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PROXIMITY BIFRAMES AND COMPACTIFICATIONS OF COMPLETELY REGULAR ORDERED SPACES

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Abstract. We generalize the concept of a strong inclusion on a biframe [Sch93] to that of a proximity on a biframe, which is related to the concept of a strong bi-inclusion on a frame introduced in [PP12b]. We also generalize the concept of a bi-compactification of a biframe [Sch93] to that of a compactification of a biframe, and prove that the poset of compactifications of a biframe L is isomorphic to the poset of proximities on L. As a corollary, we obtain Schauerte's characterization of bi-compactifications of a biframe [Sch93]. In the spatial case this yields Blatter and Seever's characterization of compactifications of completely regular ordered spaces [BS76] and a characterization of bi-compactifications of completely regular bispaces.

1. Introduction

The theory of compactifications of completely regular spaces was generalized in three directions. Nachbin [Nac65] introduced completely regular ordered spaces and Blatter [Bla75] and Blatter and Seever [BS76] developed the theory of compactifications for them. Salbany [Sal74] developed the theory of compactifications for completely regular bispaces, and Smyth [Smy92] developed the theory of stable compactifications for T_0-spaces. There are close connections between these three theories. Let Nach be the category of Nachbin spaces (compact order-Hausdorff spaces) and continuous order-preserving maps, let KRBSp be the category of compact regular bispaces and bicontinuous maps, and let StKSp be the category of stably compact spaces and proper maps. It is well known that the three categories are isomorphic (see, e.g., [GHKLMS03]). In fact, an isomorphism between Nach and KRBSp is obtained by sending a Nachbin space $(X, \tau, \leq)$ to the compact regular bispace (X, τ_u, τ_d), where τ_u is the topology of open upsets and τ_d is the topology of open downsets; and an isomorphism between KRBSp and StKSp is obtained by sending a compact regular bispace (X, τ, σ) to the stably compact space (X, τ).

The isomorphism between Nach and KRBSp does not extend to the completely regular setting. Indeed, the concept of a completely regular ordered space is more general than that of a completely regular bispace. In fact, completely regular bispaces correspond to strictly completely regular ordered spaces [Law91], and there exist completely regular ordered spaces that are not strictly completely regular [Kun90]. On the other hand, for each T_0-space (X, τ), there is a companion topology τ^* of τ such that (X, τ, τ^*)

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is a completely regular bispace (see, e.g., [Sal74]). This indicates that the theory of compactifications of completely regular ordered spaces contains as particular cases both the theory of compactifications of completely regular bispaces and the theory of stable compactifications of T_0-spaces.

In [Ban90] Banaschewski generalized the theory of compactifications of completely regular spaces to the pointfree setting. The theory of compactifications of completely regular bispaces was generalized to the pointfree setting in [Sch93], and [BH14] generalized the theory of stable compactifications of T_0-spaces to the pointfree setting. The aim of this paper is to do the same with the theory of compactifications of completely regular ordered spaces.

We generalize the concept of a compactification of a biframe introduced by Schauerte [Sch93]. In order to distinguish the two concepts, we refer to Schauerte's compactifications as bi-compactifications since they generalize to the pointfree setting the concept of a compactification of a bispace. We also generalize the concept of a strong inclusion on a biframe introduced in [Sch93] to that of a proximity on a biframe, which is related to the concept of a strong bi-inclusion on a frame introduced by Picado and Pultr [PP12b].

We prove that the poset of compactifications of a biframe L is isomorphic to the poset of proximities on L. As a corollary, we obtain Schauerte's result [Sch93] that the poset of bi-compactifications of a biframe L is isomorphic to the poset of strong inclusions on L. Restricting to the spatial case yields Blatter and Seever's characterization of compactifications of completely regular ordered spaces [BS76] and a characterization of compactifications of completely regular bispaces.

The paper is organized as follows. Section 2 consists of the necessary background. In Section 3 we recall Schauerte's description [Sch93] of bi-compactifications of biframes by means of strong inclusions. In Section 4 we generalize the concept of a bi-compactification to that of a compactification, the concept of a strong inclusion to that of a proximity on a biframe, and show that each compactification gives rise to a proximity. In Section 5 we introduce the concept of a round ideal, prove that the round ideals of a proximity biframe form a compactification of the biframe, and that each compactification of a biframe L arises up to isomorphism as the biframe of round ideals of some proximity on L. This generalizes Schauerte's description of bi-compactifications. In Section 6 we show that our results of the previous section provide a pointfree characterization of compactifications of ordered topological spaces. This gives an alternate and pointfree proof of a result of Blatter and Seever [BS76]. Finally, in Section 7 we introduce completely regular and strictly completely regular biframes and prove that a biframe is completely regular iff it has a compactification, and that it is strictly completely regular iff it has a bi-compactification. The well-known results of Nachbin [Nac65] and Salbany [Sal74] follow.

2. Preliminaries

In this section we provide all the background needed to read this paper. More details can be found in [Nac65], [GHKLMS03], [Joh82], [PP12a].

Ordered spaces. An *ordered topological space*, or simply an *ordered space*, is a triple $(X,\tau,\leq)$, where X is a set, τ is a topology, and $\leq$ is a partial order on X. We call a subset U of X an *upper set* or an *upset* if $x \in U$ and $x \leq y$ imply $y \in U$. Downsets are defined dually. An ordered space $(X,\tau,\leq)$ is *order-Hausdorff* if from $x \nleq y$ it follows that there exist an upset neighborhood N of x and a downset neighborhood M of y such that $M \cap N = \varnothing$. Equivalently $(X,\tau,\leq)$ is order-Hausdorff if $\leq$ is closed in X^2. A *Nachbin space* is a compact order-Hausdorff space. A map $f : X \to Y$ between ordered spaces is *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$. Let Nach be the category of Nachbin spaces and continuous order-preserving maps.

Bispaces. A *bitopological space*, or a *bispace*, is a triple (X,τ_1,τ_2), where τ_1 and τ_2 are two topologies on a set X. We define the *patch topology* as $\tau = \tau_1 \vee \tau_2$. Following [Sal74], we call a bispace (X,τ_1,τ_2) *compact* if (X,τ) is compact, T_0 if (X,τ) is T_0, and *regular* if it is T_0 and for each $U \in \tau_i$, we have $U = \bigcup\{V \in \tau_i : \mathrm{cl}_k(V) \subseteq U\}$ $(i \neq k,\ i,k = 1,2)$, where $\mathrm{cl}_k(V)$ is the closure of V in the topology τ_k. A map $f : X \to Y$ between bispaces is *bicontinuous* if it is continuous with respect to both topologies. Let KRBSp be the category of compact regular bispaces and bicontinuous maps. The category KRBSp is isomorphic to the category Nach. Indeed, if $(X,\tau,\leq)$ is a Nachbin space, then the open upsets and open downsets form a compact regular bispace; conversely, if (X,τ_1,τ_2) is a compact regular bispace, then $(X,\tau,\leq)$ is a Nachbin space, where τ is the patch topology and $\leq$ is the specialization order of τ_1, which is the dual of the specialization order of τ_2 (for details see, e.g., [GHKLMS03]).

Stably compact spaces. A topological space X is *locally compact* if for each $x \in X$ and open neighborhood U of x, there exist an open neighborhood V of x and a compact set K with $V \subseteq K \subseteq U$. A subset A of X is *saturated* if it is an intersection of open sets, and A is *irreducible* if $A \subseteq B \cup C$ with B, C closed implies $A \subseteq B$ or $A \subseteq C$. The space X is *sober* provided each closed irreducible set is the closure of a unique point. Finally, X is *stably compact* if it is compact, locally compact, sober, and the intersection of two compact saturated sets is compact. A map $f : X \to Y$ between two stably compact spaces is *proper* if it is continuous and the inverse image of each compact saturated set is compact. Let StKSp be the category of stably compact spaces and proper maps. This category is also isomorphic to Nach and StKSp. Indeed, if (X,τ_1,τ_2) is a compact regular bispace, then both topologies are stably compact topologies. Conversely, if (X,τ) is a stably compact space, then (X,τ,τ^k) is a compact regular bispace, where τ^k is the *co-compact topology*, whose closed sets are the compact saturated sets (see, e.g., [GHKLMS03]).

$$\mathsf{Nach} \longleftrightarrow \mathsf{KRBSp} \longleftrightarrow \mathsf{StKSp}$$

Stably compact frames. The pointfree analogues of stably compact spaces are stably compact frames. We recall that a *frame* is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$, where $a \in L$ and $S \subseteq L$. For $a, b \in L$, we say that a is *way below* b and write $a \ll b$ if for any $T \subseteq L$ with $b \leq \bigvee T$ there is a finite subset $S \subseteq T$ with $a \leq \bigvee S$. We call L *compact* if $1 \ll 1$, *locally compact* if $a = \bigvee\{b \in L : b \ll a\}$

for all $a \in L$, and *stable* if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$. A frame L is *stably compact* if it is compact, locally compact, and stable.

A map $f : L \to M$ between two frames is a *frame homomorphism* if it preserves finite meets and arbitrary joins, and a frame homomorphism $f : L \to M$ is *proper* if $a \ll b$ implies $f(a) \ll f(b)$ for all $a, b \in L$. Let StKFrm be the category of stably compact frames and proper frame homomorphisms. Then StKFrm is dually equivalent to StKSp. If X is a stably compact space, then the frame $\Omega(X)$ of opens of X is a stably compact frame and if $f : X \to Y$ is a proper map, then $f^{-1} : \Omega(Y) \to \Omega(X)$ is a proper frame homomorphism. This defines a contravariant functor $\Omega : \mathsf{StKSp} \to \mathsf{StKFrm}$.

If L is a frame, then a frame homomorphism from L to the two-element frame $\mathbf{2}$ is called a *point* of L. Let $pt(L)$ be the set of points of L, and for $a \in L$, let $\varphi(a) = \{p \in pt(L) : p(a) = 1\}$. Then $\{\varphi(a) : a \in L\}$ is a topology on $pt(L)$, and L is stably compact iff $pt(L)$ is stably compact. For a frame homomorphism $f : L \to M$, let $pt(f) : pt(M) \to pt(L)$ be given by $pt(f)(p) = p \circ f$. Then $pt(f)$ is continuous, and it is proper iff f is proper. This defines a contravariant functor $pt : \mathsf{StKFrm} \to \mathsf{StKSp}$, and the functors Ω, pt establish a dual equivalence of StKSp and StKFrm (for details see, e.g., [GHKLMS03, Joh82]). This dual equivalence is the restriction of the well-known dual equivalence between the categories SFrm of spatial frames and Sob of sober spaces (see, e.g., [Joh82, PP12a]), where a frame is spatial if $a \nleq b$ implies $p(a) = 1$ and $p(b) = 0$ for some $p \in pt(L)$. However, StKFrm is not a full subcategory of SFrm because not every frame homomorphism is proper, and similarly, StKSp is not a full subcategory of Sob.

Biframes. The pointfree analogues of bispaces are biframes introduced by Banaschewski, Brümmer, and Hardie [BBH83]. We recall that a *biframe* is a triple $L = (L_0, L_1, L_2)$, where L_0 is a frame, L_1 and L_2 are subframes of L_0, and L_0 is generated by $L_1 \cup L_2$ (that is, each $a \in L$ is a join of elements of the form $a_1 \wedge a_2$ with $a_i \in L_i$, $i = 1, 2$). A biframe L is *compact* if L_0 is compact. For $a \in L_0$ and $i = 1, 2$, set $\neg_i a = \bigvee\{b \in L_i : a \wedge b = 0\}$. For $a, b \in L_0$, define $a \prec_i b$ if $\neg_k a \vee b = 1$ ($i \neq k$, $i, k = 1, 2$). The pair $\triangleleft = (\prec_1, \prec_2)$ is called the *well inside* relation on L. We call L *regular* provided $a = \bigvee\{b \in L_i : b \prec_i a\}$ for each $a \in L_i$ and $i = 1, 2$.

A *biframe homomorphism* is a frame homomorphism $f : L_0 \to M_0$ such that $f(L_i) \subseteq M_i$ for $i = 1, 2$. We follow the terminology of [BBH83] and write this $f : L \to M$. Let KRBFrm be the category of compact regular biframes and biframe homomorphisms. Then KRBFrm is dually equivalent to KRBSp. The compact regular biframe associated with a compact regular bispace (X, τ_1, τ_2) is the biframe $\Omega(X) = (\tau, \tau_1, \tau_2)$, where $\tau = \tau_1 \vee \tau_2$ is the patch topology. If $f : X \to Y$ is bicontinuous, then f^{-1} is a biframe homomorphism. This defines a contravariant functor $\Omega : \mathsf{KRBSp} \to \mathsf{KRBFrm}$. The compact regular bispace associated with a compact regular biframe $L = (L_0, L_1, L_2)$ is the bispace $pt(L) = (pt(L_0), \tau_1, \tau_2)$, where $\tau_i = \{\varphi(a) : a \in L_i\}$, $i = 1, 2$. If $f : L_0 \to M_0$ is a biframe homomorphism, then $pt(f) : pt(M_0) \to pt(L_0)$ is a bicontinuous map. This defines a contravariant functor $pt : \mathsf{KRBFrm} \to \mathsf{KRBSp}$, and the functors Ω, pt establish a dual equivalence of KRBSp and KRBFrm. This dual equivalence is the restriction of the dual equivalence between the categories of spatial biframes and sober bispaces, where a biframe L is spatial if L_0 is a spatial frame, and a bispace (X, τ_1, τ_2) is sober if the patch topology τ is sober (for details see [BBH83]).

As a result, we obtain that KRBFrm is equivalent to StKFrm, and dually equivalent to StKSp and Nach. For a direct pointfree proof of the equivalence of KRBFrm and StKFrm see [BB88]. The functors establishing directly the dual equivalence of KRBFrm and Nach can be constructed as follows. If $(X, \tau, \leq)$ is a Nachbin space, then $\Omega(X) = (\tau, \tau_u, \tau_d)$ is a compact regular biframe, where τ is the frame of opens, τ_u is the frame of open upsets, and τ_d is the frame of open downsets. (Note that $\tau = \tau_u \vee \tau_d$, for $U \in \tau_u$ we have $U = \{V \in \tau_u : \uparrow\mathrm{cl}(V) \subseteq U\}$, and dually for $U \in \tau_d$.) If $f : X \to Y$ is continuous and order-preserving, then $f^{-1} : \Omega(Y) \to \Omega(X)$ is a biframe homomorphism. If $L = (L_0, L_1, L_2)$ is a compact regular biframe, then $(pt(L_0), \tau, \leq)$ is a Nachbin space, where $\tau = \{\varphi(a) : a \in L_0\}$, $\tau_1 = \{\varphi(a) : a \in L_1\}$, $\tau_2 = \{\varphi(a) : a \in L_2\}$, and $\leq$ is the specialization order of τ_1. (Note that τ is the patch topology of τ_1 and τ_2, and $\leq$ is the dual of the specialization order of τ_2.) If $f : L_0 \to M_0$ is a biframe homomorphism, then $pt(f) : pt(M_0) \to pt(L_0)$ is continuous and order-preserving. The correspondence of these categories can be depicted as follows.

$$\begin{tikzcd}
\mathsf{Nach} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \mathsf{KRBSp} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathsf{StKSp} \arrow[d, leftrightarrow] \\
& \mathsf{KRBFrm} \arrow[r, leftrightarrow] & \mathsf{StKFrm}
\end{tikzcd}$$

3. Compactifications and strong inclusions

We recall that a *compactification* of a topological space X is a pair (Y, e), where Y is a compact Hausdorff space and e is an embedding of X into Y such that $e(X)$ is dense in Y. Smirnov [Smi52] described the compactifications of X by means of proximities on X compatible with the topology on X.

Banaschewski [Ban90] generalized the concept of a compactification to the pointfree setting. Let L be a frame. For $a \in L$, the *pseudocomplement* of a is $\neg a = \bigvee\{b \in L : a \wedge b = 0\}$. For $a, b \in L$, we say that a is *well inside* b and write $a \prec b$ if $\neg a \vee b = 1$. A frame is *regular* provided $a = \bigvee\{b \in L : b \prec a\}$ for all $a \in L$. A frame homomorphism $f : M \to L$ is *dense* if $f(a) = 0$ implies $a = 0$. A *compactification* of a frame L is a pair (M, f) such that M is compact regular and $f : M \to L$ is an onto dense frame homomorphism. Banaschewski [Ban90] characterized the compactifications of L by means of strong inclusions on L. A *strong inclusion* on L is a binary relation $\triangleleft$ on L such that

(B1) $0 \triangleleft 0$ and $1 \triangleleft 1$.

(B2) If $a \triangleleft b$, then $a \prec b$.

(B3) If $a \leq b \triangleleft c \leq d$, then $a \triangleleft d$.

(B4) If $a, b \triangleleft c$, then $a \vee b \triangleleft c$.

(B5) If $a \triangleleft b, c$, then $a \triangleleft b \wedge c$.

(B6) If $a \lhd c$, then there is $b \in L$ with $a \lhd b \lhd c$.

(B7) If $a \lhd b$, then $\neg b \lhd \neg a$.

(B8) If $b \in L$, then $b = \bigvee\{a \in L : a \lhd b\}$.

Banaschewski [Ban90] proved that the poset of compactifications of a frame L is isomorphic to the poset of strong inclusions on L. Smirnov's characterization of compactifications of a space X follows as a corollary. Banaschewski's result was generalized in two directions.

In [BH14] compactifications of a frame L were generalized to stable compactifications of L and strong inclusions on L to proximities on L. For a frame homomorphism $f : M \to L$, let $r : L \to M$ be the right adjoint of f; that is, $r(a) = \bigvee\{x \in M : f(x) \le a\}$. A *stable compactification* of a frame L is a pair (M, f), where M is a stably compact frame and $f : M \to L$ is an onto frame homomorphism satisfying $a \ll b$ implies $r(f(a)) \ll b$. (Note that such an f is always dense.) A *proximity* on L is a binary relation $\lhd$ on L satisfying all the above axioms except (B2) and (B7). By [BH14], the poset of stable compactifications of a frame L is isomorphic to the poset of proximities on L. Both Banaschewski's characterization of compactifications of L as well as Smyth's characterization [Smy92] of stable compactifications of a T_0-space X follow as corollaries.

Schauerte [Sch93] generalized compactifications of a frame to bi-compactifications of a biframe. Let $L = (L_0, L_1, L_2)$ be a biframe. A *bi-compactification* of L is a pair (M, f), where $M = (M_0, M_1, M_2)$ is a compact regular biframe and $f : M_0 \to L_0$ is a dense biframe homomorphism such that $f(M_i) = L_i$ for $i = 1, 2$. (In particular, f is onto.) A *strong inclusion* on a biframe L is a pair $\lhd = (\lhd_1, \lhd_2)$ such that $\lhd_i$ is a binary relation on L_i satisfying for $i, k = 1, 2$ and $i \ne k$:

(S1) $0 \lhd_i 0$ and $1 \lhd_i 1$.

(S2) If $a \lhd_i b$, then $a \prec_i b$.

(S3) If $a \le b \lhd_i c \le d$, then $a \lhd_i d$.

(S4) If $a, b \lhd_i c$, then $a \vee b \lhd_i c$.

(S5) If $a \lhd_i b, c$, then $a \lhd_i b \wedge c$.

(S6) If $a \lhd_i c$, then there is $b \in L_i$ with $a \lhd_i b \lhd_i c$.

(S7) If $a \lhd_i b$, then $\neg_k b \lhd_k \neg_k a$.

(S8) If $b \in L_i$, then $b = \bigvee\{a \in L_i : a \lhd_i b\}$.

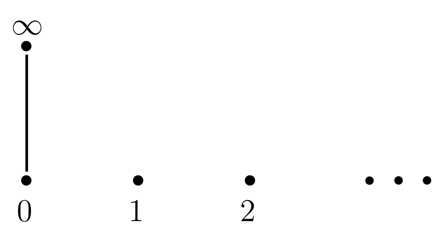
Schauerte [Sch93] proved that the poset of bi-compactifications of a biframe L is isomorphic to the poset of strong inclusions on L. As a corollary we obtain a characterization of bi-compactifications of a bispace X by means of strong inclusions on the biframe $\Omega(X)$, where a *bi-compactification* of a bispace (X,τ_1,τ_2) is a pair (Y,e) such that $Y=(Y,\pi_1,\pi_2)$ is a compact regular bispace and $e:X\to Y$ is a bispace embedding with $e(X)$ dense in the patch topology.

Neither [BH14] nor [Sch93] is directly applicable to characterize compactifications of an ordered space $(X,\tau,\le)$. We recall [Bla75, BS76] that a *compactification* of $(X,\tau,\le)$ is a pair (Y,e), where $Y=(Y,\pi,\le)$ is a Nachbin space and $e:X\to Y$ is an order-embedding such that $e(X)$ is dense in Y. As follows from [Nac65, Bla75, BS76], an ordered space $(X,\tau,\le)$ has a compactification iff it is a completely regular ordered space (see Section 7 for details). In particular, if τ_u is the topology of open upsets and τ_d is the topology of open downsets, then $\tau=\tau_u\vee\tau_d$. In addition, $(X,\tau,\le)$ is order-Hausdorff, and hence $\uparrow x,\downarrow x$ are closed for any $x\in X$. Ordered spaces satisfying this condition are called *order-T_1* (see, e.g., [McC68]) and ordered spaces satisfying $\tau=\tau_u\vee\tau_d$ are called *strongly order convex* (see, e.g., [Law91]). For an ordered space $(X,\tau,\le)$, it is clear that (X,τ_u,τ_d) is a bispace (and hence $\Omega(X)=(\tau,\tau_u,\tau_d)$ is a biframe) iff $(X,\tau,\le)$ is strongly order convex. In addition, the specialization order of τ_u is the dual of the specialization order of τ_d iff $(X,\tau,\le)$ is order-T_1 (see, e.g., [Law91, Sec. 7.3]). Because of this, we will be interested only in ordered spaces that are order-T_1 and strongly order convex. Therefore, we make the following convention.

3.1. Convention. *Throughout the paper, by an ordered space we mean an order-T_1 strongly order convex space.*

As was noted in [BH14, Ex. 3.13], a compactification of an ordered space $(X,\tau,\le)$ is not necessarily a bi-compactification of the bispace (X,τ_u,τ_d) or a stable compactification of the T_0-space (X,τ_u).

3.2. Example. Suppose X is the set of natural numbers, τ is the discrete topology on X, and $\le$ is the trivial order on X. We let τ_u be the topology of open upsets and τ_d the topology of open downsets of $(X,\tau,\le)$. Then $\tau_u=\tau_d=\tau$. Let (Y,π) be the one-point compactification of (X,τ), with order on $Y=X\cup\{\infty\}$ given by $0\le\infty$ as the only nontrivial inequality.



It is easy to see that $(Y,\pi,\le)$ is a Nachbin space and the inclusion $X\hookrightarrow Y$ is an order embedding of (X,τ) into a dense subspace of (Y,π). Therefore, $(Y,\pi,\le)$ is a compactification of $(X,\tau,\le)$.

Let π_u be the topology of open upsets and π_d the topology of open downsets of $(Y,\pi,\le)$. Then (Y,π_u,π_d) is a compact regular bispace and (Y,π_u) is a stably compact space.

However, since $\{0\} \in \tau_u$, but there is no $U \in \pi_u$ with $\{0\} = U \cap X$, we see that $X \hookrightarrow Y$ is not an embedding of (X, τ_u) into (Y, π_u). Thus, neither (Y, π_u, π_d) is a bi-compactification of (X, τ_u, τ_d) nor (Y, π_u) is a stable compactification of (X, τ_u).

3.3. Remark. Let (Y, e) be a compactification of the ordered space $(X, \tau, \le)$. Then each open set U of X has the form $e^{-1}(V)$ for some open set V of Y, each upset U of X has the form $e^{-1}(V)$ for some upset V of Y, and the same is true for downsets. However, as Example 3.2 shows, an open upset U of X may not have the form $e^{-1}(V)$ for some open upset V of Y. If in Example 3.2 we take the dual of $\le$, then we obtain an example of an ordered space, where not every open downset of X is of the form $e^{-1}(V)$ for some open downset V of Y. This distinguishes compactifications and bi-compactifications and is the main theme of this paper.

3.4. Remark. Let X be as in Example 3.2. If (Y, π) is the one-point compactification of X, then there are many closed orders $\le$ on Y that make $(Y, \pi, \le)$ a compactification of $(X, \tau, \le)$ (see, e.g., [BM11]). We note that if $\le$ is any such order for which ∞ is comparable to some $n \in X$, then an argument similar to the one given in Example 3.2 shows that (Y, π_u, π_d) is not a bi-compactification of (X, τ_u, τ_d).

On the other hand, as follows from [BH14, Sec. 3], each stable compactification of a T_0-space or a bi-compactification of a bispace can be viewed as a compactification of the corresponding ordered space. This indicates that the theory of compactifications of ordered spaces is more general than those of stable compactifications of T_0-spaces and bi-compactifications of bispaces. Our aim is to develop the pointfree version of the compactification theory for ordered spaces.

4. Compactifications of and proximities on biframes

In this section we introduce our two main concepts, that of a compactification of a biframe and a proximity on a biframe. The concept of a compactification of a biframe generalizes that of a bi-compactification, and the concept of a proximity that of a strong inclusion of [Sch93]. We recall that if (M, f) is a bi-compactification of a biframe L, then the restriction of $f : M_0 \to L_0$ to M_i is onto L_i for $i = 1, 2$. As follows from Example 3.2, this condition is too strong for our purposes.

4.1. Example. Let L be the biframe associated with the bispace (X, τ_u, τ_d) and M be the biframe associated with the bispace (Y, π_u, π_d) of Example 3.2; that is, $L = (L_0, L_1, L_2)$, where $L_0 = L_1 = L_2 = \tau$ and $M = (M_0, M_1, M_2)$, where $M_0 = \pi$, $M_1 = \pi_u$, and $M_2 = \pi_d$. Then M is a compact regular biframe. As $(X, \tau, \le)$ embeds in $(Y, \pi, \le)$, we see that the biframe homomorphism $f : M_0 \to L_0$, given by $f(U) = U \cap X$, is onto. It is dense because (X, τ) is a dense subspace of (Y, π). However, since (X, τ_u) does not embed into (Y, π_u), we see that the restriction of f to M_1 is not onto. Thus, (M, f) is not a bi-compactification of L.

On the other hand, the next example shows that we cannot simply drop the condition that $f(M_i) = L_i$ for $i = 1, 2$.

4.2. EXAMPLE. Let $X = \{x, y\}$ be the two-point set with the discrete topology τ and trivial order $\leq$. Then $(X, \tau, \leq)$ is an ordered space. Let $L = (L_0, L_1, L_2)$ be the corresponding biframe. Clearly $L_0 = L_1 = L_2$ is isomorphic to the four-element Boolean algebra. Let $(Y, \pi) = (X, \tau)$ and define order on Y by letting $x \leq y$ as the only nontrivial inequality. Let $M = (M_0, M_1, M_2)$ be the corresponding biframe. Then M_0 is isomorphic to the four-element Boolean algebra, while M_1 and M_2 are isomorphic to the three-element chain. Let $e : X \to Y$ be the identity map, and let $f = e^{-1}$. Then it is obvious that (Y, e) is a compactification of the space (X, τ), but (Y, e) is *not* a compactification of the ordered space $(X, \tau, \leq)$ because $x \nleq y$ in X, but $e(x) \leq e(y)$ in Y. Since (Y, e) is a compactification of (X, τ), we see that (M_0, f) is a compactification of L_0. Therefore, $f : M_0 \to L_0$ is an onto dense frame homomorphism. But $f(M_i)$ is properly contained in L_i for $i = 1, 2$. Thus, simply dropping the condition that $f(M_i) = L_i$ for $i = 1, 2$ does not capture the concept of compactification of an ordered space.

Instead, we require an appropriate weakening of the condition $f(M_i) = L_i$ for $i = 1, 2$. This is done in the next lemma, for which we recall our convention that ordered spaces are assumed to be order-T_1 and strongly order convex. For an ordered space $(X, \tau, \leq)$, let $L = (L_0, L_1, L_2)$ be the corresponding biframe, where $L_0 = \tau$, $L_1 = \tau_u$, and $L_2 = \tau_d$.

4.3. LEMMA. *Let $(X, \tau, \leq)$ be an ordered space, $(Y, \pi, \leq)$ be a Nachbin space, and $e : X \to Y$ be continuous and order-preserving. If $L = (L_0, L_1, L_2)$ and $M = (M_0, M_1, M_2)$ are the biframes corresponding to $(X, \tau, \leq)$ and $(Y, \pi, \leq)$, and $f : M \to L$ is the frame homomorphism associated with e, then the following are equivalent.*

(1) *e is order-reflecting.*

(2) *If $a \in L_2$ and $b \in L_0$ with $a \nleq b$, then there is $u \in M_1$ with $a \vee f(u) = 1$ and $b \vee f(u) \neq 1$.*

(3) *If $a \in L_1$ and $b \in L_0$ with $a \nleq b$, then there is $v \in M_2$ with $a \vee f(v) = 1$ and $b \vee f(v) \neq 1$.*

PROOF. (1)⇒(2): Let $a \in L_2$ and $b \in L_0$ with $a \nleq b$. Fix $y \in a \setminus b$. If $x \notin a$, then $x \nleq y$. Since e is order-reflecting, $e(x) \nleq e(y)$. Because $(Y, \pi, \leq)$ is a Nachbin space, there is $u_{e(x)} \in M_1$ with $e(x) \in u_{e(x)}$ and $e(y) \notin u_{e(x)}$. Let $u = \bigvee\{u_{e(x)} : x \notin a\}$. Then $u \in M_1$, $X \setminus a \subseteq e^{-1}(u)$, and $y \notin e^{-1}(u)$. Therefore, $a \vee f(u) = 1$ and $b \vee f(u) \neq 1$.

(2)⇒(1): Let $x \nleq y$. Set $a := X \setminus {\uparrow}x$ and $b := X \setminus \{y\}$. Then $a \in L_2$. Because $y \in a$ but $y \notin b$, we see that $a \nleq b$. By (2), there is $u \in M_1$ with $a \vee f(u) = 1$ and $b \vee f(u) \neq 1$. Therefore, $a \vee e^{-1}(u) = X$ and $b \vee e^{-1}(u) \neq X$. Thus, $e(x) \in u$ and $e(y) \notin u$. This forces $e(x) \nleq e(y)$. Consequently, e is order-reflecting.

(1)⇒(3): If $a \nleq b$, then dualizing the proof of (1)⇒(2) yields $v \in M_2$ with $a \vee f(v) = 1$ and $b \vee f(v) \neq 1$.

(3)⇒(1): If $x \nleq y$, then set $a := X \setminus {\downarrow}y$ and $b := X \setminus \{x\}$. Then $a \in L_1$ and $a \nleq b$. By (3), there is $v \in M_2$ with $a \vee f(v) = 1$ and $b \vee f(v) \neq 1$. Therefore, $e(y) \in v$ but $e(x) \notin v$. Thus, e is order-reflecting. ∎

4.4. Remark. Generalizing the notion of subfit frames (see, e.g., [PP12a, Sec. V.1]), it is natural to call a biframe L *subfit* if for $a \in L_k$ and $b \in L_0$ with $a \nleq b$, there is $c \in L_i$ with $a \vee c = 1$ and $b \vee c \neq 1$ ($i, k = 1, 2,\ i \neq k$). Conditions (2) and (3) of Lemma 4.3 provide a strengthening of the notion of subfitness, which as we will see, is fundamental for developing the theory of compactifications of biframes. For a different strengthening of the subfitness condition for biframes see [PP14].

4.5. Remark. While Conditions (2) and (3) of Lemma 4.3 are equivalent when L is the biframe corresponding to $(X, \tau, \leq)$, the next example shows that they are not equivalent in general. Let $e : X \to Y$ and $f : M \to L$ satisfy Lemma 4.3. We assume that $\leq$ is not trivial; that is, there are $x, y \in X$ with $x \leq y$ and $x \neq y$. Set $K = (L_0, L_0, L_2)$. Then K is a biframe and $f : M \to K$ is a biframe homomorphism. Because $K_2 = L_2$, we see that $f : M \to K$ satisfies Condition (2). On the other hand, let $a := X - \{y\}$ and $b := X - \{x\}$. Then $a \nleq b$ and $a \in L_0 = K_1$. If $v \in M_2$ with $a \vee f(v) = 1$, then $y \in f(v)$. Since v is a downset, so is $f(v)$, which means $x \in f(v)$. Therefore, $f(v) \vee b = 1$. Thus, $f : M \to K$ does not satisfy Condition (3).

Lemma 4.3 (together with Remark 4.5) motivates the following definition of a compactification of a biframe. For an ease of formulation, we make the following assumption, which is standard in the biframe literature.

4.6. Convention. *Throughout the paper, for a biframe $L = (L_0, L_1, L_2)$, we use L_i, L_k to denote L_1 or L_2, always assuming that $i, k = 1, 2$ and $i \neq k$.*

4.7. Definition. *A* compactification *of a biframe $L = (L_0, L_1, L_2)$ is a pair (M, f) such that*

(1) *$M = (M_0, M_1, M_2)$ is a compact regular biframe.*

(2) *$f : M \to L$ is an onto dense biframe homomorphism.*

(3) *If $a \in L_k$ and $b \in L_0$ with $a \nleq b$, then there is $u \in M_i$ with $a \vee f(u) = 1$ and $b \vee f(u) \neq 1$.*

4.8. Remark. Suppose $(X, \tau, \leq)$ is an ordered space and (Y, e) is a compactification of $(X, \tau, \leq)$. Let L be the biframe corresponding to $(X, \tau, \leq)$, M be the biframe corresponding to $(Y, \pi, \leq)$, and f be the biframe homomorphism associated with e. Then M is a compact regular biframe. Since e is a topological embedding, f is onto, and as e is dense, so is f. Therefore, by Lemma 4.3, (M, f) is a compactification of L.

Conversely, suppose (M, f) is a compactification of a biframe L. Let $X = pt(L_0)$, $\tau_0 = \{\varphi(a) : a \in L_0\}$, $\tau_i = \{\varphi(a) : a \in L_i\}$, and $\leq$ be the specialization order of τ_1. Similarly, let $Y = pt(M_0)$, $\pi_0 = \{\varphi(u) : u \in M_0\}$, $\pi_i = \{\varphi(u) : u \in M_i\}$, and $\leq$ be the specialization order of π_1. Since M is compact regular, $(Y, \pi, \leq)$ is a Nachbin space, and $\leq$ is the dual of the specialization order of π_2. Let $\leq'$ be the specialization order of τ_2. We show that $\leq'$ is the dual of $\leq$; that is, we show that $p \leq q$ iff $q \leq' p$ for all $p, q \in X$.

Suppose there are $p, q \in X$ with $p \leq q$ and $q \nleq' p$. Since (M, f) is a compactification of the biframe L, it is easy to see that (M_0, f) is a compactification of the frame L_0.

Therefore, L_0 is a regular frame, and so (X,τ) is a regular space. Clearly it is also T_0, and hence it is a T_1-space. Thus, there is $b \in L_0$ with $X - \{q\} = \varphi(b)$. From $q \nleq' p$ it follows that there is $a \in L_2$ with $q(a) = 1$ and $p(a) = 0$. Then $a \nleq b$ as $q(a) = 1$ but $q(b) = 0$. Consequently, there is $u \in M_1$ with $a \vee f(u) = 1$ and $b \vee f(u) \neq 1$. Since $p(a) = 0$, this forces $p(f(u)) = 1$, so $q(f(u)) = 1$ as $p \leq q$. But $q(f(u)) = 1$ implies $q \in \varphi(f(u))$, so $\varphi(b) \cup \varphi(f(u)) = X$, and so $b \vee f(u) = 1$, a contradiction. Therefore, $p \leq q$ implies $q \leq' p$, and a dual argument shows that $q \leq' p$ implies $p \leq q$. Thus, $\leq$ is the dual of $\leq'$.

This yields that $(X,\tau,\leq)$ is order-T_1. It is also clearly strongly order convex, and so $(X,\tau,\leq)$ is an ordered space in the sense of Convention 3.1. Since f is an onto biframe homomorphism, the corresponding map $e : X \to Y$ is an order-preserving topological embedding. Therefore, by Lemma 4.3, e is order-reflecting. Thus, if $e(X)$ is dense in Y, then (Y,e) is a compactification of $(X,\tau,\leq)$. For example, if L is a spatial biframe, then L_0 is a spatial frame, and as f is dense, so is e, yielding that (Y,e) is a compactification of $(X,\tau,\leq)$.

4.9. Remark. We show that each bi-compactification is a compactification. Suppose (M,f) is a bi-compactification of L. Then M is compact regular, $f : M \to L$ is a dense biframe homomorphism, and $f(M_i) = L_i$. In particular, f is onto. Suppose that $a \in L_k$ and $b \in L_0$ with $a \nleq b$. Since $a \in L_k = f(M_k)$, we have $a = f(x)$ for some $x \in M_k$. As M is regular, $x = \bigvee\{y \in M_k : y \prec_k x\}$. From $f(x) \nleq b$ it follows that there is $y \in M_k$ with $y \prec_k x$ and $f(y) \nleq b$. Set $u := \neg_i y \in M_i$. Since $u \vee x = 1$, we see that $f(u) \vee a = 1$. If $f(u) \vee b = 1$, then $f(\neg_i y) \vee b = 1$. Because $f(\neg_i y) \leq \neg_i f(y)$, we obtain $\neg_i f(y) \vee b = 1$, yielding $f(y) \leq b$. The obtained contradiction proves that $f(u) \vee b \neq 1$. Thus, (M,f) is a compactification of L. On the other hand, as follows from Example 4.1, not every compactification is a bi-compactification.

Next we generalize the concept of a strong inclusion on a biframe to that of a proximity. Our definition is related to the concept of a strong bi-inclusion on a frame studied in [PP12b]. Let $L = (L_0, L_1, L_2)$ be a biframe. If $(\lhd_1, \lhd_2)$ is a pair of relations on L_0, we define

$$L'_i = \{b \in L_i : b = \bigvee\{a \in L_i : a \lhd_i b\}\}.$$

4.10. Remark. If $\lhd_i$ satisfies (P1), (P2), (P3), and (P5) of Definition 4.11 below, then L'_i is a subframe of L_i. Indeed, $0, 1 \in L'_i$ by (P1). By (P2), if $a \lhd_i b$, then $a \leq b$. By (P3) and (P5), if $a \lhd_i x$ and $b \lhd_i y$, then $a \wedge b \lhd_i x \wedge y$. Therefore, for $x, y \in L'_i$, we have

$$\begin{aligned} x \wedge y &= \bigvee\{a \in L_i : a \lhd_i x\} \wedge \bigvee\{b \in L_i : b \lhd_i y\} \\ &= \bigvee\{a \wedge b : a, b \in L_i, a \lhd_i x, b \lhd_i y\} = \bigvee\{c \in L_i : c \lhd_i x \wedge y\}. \end{aligned}$$

Thus, $x \wedge y \in L'_i$. Next, if $\{x_\alpha\} \in L'_i$ and $x = \bigvee_\alpha x_\alpha$, then by (P2), we have

$$x = \bigvee_\alpha x_\alpha = \bigvee_\alpha \bigvee\{a \in L_i : a \lhd_i x_\alpha\} = \bigvee\{a \in L_i : \exists \alpha \text{ with } a \lhd_i x_\alpha\} \leq \bigvee\{a \in L_i : a \lhd_i x\} \leq x.$$

Therefore, $x = \bigvee\{a \in L_i : a \lhd_i x\}$, so $x \in L'_i$. Thus, L'_i is a subframe of L_i.

We let $\lhd_0$ be the relation on L_0 given by

$$a \lhd_0 b \text{ iff there are } u_i \in L_i' \text{ with } a \lhd_i u_i \text{ and } u_1 \wedge u_2 \le b.$$

4.11. Definition. *A* proximity *on a biframe L is a pair $\lhd = (\lhd_1, \lhd_2)$ of relations on L_0 satisfying the following axioms for $a, b, c, d \in L_0$.*

(P1) $0 \lhd_i 0$ *and* $1 \lhd_i 1$.

(P2) *If $a \lhd_i b$, then $a \prec_i b$.*

(P3) *If $a \le b \lhd_i c \le d$, then $a \lhd_i d$.*

(P4) *If $a, b \lhd_i c$, then $a \vee b \lhd_i c$.*

(P5) *If $a \lhd_i b, c$, then $a \lhd_i b \wedge c$.*

(P6) *If $a \lhd_i c$, then there is $b \in L_i'$ with $a \lhd_i b \lhd_i c$.*

(P7) *If $a \lhd_i b$, then $\neg_k b \lhd_k \neg_k a$.*

(P8) $b = \bigvee \{a \in L_0 : a \lhd_0 b\}$.

(P9) *If $a \in L_k$ and $a \nleq b$, then there is $u \in L_i'$ with $a \vee u = 1$ and $b \vee u \ne 1$.*

If $\lhd$ is a proximity on a biframe L, then we call the pair $(L, \lhd)$ a proximity biframe.

4.12. Remark.

(1) By an argument similar to [PP12b, Rem. 3.3(2)], (P7) can equivalently be stated as

$$\text{(P7}'\text{) If } a \lhd_i b, \text{ then } \neg b \lhd_k \neg a,$$

where we write $\neg$ for the pseudocomplement in L_0.

(2) Most axioms of Definition 4.11 are repeats of the axioms in [Sch93, Def. 2] defining a strong inclusion on a biframe, but with a slight difference that in Definition 4.11 both $\lhd_1$ and $\lhd_2$ are relations on L_0, while in [Sch93, Def. 2] $\lhd_1$ is a relation on L_1 and $\lhd_2$ is a relation on L_2. However, if $\lhd_i$ is a relation on L_0, then $\lhd_i|_{L_i}$ is a relation on L_i, and conversely, if $\lhd_i$ is a relation on L_i, then $\lhd_i$ can be extended to a relation $\lhd_i'$ on L_0 by setting $a \lhd_i' b$ iff there exist $a', b' \in L_i$ such that $a \le a' \lhd_i b' \le b$. Taking this into account, we view Schauerte's strong inclusions as pairs of relations on L_0. Then a strong inclusion is nothing but a proximity in which (P8) is strengthened to (S8). This strengthening is equivalent to $L_i' = L_i$, and so (P9) follows. The proof is similar to the proof given in Remark 4.9. Suppose that $a \in L_k$ and $b \in L_0$ with $a \nleq b$. Since $L_k' = L_k$, there is $x \in L_k$ with $x \lhd_k a$ and $x \nleq b$. By (P2), $x \prec_k a$, so $\neg_i x \vee a = 1$. Set $u := \neg_i x$. Then $u \in L_i = L_i'$ and $u \vee a = 1$. If $u \vee b = 1$, then $\neg_i x \vee b = 1$, which implies $x \le b$, a contradiction. Thus, $u \vee b \ne 1$, and hence (P9) holds.

(3) Although in general a proximity biframe L does not have to be regular (see Remark 7.7 and Theorem 7.10), it is easy to see that if $L_i' = L_i$ for $i = 1, 2$, then L is regular.

(4) Our definition of L_i' differs from that of [PP12b] due to us working with biframes rather than frames. It is not difficult to see that if $(L, \triangleleft)$ is a proximity biframe, then (L_0, L_1', L_2') is a biframe, and $\triangleleft$ is a strong inclusion on (L_0, L_1', L_2'). Moreover, $\triangleleft$ is a strong inclusion on (L_0, L_1, L_2) iff $L_i' = L_i$. The proof of this is mostly straightforward, but it requires to verify that $a \triangleleft_i b$ implies $a \prec_i' b$, where $\prec_i'$ is the well inside relation for L_i'. This is proved in Lemma 4.14(4) below.

(5) If $a \triangleleft_i b$ $(i = 1, 2)$, then $a \triangleleft_0 b$. For, since $a \triangleleft_k 1$ and $b \wedge 1 \le b$, it follows that $a \triangleleft_0 b$.

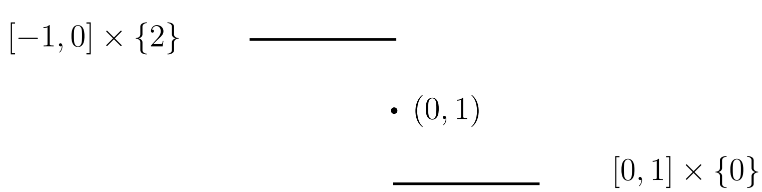
(6) If $\triangleleft = (\triangleleft_1, \triangleleft_2)$ is a proximity on a biframe L, then the two relations $\triangleleft_1$ and $\triangleleft_2$ determine each other. To see this, define two auxiliary relations tot and con on L_0, where $a \mathrel{\mathsf{tot}} b$ if $a \vee b = 1$ and $a \mathrel{\mathsf{con}} b$ if $a \wedge b = 0$. Then $a \triangleleft_k b$ iff there exist x, y such that $b \mathrel{\mathsf{tot}} x \triangleleft_i y \mathrel{\mathsf{con}} a$. For one implication, suppose $a \triangleleft_k b$. By (P6) and (P2), find c such that $a \triangleleft_k c \prec_k b$. Therefore, $\neg_i c \vee b = 1$, and so $b \mathrel{\mathsf{tot}} \neg_i c$. By (P7), $\neg_i c \triangleleft_i \neg_i a$. It is clear that $\neg_i a \mathrel{\mathsf{con}} a$. Set $x := \neg_i c$ and $y := \neg_i a$. Thus, $b \mathrel{\mathsf{tot}} x \triangleleft_i y \mathrel{\mathsf{con}} a$. For the other implication, suppose there are x, y such that $b \mathrel{\mathsf{tot}} x \triangleleft_i y \mathrel{\mathsf{con}} a$. By (P6) and (P2), find z with $x \triangleleft_i z \prec_i y$. Therefore, $\neg_k z \vee y = 1$ and $y \wedge a = 0$, yielding $a \le \neg_k z$. By (P7), $\neg_k z \triangleleft_k \neg_k x$. Also, since $b \vee x = 1$, we have $\neg_k x \le b$. Thus, $a \le \neg_k z \triangleleft_k \neg_k x \le b$, and by (P3), we conclude that $a \triangleleft_k b$.

4.13. Lemma. *Let* $(L, \triangleleft)$ *be a proximity biframe. Then* $\triangleleft_0$ *satisfies Axioms* (B1), (B2), (B3), (B5), (B6), (B8) *of a strong inclusion on* L_0. *However, it may not satisfy* (B4) *and* (B7).

Proof. It is trivial to see that (B1), (B3), and (B8) hold. To prove (B2), let $a \triangleleft_0 b$. Then there are $u_i \in L_i'$ with $a \triangleleft_i u_i$ and $u_1 \wedge u_2 \le b$. By (P2), $a \prec_i u_i$, so $\neg_k a \vee u_i = 1$. But $\neg_k a \le \neg a$, so $\neg a \vee u_i = 1$. Therefore, $\neg a \vee (u_1 \wedge u_2) = 1$, and so $\neg a \vee b = 1$. This shows $a \prec b$. To see (B5), suppose that $a \triangleleft_0 b, c$. Then there are $u_i, v_i \in L_i'$ with $a \triangleleft_i u_i, v_i$, $u_1 \wedge u_2 \le b$, and $v_1 \wedge v_2 \le c$. By (P5), $a \triangleleft_i u_i \wedge v_i$, and since $(u_1 \wedge v_1) \wedge (u_2 \wedge v_2) \le b \wedge c$, we conclude that $a \triangleleft_0 b \wedge c$. To verify (B6), let $a \triangleleft_0 b$. Then there are $u_i \in L_i'$ with $a \triangleleft_i u_i$ and $u_1 \wedge u_2 \le b$. By (P6) find $x_i \in L_i'$ with $a \triangleleft_i x_i \triangleleft_i u_i$. Therefore, $a \triangleleft_0 x_1 \wedge x_2 \triangleleft_0 b$.

To see that $\triangleleft_0$ need not satisfy (B4) and (B7), let

$$X = ([-1, 0] \times \{2\}) \cup (\{(0, 1)\}) \cup ([0, 1] \times \{0\}).$$



It is easy to see that X is a Nachbin space with the topology and order inherited from $\mathbb{R}^2$. Let L be the biframe associated with X. Then L is compact regular. Let $\lhd = (\prec_1, \prec_2)$ be the well inside relation on L. Set $U := [-1,0] \times \{2\}$, $V := [0,1] \times \{0\}$, and $W := U \cup V$. Then $U \prec_1 W$ and $V \prec_2 W$, so by Remark 4.12(5), $U, V \prec_0 W$. On the other hand, if $W \prec_i T$, then $p := (0,1) \in T$. Consequently, $W \not\prec_0 W$, and so $\prec_0$ does not satisfy (B4). Similarly, setting $W' = \{p\} = X \smallsetminus W$, we see that $W' \prec_0 W'$. But $\neg W' = W$ and $W \not\prec_0 W$, hence $\prec_0$ does not satisfy (B7). ∎

Let $M \in \mathsf{KRBFrm}$ and let $\lhd = (\prec_1, \prec_2)$ be the well inside relation on M. By [Sch93, Lem. 1], $\lhd$ is a strong inclusion on M, hence $\lhd$ is a proximity on M. By [FS09, Lem. 4.2], $\lhd$ is the unique strong inclusion on M. We show that $\lhd$ is the unique proximity on M. For this we require the following lemma.

4.14. LEMMA. *Let $(L, \lhd)$ be a proximity biframe.*

(1) *If $a \in L_i$, then $\neg_k a = \neg_k \neg_i \neg_k a$.*

(2) *If $a \in L_i'$, then $\neg_k' a = \neg_k' \neg_i' \neg_k' a$.*

(3) *If $a \lhd_i b$, then $\neg_i \neg_k a \lhd_i b$.*

(4) *If $a \lhd_i b$, then $a \prec_i' b$.*

(5) *If $a \lhd_i b$, then $\neg_i' \neg_k' a \lhd_i b$.*

(6) *If $L \in \mathsf{KRBFrm}$, then $a \prec_i b$ iff there are $u, v \in L_i$ with $a \le u \prec_i v \le b$.*

PROOF. (1) If $a \in L_i$, then $a \le \neg_i \neg_k a$. Therefore, $\neg_k \neg_i \neg_k a \le \neg_k a$. Conversely, since $\neg_k a \in L_k$ and $\neg_k a \wedge \neg_i \neg_k a = 0$, we see that $\neg_k a \le \neg_k \neg_i \neg_k a$. Thus, $\neg_k a = \neg_k \neg_i \neg_k a$.

(2) is proved similarly to (1).

(3) Suppose that $a \lhd_i b$. By (P6), there are $c, d \in L_i' \subseteq L_i$ with $a \lhd_i c \lhd_i d \lhd_i b$. By (P2), $c \prec_i d$, so $\neg_k c \vee d = 1$. By (1), $\neg_k \neg_i \neg_k c \vee d = 1$. Therefore, $\neg_i \neg_k c \le d$. Thus, $\neg_i \neg_k a \le \neg_i \neg_k c \le d \lhd_i b$, so by (P3), $\neg_i \neg_k a \lhd_i b$.

(4) Let $a \lhd_i b$. By (P6), there is $d \in L_i' \subseteq L_i$ with $a \lhd_i d \lhd_i b$. By (3), $a \lhd_i d \le \neg_i \neg_k d \lhd_i b$. By (P3) and (P7), $\neg_k \neg_i \neg_k d \lhd_k \neg_k a$. By (P6), there is $c \in L_k' \subseteq L_k$ with $\neg_k \neg_i \neg_k d \lhd_k c \lhd_k \neg_k a$. By (P2), $\neg_k \neg_i \neg_k d \prec_k c$, so $\neg_i \neg_k \neg_i \neg_k d \vee c = 1$. By (1), $\neg_i \neg_k \neg_i \neg_k d = \neg_i \neg_k d$, so $\neg_i \neg_k d \vee c = 1$. This implies $b \vee c = 1$. Now, $c \prec_k \neg_k a$ and $c \in L_k'$ yield $c \le \neg_k' a$. Thus, $\neg_k' a \vee b = 1$, so $a \prec_i' b$.

(5) is proved similarly to (3), but uses (4) instead of (P2).

(6) One direction is clear. For the other, suppose that $a \prec_i b$. Then $\neg_k a \vee b = 1$. By compactness and regularity, there is $x \in L_k$ with $x \prec_k \neg_k a$ and $x \vee b = 1$. Therefore, there is $y \in L_k$ with $x \prec_k y \prec_k \neg_k a$. This yields $\neg_i y \prec_i \neg_i x$. From $x \vee b = 1$, we get $\neg_i x \le b$. Because $y \prec_k \neg_k a$ we have $\neg_i y \vee \neg_k a = 1$. This gives $\neg(\neg_k a) \le \neg_i y$. But $a \le \neg(\neg_k a)$, so $a \le \neg_i y$. If $u := \neg_i y$ and $v := \neg_i x$, then we have $u, v \in L_i$ with $a \le u \prec_i v \le b$. ∎

4.15. PROPOSITION. *Let M be a compact regular biframe. Then $\lhd = (\prec_1, \prec_2)$ is the unique proximity on M.*

PROOF. By [Sch93, Lem. 1], $(\prec_1, \prec_2)$ is a strong inclusion on M. Therefore, by Remark 4.12(2), $(\prec_1, \prec_2)$ is a proximity on M. Let $(\lhd_1, \lhd_2)$ be a proximity on M. We show that $(\lhd_1, \lhd_2)$ is a strong inclusion on M. For this it is sufficient to show that $M_i \subseteq M_i'$ for $i = 1, 2$. Let $a \in M_i$ and let $b = \bigvee\{x \in M_i' : x \le a\}$. Then $b \le a$. If $a \nleq b$, then by (P9), there is $c \in M_k'$ with $a \vee c = 1$ and $b \vee c \ne 1$. Since $c \in M_k'$, we have $c = \bigvee\{y \in M_k' : y \lhd_k c\}$. Therefore, by compactness, there are $y, z \in M_k'$ with $y \lhd_k z \lhd_k c$ and $a \vee y = 1$. By Lemma 4.14(5), $\neg_k'\neg_i' y \lhd_k \neg_k'\neg_i' z \lhd_k c$. Set $u := \neg_k'\neg_i' y$ and $v := \neg_k'\neg_i' z$. By Lemma 4.14(2), $u = \neg_k'\neg_i' u$, $v = \neg_k'\neg_i' v$, and $y \le u \lhd_k v \le c$. Thus, $a \vee u = 1$ and $b \vee v \ne 1$.

As $a \vee u = 1$ and $u = \neg_k'\neg_i' u$, we have $\neg_k'\neg_i' u \vee a = 1$, hence $\neg_i' u \prec_i' a$. Since $\neg_i' u \in M_i'$, we obtain $\neg_i' u \le b$. By Lemma 4.14(4), $u \lhd_k v$ implies $u \prec_k' v$. Therefore, $\neg_i' v \prec_i' \neg_i' u$. This yields $\neg_i' v \prec_i' b$. Thus, $\neg_k'\neg_i' v \vee b = 1$. Since $\neg_k'\neg_i' v = v$, we conclude that $v \vee b = 1$. The obtained contradiction proves that $b = a$. Consequently, $M_i' = M_i$, and so $(\lhd_1, \lhd_2)$ is a strong inclusion on M. Now apply [FS09, Lem. 4.2] to conclude that $(\lhd_1, \lhd_2) = (\prec_1, \prec_2)$. ■

Next we show how to construct proximities from compactifications. Let (M, f) be a compactification of L. Then f has the right adjoint $r : L_0 \to M_0$ given by

$$r(a) = \bigvee\{x \in M_0 : f(x) \le a\}.$$

The next lemma gives some basic properties of r. For $x, y \in M_0$, we recall that $x \prec y$ if $\neg x \vee y = 1$.

4.16. LEMMA. *Let (M, f) be a compactification of a biframe L, and let $r : L_0 \to M_0$ be the right adjoint of $f : M_0 \to L_0$.*

(1) *$x \le r(a)$ iff $f(x) \le a$ for $x \in M_0$ and $a \in L_0$.*

(2) *$r(a \wedge b) = r(a) \wedge r(b)$ for $a, b \in L_0$.*

(3) *$fr(a) = a$ for $a \in L_0$.*

(4) *$r(\neg a) = \neg r(a)$ and $r(\neg_i a) = \neg_i r(a)$ for $a \in L_0$ and $i = 1, 2$.*

(5) *Suppose $x, y \in M_0$. If $x \prec_i y$, then $rf(x) \prec_i y$ for $i = 1, 2$, and if $x \prec y$, then $rf(x) \prec y$.*

(6) *Suppose $x, y \in M_0$. If $x \prec_0 y$, then $rf(x) \prec_0 y$.*

PROOF. The first two properties are obvious because r is the right adjoint of f. The third property holds because f is onto. That $r(\neg a) = \neg r(a)$ is proved in [Ban90, Lem. 1], and that $r(\neg_i a) = \neg_i r(a)$ is proved similarly. To see (5), let $x \prec_i y$. Then $\neg_k x \vee y = 1$. Set $a := f(x)$. We have

$$f(r(a) \wedge \neg_k x) = fr(a) \wedge f(\neg_k x) \le fr(a) \wedge \neg_k f(x) = a \wedge \neg_k a = 0.$$

Since f is dense, we obtain $r(a) \wedge \neg_k x = 0$. Therefore, $\neg_k x \leq \neg_k r(a)$. Thus, $\neg_k r(a) \vee y = 1$, so $r(a) \prec_i y$. A similar argument gives that $x \prec y$ implies $rf(x) \prec y$. To see (6), let $x \prec_0 y$. Then there are u_i with $x \prec_i u_i$ and $u_1 \wedge u_2 \leq y$. By (5), $x \prec_i u_i$ implies $rf(x) \prec_i u_i$. Therefore, by Lemma 4.13, $rf(x) \prec_0 u_1 \wedge u_2 \leq y$. Thus, $rf(x) \prec_0 y$. ∎

4.17. Remark. It is an easy consequence of Lemma 4.16(5) that for $a, b \in L_0$ we have $r(a) \prec_i r(b)$ iff there are $x, y \in M_0$ such that $a \leq f(x)$, $x \prec_i y$, and $f(y) \leq b$.

4.18. Proposition. *Let (M, f) be a compactification of L. For $a, b \in L_0$ and $i = 1, 2$, define $a \lhd_i b$ if $r(a) \prec_i r(b)$. Then $\lhd = (\lhd_1, \lhd_2)$ is a proximity on L and $L_i' = f(M_i)$.*

Proof. We first prove that $L_i' = f(M_i)$. Let $y \in M_i$. Since M is regular, $y = \bigvee\{x \in M_i : x \prec_i y\}$. Because f is a frame homomorphism, $f(y) = \bigvee\{f(x) : x \in M_i \text{ and } x \prec_i y\}$. By Lemma 4.16(5), if $x \prec_i y$, then $rf(x) \prec_i y \leq rf(y)$, so $f(x) \lhd_i f(y)$. Thus, $f(y) \in L_i'$, so $f(M_i) \subseteq L_i'$. Conversely, if $b \in L_i'$, then $b = \bigvee\{a \in L_i : a \lhd_i b\}$. From $a \lhd_i b$ it follows that $r(a) \prec_i r(b)$. Since M is compact regular, there is $z \in M_i$ with $r(a) \prec_i z \prec_i r(b)$. By Lemma 4.16(5), $rf(z) \prec_i r(b)$. Therefore, $a \leq f(z) \lhd_i b$. Thus, b is the join of those $f(z)$ for $z \in M_i$ with $f(z) \lhd_i b$, so $b \in f(M_i)$. Consequently, $L_i' = f(M_i)$.

We now prove that $\lhd$ is a proximity on L.

(P1) Since f is dense, $r(0) = 0$. Thus, as $0 \prec_i 0$, we have $0 \lhd_i 0$. Also, $1 \lhd_i 1$ because $r(1) = 1$ and $1 \prec_i 1$.

(P2) Suppose $a \lhd_i b$. Then $r(a) \prec_i r(b)$. Since f preserves $\prec_i$, we have $fr(a) \prec_i fr(b)$, so $a \prec_i b$.

(P3) Suppose $a \leq b \lhd_i c \leq d$. Then $r(b) \prec_i r(c)$. Since r is order-preserving, we have $r(a) \leq r(b) \prec_i r(c) \leq r(d)$, so $r(a) \prec_i r(d)$. Thus, $a \lhd_i d$.

(P4) Suppose $a, b \lhd_i c$. Then $r(a), r(b) \prec_i r(c)$. Therefore, $r(a) \vee r(b) \prec_i r(c)$. By Lemma 4.16(5), $rf(r(a) \vee r(b)) \prec_i r(c)$. But $rf(r(a) \vee r(b)) = r(fr(a) \vee fr(b)) = r(a \vee b)$, so $r(a \vee b) \prec_i r(c)$, and hence $a \vee b \lhd_i c$.

(P5) Suppose $a \lhd_i b, c$. Then $r(a) \prec_i r(b), r(c)$. Therefore, $r(a) \prec_i r(b) \wedge r(c) = r(b \wedge c)$. Thus, $a \lhd_i b \wedge c$.

(P6) Suppose $a \lhd_i c$. Then $r(a) \prec_i r(c)$. Therefore, there is $x \in M_i$ with $r(a) \prec_i x \prec_i r(c)$. By Lemma 4.16(5), $r(a) \prec_i x \leq rf(x) \prec_i r(c)$. Thus, if $b = f(x)$, then $r(a) \prec_i r(b) \prec_i r(c)$, so $a \lhd_i b \lhd_i c$.

(P7) Suppose $a \lhd_i b$. Then $r(a) \prec_i r(b)$, so $\neg_k r(b) \prec_k \neg_k r(a)$. Lemma 4.16(4) yields $r(\neg_k b) \prec_k r(\neg_k a)$. Thus, $\neg_k b \lhd_k \neg_k a$.

(P8) Let $b \in L_0$. Then $r(b) \in M_0$, and since M_0 is generated by $M_1 \cup M_2$, we have $r(b) = \bigvee\{v_1 \wedge v_2 : v_i \in M_i, v_1 \wedge v_2 \leq r(b)\}$. As M is regular, we may write $r(b) = \bigvee\{u_1 \wedge u_2 : \exists v_i \in M_i, u_i \prec_i v_i, v_1 \wedge v_2 \leq r(b)\}$. Since f is a frame homomorphism, $b = fr(b) = \bigvee\{f(u_1) \wedge f(u_2) : \exists v_i \in M_i, u_i \prec_i v_i, v_1 \wedge v_2 \leq r(b)\}$. By Lemma 4.16(5), $u_i \prec_i v_i$ implies $rf(u_i) \prec_i rf(v_i)$, so $f(u_i) \lhd_i f(v_i)$. Therefore, $f(u_1) \wedge f(u_2) \lhd_i f(v_i)$ and $f(v_1) \wedge f(v_2) \leq b$. Thus, $f(u_1) \wedge f(u_2) \lhd_0 b$, and so $b = \bigvee\{a \in L_0 : a \lhd_0 b\}$.

(P9) Suppose that $a \in L_k$, $b \in L_0$, and $a \nleq b$. Since (M, f) is a compactification of L, there is $u \in M_i$ with $a \vee f(u) = 1$ and $b \vee f(u) \neq 1$. Because $L_i' = f(M_i)$, we see that (P9) holds. ∎

4.19. Remark. In [Sch93, Lem. 2] Schauerte proves that if $f : M \to L$ is a bicompactification, then L has a strong inclusion. Proposition 4.18 generalizes her result.

4.20. Example. We show that there exist proximities on biframes that are not strong inclusions. For this it is convenient to interpret Proposition 4.18 in the spatial case. Let (Y, e) be a compactification of an ordered space $(X, \tau, \leq)$. By Remark 4.8, $(\Omega(Y), e^{-1})$ is a compactification of the biframe $\Omega(X)$. By Proposition 4.18, there is a proximity $\lhd = (\lhd_1, \lhd_2)$ on $\Omega(X)$. For simplicity we identify X with its image in Y and let $\Delta = Y \smallsetminus X$. For open sets U, V in X, we show that $U \lhd_1 V$ iff ${\uparrow}\operatorname{cl}(U) \subseteq \operatorname{int}(V \cup \Delta)$ and $U \lhd_2 V$ iff ${\downarrow}\operatorname{cl}(U) \subseteq \operatorname{int}(V \cup \Delta)$, where cl and int are the closure and interior operators in Y. For this we first describe $\prec = (\prec_1, \prec_2)$ on $\Omega(Y)$. If U is open in Y, then $\neg_2 U$ is the largest open downset disjoint from U. Therefore, $Y \smallsetminus \neg_2 U$ is the smallest closed upset containing U. Since ${\uparrow}\operatorname{cl}(U)$ is the smallest closed upset containing U ([Nac65, Prop. 4]), we obtain $\neg_2 U = Y \smallsetminus {\uparrow}\operatorname{cl}(U)$. Thus, for U, V open in Y, we have $U \prec_1 V$ iff $\neg_2 U \cup V = Y$ iff ${\uparrow}\operatorname{cl}(U) \subseteq V$. Similarly, $U \prec_2 V$ iff ${\downarrow}\operatorname{cl}(U) \subseteq V$.

We now describe $\lhd_1$. By Proposition 4.18, if U, V are open in X, then $U \lhd_1 V$ iff $r(U) \prec_1 r(V)$. For each open W in X, it is easy to see that $r(W) = \operatorname{int}(W \cup \Delta)$. Therefore, $U \lhd_1 V$ iff ${\uparrow}\operatorname{cl}(\operatorname{int}(U \cup \Delta)) \subseteq \operatorname{int}(V \cup \Delta)$. But since X is dense in Y, we have $\operatorname{cl}(\operatorname{int}(U \cup \Delta)) = \operatorname{cl}(\operatorname{int}(U \cup \Delta) \cap X) = \operatorname{cl}(U)$. Thus, $U \lhd_1 V$ iff ${\uparrow}\operatorname{cl}(U) \subseteq \operatorname{int}(V \cup \Delta)$. Similarly, $U \lhd_2 V$ iff ${\downarrow}\operatorname{cl}(U) \subseteq \operatorname{int}(V \cup \Delta)$.

We are ready to give an example of a proximity which is not a strong inclusion. Let X and Y be the ordered spaces of Example 3.2, and let $\lhd$ be the induced proximity on $\Omega(X)$ described above. We let $V = \{0\}$. Then V is an open upset of X. If U is open in X with $U \lhd_1 V$, then $U \subseteq V$. If $U \neq \varnothing$, then $U = V$, so $\infty \in {\uparrow}\operatorname{cl}(U) \subseteq \operatorname{int}(V \cup \{\infty\})$. This forces V to be cofinite, a contradiction. Therefore, $U = \varnothing$. Thus, V is not the join of those U with $U \lhd_1 V$. Consequently, $\lhd$ is not a strong inclusion on $\Omega(X)$.

5. Round ideals and compactifications

In Proposition 4.18 we saw how a compactification of a biframe L gives rise to a proximity on L. In this section we prove the converse, that a proximity on L gives rise to a compactification of L. We also prove that the poset of compactifications of L is isomorphic to the poset of proximities on L. This generalizes [Sch93, Prop. 1].

5.1. Definition. *Let $(L, \lhd)$ be a proximity biframe. We call an ideal I of L_0 an* i-round ideal *if for each $a \in I$ there is $b \in I$ with $a \lhd_i b$. Let $\mathcal{R}_i$ be the set of all i-round ideals of L_0.*

5.2. Remark. If I is an i-round ideal in L_0, then $I \cap L_i$ is an i-round ideal in L_i, and I is generated by $I \cap L_i$. It is easy to see that this yields an isomorphism between the i-round ideals of L_0 and the i-round ideals of L_i.

5.3. Lemma. *Each $\mathcal{R}_i$ $(i = 1, 2)$ is a subframe of the frame $\mathcal{I}(L_0)$ of all ideals of L_0.*

Proof. Let $I, J \in \mathcal{R}_i$ and let $a \in I \cap J$. Then there are $b \in I$ and $c \in J$ with $a \lhd_i b, c$. By (P5), $a \lhd_i b \wedge c$. Since $b \wedge c \in I \cap J$, we conclude that the ideal $I \cap J$ is i-round. Next, let I_α be a family of i-round ideals, and set $I := \bigvee I_\alpha$ (where the join is taken in $\mathcal{I}(L_0)$). Let $a \in I$. Then $a = x_1 \vee \cdots \vee x_n$ for some $x_t \in I_{\alpha_t}$. For each t there is $y_t \in I_{\alpha_t}$ with $x_t \lhd_i y_t$. Therefore, $a \lhd_i y_1 \vee \cdots \vee y_n$ and $y_1 \vee \cdots \vee y_n \in I$. Thus, I is i-round. Consequently, $\mathcal{R}_i$ is a subframe of $\mathcal{I}(L_0)$. ∎

5.4. Definition. *Let $\mathcal{R}_0$ be the subframe of $\mathcal{I}(L_0)$ generated by $\mathcal{R}_1 \cup \mathcal{R}_2$, and set $\mathcal{R} := (\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2)$. We call an ideal I in $\mathcal{R}_0$ a* round *ideal.*

5.5. Lemma. *Let $(L, \lhd)$ be a proximity biframe.*

(1) *If $b \in L_0$, then $\twoheaddownarrow_i b := \{a \in L_0 : a \lhd_i b\}$ is an i-round ideal.*

(2) *If I is an i-round ideal, then $I = \bigcup\{\twoheaddownarrow_i b : b \in I\}$.*

(3) *If $a \lhd_i b$, then $\twoheaddownarrow_i a \prec_i \twoheaddownarrow_i b$, where $\prec_i$ is the well inside relation on $\mathcal{R}_i$.*

(4) *If I, J are round ideals, then $I \prec_i J$ iff there are $a, b \in J$ with $a \lhd_i b$ and $I \subseteq \twoheaddownarrow_i a$.*

Proof. (1) It is easy to see that $\twoheaddownarrow_i b$ is an ideal of L_0. To see it is i-round, let $a \in \twoheaddownarrow_i b$. Then $a \lhd_i b$. By (P6), there is c with $a \lhd_i c \lhd_i b$. Therefore, $c \in \twoheaddownarrow_i b$ and $a \lhd_i c$. Thus, $\twoheaddownarrow_i b$ is i-round.

(2) It is clear that $\bigcup\{\twoheaddownarrow_i b : b \in I\} \subseteq I$. For the reverse inclusion, let $a \in I$. Since I is i-round, there is $b \in I$ with $a \lhd_i b$. Thus, $a \in \twoheaddownarrow_i b \subseteq \bigcup\{\twoheaddownarrow_i b : b \in I\}$.

(3) Find c, d with $a \lhd_i c \lhd_i d \lhd_i b$. By (P2), $c \prec_i d$, so $\neg_k c \vee d = 1$. By (P7), $\neg_k c \lhd_k \neg_k a$. So $\neg_k c \in \twoheaddownarrow_k \neg_k a \subseteq \neg_k \twoheaddownarrow_i a$ and $d \in \twoheaddownarrow_i b$. Thus, $\neg_k \twoheaddownarrow_i a \vee \twoheaddownarrow_i b = L_0$, so $\twoheaddownarrow_i a \prec_i \twoheaddownarrow_i b$.

(4) We prove (4) for $i = 1$. The proof for $i = 2$ is similar. Suppose that $I \prec_1 J$. Then $\neg_2 I \vee J = L_0$. By (2), $\neg_2 I = \bigcup\{\twoheaddownarrow_2 c : c \in \neg_2 I\}$. Since $\mathcal{I}(L_0)$ is compact and $\{\twoheaddownarrow_2 c : c \in \neg_2 I\}$ is directed, there is $c \in \neg_2 I$ with $\twoheaddownarrow_2 c \vee J = L_0$. As $\neg_2 I$ is 2-round, there is $d \in \neg_2 I$ with $c \lhd_2 d$. Applying (3) gives $\twoheaddownarrow_2 c \prec_2 \neg_2 I$, and a similar argument shows $\twoheaddownarrow_2 d \prec_2 \neg_2 I$. Thus, $\neg_1 \twoheaddownarrow_2 d \vee \neg_2 I = L_0$. By a repeat of the arguments above but replacing $\neg_2 I$ with $\neg_1 \twoheaddownarrow_2 d$, there are $a, b \in \neg_1 \twoheaddownarrow_2 d$ with $a \lhd_1 b$ and $\twoheaddownarrow_1 a \vee \neg_2 I = L_0$. This yields $I \subseteq \neg(\neg_2 I) \subseteq \twoheaddownarrow_1 a$. Also, since $c \lhd_2 d$, we get $\twoheaddownarrow_2 d \vee J = L_0$, so $\neg_1 \twoheaddownarrow_2 d \subseteq J$. Therefore, $a, b \in J$. Thus, there are $a, b \in J$ with $a \lhd_1 b$ and $I \subseteq \twoheaddownarrow_1 a$. Conversely, suppose there are $a, b \in J$ with $a \lhd_1 b$ and $I \subseteq \twoheaddownarrow_1 a$. By (3), $I \subseteq \twoheaddownarrow_1 a \prec_1 \twoheaddownarrow_1 b \subseteq J$. Thus, $I \prec_1 J$. ∎

5.6. Remark. If I, J are i-round ideals, then $I \prec_i J$ iff there is $a \in J$ with $I \subseteq \twoheaddownarrow_i a$. Thus, for i-round ideals, the description of $\prec_i$ is simpler than that for round ideals.

5.7. Proposition. $\mathcal{R} \in \mathsf{KRBFrm}$.

Proof. It follows from the definition that $\mathcal{R}$ is a biframe. Since $\mathcal{R}_0$ is a subframe of $\mathcal{I}(L_0)$ and $\mathcal{I}(L_0)$ is compact, $\mathcal{R}_0$ is compact. For regularity, let $I \in \mathcal{R}_i$. By Lemma 5.5, I is the join of $\twoheaddownarrow_i a$ for $a \in I$, each $\twoheaddownarrow_i a \in \mathcal{R}_i$, and $\twoheaddownarrow_i a \prec_i I$. Thus, I is the join of the elements of $\mathcal{R}_i$ well inside it, hence $\mathcal{R}$ is a regular biframe. ∎

Let $(L, \triangleleft)$ be a proximity biframe and let $a \in L_0$. We define

$$\mathord{\downarrow\!\!\!\!\!\downarrow}_0 a = \bigvee\{\mathord{\downarrow\!\!\!\!\!\downarrow}_1 u_1 \cap \mathord{\downarrow\!\!\!\!\!\downarrow}_2 u_2 : u_1 \wedge u_2 \le a\}.$$

It is clear from the definitions of $\mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$ and $\mathcal{R}_0$ that $\mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$ is a round ideal of $(L, \triangleleft)$.

5.8. LEMMA. *Let $(L, \triangleleft)$ be a proximity biframe.*

(1) *If $a \triangleleft_0 b$, then $a \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b$.*

(2) *If $b \in L_0$, then $\bigvee \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b = b$.*

(3) *If I is a round ideal, then $I = \bigvee\{\mathord{\downarrow\!\!\!\!\!\downarrow}_0 a : \exists b \in I$ with $a \triangleleft_0 b\}$.*

(4) *If I is a round ideal, then I is generated by $\{a \in I : \exists b \in I$ with $a \triangleleft_0 b\}$.*

(5) *If I, J are round ideals, then $I \prec_0 J$ iff there are $a, b \in J$ with $a \triangleleft_0 b$ and $I \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$.*

PROOF. (1) Suppose $a \triangleleft_0 b$. Then there are u_i with $a \triangleleft_i u_i$ and $u_1 \wedge u_2 \le b$. Therefore, $a \in \mathord{\downarrow\!\!\!\!\!\downarrow}_1 u_1 \cap \mathord{\downarrow\!\!\!\!\!\downarrow}_2 u_2$, and since $u_1 \wedge u_2 \le b$, we get $a \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b$.

(2) By (P8), $b = \bigvee\{a : a \triangleleft_0 b\}$. Take a with $a \triangleleft_0 b$. By (1), $a \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b$. Therefore, $b = \bigvee\{a : a \triangleleft_0 b\} \le \bigvee \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b \le b$, which yields (2).

(3) The $\supseteq$ inclusion is clear. We show the $\subseteq$ inclusion. Let $x \in I$. Since I is round, there are i-round ideals $I_{i\alpha}$ with $I = \bigvee(I_{1\alpha} \cap I_{2\alpha})$. So $x = x_1 \vee \cdots \vee x_n$ for some $x_t \in I_{1\alpha_t} \cap I_{2\alpha_t}$, $t = 1, \dots, n$. Therefore, there are $c_t, d_t \in I_{1\alpha_t}$ and $e_t, f_t \in I_{2\alpha_t}$ with $x_t \triangleleft_1 c_t \triangleleft_1 d_t$ and $x_t \triangleleft_2 e_t \triangleleft_2 f_t$. Set $a_t := c_t \wedge e_t$ and $b_t := d_t \wedge f_t$. Then $a_t, b_t \in I_{1\alpha_t} \cap I_{2\alpha_t} \subseteq I$ and $x_t \triangleleft_0 a_t \triangleleft_0 b_t$. Thus, by (1), $x_t \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 a_t$ and $a_t \triangleleft_0 b_t$. Consequently, $x \in \bigvee\{\mathord{\downarrow\!\!\!\!\!\downarrow}_0 a : \exists b \in I$ with $a \triangleleft_0 b\}$.

(4) Let I be a round ideal and let $x \in I$. By the proof of (3), $x = x_1 \vee \cdots \vee x_n$ with $x_t \triangleleft_0 a_t$ and $a_t \in I$, $t = 1, \dots, n$. Thus, I is generated by $\{a \in I : \exists b \in I$ with $a \triangleleft_0 b\}$.

(5) Let I, J be round ideals. First suppose that $I \prec_0 J$. Then there are i-round ideals K_i with $I \prec_i K_i$ and $K_1 \cap K_2 \subseteq J$. By Lemma 5.5(4), there are $c_i, b_i \in K_i$ with $c_i \triangleleft_i b_i$ and $I \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_i c_i$. By (P6), there are a_i with $c_i \triangleleft_i a_i \triangleleft_i b_i$. Set $a := a_1 \wedge a_2$ and $b := b_1 \wedge b_2 \in K_1 \cap K_2 \subseteq J$. Then $c_1 \wedge c_2 \triangleleft_0 a \triangleleft_0 b$. Therefore, $\mathord{\downarrow\!\!\!\!\!\downarrow}_1 c_1 \cap \mathord{\downarrow\!\!\!\!\!\downarrow}_2 c_2 \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$. Thus, $I \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$. Conversely, suppose there are $a, b \in J$ with $a \triangleleft_0 b$ and $I \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$. Then there are c_i with $a \triangleleft_i c_i$ and $c_1 \wedge c_2 \le b$. By (P6), there are x_i, y_i with $a \triangleleft_i x_i \triangleleft_i y_i \triangleleft_i c_i$. Set $K_i := \mathord{\downarrow\!\!\!\!\!\downarrow}_i c_i$. Then $x_i, y_i \in K_i$ and $\mathord{\downarrow\!\!\!\!\!\downarrow}_0 a \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_i x_i$. Since $I \subseteq \mathord{\downarrow\!\!\!\!\!\downarrow}_0 a$, by Lemma 5.5(4), $I \prec_i K_i$. Moreover, $K_1 \cap K_2 \subseteq {\downarrow} c_1 \cap {\downarrow} c_2 \subseteq {\downarrow} b \subseteq J$. Thus, $I \prec_0 J$. ∎

5.9. REMARK. It is easy to see that I is a round ideal iff for each $a \in I$ there is $b \in I$ with $a \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b$. On the other hand, for a round ideal I it may be false that for each $a \in I$ there is $b \in I$ with $a \triangleleft_0 b$. For example, following the notation of the second half of the proof of Lemma 4.13, $\mathord{\downarrow\!\!\!\!\!\downarrow}_0 W$ is a round ideal of L_0. Moreover, as $W = U \cup V$ and $U, V \triangleleft_0 W$, we have $W \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 W$. However, there is no T with $W \triangleleft_0 T \in \mathord{\downarrow\!\!\!\!\!\downarrow}_0 W$ since $W \not\triangleleft_0 W$.

5.10. PROPOSITION. *Let $\triangleleft$ be a proximity on a biframe L and let $\mathcal{R}$ be the biframe of round ideals of $(L, \triangleleft)$. Define $f : \mathcal{R}_0 \to L_0$ by $f(I) = \bigvee I$. Then $(\mathcal{R}, f)$ is a compactification of L. Moreover, the right adjoint r of f is given by $r(b) = \mathord{\downarrow\!\!\!\!\!\downarrow}_0 b$ for all $b \in L_0$.*

Proof. By Proposition 5.7, $\mathcal{R}$ is a compact regular biframe. Since $\bigvee : \mathcal{I}(L_0) \to L_0$ is a frame homomorphism, $\mathcal{R}_0$ is a subframe of $\mathcal{I}(L_0)$, and f is the restriction of $\bigvee$ to $\mathcal{R}_0$, it follows that f is a frame homomorphism. By Lemma 5.8(2), f is onto. It is dense because if $I \in \mathcal{R}_0$ with $\bigvee I = 0$, then $I = \{0\}$. We next show that $f(\mathcal{R}_i) = L_i'$. Let I be an i-round ideal. If $a \in I$, then there is $b \in I$ with $a \lhd_i b$. Therefore, there is $c \in L_i'$ with $a \lhd_i c \lhd_i b$. Thus, $\bigvee I$ is the join of $I \cap L_i'$, yielding that $\bigvee I \in L_i'$. This gives $f(\mathcal{R}_i) \subseteq L_i'$. Conversely, if $a \in L_i'$, then $a = \bigvee {\downarrow\!\!\!\!\!\!\not}_i a$. Since ${\downarrow\!\!\!\!\!\!\not}_i a \in \mathcal{R}_i$, we conclude that $L_i' \subseteq f(\mathcal{R}_i)$, hence the equality. In particular, f is a biframe homomorphism. To conclude that $(\mathcal{R}, f)$ is a compactification of L, let $a \in L_k$, $b \in L_0$, and $a \nleq b$. By (P9), there is $u \in L_i'$ with $a \vee u = 1$ and $b \vee u \neq 1$. Since $f(\mathcal{R}_i) = L_i'$, there is an i-round ideal I such that $u = f(I)$. Therefore, $a \vee f(I) = 1$ and $b \vee f(I) \neq 1$. Thus, $(\mathcal{R}, f)$ is a compactification of L.

Finally, to see that $r(b) = {\downarrow\!\!\!\!\!\!\not}_0 b$, it is sufficient to see that ${\downarrow\!\!\!\!\!\!\not}_0 b$ is the largest round ideal I of $(L, \lhd)$ satisfying $\bigvee I \leq b$. By Lemma 5.8(2), $\bigvee {\downarrow\!\!\!\!\!\!\not}_0 b = b$. Suppose that I is a round ideal with $\bigvee I \leq b$. By Lemma 5.8(3), I is generated by $\{a : \exists x \in I \text{ with } a \lhd_0 x\}$. Let $a \lhd_0 x \in I$. Then $x \leq \bigvee I \leq b$, so ${\downarrow\!\!\!\!\!\!\not}_0 x \subseteq {\downarrow\!\!\!\!\!\!\not}_0 b$, and hence $a \in {\downarrow\!\!\!\!\!\!\not}_0 b$. This yields $I \subseteq {\downarrow\!\!\!\!\!\!\not}_0 b$. Thus, $r(b) = {\downarrow\!\!\!\!\!\!\not}_0 b$. ∎

5.11. Corollary. *A biframe has a compactification iff it has a proximity.*

Proof. Apply Propositions 4.18 and 5.10. ∎

Let L be a biframe. Two compactifications (M, f) and (M', f') of L are *equivalent* if there is a biframe isomorphism $k : M \to M'$ with $f = f' \circ k$. Let $\mathfrak{C}(L)$ be the set of all equivalence classes of compactifications of L. We denote by $[M, f]$ the equivalence class of (M, f), and define a partial order on $\mathfrak{C}(L)$ by $[M, f] \leq [M', f']$ if there is a biframe homomorphism $k : M \to M'$ with $f = f' \circ k$. It is obvious that such a k is dense. Therefore, since M_0, M_0' are compact regular frames, it follows that k is 1-1 (see, e.g., [Ban90, p. 106]).

We also let $\mathfrak{P}(L)$ be the set of all proximities on L. Then $\mathfrak{P}(L)$ is a poset by setting $\lhd \leq \lhd'$ if $\lhd_i \subseteq \lhd_i'$ for $i = 1, 2$. Our goal is to prove that the posets $\mathfrak{C}(L)$ and $\mathfrak{P}(L)$ are isomorphic. For this we need the following lemma.

5.12. Lemma. *Let L be a biframe, let (M, f) be a compactification of L, and let r be the right adjoint of f. Define $\lhd$ to be the proximity on L arising from (M, f) as in Proposition 4.18.*

(1) *For $x, y \in M_0$, if $x \prec_0 y$, then $x \prec y$.*

(2) *If $x \in M_0$ and $y \in M_i$, then $x \prec_0 y$ iff $x \prec_i y$.*

(3) *If $y \in M_i$, then ${\downarrow\!\!\!\!\!\!\not}_0 y = {\downarrow\!\!\!\!\!\!\not}_i y$.*

(4) *For $a_1, \dots, a_n \in L_0$, if $a_t \lhd_0 b_t$ for $t = 1, \dots, n$, then $r(a_1 \vee \cdots \vee a_n) \leq r(b_1) \vee \cdots \vee r(b_n)$.*

(5) *For $x, y \in M_0$, if $x \prec_i y$, then $f(x) \prec_i f(y)$ for $i = 1, 2$.*

(6) *For $x, y \in M_0$, if $x \prec_0 y$, then $f(x) \prec_0 f(y)$.*

(7) *If I is a round ideal of M, then $x \in I$ implies $rf(x) \in I$.*

PROOF. (1) If $x \prec_0 y$, then there exist $u_i \in M_i$ with $x \prec_i u_i$ and $u_1 \wedge u_2 \le y$. Therefore, $\neg_2 x \vee u_1 = 1 = \neg_1 x \vee u_2$. Since $\neg_i x \le \neg x$ for $i = 1, 2$, we get $\neg x \vee u_1 = \neg x \vee u_2 = 1$. Thus, $\neg x \vee (u_1 \wedge u_2) = 1$, so $\neg x \vee y = 1$, yielding $x \prec y$.

(2) If $x \prec_i y$, then by Remark 4.12(5), $x \prec_0 y$. Conversely, suppose that $x \prec_0 y$. By (1), $x \prec y$. Therefore, $\neg x \vee y = 1$. Since M is regular, $y \in M_i$ implies $y = \bigvee\{z \in M_i : z \prec_i y\}$. Because M is compact, there is z with $\neg x \vee z = 1$ and $z \prec_i y$. Thus, $x \le z \prec_i y$, so $x \prec_i y$.

(3) Clearly $\Downarrow_i y \subseteq \Downarrow_0 y$. Conversely, by Lemma 5.8(4), $\Downarrow_0 y$ is the ideal generated by $\{x \in M_0 : x \prec_0 y\}$, so by (2), $\Downarrow_0 y \subseteq \Downarrow_i y$.

(4) Let $a_t \lhd_0 b_t$. Then $r(a_t) \prec_0 r(b_t)$. By (1), $r(a_t) \prec r(b_t)$. Therefore, $r(a_1) \vee \cdots \vee r(a_n) \prec r(b_1) \vee \cdots \vee r(b_n)$. Applying Lemma 4.16(5) then yields $rf(r(a_1) \vee \cdots \vee r(a_n)) \prec r(b_1) \vee \cdots \vee r(b_n)$. Since $rf(r(a_1) \vee \cdots \vee r(a_n)) = r(a_1 \vee \cdots \vee a_n)$, we obtain $r(a_1 \vee \cdots \vee a_n) \prec r(b_1) \vee \cdots \vee r(b_n)$, from which (4) follows.

(5) Suppose that $x \prec_i y$. Then $\neg_k x \vee y = 1$. Since f is a frame homomorphism, $f(\neg_k x) \vee f(y) = 1$. But $f(\neg_k x) \le \neg_k f(x)$. Therefore, $\neg_k f(x) \vee f(y) = 1$, yielding $f(x) \prec_i f(y)$.

(6) Let $x \prec_0 y$. Then there are u_i with $x \prec_i u_i$ and $u_1 \wedge u_2 \le y$. By (6), $x \prec_i u_i$ implies $f(x) \prec_i f(u_i)$. Also, as f preserves finite meets, $u_1 \wedge u_2 \le y$ implies $f(u_1) \wedge f(u_2) \le f(y)$. Thus, $f(x) \prec_0 f(y)$.

(7) Let $x \in I$. Since I is round, by Lemma 5.8(4), we may write $x = x_1 \vee \cdots \vee x_n$ with $x_s \prec_0 y_s \in I$ for each s. Find z_s with $x_s \prec_0 z_s \prec_0 y_s$. By Lemma 4.16(6), $x_s \le rf(x_s) \prec_0 z_s \le rf(z_s) \prec_0 y_s \le rf(y_s)$. So $f(x) = f(x_1) \vee \cdots \vee f(x_n)$ and $f(x_s) \lhd_0 f(z_s) \lhd_0 f(y_s)$. Therefore, by (4), $r(f(x_1) \vee \cdots \vee f(x_n)) \le rf(z_1) \vee \cdots \vee rf(z_n)$. Thus, $rf(x) \le rf(z_1) \vee \cdots \vee rf(z_n) \le y_1 \vee \cdots \vee y_n \in I$, and so $rf(x) \in I$. ∎

5.13. PROPOSITION. *Let (M, f) be a compactification of a biframe L, and define $\lhd = (\lhd_1, \lhd_2)$ on L by $a \lhd_i b$ if $r(a) \prec_i r(b)$. Let $\mathcal{R}$ be the biframe of round ideals of $(L, \lhd)$. Then (M, f) and $(\mathcal{R}, \bigvee)$ are equivalent compactifications of L.*

PROOF. We show that $f : M \to L$ induces a biframe isomorphism $\varphi : \mathcal{R}(M, \prec) \to \mathcal{R}(L, \lhd)$ such that the diagram

$$\begin{tikzcd} \mathcal{R}(M,\prec) \arrow[r, "\varphi"] \arrow[d, "\vee"'] & \mathcal{R}(L,\lhd) \arrow[d, "\vee"] \\ (M,\prec) \arrow[r, "f"'] & (L,\lhd) \end{tikzcd}$$

commutes. It is well known (see, e.g., [BM80]) that there is a frame homomorphism φ from the frame $\mathcal{I}(M_0)$ of ideals of M_0 to the frame of ideals $\mathcal{I}(L_0)$ of L_0 given by $\varphi(I) = \downarrow f(I)$. To see that φ maps i-round ideals of $(M, \prec)$ to i-round ideals of $(L, \lhd)$, it suffices to show that if I is i-round in M, then $\downarrow f(I) = \Downarrow_i f(I)$. One inclusion is obvious. For the reverse inclusion, if $a \le f(x)$ for some $x \in I$, then there is $y \in I$ with $x \prec_i y$. Therefore, $f(x) \lhd_i f(y)$, so $a \lhd_i f(y)$, and hence $a \in \Downarrow_i f(I)$. Thus, φ restricts to a frame homomorphism from $\mathcal{R}(M, \prec)$ to $\mathcal{R}(L, \lhd)$, and so $\varphi : \mathcal{R}(M, \prec) \to \mathcal{R}(L, \lhd)$ is a biframe homomorphism. It remains to see that φ is 1-1 and onto. For onto, it suffices to show that all i-round ideals are in the image of φ. Let J be an i-round ideal of $(L, \lhd)$. Set

$I = \twoheaddownarrow_i f^{-1}(J)$. Then I is an i-round ideal of M. We claim that $\varphi(I) = J$. If $x \in I$, then $x \prec_i y$ for some y with $f(y) \in J$, so $f(x) \le f(y)$, and hence $f(x) \in J$. Thus, $\varphi(I) = \downarrow f(I) \subseteq J$. Conversely, let $a \in J$. Then there is $b \in J$ with $a \lhd_i b$. Since f is onto, there are $x, y \in M_0$ with $f(x) = a$ and $f(y) = b$. From $f(x) \lhd_i f(y)$ we get $rf(x) \prec_i rf(y)$. As $f(rf(y)) = f(y) = b \in J$, we have $rf(y) \in f^{-1}(J)$. Therefore, $rf(x) \in \twoheaddownarrow_i f^{-1}(J) = I$. Thus, $a = f(x) = frf(x) \in f(I) \subseteq \downarrow f(I)$.

To see that φ is 1-1, let I, J be round ideals of M with $\varphi(I) = \varphi(J)$. So $\downarrow f(I) = \downarrow f(J)$. If $I \neq J$, then without loss of generality we may assume that $I \nsubseteq J$. By Lemma 5.8(4), there are $x \in I \setminus J$ and $y \in I$ with $x \prec_0 y$. Find z with $x \prec_0 z \prec_0 y$. By Lemma 4.14(6), $rf(z) \prec_0 rf(y)$. Therefore, $f(z) \lhd_0 f(y) \in \downarrow f(I)$, and hence $f(z) \in \downarrow f(J)$. Thus, $f(z) \le f(u)$ for some $u \in J$. Since J is round, by Lemma 5.12(7), $rf(u) \in J$. This yields $x \le rf(z) \le rf(u) \in J$, so $x \in J$. The obtained contradiction proves that $I = J$.

Thus, $\varphi : \mathcal{R}(M, \prec) \to \mathcal{R}(L, \lhd)$ is a biframe isomorphism, and $\bigvee \circ \varphi = f \circ \bigvee$ by the definition of φ. To complete the proof, it is sufficient to note that $\bigvee : \mathcal{R}(M, \prec) \to M$ is a biframe isomorphism (this is contained in the proof of [Sch93, Prop. 1]). ■

5.14. Theorem. *Let L be a biframe. Then the posets $\mathfrak{C}(L)$ and $\mathfrak{P}(L)$ are isomorphic.*

Proof. Let (M, f) be a compactification of L and let r be the right adjoint of f. By Proposition 4.18, $\lhd = (\lhd_1, \lhd_2)$, given by $a \lhd_i b$ if $r(a) \prec_i r(b)$, is a proximity on L. We first show that if (M, f) and (M', f') are equivalent compactifications, then $\lhd = \lhd'$. Let $k : M_0 \to M_0'$ be a biframe isomorphism with $f = f' \circ k$. If $a, b \in L_0$ with $a \lhd_i b$, then $r(a) \prec_i r(b)$. Therefore, $k(r(a)) \prec_i k(r(b))$. By Lemma 4.16(5), $r'f'(kr(a)) \prec_i r'f'(kr(b))$. But $r'f'(kr(a)) = r'fr(a) = r'(a)$ and $r'f'(kr(b)) = r'(b)$. Thus, $r'(a) \prec_i' r'(b)$, so $a \lhd_i' b$. Consequently, $\lhd \le \lhd'$. The same argument with k^{-1} replacing k shows that $\lhd' \le \lhd$. Therefore, $\lhd = \lhd'$. We thus have a well-defined map $\alpha : \mathfrak{C}(L) \to \mathfrak{P}(L)$ given by $\alpha([M, f]) = \lhd$. The argument above shows that if $k : M \to M'$ is a biframe homomorphism with $f = f' \circ k$, then $\lhd \le \lhd'$. Thus, α is order-preserving.

If $\lhd$ is a proximity on L, then by Proposition 5.10, $(\mathcal{R}, \bigvee)$ is a compactification of L, where $\mathcal{R}$ is the biframe of round ideals of $(L, \lhd)$. Thus, we have a map $\beta : \mathfrak{P}(L) \to \mathfrak{C}(L)$, given by $\beta(\lhd) = [\mathcal{R}, \bigvee]$. If $\lhd \le \lhd'$, then $\lhd_i \subseteq \lhd_i'$ for $i = 1, 2$. Therefore, an i-round ideal of $(L, \lhd)$ is i-round for $(L, \lhd')$. Thus, the inclusion map $\mathcal{R}(L, \lhd) \to \mathcal{R}(L, \lhd')$ yields $[\mathcal{R}(L, \lhd), \bigvee] \le [\mathcal{R}(L, \lhd'), \bigvee]$, and so β is order-preserving.

It remains to show that α and β are inverses of each other. We first show that $\alpha \circ \beta$ is the identity. Let $\lhd$ be a proximity on L. Then $\beta(\lhd) = [\mathcal{R}, \bigvee]$. By Proposition 5.10, $\twoheaddownarrow_0 : L_0 \to \mathcal{R}_0$ is the right adjoint of $\bigvee : \mathcal{R}_0 \to L_0$. Therefore, to see that $\alpha\beta(\lhd) = \lhd$, we must show that $a \lhd_i b$ iff $\twoheaddownarrow_0 a \prec_i \twoheaddownarrow_0 b$. Suppose that $a \lhd_i b$. Then there are $c, d \in L_i'$ with $a \lhd_i c \lhd_i d \lhd_i b$. We have $\twoheaddownarrow_0 a \subseteq \downarrow a \subseteq \twoheaddownarrow_i c$. Also, since $d \lhd_i b$, Remark 4.12(5) yields $d \lhd_0 b$, so $d \in \twoheaddownarrow_0 b$ by Lemma 5.8(1). Thus, by Lemma 5.5(4), $\twoheaddownarrow_0 a \prec_i \twoheaddownarrow_0 b$. Conversely, suppose that $\twoheaddownarrow_0 a \prec_i \twoheaddownarrow_0 b$. By Lemma 5.5(4), there are $c \lhd_i d \in \twoheaddownarrow_0 b$ with $\twoheaddownarrow_0 a \subseteq \twoheaddownarrow_i c$. By Lemma 5.8(2), $a = \bigvee \twoheaddownarrow_0 a \le \bigvee \twoheaddownarrow_i c \le c$, so $a \le c \lhd_i d \le b$. Thus, $a \lhd_i b$.

Finally, to show that $\beta \circ \alpha$ is the identity, let (M, f) be a compactification of L. Let $\lhd = \alpha[M, f]$ and $\mathcal{R}$ be the biframe of round ideals of $(L, \lhd)$. By Proposition 5.13, (M, f)

is equivalent to $(\mathcal{R}, \bigvee)$. Thus, $\beta(\alpha([M,f])) = \beta(\triangleleft) = [\mathcal{R}, \bigvee] = [M,f]$. This finishes the proof. ∎

For a biframe L, let $\mathfrak{B}(L)$ be the subposet of $\mathfrak{C}(L)$ consisting of equivalence classes of bi-compactifications of L, and let $\mathfrak{S}(L)$ be the subposet of $\mathfrak{P}(L)$ consisting of strong inclusions on L. As a consequence of Theorem 5.14, we obtain the following result of Schauerte [Sch93, Prop. 1].

5.15. Corollary. *For a biframe L, the poset $\mathfrak{B}(L)$ is isomorphic to the poset $\mathfrak{S}(L)$.*

Proof. By Theorem 5.14 it is sufficient to show that if (M,f) is a bi-compactification of L, then its corresponding proximity $\triangleleft$ is a strong inclusion on L, and conversely, if $\triangleleft$ is a strong inclusion on L, then $[\mathcal{R}, \bigvee]$ is a bi-compactification of L, where $\mathcal{R}$ is the biframe of round ideals of $(L, \triangleleft)$.

First suppose that (M,f) is a bi-compactification of L. Then $f(M_i) = L_i$. By Proposition 4.18, $f(M_i) = L_i'$. Thus, $L_i = L_i'$, and hence $\triangleleft$ is a strong inclusion on L.

Conversely, suppose that $\triangleleft$ is a strong inclusion on L. Let $b \in L_i$. Then $b = \bigvee\{a \in L_i : a \triangleleft_i b\}$. Therefore, $b = \bigvee \twoheaddownarrow_i b$. Thus, b is the join of the i-round ideal $\twoheaddownarrow_i b$, so the image of $\mathcal{R}_i$ under $\bigvee$ is L_i. This means that $[\mathcal{R}, \bigvee]$ is a bi-compactification of L. ∎

6. Compactifications of ordered spaces

In this section we apply the results of the previous sections to characterize compactifications of ordered spaces and bispaces. Let $(X, \tau, \le)$ be an ordered space. Recalling our convention that $(X, \tau, \le)$ is assumed to be order-T_1 and strongly order convex, $\Omega(X) = (\tau, \tau_u, \tau_d)$ is a biframe, where τ_u is the frame of open upsets and τ_d is the frame of open downsets of $(X, \tau, \le)$.

We say that two compactifications (Y, e) and (Y', e') of $(X, \tau, \le)$ are *equivalent* if there is an order-homeomorphism $f : Y \to Y'$ such that $f \circ e = e'$. Let $\mathfrak{C}(X)$ be the set of all equivalence classes of compactifications of $(X, \tau, \le)$. We denote by $[Y, e]$ the equivalence class of (Y, e), and define a partial order on $\mathfrak{C}(X)$ by $[Y', e'] \le [Y, e]$ if there is a continuous order-preserving map $f : Y \to Y'$ with $f \circ e = e'$. It is easy to see that such an f is onto.

6.1. Theorem. *Let $(X, \tau, \le)$ be an ordered space.*

(1) *Let $(Y, \pi, \le)$ be a Nachbin space and let $e : X \to Y$ be an order-preserving continuous map. Then (Y, e) is a compactification of $(X, \tau, \le)$ iff $(\Omega(Y), e^{-1})$ is a compactification of the biframe $\Omega(X)$.*

(2) *The poset $\mathfrak{C}(X)$ of compactifications of $(X, \tau, \le)$ is isomorphic to the poset $\mathfrak{C}(\Omega(X))$ of compactifications of the biframe $\Omega(X)$.*

Proof. (1) Since $(Y,\pi,\le)$ is a Nachbin space, the biframe $\Omega(Y)$ is compact and regular. Because e is continuous and order-preserving, $e^{-1}:\Omega(Y)\to\Omega(X)$ is a biframe homomorphism. Now, e is a topological embedding iff e^{-1} is onto, and $e(X)$ is dense in Y iff e^{-1} is a dense frame homomorphism. In addition, by Lemma 4.3, e is an order-embedding iff Condition (3) of Definition 4.7 is satisfied. Thus, (Y,e) is a compactification of $(X,\tau,\le)$ iff $(\Omega(Y),e^{-1})$ is a compactification of $\Omega(X)$.

(2) By (1), if (Y,e) is a compactification of $(X,\tau,\le)$, then $(\Omega(Y),e^{-1})$ is a compactification of $\Omega(X)$. Suppose that (Y,e) and (Y',e') are equivalent compactifications of X. Then there is an order-homeomorphism $f:Y\to Y'$ with $f\circ e=e'$.

$$\begin{tikzcd} Y \arrow[rr, "f"] & & Y' \\ & X \arrow[ul, "e"] \arrow[ur, "e'"'] & \end{tikzcd}$$

Therefore, f^{-1} is an isomorphism of biframes such that $e^{-1}\circ f^{-1}=(e')^{-1}$. Thus, $(\Omega(Y),e^{-1})$ and $(\Omega(Y'),(e')^{-1})$ are equivalent compactifications of $\Omega(X)$. This yields a well-defined map $\Phi:\mathfrak{C}(X)\to\mathfrak{C}(\Omega(X))$, sending $[Y,e]$ to $[\Omega(Y),e^{-1}]$. To see this is order-preserving, suppose that $[Y',e']\le[Y,e]$. Then there is a continuous order-preserving map $f:Y\to Y'$ with $f\circ e=e'$. Therefore, $f^{-1}:\Omega(Y')\to\Omega(Y)$ is a biframe homomorphism with $e^{-1}\circ f^{-1}=(e')^{-1}$, so $[\Omega(Y'),(e')^{-1}]\le[\Omega(Y),e^{-1}]$. To see the map is order-reflecting, suppose that $[\Omega(Y'),(e')^{-1}]\le[\Omega(Y),e^{-1}]$. Then there is a biframe homomorphism $\sigma:\Omega(Y')\to\Omega(Y)$ with $e^{-1}\circ\sigma=(e')^{-1}$.

$$\begin{tikzcd} \Omega(Y) \arrow[dr, "e^{-1}"'] & & \Omega(Y') \arrow[ll, "\sigma"'] \arrow[dl, "(e')^{-1}"] \\ & \Omega(X) & \end{tikzcd}$$

From the dual equivalence of Nach and KRBFrm, there is a continuous order-preserving map $f:Y\to Y'$ with $\sigma=f^{-1}$, and $e^{-1}\circ\sigma=(e')^{-1}$ yields $f\circ e=e'$. Thus, $[Y',e']\le[Y,e]$.

It is left to show that Ψ is onto. Suppose (M,h) is a compactification of $\Omega(X)$. Let $Y=pt(M_0)$, $\pi_0=\{\varphi(u):u\in M_0\}$, $\pi_i=\{\varphi(u):u\in M_i\}$, and $\le$ be the specialization order of π_1. Then $\pi=\pi_1\vee\pi_2$, $\le$ is the dual of the specialization order of π_2, and $(Y,\pi,\le)$ is a Nachbin space. Moreover, $M\cong\Omega(Y)$. Define $e:X\to Y$ by $e(x)(m)=1$ iff $x\in h(m)$. Since $h(m)=e^{-1}(\varphi(m))$, we see that e is continuous. Moreover, e is a topological embedding since if U is open in X, write $U=h(m)$ for some $m\in M_0$. Then $e(U)=e(e^{-1}(\varphi(m)))\subseteq\varphi(m)\cap e(X)$. For the reverse inclusion, if $e(x)\in\varphi(m)$, then $x\in e^{-1}(\varphi(m))=h(m)=U$. Thus, $e(U)=\varphi(m)\cap e(X)$, which shows that e is a topological embedding. To see that e is order-preserving, let $x\le x'$, $m\in M_1$, and $e(x)(m)=1$. Then $h(m)$ is an open upset of X and $x\in h(m)$, yielding $x'\in h(m)$. Therefore, $e(x')(m)=1$, giving $e(x)\le e(x')$. Finally, since $(\Omega(Y),e^{-1})$ is a compactification of $\Omega(X)$, Condition (3) of Definition 4.7 and Lemma 4.3 show that e is order-reversing. Thus, (Y,e) is a compactification of X, so $\Psi[Y,e]=[M,h]$, and hence Ψ is an isomorphism of $\mathfrak{C}(X)$ and $\mathfrak{C}(\Omega(X))$. ■

Applying Theorem 5.14 yields

6.2. Corollary. *The poset* $\mathfrak{C}(X)$ *of compactifications of* $(X, \tau, \leq)$ *is isomorphic to the poset* $\mathfrak{P}(\Omega(X))$ *of proximities on the biframe* $\Omega(X)$.

6.3. Remark. In [BS76, Thm. 5.25] Blatter and Seever prove that the poset of compactifications of an ordered space $(X, \tau, \leq)$ is isomorphic to the poset of quasi-proximities on the powerset $\mathcal{P}(X)$ which are compatible with τ and $\leq$. We point out that if $\triangleleft_1$ is a quasi-proximity on $\mathcal{P}(X)$, then there is a dual quasi-proximity $\triangleleft_2$ given by $A \triangleleft_2 B$ iff $X \smallsetminus B \triangleleft_1 X \smallsetminus A$. Restricting the $\triangleleft_i$ to open sets yields a proximity on $\Omega(X) = (\tau, \tau_u, \tau_d)$. To see this, the key observation is that compatibility with τ is equivalent to (P8), and compatibility with $\leq$ is equivalent to (P9). The proofs of the other axioms are straightforward after showing that if $\triangleleft_1$ is a quasi-proximity compatible with τ and $\leq$ and $A \triangleleft_1 B$, then $A \triangleleft_1 \bigcup\{U \in \tau_u' : U \subseteq B\}$. Conversely, given a proximity $\triangleleft$ on $\Omega(X)$, we obtain a quasi-proximity on $\mathcal{P}(X)$ by defining $A \triangleleft_1 B$ if there are $U, V \in \tau_1$ with $A \subseteq U \triangleleft_1 V \subseteq B$. Again, (P8) is equivalent to compatibility of $\triangleleft$ with τ, and (P9) is equivalent to compatibility of $\triangleleft$ with $\leq$. The arguments in [BS76] are analytic in nature. Corollary 6.2 yields an alternate proof of their result.

For a bispace (X, τ_1, τ_2), we say that two bi-compactifications (Y, e) and (Y', e') are *equivalent* if there is a bi-homeomorphism $f : Y \to Y'$ such that $f \circ e = e'$. Let $\mathfrak{B}(X)$ be the set of all equivalence classes of bi-compactifications of (X, τ_1, τ_2). We denote by $[Y, e]$ the equivalence class of (Y, e), and define a partial order on $\mathfrak{B}(X)$ by $[Y', e'] \leq [Y, e]$ if there is a bi-continuous map $f : Y \to Y'$ with $f \circ e = e'$. The following result is a spatial version of [Sch93, Prop. 1]. Parts (1) and (2) are easy consequences of Theorem 6.1, and Part (3) follows from Corollary 5.15.

6.4. Theorem. *Let* (X, τ_1, τ_2) *be a bispace.*

(1) *Let* (Y, π_1, π_2) *be a compact regular bispace and let* $e : X \to Y$ *be a bicontinuous map. Then* (Y, e) *is a bi-compactification of* (X, τ_1, τ_2) *iff* $(\Omega(Y), e^{-1})$ *is a bi-compactification of the biframe* $\Omega(X)$.

(2) *The poset* $\mathfrak{B}(X)$ *of bi-compactifications of the bispace* (X, τ_1, τ_2) *is isomorphic to the poset* $\mathfrak{B}(\Omega(X))$ *of bi-compactifications of the biframe* $\Omega(X)$.

(3) $\mathfrak{B}(X)$ *is isomorphic to the poset* $\mathfrak{S}(\Omega(X))$ *of strong inclusions on the biframe* $\Omega(X)$.

7. Complete regularity and strict complete regularity

As we saw in the previous sections, a biframe L has a compactification iff there is a proximity on L, and an ordered space $(X, \tau, \leq)$ has a compactification iff the corresponding biframe $\Omega(X)$ has a proximity. In this section we show that these are equivalent to L and X being completely regular. The concept of completely regular ordered spaces was introduced by Nachbin [Nac65]. Completely regular bispaces were defined in [Lan67] (see also [Sal74]). As was shown in [Law91], completely regular bispaces correspond to strictly completely regular ordered spaces, which by [Kun90] form a proper subcategory of the category of completely regular ordered spaces. The pointfree analogues of completely regular bispaces were discussed in [BBH83]. We provide the pointfree analogues of completely regular ordered spaces. To distinguish between the two notions, we call the latter biframes completely regular and the former ones strictly completely regular. We prove that a biframe L is completely regular iff L has a compactification, and that L is strictly completely regular iff L has a bi-compactification.

7.1. Definition. *Let $(X, \tau, \leq)$ be an ordered space.*

(1) [Nac65, Ch. II, §1] X *is a* completely regular ordered space *if*

- (a) $x \nleq y$ *implies there is a continuous order-preserving* $f : X \to [0,1]$ *with* $f(x) = 1$ *and* $f(y) = 0$,
- (b) $x \notin F$ *and* F *closed imply that there are continuous order-preserving* $f, g : X \to [0,1]$ *with* $f(x) = 1$, $g(x) = 0$, *and* $F \subseteq f^{-1}(0) \cup g^{-1}(1)$.

(2) [Law91, Def. 9] X *is a* strictly completely regular ordered space *if*

- (a) $x \notin F$ *and* F *a closed downset imply that there is a continuous order-preserving* $f : X \to [0,1]$ *with* $f(x) = 1$ *and* $F \subseteq f^{-1}(0)$,
- (b) $x \notin F$ *and* F *a closed upset imply that there is a continuous order-preserving* $f : X \to [0,1]$ *with* $f(x) = 0$ *and* $F \subseteq f^{-1}(1)$.

7.2. Remark.

(1) In [Nac65, Ch. II, §1], Definition 7.1(1b) is phrased in terms of order preserving $f : X \to [0,1]$ and order-reversing $g : X \to [0,1]$. However, $g : X \to [0,1]$ is order-reversing iff $(1-g) : X \to [0,1]$ is order-preserving. Therefore, the above formulation is equivalent to the original one.

(2) In [Nac65, Ch. II, §1], it is not assumed that $(X, \tau, \leq)$ is order-T_1 and strongly order convex. However, these two properties follow from Conditions (1a) and (1b). Therefore, our convention does not affect the definition.

As follows from [Law91, Rem. 10], each strictly completely regular ordered space is a completely regular ordered space. An example of a completely regular ordered space that is not strictly completely regular was given in [Kun90, Ex. 6]. We recall (see [Law91, Thm. 12]) that $(X, \tau, \leq)$ is a strictly completely regular order space iff (X, τ_u, τ_d) is a completely regular bispace, which happens iff (X, τ_u, τ_d) has a bi-compactification.

Let L be a biframe. We define $\lll_i$ on L_0 by $a \lll_i b$ if there is a family $\{c_p\} \subseteq L_i$ for $p \in \mathbb{Q} \cap [0,1]$ such that $a \leq c_0$, $c_1 \leq b$, and $c_p \prec_i c_q$ whenever $p < q$. Let

$$L_i' = \{b \in L_i : b = \bigvee \{a \in L_i : a \lll_i b\}\}.$$

Then L_i' is a subframe of L_i. Set

$$a \lll_0 b \text{ iff there are } u_i \in L_i' \text{ with } a \lll_i u_i \text{ and } u_1 \wedge u_2 \leq b.$$

7.3. Definition. *Let L be a biframe.*

(1) *We call L* completely regular *provided*

   (*a*) *If $a \in L_k$ and $b \in L_0$ with $a \nleq b$, then there is $u \in L_i'$ with $a \vee u = 1$ and $b \vee u \neq 1$.*

   (*b*) *$b = \bigvee\{a \in L_0 : a \ll_0 b\}$ for each $b \in L_0$.*

(2) *We call L* strictly completely regular *if $b = \bigvee\{a \in L_i : a \ll_i b\}$ for each $b \in L_i$.*

7.4. Remark.

(1) As we will see in the proof of Proposition 7.6, Conditions (1a) and (1b) of Definition 7.1 correspond to Conditions (1a) and (1b) of Definition 7.3, respectively.

(2) What we call strictly completely regular is usually called completely regular in the biframe literature.

Let L be strictly completely regular. For $a, b \in L_0$, since $a \ll_i b$ implies $a \ll_0 b$, it is obvious that L satisfies Condition (1b) of Definition 7.3. Also, as $L_i' = L_i$, Condition (1a) can be proved as in Remark 4.12(2). Therefore, each strictly completely regular biframe is completely regular. On the other hand, as we will see shortly, there exist completely regular biframes that are not strictly completely regular.

Let $(X, \tau, \leq)$ be an ordered space. Suppose that for each $p \in \mathbb{Q} \cap [0,1]$ there is an open downset (resp. open upset) U_p such that ${\downarrow}\operatorname{cl}(U_p) \subseteq U_q$ (resp. ${\uparrow}\operatorname{cl}(U_p) \subseteq U_q$) whenever $p < q$. Define $f : X \to [0,1]$ by $f(x) = \sup\{p : x \notin U_p\} = \inf\{p : x \in U_p\}$. (The supremum of $\varnothing$ is assumed to be 0, and the infimum 1.) It follows from the proof of [Nac65, Thm. 2], which is an adaptation of Urysohn's Lemma to the setting of ordered spaces, that f is continuous and order-preserving (resp. order-reversing). Moreover, $U_0 \subseteq f^{-1}(0)$ and $X \smallsetminus U_1 \subseteq f^{-1}(1)$. We call $\{U_p\}$ a *Urysohn family* and f the *Urysohn function* associated to $\{U_p\}$.

7.5. Lemma. *Let $(X, \tau, \leq)$ be an ordered space.*

(1) *Suppose that U, V are open downsets of X with $U \ll_2 V$. Then there is a continuous order-preserving map $f : X \to [0,1]$ with $U \subseteq f^{-1}(0)$ and $X \smallsetminus V \subseteq f^{-1}(1)$.*

(2) *Let $f : X \to [0,1]$ be continuous and order-preserving. Then $f^{-1}[0,r)$ is an open downset for each $r \in [0,1]$ and if $r < s$, then $f^{-1}[0,r) \ll_2 f^{-1}[0,s)$. Moreover, $f^{-1}[0,r) \in \tau_d'$ for each r.*

(3) *Suppose that U, V are open upsets of X with $U \ll_1 V$. Then there is a continuous order-preserving map $f : X \to [0,1]$ with $U \subseteq f^{-1}(1)$ and $X \smallsetminus V \subseteq f^{-1}(0)$.*

(4) *Let $f : X \to [0,1]$ be continuous and order-preserving. Then $f^{-1}(r,1]$ is an open upset of X for each $r \in [0,1]$ and if $r < s$, then $f^{-1}(s,1] \ll_1 f^{-1}(r,1]$. Moreover, $f^{-1}(r,1] \in \tau_u'$ for each r.*

Proof. (1) From $U \lll_2 V$ it follows that there are open downsets W_p for $p \in \mathbb{Q} \cap [0,1]$ such that $U \subseteq W_0$, $W_1 \subseteq V$, and $W_p \prec_2 W_q$ whenever $p < q$. We show that $W_p \prec_2 W_q$ implies ${\downarrow}\operatorname{cl}(W_p) \subseteq W_q$. For, if $W_p \prec_2 W_q$, then $\neg_1 W_p \cup W_q = X$. Thus, $X \smallsetminus (\neg_1 W_p) \subseteq W_q$. Since $W_p \cap \neg_1 W_p = \varnothing$, we have $W_p \subseteq X \smallsetminus (\neg_1 W_p)$. As $X \smallsetminus (\neg_1 W_p)$ is closed, $\operatorname{cl}(W_p) \subseteq X \smallsetminus (\neg_1 W_p)$. Therefore, because $X \smallsetminus (\neg_1 W_p)$ is a downset, ${\downarrow}\operatorname{cl}(W_p) \subseteq X \smallsetminus (\neg_1 W_p) \subseteq W_q$. Let f be the Urysohn function associated to $\{W_p\}$. Then f is continuous, order-preserving, $U \subseteq W_0 \subseteq f^{-1}(0)$, and $X \smallsetminus V \subseteq X \smallsetminus W_1 \subseteq f^{-1}(1)$.

(2) First note that $\neg_1 f^{-1}[0,r)$ contains $f^{-1}(r,1]$ since $f^{-1}(r,1]$ is an open upset disjoint from $f^{-1}[0,r)$. Therefore, if $r < s$, then as $f^{-1}[0,s) \cup f^{-1}(r,1] = X$, we see that $f^{-1}[0,r) \prec_2 f^{-1}[0,s)$. To show that $f^{-1}[0,r) \lll_2 f^{-1}[0,s)$ we need to produce an appropriate Urysohn family $\{U_p\}$, which we do by scaling the unit interval to $[r,s]$. If we set $U_p = f^{-1}[0, r + (s-r)p)$, then $U_0 = f^{-1}[0,r)$ and $U_1 = f^{-1}[0,s)$. Moreover, $p < q$ implies $U_p \prec_2 U_q$ by the previous observation. Thus, $f^{-1}[0,r) \lll_2 f^{-1}[0,s)$.

Furthermore, if $r \in [0,1]$, then as $f^{-1}[0,r) = \bigcup\{f^{-1}[0,t) : t < r\}$ and $f^{-1}[0,t) \lll_2 f^{-1}[0,r)$ whenever $t < r$, we see that $f^{-1}[0,r) \in \tau_d'$.

The proofs of (3) and (4) are analogous to those of (1) and (2), respectively. We leave out the details, but point out that for (3), producing an appropriate Urysohn family of open upsets will yield an order-reversing map g. The map $f = 1 - g$ will then be the desired map. ∎

7.6. Proposition. *Let $(X, \tau, \leq)$ be an ordered space.*

(1) *$\Omega(X)$ is a completely regular biframe iff X is a completely regular ordered space.*

(2) *$\Omega(X)$ is a strictly completely regular biframe iff X is a strictly completely regular ordered space.*

Proof. (1) Suppose that X is a completely regular ordered space. Let $A \in \tau_d$ and $B \in \tau$ with $A \notin B$. Then there is $y \in A \smallsetminus B$. If $x \notin A$, then $x \nleq y$. Therefore, there is a continuous order-preserving $f : X \to [0,1]$ with $f(x) = 1$ and $f(y) = 0$. Set $U_x := f^{-1}(\frac{1}{2}, 1]$. By Lemma 7.5(4), $U_x \in \tau_u'$, and it is clear that $x \in U_x$ and $y \notin U_x$. Let U be the union of the U_x as x ranges over $X \smallsetminus A$. Then $U \in \tau_u'$, $A \cup U = X$, and $B \cup U \neq X$. If $A \in \tau_u$ with $A \notin B$, then dualizing the argument above and using the map f obtained from Lemma 7.5(3) yields $V \in \tau_d'$ with $A' \cup V = X$ and $B \cup V \neq X$. Thus, $\Omega(X)$ satisfies Condition (1a) of Definition 7.3.

Next let U be an open subset of X and set $F := X \smallsetminus U$. Since X is a completely regular ordered space, for each $x \in U$, there are continuous order-preserving $f, g : X \to [0,1]$ with $f(x) = 1$, $g(x) = 0$, and $F \subseteq f^{-1}(0) \cup g^{-1}(1)$. Let $U_1 = f^{-1}(\frac{1}{2}, 1]$ and $U_2 = g^{-1}[0, \frac{1}{2})$. Then $U_1 \in \tau_u$, $U_2 \in \tau_d$, and $U_1 \cap U_2 \subseteq U$. Moreover, if $W_1 = f^{-1}(\frac{2}{3}, 1]$ and $W_2 = g^{-1}[0, \frac{1}{3})$, then $W_i \lll_i U_i$ by Lemma 7.5. Therefore, $x \in W_1 \cap W_2 \lll_0 U$. Thus, $U = \bigcup\{W \in \tau : W \prec\prec_0 U\}$, and so $\Omega(X)$ satisfies Condition (1b) of Definition 7.3. Consequently, $\Omega(X)$ is a completely regular biframe.

Conversely, let $\Omega(X)$ be a completely regular biframe. First, suppose that $x, y \in X$ with $x \nleq y$. Let $A = X \smallsetminus {\uparrow}x$ and $B = X \smallsetminus \{y\}$. Since ${\uparrow}x$ is closed, $A \in \tau_d$. Also, since

$\uparrow y, \downarrow y$ are closed, $\{y\}$ is closed, so B is open. From $x \nleq y$ it follows that $A \notin B$. Thus, by Condition (1a) of Definition 7.3, there is $U \in \tau_u'$ with $A \cup U = X$ and $B \cup U \neq X$. This means that $x \in U$ and $y \notin U$. Since $U \in \tau_u'$, there is $W \in \tau_u$ with $x \in W$ and $W \ll_1 U$. By Lemma 7.5(3), there is a continuous order-preserving $f : X \to [0,1]$ with $W \subseteq f^{-1}(1)$ and $X \smallsetminus U \subseteq f^{-1}(0)$. Because $y \in X \smallsetminus U$, we conclude that $f(x) = 1$ and $f(y) = 0$.

Next, let $x \in X$ and let F be a closed subset of X with $x \notin F$. Set $U \coloneqq X \smallsetminus F$. Then U is an open neighborhood of x. By Condition (1b) of Definition 7.3, there is $W \in \tau$ with $x \in W$ and $W \ll_0 U$. Therefore, there are $U_1 \in \tau_u'$ and $U_2 \in \tau_d'$ with $W \ll_i U_i$ and $U_1 \cap U_2 \subseteq U$. By Lemma 7.5, there are continuous order-preserving $f, g : X \to [0,1]$ with $W \subseteq f^{-1}(1) \cap g^{-1}(0)$, $X \smallsetminus U_1 \subseteq f^{-1}(0)$, and $X \smallsetminus U_2 \subseteq g^{-1}(1)$. Since $x \in W$, we see that $f(x) = 1$ and $g(x) = 0$. Moreover, $F = X \smallsetminus U \subseteq (X \smallsetminus U_1) \cup (X \smallsetminus U_2) \subseteq f^{-1}(0) \cup g^{-1}(1)$. Thus, X is a completely regular ordered space.

(2) Suppose X is a strictly completely regular ordered space. Let $x \in U \in \tau_u$ and set $F \coloneqq X \smallsetminus U$. Clearly F is a closed downset not containing x. Since X is strictly completely regular, there is a continuous order-preserving $f : X \to [0,1]$ with $f(x) = 1$ and $F \subseteq f^{-1}(0)$. Therefore, $x \in f^{-1}(\frac{1}{3}, 1] \ll_1 f^{-1}(\frac{1}{2}, 1] \subseteq U$. Thus, $U = \bigcup\{V \in \tau_u : V \ll_1 U\}$. A similar argument for open downsets then yields $\Omega(X)$ is strictly completely regular.

Conversely, suppose that $\Omega(X)$ is strictly completely regular. Let F be a closed downset and $x \notin F$. Set $U \coloneqq X \smallsetminus F$. Then $U \in \tau_u$ and $x \in U$. Since $\Omega(X)$ is strictly completely regular, there is $V \in \tau_u$ with $x \in V$ and $V \ll_1 U$. By Lemma 7.5(3), there is a continuous order-preserving $f : X \to [0,1]$ with $V \subseteq f^{-1}(1)$ and $X \smallsetminus V \subseteq f^{-1}(0)$. Thus, $f(x) = 1$ and $F \subseteq X \smallsetminus V \subseteq f^{-1}(0)$. A dual argument for closed upsets then shows that X is strictly completely regular. ∎

7.7. Remark. It is an immediate consequence of Proposition 7.6 and Künzi's example [Kun90, Ex. 6] of a completely regular ordered space which is not strictly completely regular that there exist completely regular biframes which are not strictly completely regular. In fact, Künzi's example yields a completely regular biframe which is not regular. On the other hand, it is obvious that every strictly completely regular biframe is regular.

7.8. Theorem.

(1) *If L is a completely regular biframe, then $\lhd = (\ll_1, \ll_2)$ is the largest proximity on L.*

(2) *If L is a strictly completely regular biframe, then $\lhd = (\ll_1, \ll_2)$ is the largest strong inclusion on L.*

Proof. (1) It is easy to see that $\lhd$ satisfies (P1)–(P5) and (P7). For (P6), let $a \ll_i b$. Then there is a family $\{c_p\} \subseteq L_i$ for $p \in \mathbb{Q} \cap [0,1]$ such that $a \le c_0$, $c_1 \le b$, and $c_p \prec_i c_q$ whenever $p < q$. Let $y = \bigvee\{x \in L_i : x \ll_i c_{1/2}\}$. Given $x \ll_i c_{1/2}$, there is $d \in L_i$ with $x \ll_i d \ll_i c_{1/2}$. From this it follows that if $x \ll_i c_{1/2}$, then $x \ll_i y$. Consequently, $y = \bigvee\{x \in L_i : x \ll_i y\}$, and so $y \in L_i'$. Furthermore, $a \ll_i y \ll_i b$. Thus, $\lhd$ satisfies (P6). It is also clear that Condition (1b) of Definition 7.3 yields (P8) and Condition (1a) yields (P9). Thus, $\lhd$ is a proximity on L. If $\lhd' = (\lhd_1, \lhd_2)$ is a proximity on L, then by (P2), $\lhd_i$ is contained in $\prec_i$. Therefore, (P6) yields that $\lhd_i$ is contained in $\ll_i$. Thus, $\lhd$ is the largest proximity on L.

(2) By (1), $\lhd = (\ll_1, \ll_2)$ satisfies (P1)–(P7). Moreover, since L is strictly completely regular, $\lhd$ satisfies (S8). Thus, $\lhd$ is a strong inclusion on L. Now apply (1) to conclude that $\lhd$ is the largest strong inclusion on L. ∎

7.9. Remark.

(1) By Theorem 7.8(1), if L is a completely regular biframe, then $\lhd = (\ll_1, \ll_2)$ is the largest proximity on L. Therefore, the compactification of L corresponding to $\lhd = (\ll_1, \ll_2)$ is the largest compactification of L. If L corresponds to a completely regular ordered space $(X, \tau, \leq)$, then the largest compactification of L corresponds to the Nachbin compactification of $(X, \tau, \leq)$, which is the largest compactification of $(X, \tau, \leq)$ [Nac65] (see also [Cho79]).

(2) Similarly, if L is strictly completely regular, then $\lhd = (\ll_1, \ll_2)$ is the largest strong inclusion on L. Therefore, the bi-compactification of L corresponding to $\lhd = (\ll_1, \ll_2)$ is the largest bi-compactification of L. If L corresponds to a completely regular bispace (X, τ_1, τ_2), then the largest bi-compactification of L corresponds to the largest compactification of (X, τ_1, τ_2) [Sal74].

The next theorem is now an easy consequence of Theorem 7.8.

7.10. Theorem. *Let L be a biframe.*

(1) *L has a compactification iff L is completely regular.*

(2) *L has a bi-compactification iff L is strictly completely regular.*

Proof. (1) By Corollary 5.11, L has a compactification iff L has a proximity. Therefore, it suffices to prove that L has a proximity iff it is completely regular. If L is completely regular, then by Theorem 7.8(1), $\lhd = (\ll_1, \ll_2)$ is a proximity on L. Conversely, if L has a proximity $\lhd = (\lhd_1, \lhd_2)$, then (P2) and (P6) yield that $\lhd_i$ is contained in $\ll_i$. Thus, (P8) yields Condition (1b) of Definition 7.3 an (P9) yields Condition (1a). Consequently, L is completely regular.

(2) By [Sch93, Prop. 1], L has a bi-compactification iff L has a strong inclusion. Therefore, it is sufficient to show that L has a strong inclusion iff it is strictly completely regular. If L is strictly completely regular, then by Theorem 7.8(2), $\lhd = (\ll_1, \ll_2)$ is a strong inclusion on L. Conversely, if L has a strong inclusion $\lhd = (\lhd_1, \lhd_2)$, then $\lhd_i$ is contained in $\ll_i$, and so (S8) yields that L is strictly completely regular. ∎

The following well-known theorem is now an immediate consequence of our results. Part (1) follows from Theorems 6.1, 7.10, and Proposition 7.6, and Part (2) from Theorems 6.4, 7.10, and Proposition 7.6.

7.11. Corollary.

(1) [Nac65] *An ordered space $(X, \tau, \leq)$ has a compactification iff $(X, \tau, \leq)$ is completely regular.*

(2) [Sal74] *A bispace (X, τ_1, τ_2) has a compactification iff (X, τ_1, τ_2) is completely regular.*

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References

B. Banaschewski, *Compactification of frames*, Math. Nachr. **149** (1990), 105–115.

B. Banaschewski and G. C. L. Brümmer, *Stably continuous frames*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 7–19.

B. Banaschewski, G. C. L. Brümmer, and K. A. Hardie, *Biframes and bispaces*, Proceedings of the Symposium on Categorical Algebra and Topology (Cape Town, 1981), vol. 6, 1983, pp. 13–25.

B. Banaschewski and C. J. Mulvey, *Stone-Čech compactification of locales. I*, Houston J. Math. **6** (1980), no. 3, 301–312.

G. Bezhanishvili and J. Harding, *Stable compactifications of frames*, Cah. Topol. Geom. Differ. Categ. **55** (2014), no. 1, 37–65.

G. Bezhanishvili and P. J. Morandi, *One-point order-compactifications*, Houston J. Math. **37** (2011), no. 3, 699–713.

J. Blatter, *Order compactifications of totally ordered topological spaces*, J. Approximation Theory **13** (1975), 56–65.

J. Blatter and G. L. Seever, *Interposition and lattice cones of functions*, Trans. Amer. Math. Soc. **222** (1976), 65–96.

T. H. Choe, *Partially ordered topological spaces*, An. Acad. Brasil. Ciênc. **51** (1979), no. 1, 53–63.

J. Frith and A. Schauerte, *The Samuel compactification for quasi-uniform biframes*, Topology Appl. **156** (2009), 2116–2122.

G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous lattices and domains*, Cambridge University Press, Cambridge, 2003.

P. T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge, 1982.

H.-P. A. Künzi, *Completely regular ordered spaces*, Order **7** (1990), 283–293.

E. P. Lane, *Bitopological spaces and quasi-uniform spaces*, Proc. London Math. Soc. **17** (1967), 241–256.

J. D. Lawson, *Order and strongly sober compactifications*, Topology and category theory in computer science (Oxford, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, 1991, pp. 179–205.

S. D. McCartan, *Separation axioms for topological ordered spaces*, Proc. Cambridge Philos. Soc. **64** (1968), 965–973.

L. Nachbin, *Topology and order*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1965.

J. Picado and A. Pultr, *Frames and locales*, Frontiers in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2012.

J. Picado and A. Pultr, *On strong inclusions and asymmetric proximities in frames*, Order **29** (2012), 513–531.

J. Picado and A. Pultr, *(Sub)fit biframes and non-symmetric nearness*, Topology Appl. **168** (2014), 66–81.

S. Salbany, *Bitopological spaces, compactifications and completions*, Mathematical Monographs of the University of Cape Town, No. 1, Department of Mathematics, University of Cape Town, Cape Town, 1974.

A. Schauerte, *Biframe compactifications*, Comment. Math. Univ. Carolin. **34** (1993), 567–574.

Y. M. Smirnov, *On proximity spaces*, Mat. Sbornik N.S. **31(73)** (1952), 543–574, (Russian).

M. B. Smyth, *Stable compactification. I*, J. London Math. Soc. **45** (1992), 321–340.

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