoint, but the explicit definition of the right adjoints in terms of codiscrete fiberings will play a very important role here. In any case, let us summarize the above in standard terminology.

5.5. Lemma. *If $p : \mathcal{E} \to \mathcal{S}$ is local and essential then so is $p/B : \mathcal{E}/p^*B \to \mathcal{S}/B$ for any B in $\mathcal{S}$.*

Hyperconnectedness also behaves well under slicing. We formulate this in terms of the Nullstellensatz (in its θ-formulation).

5.6. Lemma. *Let $p : \mathcal{E} \to \mathcal{S}$ be connected essential (so that we have a string $p_! \dashv p^* \dashv p_*$ with fully faithful p^*). If the unit of $p_! \dashv p^*$ is epi then so is the unit of $(p/B)_! \dashv (p/B)^*$ for each B in $\mathcal{S}$.*

Proof. Lemmas 5.1 and 5.2 imply that $(p/B)^* : \mathcal{S}/B \to \mathcal{E}/p^*B$ is fully faithful and has a left adjoint $(p/B)_!$. In particular, if we let σ be the unit of $p_! \dashv p^*$ then the unit of $(p/B)_! \dashv (p/B)^*$ 'coincides' with σ if understood from x in $\mathcal{E}/p^*B$ as suggested in the diagram below

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "x"'] & p^*(p_! X) \arrow[d, "p^*(p_! x)"] \\ p^*B & p^*(p_!(p^*B)) \arrow[l, "p^*\tau"] \end{tikzcd}$$

for $x : X \to p^*B$ a map in $\mathcal{E}$. Hence, if σ is epi, so is the unit of $(p/B)_! \dashv (p/B)^*$. ∎

Combining Lemmas 5.5 and 5.6 we may conclude that if p is pre-cohesive then p/B is 'almost pre-cohesive'. The only thing that may be missing is finite-product preservation of $(p/B)_! : \mathcal{E}/p^*B \to \mathcal{S}/B$. We will return to this issue in Section 10. For the moment, it is not unreasonable to introduce the following.

5.7. Definition. A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ will be called *stably pre-cohesive* if the slice $p/B : \mathcal{E}/p^*B \to \mathcal{S}/B$ is pre-cohesive for every B in $\mathcal{S}$.

Of course, a stably pre-cohesive geometric morphism is pre-cohesive.

It is also relevant to stress that Sufficient Cohesion is stable under slicing, at least, in its 'connected Ω' formulation and under the assumption that $p_!$ preserves finite products.

5.8. Lemma. *Let $p : \mathcal{E} \to \mathcal{S}$ be connected essential and such that $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. If $p_!\Omega = 1$ then, for every B in $\mathcal{S}$, $(p/B)_!(\Omega_B) = 1$ where Ω_B is the subobject classifier of $\mathcal{E}/p^*B$.*

Proof. It is well known that Ω_B is the projection $\Omega \times p^*B \to p^*B$, where Ω is the subobject classifier of $\mathcal{E}$. If we apply $(p/B)_! : \mathcal{E}/p^*B \to \mathcal{S}/B$ then Ω_B is sent to the composite on the left below

$$p_!(\Omega \times p^*B) \xrightarrow{p_!\pi_1} p_!(p^*B) \xrightarrow{\tau} B \qquad\qquad p_!\Omega \times p_!(p^*B) \xrightarrow{\pi_1} p_!(p^*B) \xrightarrow{\tau} B$$

and, since $p_!$ preserves finite products, this is isomorphic (as an object of $\mathcal{S}/B$) to the composite on the right above. So, if $p_!\Omega = 1$, $(p/B)_!(\Omega_B)$ is isomorphic to the identity on B. That is, $(p/B)_!(\Omega_B) \cong 1$ in $\mathcal{S}/B$. ∎

Stronger hypotheses allow us to derive the next simpler statement.

5.9. Corollary. *Let $p : \mathcal{E} \to \mathcal{S}$ be stably pre-cohesive. If p is sufficiently cohesive then so is $p/B : \mathcal{E}/p^*B \to \mathcal{S}/B$ for every B in $\mathcal{S}$.*

On the other hand, we have to admit that we do not know an example of a pre-cohesive geometric morphism that is not stably so. In spite of this, we can address the question of characterizing the stably pre-cohesive geometric morphisms among the pre-cohesive ones. As suggested after Lemma 5.6 this must be related to product-preservation of the functors $(p/B)_!$. We will do this in Section 10 where we also relate the issue with molecularity/local-connectedness.

6. Connected Codiscreteness

Let $p : \mathcal{E} \to \mathcal{S}$ be a geometric morphism such that the adjunction $p^* \dashv p_*$ extends to a string of adjoints

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p_!"'] \arrow[d, shift left=4, leftarrow, "p^*"] \arrow[d, shift left=8, "p_*"] \arrow[d, shift left=12, leftarrow, "p^!"] \\ \mathcal{S} \end{tikzcd}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$. (We are not even assuming that $p^!$ is fully faithful.)

6.1. Definition. We say that p satisfies *Connected Codiscreteness (CC)* if for every S in $\mathcal{S}$, the unique map $p_!(p^! S) \to 1$ is mono.

We understand *CC* as a precise formulation of the intuition suggesting that codiscrete objects are connected or 'as connected as possible'.

During CT2014 Johnstone observed that over a De Morgan base, Sufficient Cohesion implies CC. The study of his proof leads to the following.

6.2. Lemma. *If $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products then, CC holds if and only if $p^! 2$ is connected.*

Proof. One direction is trivial; for the other assume that $p_!(p^! 2) = 1$ and let S be an object in $\mathcal{S}$. To prove that $p_!(p^! S) \to 1$ is mono, it is enough to show that the projections $p_!(p^! S) \times p_!(p^! S) \to p_!(p^! S)$ are equal. The projections $\pi_0, \pi_1 : S^2 = S \times S \to S$ in $\mathcal{S}$ determine a map $[\pi_0, \pi_1] : S^2 + S^2 \to S$ and, by distributivity, a map $h : 2 \times S^2 \to S$ such that the following diagram

$$\begin{tikzcd} 1 \times S^2 \arrow[r, "in_0 \times id"] \arrow[d, "\cong"'] & 2 \times S^2 \arrow[d, "h"'] & 1 \times S^2 \arrow[l, "in_1 \times id"'] \arrow[d, "\cong"] \\ S^2 \arrow[r, "\pi_0"'] & S & S^2 \arrow[l, "\pi_1"] \end{tikzcd}$$

commutes, where $in_0, in_1 : 1 \to 1 + 1 = 2$ are the coproduct injections and the vertical isos are the obvious projections. Since the rightmost adjoint $p^! : \mathcal{S} \to \mathcal{E}$ preserves products, there exists an $h' : (p^! 2) \times (p^! S) \times (p^! S) \to p^! S$ such that the following diagram

$$\begin{tikzcd} p^! 1 \times (p^! S)^2 \arrow[r, "(p^! in_0) \times id"] \arrow[d, "\cong"'] & p^! 2 \times (p^! S)^2 \arrow[d, "h'"'] & p^! 1 \times (p^! S)^2 \arrow[l, "(p^! in_1) \times id"'] \arrow[d, "\cong"] \\ (p^! S)^2 \arrow[r, "\pi_0"'] & p^! S & (p^! S)^2 \arrow[l, "\pi_1"] \end{tikzcd}$$

commutes. As the leftmost adjoint $p_! : \mathcal{E} \to \mathcal{S}$ also preserves finite products, there exists a map $H : p_!(p^!2) \times (p_!(p^!S))^2 \to p_!(p^!S)$ such that the next diagram

$$\begin{tikzcd}
p_!(p^!1) \times (p_!(p^!S))^2 \arrow[r, "p_!(p^!in_0)\times id"] \arrow[d, "\cong"'] & p_!(p^!2) \times (p_!(p^!S))^2 \arrow[d, "H"'] & p_!(p^!1) \times (p_!(p^!S))^2 \arrow[l, "p_!(p^!in_1)\times id"'] \arrow[d, "\cong"] \\
(p_!(p^!S))^2 \arrow[r, "\pi_0"'] & p_!(p^!S) & (p_!(p^!S))^2 \arrow[l, "\pi_1"]
\end{tikzcd}$$

commutes, but since $p_!(p^!2) = 1$, $p_!(p^!in_0) = p_!(p^!in_1) : 1 = p_!(p^!1) \to p_!(p^!2) = 1$ and so, $\pi_0 = \pi_1 : (p_!(p^!S))^2 \to p_!(p^!S)$. Hence, $(p_!(p^!S)) \to 1$ is mono. ∎

In general, CC seems stronger than Sufficient Cohesion.

6.3. LEMMA. *If $p : \mathcal{E} \to \mathcal{S}$ is an equivalence and satisfies CC then $\mathcal{S}$ is inconsistent.*

PROOF. The hypotheses imply that for every S in $\mathcal{S}$ the unique $S \to 1$ is mono. ∎

The similarity with Proposition 3 in [8] is explained by the following fact.

6.4. LEMMA. *If p^* is fully faithful and both the Nullstellensatz and CC hold, then for every object Y in $\mathcal{E}$ there is a mono $Y \to Z$ such that $p_!Z = 1$.*

PROOF. Let X in $\mathcal{E}$ and define $\overline{X}$ by the pushout on the left below

$$\begin{tikzcd}
p^*(p_*X) \arrow[r, "\phi"] \arrow[d, "\beta"'] & p^!(p_*X) \arrow[d, "\iota_1"] \\
X \arrow[r, "\iota_0"'] & \overline{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
p_!(p^*(p_*X)) \arrow[r, "p_!\phi"] \arrow[d, "p_!\beta"'] & p_!(p^!(p_*X)) \arrow[d, "p_!\iota_1"] \\
p_!X \arrow[r, "p_!\iota_0"'] & p_!\overline{X}
\end{tikzcd}$$

in $\mathcal{E}$. By the Nullstellensatz both β and ϕ are mono so ι_0 and ι_1 are mono. Applying the leftmost adjoint we obtain a pushout in $\mathcal{S}$ as on the right above. Since $\theta = (p_!\beta)(\tau_{p_*}^{-1})$, the Nullstellensatz implies that $p_!\iota_1$ is epi and CC implies that its domain is subterminal so $p_!\iota_1 : p_!(p^!(p_*X)) \to p_!\overline{X}$ is an iso. There is a clear geometric intuition: $\overline{X}$ is the result of gluing, along bare points, whatever cohesion binds X and the codiscrete cohesion binding $p^!(p_*X)$. Regardless of this intuitive explanation, we have proved that every X embeds in an object $\overline{X}$ such that $p_!\overline{X}$ is subterminal. To complete the proof notice that for any Y in $\mathcal{E}$ we have a mono $Y \to Y + 1 \to \overline{Y+1}$ and that $p_!(\overline{Y+1})$ is subterminal and has a point. ∎

Together with Proposition 3.3 we obtain the following.

6.5. COROLLARY. *If p is pre-cohesive then* CC *implies Sufficient Cohesion.*

We don't know if the converse holds in general. Direct calculation shows that the two conditions are equivalent over **Set** but the following is more elegant.

6.6. COROLLARY. [Johnstone 2014] *If $\mathcal{S}$ is De Morgan and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then Sufficient Cohesion is equivalent to* CC.

PROOF. Combine Proposition 3.4 and Lemma 6.2. ∎

7. Explicit Connected Codiscreteness

(The results in this section will not be used in the rest of the paper but they give a concrete idea of what is the value of $p_!(p^! S)$.) Let $p : \mathcal{E} \to \mathcal{S}$ be as in the beginning of Section 6 and recall that the *support* of an object X in a topos is the image $\mathbf{supp}(X) \to 1$ of the unique map $X \to 1$.

7.1. Definition. Say that $p : \mathcal{E} \to \mathcal{S}$ satisfies *Explicit Connected Codiscreteness (ECC)* if $p_!(p^! S) = \mathbf{supp}(S)$.

ECC is a strengthening of *CC*.

7.2. Lemma. *If ECC holds then CC holds and* $p^! 0 = 0$.

Proof. If $p_!(p^! S) = \mathbf{supp}(S)$ then $p_!(p^! S) \to 1$ is clearly mono. Also clearly, $\mathbf{supp}(0) = 0$; so *ECC* implies the existence of a map $p_!(p^! 0) \to \mathbf{supp}(0) = 0$ and by adjointness, a map $p^! 0 \to p^* 0 = 0$. ∎

Assume from now on that p is local and essential so that we have access to the natural transformations $\theta : p_* \to p_!$ and $\phi : p^* \to p^!$.

7.3. Lemma. *If the Nullstellensatz holds for p then, CC is equivalent to ECC.*

Proof. Assume that *CC* holds. Then we have commutative rectangle as below

$$
\begin{tikzcd}
S \arrow[r, "\tau^{-1}"] \arrow[d] & p_!(p^* S) \arrow[r, "p_!\phi"] & p_!(p^! S) \arrow[d] \\
\mathbf{supp}(S) \arrow[rr] \arrow[urr, dotted] & & 1
\end{tikzcd}
$$

and, since the left vertical map is epi (by definition) and the right vertical map is mono (by *CC*), there exists a unique diagonal map making the triangles commute. This diagonal map is clearly mono. To prove that it is an iso it is enough to show that the top horizontal map is epi, but this follows from Lemma 2.3. ∎

Combining the above with Proposition 4.4 we obtain the following.

7.4. Corollary. *If* $\mathcal{S}$ *is Boolean and* $p : \mathcal{E} \to \mathcal{S}$ *is connected essential then ECC holds if and only if both CC and the Nullstellensatz hold.*

8. Stable Connected Codiscreteness

In this section we let $p : \mathcal{E} \to \mathcal{S}$ be local and essential so that each $p/B : \mathcal{E}/p^* B \to \mathcal{S}/B$ is also local an essential (Lemma 5.5).

8.1. Definition. The geometric morphism p is said to satisfy *Stable Connected Codiscreteness (SCC)* if $p/B : \mathcal{E}/p^*B \to \mathcal{S}/B$ satisfies CC for each B in $\mathcal{S}$.

We stress that we are not assuming that p is pre-cohesive. We are only assuming that p^* and $p^!$ are fully faithful. We are not assuming that $p_!$ preserves finite products nor that the Nullstellensatz holds.

To characterize SCC in terms of p recall (Definition 5.3) that the codiscrete fibering of a map $f : A \to B$ in $\mathcal{S}$ is the map $f_1 : \Phi f \to p^*B$ in $\mathcal{E}$ that appears in the square below

$$\begin{tikzcd}
\Phi f \arrow[r, "f_1"] \arrow[d, "f_0"'] & p^*B \arrow[d, "\phi"] \\
p^!A \arrow[r, "p^!f"'] & p^!B
\end{tikzcd}$$

which is assumed to be a pullback in $\mathcal{E}$.

8.2. Lemma. *The following items are equivalent:*

1. *SCC holds,*
2. *for every $f : A \to B$ in $\mathcal{S}$ the transposition $p_!(\Phi f) \to B$ of $f_1 : \Phi f \to p^*B$ is mono in $\mathcal{S}$,*
3. *for every $f : A \to B$ in $\mathcal{S}$, $p_!f_1 : p_!(\Phi f) \to p_!(p^*B)$ is mono in $\mathcal{S}$.*

Proof. The composite functor on the left below

$$\mathcal{S}/B \xrightarrow{(p/B)^!} \mathcal{E}/p^*B \xrightarrow{(p/B)_!} \mathcal{S}/B \qquad\qquad p_!(\Phi f) \xrightarrow{p_!f_1} p_!(p^*B) \xrightarrow{\tau} B$$

sends $f : A \to B$ in $\mathcal{S}/B$ to the composite on the right above; so the relation with SCC is clear. Also, since τ is iso, the above is equivalent to $p_!f_1$ being mono. ∎

In other words, SCC holds if and only if $p_!$ sends codiscrete fiberings to monos. Intuitively, f_1 acts on points just as f does, but the fibres of f_1 are codiscrete/connected. Let us make this more precise.

8.3. Lemma. *The rectangle below*

$$\begin{tikzcd}
p_*(\Phi f) \arrow[r, "p_*f_0"] \arrow[d, "p_*f_1"'] & p_*(p^!A) \arrow[r, "\epsilon"] \arrow[d, "p_*(p^!f)"] & A \arrow[d, "f"] \\
p_*(p^*B) \arrow[r, "p_*\phi"'] & p_*(p^!B) \arrow[r, "\epsilon"'] & B
\end{tikzcd}$$

is a pullback and the horizontal maps are isos.

Proof. The left square is a pullback because p_* is a right adjoint, the right square is a pullback because ϵ is an iso and, finally, the bottom composite equals $\alpha^{-1} : p_*(p^*B) \to B$ by Lemma 2.1. ∎

That is, f and $p_* f_1$ are isomorphic as objects of the category $\mathcal{S}^{\rightarrow}$ of maps in $\mathcal{S}$.

The assignment $X \mapsto \theta_X$ extends to a functor $\Theta : \mathcal{E} \to \mathcal{S}^{\rightarrow}$ and allows an alternative view of SCC.

8.4. Lemma. *The following diagram*

$$\begin{tikzcd}
p_*(\Phi f) \arrow[r, "p_* f_0"] \arrow[d, "\theta"'] & p_*(p^! A) \arrow[r, "\epsilon"] & A \arrow[d, "f"] \\
p_!(\Phi f) \arrow[r, "p_! f_1"'] & p_!(p^* B) \arrow[r, "\tau"'] & B
\end{tikzcd}$$

commutes and, as a map in $\mathcal{S}^{\rightarrow}$ from $\Theta(\Phi f)$ to f, it is universal from Θ to f.

Proof. Contemplate the following diagram

$$\begin{tikzcd}
p_*(\Phi f) \arrow[rr, "p_* f_0"] \arrow[dr, "p_* f_1"] \arrow[dd, "\theta"'] & & p_*(p^! A) \arrow[r, "\epsilon"] \arrow[d, phantom, "1"] & A \arrow[dd, "f"] \\
& p_*(p^* B) \arrow[r, "p_* \phi"'] \arrow[d, "\theta"] \arrow[dr, phantom, "2"] & p_*(p^! B) \arrow[dr, "\epsilon"] & \\
p_!(\Phi f) \arrow[r, "p_! f_1"'] & p_!(p^* B) \arrow[rr, "\tau"'] & & B
\end{tikzcd}$$

and notice that inner diagram 1 commutes by Lemma 8.3, that inner 2 commutes by Lemma 2.1 and that the remaining inner diagram commutes by naturality. Let us denote the resulting map in $\mathcal{S}^{\rightarrow}$ by $\kappa : \Theta(\Phi f) \to f$.

To prove universality from Θ to f let X in $\mathcal{E}$ and a map (a, b) from ΘX to f in $\mathcal{S}^{\rightarrow}$ as on the left below

$$\begin{tikzcd}
p_* X \arrow[r, "a"] \arrow[d, "\theta"'] & A \arrow[d, "f"] \\
p_! X \arrow[r, "b"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "\sigma"'] & p^!(p_* X) \arrow[r, "p^! a"] \arrow[d, "p^! \theta"'] & p^! A \arrow[d, "p^! f"] \\
p^*(p_! X) \arrow[r, "\phi"] \arrow[dr, "p^* b"'] & p^!(p_! X) \arrow[r, "p^! b"] & p^! B \\
& p^* B \arrow[ur, "\phi"'] &
\end{tikzcd}$$

apply $p^!$ to it, pre-compose with η and calculate as on the right above using Lemma 2.1. The pullback defining Φf implies the existence of a unique map $c : X \to \Phi f$ such that the following diagram

$$\begin{tikzcd}
p^!(p_* X) \arrow[d, "p^! a"'] & X \arrow[l, "\eta"'] \arrow[r, "\sigma"] \arrow[d, dotted, "c"] & p^*(p_! X) \arrow[d, "p^* b"] \\
p^! A & \Phi f \arrow[l, "f_0"] \arrow[r, "f_1"'] & p^* B
\end{tikzcd}$$

commutes. It is easy to check that c is the unique map such that $\kappa(\Theta c) = (a, b) : \Theta X \to f$. ∎

As usual, Lemma 8.4 determines a right adjoint $\Phi : \mathcal{S}^{\to} \to \mathcal{E}$ with $\kappa_f : \Theta(\Phi f) \to f$ as counit.

8.5. Proposition. *SCC holds for p if and only if the counit of $\Theta \dashv \Phi : \mathcal{S}^{\to} \to \mathcal{E}$ is mono.*

Proof. The counit $\kappa_f : \Theta(\Phi f) \to f$ consists of the two maps displayed horizontally in the statement of Lemma 8.4. The top horizontal map is an iso by Lemma 8.3 so, κ_f is mono if and only if the bottom map $\tau(p_! f_1) : p_!(\Phi f) \to B$, which is the transposition of f_1, is mono. ∎

The next result exploits the fact that epic monics are isos in toposes.

8.6. Lemma. *If SCC holds then for every epi $f : A \to B$ in $\mathcal{S}$, $\kappa_f : \Theta(\Phi f) \to f$ is an iso in $\mathcal{S}^{\to}$.*

Proof. It is clear from Lemma 8.4 that for every epimorphism $f : A \to B$ in $\mathcal{S}$ the counit $\kappa_f : \Theta(\Phi f) \to f$ is epi in $\mathcal{S}^{\to}$. ∎

Loosely speaking, SCC implies that every epi in $\mathcal{S}$ is a θ. We stress that the previous result holds without assuming the Nullstellensatz.

Again, as in the comment before Corollary 6.6, we don't know if *CC* is equivalent to *SCC*; but, as observed by Johnstone, the relations are clear over a De Morgan base, essentially because the class of De Morgan toposes is closed under slicing.

8.7. Corollary. [Johnstone 2014] *If $\mathcal{S}$ is De Morgan and $p : \mathcal{E} \to \mathcal{S}$ is stably pre-cohesive then: Sufficient Cohesion,* CC *and* SCC *are all equivalent.*

Proof. Of course, *SCC* implies *CC*, and the latter is equivalent to Sufficient Cohesion by Corollary 6.6. So assume that p is sufficiently cohesive. Then, for any object B in $\mathcal{S}$, the pre-cohesive $p/B : \mathcal{E}/p^*B \to \mathcal{S}/B$ is sufficiently cohesive by Lemma 5.8. Moreover, $\mathcal{S}/B$ is De Morgan, so *CC* must hold (for p/B) by Corollary 6.6 again. ∎

9. Epi-continuity and Internal Choice

Let $p : \mathcal{E} \to \mathcal{S}$ be a geometric morphism such that the adjunction $p^* \dashv p_*$ extends to a string of adjoints

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift left=2, "p_*"] \\ \mathcal{S} \arrow[u, shift right=2, "p^*"'] \arrow[u, shift left=6, "p^!"] \end{tikzcd}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$. Assume also that p^* is fully faithful and that $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products, so that we have the canonical natural transformation $p_!(X^{p^*S}) \to (p_! X)^S$ whose transposition is the map

$$p_!(X^{p^*S}) \times S \xrightarrow{id \times \tau^{-1}} p_!(X^{p^*S}) \times p_!(p^*S) \xrightarrow{\cong} p_!(X^{p^*S} \times p^*S) \xrightarrow{ev} p_! X$$

in $\mathcal{S}$.

Since $p^* : \mathcal{S} \to \mathcal{E}$ preserves finite products the canonical $p_*(X^{p^*S}) \to (p_*X)^S$ is an isomorphism.

9.1. Lemma. *For any X in $\mathcal{E}$ and S in $\mathcal{S}$ the following diagram*

$$\begin{tikzcd}
p_*(X^{p^*S}) \arrow[r] \arrow[d, "\theta"'] & (p_*X)^S \arrow[d, "\theta^S"] \\
p_!(X^{p^*S}) \arrow[r] & (p_!X)^S
\end{tikzcd}$$

commutes. Hence, if the Nullstellensatz holds, $\theta^S : (p_*X)^S \to (p_!X)^S$ is epi if and only if the canonical $p_!(X^{p^*S}) \to (p_!X)^S$ is epi.

Proof. Transposing and calculating one ends up reducing commutativity of the square in the statement to commutativity of the following rectangle

$$\begin{tikzcd}
p_*(X^{p^*S}) \times S \arrow[r, "id\times\alpha"] \arrow[d, "\theta\times id"'] & p_*(X^{p^*S}) \times p_*(p^*S) \arrow[r, "\cong"] \arrow[d, "\theta\times\theta"] & p_*(X^{p^*S} \times p^*S) \arrow[d, "\theta"] \\
p_!(X^{p^*S}) \times S \arrow[r, "id\times\tau^{-1}"'] & p_!(X^{p^*S}) \times p_!(p^*S) \arrow[r, "\cong"'] & p_!(X^{p^*S} \times p^*S)
\end{tikzcd}$$

but the right square commutes by naturality of θ and left square commutes because $\theta\alpha = \tau^{-1}$ by Lemma 2.1. ∎

The next corollary, formulated using epi-continuity (Definition 2.8), will be more immediately applicable.

9.2. Lemma. *If $\mathcal{E} \to \mathcal{S}$ is pre-cohesive then, p satisfies epi-continuity if and only if the map $\theta^S : (p_*X)^S \to (p_!X)^S$ is epi for every X in $\mathcal{E}$ and S in $\mathcal{S}$.*

We can now prove the main result.

9.3. Theorem. *If $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive and SCC holds then, $p : \mathcal{E} \to \mathcal{S}$ satisfies epi-continuity if and only if $\mathcal{S}$ satisfies IAC. Also, in this case, the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ coincides with $\mathcal{E}_{\neg\neg} \to \mathcal{E}$.*

Proof. Let $f : A \to B$ be epi in $\mathcal{S}$. Lemma 8.6 implies that $\kappa_f : \Theta(\Phi f) \to f$ is an iso in $\mathcal{S}^{\to}$. Hence, IAC holds in $\mathcal{S}$ if and only if $\theta^S : (p_*X)^S \to (p_!X)^S$ is epi for every X in $\mathcal{E}$ and S in $\mathcal{S}$. So the first part of the result follows by Lemma 9.2. Finally, since IAC implies that $\mathcal{S}$ is Boolean by Diaconescu's Theorem (see Remark D4.5.8 in [3]), the second part of the statement holds by Corollary 4.5. ∎

We immediately obtain the following result stated in terms of the original concepts introduced in [8].

9.4. COROLLARY. *If $p : \mathcal{E} \to \mathcal{S}$ is cohesive and SCC holds then $\mathcal{S}$ satisfies IAC and the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ coincides with $\mathcal{E}_{\neg\neg} \to \mathcal{E}$.*

One interpretation of the development above involves the conclusion that the real contrast between *Mengen* and *Kardinalen* emerges from the case of a topos whose double-negation part has additional remarkable properties. Let us summarize, in this spirit, some of our results.

9.5. COROLLARY. *Let $\mathcal{E}$ be a topos such that the adjunction $p_* \dashv p^! : \mathcal{E}_{\neg\neg} \to \mathcal{E}$ underlying the inclusion of the double-negation subtopos extends to a string of adjoints*

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift left=2, "p_*"] \\ \mathcal{E}_{\neg\neg} \arrow[u, shift right=2, "p^*"'] \arrow[u, shift left=6, "p^!"] \end{tikzcd}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$, and such that the leftmost adjoint $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. Then the Nullstellensatz holds so that $p : \mathcal{E} \to \mathcal{E}_{\neg\neg}$ is pre-cohesive. Moreover,

1. *(Sufficient Cohesion) The following are equivalent:*
   - *(a) The pre-cohesive $p : \mathcal{E} \to \mathcal{E}_{\neg\neg}$ is sufficiently cohesive.*
   - *(b) The codiscrete $p^!2$ is connected (i.e. $p_!(p^!2) = 1$).*
   - *(c) CC holds for p.*
   - *(d) ECC holds for p.*
2. *If SCC holds for p then, p satisfies epi-continuity if and only if IAC holds in $\mathcal{E}_{\neg\neg}$.*

PROOF. For $p : \mathcal{E} \to \mathcal{E}_{\neg\neg}$ as in the statement, the Nullstellensatz holds by Lemma 4.2. Concerning the items under the heading 'Sufficient Cohesion', notice that the first two are equivalent by Proposition 3.4. The second and third are equivalent by Lemma 6.2. The third and fourth items are equivalent by Lemma 7.3. Finally, if SCC holds, Theorem 9.3 implies that: p satisfies epi-continuity if and only if IAC holds in $\mathcal{E}_{\neg\neg}$. ∎

We stress that, by Corollary 9.4, for any cohesive topos $p : \mathcal{E} \to \mathcal{S}$ satisfying SCC, the domain $\mathcal{E}$ satisfies the hypotheses of Corollary 9.5 and, moreover, $\mathcal{S} = \mathcal{E}_{\neg\neg}$. Similarly, by Corollary 4.5, for any pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ with Boolean $\mathcal{S}$, $\mathcal{E}$ satisfies the hypotheses of Corollary 9.5 and, moreover, $\mathcal{S} = \mathcal{E}_{\neg\neg}$.

In particular, all the Grothendieck pre-cohesive toposes over **Set** characterized in [4], the examples over atomic toposes discussed in [10] and the cohesive examples over **Set** built in [9] satisfy the hypotheses of Corollary 9.5.

10. Molecularity/local-connectedness

Let $p : \mathcal{E} \to \mathcal{S}$ be an essential geometric morphism. The adjunction $p_! \dashv p^*$ need not be indexed and it is then natural to consider those $p : \mathcal{E} \to \mathcal{S}$ such that p^* has an $\mathcal{S}$-indexed left adjoint. These geometric morphisms are called *locally connected* or *molecular* [1].

10.1. Lemma. *The essential geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is molecular if and only if the following condition is satisfied: if the square on the left below is a pullback*

$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & p^*A \arrow[d, "p^*f"] \\ Y \arrow[r, "y"'] & p^*B \end{tikzcd} \qquad \begin{tikzcd} p_!X \arrow[r, "p_!\pi_1"] \arrow[d, "p_!\pi_0"'] & p_!(p^*A) \arrow[r, "\tau"] \arrow[d, "p_!(p^*f)"] & A \arrow[d, "f"] \\ p_!Y \arrow[r, "p_!y"'] & p_!(p^*B) \arrow[r, "\tau"'] & B \end{tikzcd}$$

then so is the rectangle the right. Hence, if p is moreover connected, then p is molecular if and only if $p_! : \mathcal{E} \to \mathcal{S}$ preserves the pullbacks on the left above.

Proof. The first statement is just the discussion between Theorems 5 and 6 in [1]. If p is also connected then the right square of the rectangle on the right above is a pullback. ∎

The next result is borrowed from Section 3.7 in [2].

10.2. Lemma. *If $L \dashv R : \mathcal{S} \to \mathcal{E}$ is a reflection with unit $\sigma : Id \to RL$ then, $L : \mathcal{E} \to \mathcal{S}$ preserves pullbacks of the form depicted on the left below*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "\sigma"'] & R(LX) \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & RA \end{tikzcd}$$

if and only if it preserves pullbacks as on the right above.

A reflection $L \dashv R : \mathcal{S} \to \mathcal{E}$ satisfying the above conditions is said to have *stable units.*

10.3. Proposition. *If $p : \mathcal{E} \to \mathcal{S}$ is connected essential then the following are equivalent:*

1. *p is molecular and $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products,*
2. *$p_! : \mathcal{E} \to \mathcal{S}$ sends pullbacks*

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & Y \arrow[d, "y"] \\ X \arrow[r, "x"'] & p^*B \end{tikzcd}$$

   in $\mathcal{E}$ to pullbacks in $\mathcal{S}$ (i.e. $p_! \dashv p^*$ has stable units),
3. *$(p/B)_! : \mathcal{E}/p^*B \to \mathcal{S}/B$ preserves finite products for every B in $\mathcal{S}$.*

PROOF. Bearing in mind the description of $(p/B)_! : \mathcal{E}/p^*B \to \mathcal{S}/B$ given in Lemma 5.2 it is clear that the last two items are equivalent. To prove that the third item implies the first notice that $(p/B)^*$ is fully faithful by Lemma 5.1. Since its left adjoint preserves finite products by hypothesis then $(p/B)^*$ is cartesian closed by generalities about adjunctions between cartesian closed categories (Corollary A1.5.9 in [3]). Cartesian closure of these functors is equivalent to p being molecular (see Proposition C3.3.1 in [3]).

Finally, assume that p is molecular and that $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. If we consider a pullback diagram as in the second item then the diagram below

$$
\begin{tikzcd}
P \arrow[r] \arrow[d, "\langle \pi_0, \pi_1 \rangle"'] & p^*B \arrow[r, "id"] \arrow[d, "\Delta"] & p^*B \arrow[d, "p^*\Delta"] \\
X \times Y \arrow[r, "x \times y"'] & p^*B \times p^*B \arrow[r, "\cong"'] & p^*(B \times B)
\end{tikzcd}
$$

is also a pullback, where the right-bottom iso is the inverse to the canonical map in the opposite direction. Lemma 10.1 implies that the following rectangle

$$
\begin{tikzcd}
p_!P \arrow[r] \arrow[d, "p_!\langle \pi_0, \pi_1 \rangle"'] & p_!(p^*B) \arrow[r, "id"] \arrow[d, "p_!\Delta"] & p_!(p^*B) \arrow[r, "\tau"] \arrow[d, "p_!(p^*\Delta)"] & B \arrow[d, "\Delta"] \\
p_!(X \times Y) \arrow[r, "p_!(x \times y)"'] & p_!(p^*B \times p^*B) \arrow[r, "\cong"'] & p_!(p^*(B \times B)) \arrow[r, "\tau"'] & B \times B
\end{tikzcd}
$$

is a pullback. The bottom composite equals

$$
p_!(X \times Y) \xrightarrow[\langle p_!\pi_0, p_!\pi_1 \rangle]{} p_!X \times p_!Y \xrightarrow[p_!x \times p_!y]{} p_!(p^*B) \times p_!(p^*B) \xrightarrow[\tau \times \tau]{} B \times B
$$

so the following rectangle

$$
\begin{tikzcd}
p_!P \arrow[r] \arrow[d, "\langle p_!\pi_0, p_!\pi_1 \rangle"'] & p_!(p^*B) \arrow[r, "\tau"] & B \arrow[d, "\Delta"] \\
p_!X \times p_!Y \arrow[r, "p_!x \times p_!y"'] & p_!(p^*B) \times p_!(p^*B) \arrow[r, "\tau \times \tau"'] & B \times B
\end{tikzcd}
$$

is also a pullback, which means that $p_!$ preserves the prescribed pullback. ∎

Notice that the second item is a strengthening of the condition that $p_! : \mathcal{E} \to \mathcal{S}$ preserves binary products. Roughly speaking, the 'pieces' functor not only preserves binary products but pullbacks over any discrete base.

Geometric morphisms p satisfying the first item of Proposition 10.3 are called *stably locally connected* in [4]. Johnstone informed us that the terminology was chosen by analogy with 'stably locally compact'. Notice the similar conditions 'compact subsets are stable under finite intersection' and 'connected objects are stable under finite product'.

Molecularity and pre-cohesiveness may be related as follows.

10.4. Corollary. *If $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then the following are equivalent:*

1. *p is stably pre-cohesive,*
2. *p is molecular,*
3. *$p_! : \mathcal{E} \to \mathcal{S}$ sends pullbacks*

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & Y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[x]{} & p^*B \end{array}$$

in $\mathcal{E}$ to pullbacks in $\mathcal{S}$.

As mentioned before, all the examples of pre-cohesive geometric morphisms we know of are stably so. So, to clarify the matter fully, it seems necessary to either find a pre-cohesive example that is not stably pre-cohesive, or prove that pre-cohesive geometric morphisms are molecular. Over **Set** the distinction clearly vanishes.

10.5. Corollary. *If $p : \mathcal{E} \to$ **Set** is pre-cohesive then it is stably pre-cohesive.*

Proof. Every essential $p : \mathcal{E} \to$ **Set** is molecular. ∎

Corollaries 8.7 and 10.5 together imply the following.

10.6. Corollary. *If $p : \mathcal{E} \to$ **Set** is pre-cohesive then: Sufficient Cohesion,* CC *and* SCC *are all equivalent.*

Allow us an informal comment: while the condition of having an $\mathcal{S}$-indexed left adjoint is technically natural, its manifestation in localic and (pre-)cohesive examples seems to be of different nature. On the one hand, for pre-cohesive geometric morphisms, molecularity is equivalent to the functor $p_!$ preserving certain pullbacks (Corollary 10.4). Moreover, in sufficiently cohesive cases, $p_!$ does not preserve equalizers. On the other hand, if a molecular localic $p : \mathcal{E} \to \mathcal{S}$ is such that $p_!$ preserves finite products then $p_!$ preserves finite limits. (See the paragraph before Axiom 2 in [7] or Lemma 1.1 in [4].)

11. Atomicity

There is another feature of categories of non-cohesion that we can tentatively address. Atomicity of a topos $\mathcal{S}$ may be partly expressed by saying that the subobject lattices have the property that inclusion can be tested by irreducibles. So one is led to formulate the conjecture that this form of atomicity holds for $\mathcal{S}$ if there is a sufficiently cohesive and cohesive $p : \mathcal{E} \to \mathcal{S}$. We don't know if this is true but we can prove a related result if we introduce into the analysis a further ingredient that appears in practice, namely a lower base $\mathcal{B}$ over which p is defined. For example, if $\mathcal{B}$ is **Set** and $\mathcal{E}$ is a presheaf topos. In particular, this is the case of the pre-cohesive topos associated to the real field discussed in [10]. It is pre-cohesive over the Galois topos and molecular over **Set**; but it is not pre-cohesive over **Set**.

With this extra ingredient $\mathcal{B}$ we will show that, under hypotheses similar to those in Theorem 9.3, $\mathcal{S}$ is atomic over $\mathcal{B}$. (Recall that a geometric morphism is *atomic* if its inverse image is logical. Recall also that if $\mathcal{S} \to \mathbf{Set}$ is atomic then the subobject lattices of $\mathcal{S}$ are atomic Boolean algebras. Hence, in this case, $\mathcal{S}$ satisfies the 'partial' atomicity formulated in the first paragraph.)

Let us assume then that the following diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "p"] \arrow[dr] & \mathcal{S} \arrow[d] \\ & \mathcal{B} \end{tikzcd}$$

commutes in the category of toposes and geometric morphisms.

11.1. Lemma. *If the composite $\mathcal{E} \to \mathcal{B}$ is molecular, $p : \mathcal{E} \to \mathcal{S}$ is connected, and $\mathcal{S}$ and $\mathcal{B}$ are Boolean, then $\mathcal{S} \to \mathcal{B}$ is atomic.*

Proof. By the 3-out-of-2 property of molecular geometric morphisms (C3.3.2 in [3]), $\mathcal{S} \to \mathcal{B}$ is molecular and since both $\mathcal{S}$ and $\mathcal{B}$ are Boolean, the inverse image preserves the subobject classifier. Hence $\mathcal{S} \to \mathcal{B}$ is atomic. ∎

We already know that Booleaness of $\mathcal{S}$ follows from epi-continuity of p.

11.2. Proposition. *Let $p : \mathcal{E} \to \mathcal{S}$ and $\mathcal{S} \to \mathcal{B}$ be geometric morphisms such that the composite $\mathcal{E} \to \mathcal{B}$ is molecular and $\mathcal{B}$ is Boolean. If p is pre-cohesive and satisfies epi-continuity and SCC then $\mathcal{S} \to \mathcal{B}$ is atomic.*

Proof. If p is pre-cohesive then it is connected. Epi-continuity and SCC imply that $\mathcal{S}$ is Boolean, so we can apply Lemma 11.1. ∎

Over $\mathcal{B} = \mathbf{Set}$ we get a shorter statement.

11.3. Corollary. *Let $\mathcal{E}$ be a molecular Grothendieck topos. If $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive and satisfies epi-continuity and SCC then $\mathcal{S}$ is atomic over* **Set**.

Proof. Since $p^! \dashv p_* : \mathcal{S} \to \mathcal{E}$ is a subtopos then $\mathcal{S}$ is a Grothendieck topos; so there is a geometric morphism $\mathcal{S} \to \mathbf{Set}$ and Proposition 11.2 is applicable. ∎

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