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SPECTRA OF COMPACT REGULAR FRAMES

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ABSTRACT. By Isbell duality, each compact regular frame L is isomorphic to the frame of opens of a compact Hausdorff space X. In this note we study the spectrum $\mathrm{Spec}(L)$ of prime filters of a compact regular frame L. We prove that X is realized as the minimum of $\mathrm{Spec}(L)$ and the Gleason cover of X as the maximum of $\mathrm{Spec}(L)$. We also characterize zero-dimensional, extremally disconnected, and scattered compact regular frames by means of $\mathrm{Spec}(L)$.

1. Introduction

By Isbell duality [I72] (see also [BM80, J82]), the category KHaus of compact Hausdorff spaces and continuous maps is dually equivalent to the category KRFrm of compact regular frames and frame homomorphisms. The functors establishing this dual equivalence are $\Omega : \mathsf{KHaus} \to \mathsf{KRFrm}$ and $\mathrm{pt} : \mathsf{KRFrm} \to \mathsf{KHaus}$. The functor Ω associates with each compact Hausdorff space X, the compact regular frame $\Omega(X)$ of open subsets of X, and the functor pt associates with each compact regular frame L, the compact Hausdorff space $\mathrm{pt}(L)$ of points of L, where we recall that a point of a frame L is a frame homomorphism from L to the two-element frame $\mathbf{2} = \{0, 1\}$.

It is well known (see, e.g., [J82, Ch. II.1.3]) that points of a frame L correspond to completely prime filters of L, and so $\mathrm{pt}(L)$ can be thought of as a subset of the set $\mathrm{Spec}(L)$ of prime filters of L, often referred to as the *spectrum* of L. Spectra play an important role in the study of distributive lattices and Heyting algebras. By Stone duality for distributive lattices [S37] (see also [J82]), the category Dist of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category Spec of spectral spaces and spectral maps, where a spectral space is a compact coherent sober space. The functors establishing this dual equivalence are $\mathrm{Spec} : \mathsf{Dist} \to \mathsf{Spec}$ and $\mathrm{KO} : \mathsf{Spec} \to \mathsf{Dist}$. The functor Spec associates with each bounded distributive lattice L, its spectrum $\mathrm{Spec}(L)$ equipped with the Stone topology τ given by letting $\{\varphi(a) : a \in L\}$ be a basis for τ, where $\varphi(a) = \{\mathfrak{p} \in \mathrm{Spec}(L) : a \in \mathfrak{p}\}$. The functor KO associates with each spectral space X, the lattice $\mathrm{KO}(X)$ of compact open sets of X.

An alternative representation of bounded distributive lattices is obtained by means of Priestley spaces. We recall that a Priestley space is a compact ordered space satisfying

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the Priestley separation axiom: If $x \nleqslant y$, then there is a clopen upset U containing x and missing y. Let Pries be the category of Priestley spaces and continuous order preserving maps. By Priestley duality [P70, P72], Dist is dually equivalent to Pries. The dual Priestley space of $L \in \mathsf{Dist}$ is the ordered space $(\mathrm{Spec}(L), \leqslant, \pi)$, where $\leqslant$ is set-theoretic inclusion and π is the patch topology of the Stone topology τ (which has $\{\varphi(a) \setminus \varphi(b) : a, b \in L\}$ as a basis); and the dual lattice of $(X, \leqslant, \pi) \in \mathsf{Pries}$ is the lattice of clopen upsets.

The categories Spec and Pries are isomorphic [C75]. For each Priestley space $(X, \leqslant, \pi)$, the topology of open upsets is a spectral topology, and each spectral topology τ can be realized this way by taking π to be the patch topology and $\leqslant$ the specialization order of τ.

The spectrum of a compact regular frame L carries all the information about L. As a result, $\mathrm{Spec}(L)$ may have a rather complicated structure. Our purpose is to study $\mathrm{Spec}(L)$ for $L \in \mathsf{KRFrm}$. We prove that the spectrum $\mathrm{Min}(L) \subseteq \mathrm{Spec}(L)$ of minimal primes of L is homeomorphic to $\mathrm{pt}(L)$, that the spectrum $\mathrm{Max}(L) \subseteq \mathrm{Spec}(L)$ of maximal filters of L is homeomorphic to the Gleason cover $\widehat{\mathrm{pt}(L)}$ of $\mathrm{pt}(L)$, and that the Gleason map $\gamma : \widehat{\mathrm{pt}(L)} \to \mathrm{pt}(L)$ is encoded in the order structure of $(\mathrm{Spec}(L), \leqslant)$. We also characterize frame homomorphisms between compact regular frames, give examples indicating the complex structure of $(\mathrm{Spec}(L), \leqslant)$, and describe zero-dimensional, extremally disconnected, and scattered frames $L \in \mathsf{KRFrm}$ by means of $\mathrm{Spec}(L)$.

2. Preliminaries

We recall that a *frame* is a complete lattice L satisfying the join-infinite distributive law (JID):

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}.$$

A *frame homomorphism* is a map $h : L \to K$ preserving finite meets and arbitrary joins. In particular, each frame homomorphism is a bounded lattice homomorphism. Let Frm be the category of frames and frame homomorphisms.

Each frame L is a Heyting algebra, where for $a, b \in L$, we have

$$a \to b = \bigvee \{x \in L : a \wedge x \leqslant b\}.$$

In particular, $\neg a = \bigvee \{x \in L : a \wedge x = 0\}$. However, frame homomorphisms need not preserve $\to$ and $\neg$.

An element a of a frame L is *compact* if $a \leqslant \bigvee S$ implies $a \leqslant \bigvee T$ for some finite subset T of S; a frame L is *compact* if its top element 1 is compact. For $a, b \in L$, we say that a is *well inside* b and write $a \prec b$ provided $\neg a \vee b = 1$. It is easily seen that $\twoheaddownarrow a := \{x \in L : x \prec a\}$ is an ideal of L. A frame L is *regular* if $a = \bigvee \twoheaddownarrow a$ for each $a \in L$. Let KRFrm be the full subcategory of Frm consisting of compact regular frames.

If X is a compact Hausdorff space, then the frame $\Omega(X)$ of opens of X is a compact regular frame, and each compact regular frame arises this way. Indeed, let $L \in \mathsf{KRFrm}$

and let $\mathrm{pt}(L)$ be the set of points of L. For $a \in L$, set $O(a) = \{p \in \mathrm{pt}(L) : p(a) = 1\}$. Then $\Omega(\mathrm{pt}(L)) = \{O(a) : a \in L\}$ is a compact Hausdorff topology on $\mathrm{pt}(L)$ and $O : L \to \Omega(\mathrm{pt}(L))$ is a frame isomorphism. This is part of Isbell duality establishing that KRFrm is dually equivalent to the category KHaus of compact Hausdorff spaces and continuous maps.

For a frame L, let $(\mathrm{Spec}(L), \leqslant, \pi)$ be the Priestley dual of L. Since each frame is a complete Heyting algebra, this forces the Priestley dual to satisfy additional conditions. Namely, by Esakia duality [E74] (which is a restricted Priestley duality), Heyting algebras dually correspond to Esakia spaces; that is, Priestley spaces that satisfy the Esakia condition: the downset ${\downarrow}U := \{x : x \leqslant u \text{ for some } u \in U\}$ is clopen for each clopen U. Therefore, since L is a Heyting algebra, we see that $(\mathrm{Spec}(L), \leqslant, \pi)$ is an Esakia space. In fact, for $a, b \in L$, we have

$$\varphi(a \to b) = \mathrm{Spec}(L) \smallsetminus {\downarrow}(\varphi(a) \smallsetminus \varphi(b)) \text{ and } \varphi(\neg a) = \mathrm{Spec}(L) \smallsetminus {\downarrow}\varphi(a).$$

In addition, since L is complete, $(\mathrm{Spec}(L), \leqslant, \pi)$ is extremally order-disconnected; that is, the closure of each open upset is clopen (see, e.g., [PS88, Sec. 2], [BB08, Rem. 2.6]).

Since $\mathrm{Spec}(L)$ is a Priestley space, there is a 1-1 correspondence between ideals of L and open upsets of $\mathrm{Spec}(L)$, and between filters of L and closed upsets of $\mathrm{Spec}(L)$ (see, e.g., [P84, Sec. 8], [BBGK10, Sec. 6]). Indeed, if I is an ideal of L, then $U(I) = \bigcup\{\varphi(a) : a \in I\}$ is an open upset of $\mathrm{Spec}(L)$; and if F is a filter of L, then $K(F) = \bigcap\{\varphi(a) : a \in F\}$ is a closed upset of $\mathrm{Spec}(L)$. Conversely, if U is an open upset of $\mathrm{Spec}(L)$, then $I(U) = \{a \in L : \varphi(a) \subseteq U\}$ is an ideal of L; and if K is a closed upset of $\mathrm{Spec}(L)$, then $F(K) = \{a \in L : K \subseteq \varphi(a)\}$ is a filter of L. Moreover, these correspondences are 1-1. In particular, each open upset is the union of clopen upsets contained in it, and each closed upset is the intersection of clopen upsets containing it.

For $S \subseteq \mathrm{Spec}(L)$, we call $\mathfrak{p} \in S$ a *maximal point* of S if $\mathfrak{p} \leqslant \mathfrak{q}$ and $\mathfrak{q} \in S$ imply $\mathfrak{p} = \mathfrak{q}$. *Minimal points* are defined dually. Let $\mathrm{Max}(S)$ and $\mathrm{Min}(S)$ be the sets of maximal and minimal points of S, respectively. If $S = \mathrm{Spec}(L)$, then we denote the sets of maximal and minimal points by $\mathrm{Max}(L)$ and $\mathrm{Min}(L)$, respectively. Clearly $\mathrm{Max}(L)$ is the set of maximal filters and $\mathrm{Min}(L)$ the set of minimal prime filters of L.

Since $\mathrm{Spec}(L)$ is a Priestley space, for each nonempty closed subset F of $\mathrm{Spec}(L)$, the sets $\mathrm{Max}(F)$ and $\mathrm{Min}(F)$ are nonempty. In fact, for each $f \in F$, there are $M \in \mathrm{Max}(F)$ and $m \in \mathrm{Min}(F)$ such that $m \leqslant f \leqslant M$ (see, e.g., [E85, Thm. III.2.1], [B06, Thm. 2.3.24]).

3. The spectrum of a compact regular frame

Let L be a frame and let $\mathrm{Spec}(L)$ be the spectrum of L. From now on we will view $\mathrm{Spec}(L)$ as a Priestley space, where $\leqslant$ is inclusion and π is the patch topology of the Stone topology τ. Then, since L is a complete Heyting algebra, $\mathrm{Spec}(L)$ is an extremally order-disconnected Esakia space.

3.1. Lemma. *For a frame L, the following are equivalent.*

1. *L is compact.*

2. *If I is an ideal of L with $\bigvee I = 1$, then $I = L$.*

3. *Each $\mathfrak{p} \in \mathrm{Min}(L)$ is an isolated point.*

4. *There are no proper dense open upsets in* $\mathrm{Spec}(L)$*.*

Proof. (1)⇒(2): Suppose I is an ideal of L with $\bigvee I = 1$. Since L is compact, there is a finite $J \subseteq I$ with $\bigvee J = 1$. But $\bigvee J \in I$. Thus, $1 \in I$, and hence $I = L$.

(2)⇒(3): Suppose there is $\mathfrak{p} \in \mathrm{Min}(L)$ that is not isolated. Let $U = \mathrm{Spec}(L) \smallsetminus \{\mathfrak{p}\}$. Then U is an open upset and $\overline{U} = \mathrm{Spec}(L)$. Let I be the ideal $I(U) = \{a : \varphi(a) \subseteq U\}$. By [BB08, Lem. 2.3], $\varphi(\bigvee I) = \overline{\bigcup \{\varphi(a) : a \in I\}}$. Therefore,

$$\varphi(\bigvee I) = \overline{\bigcup \{\varphi(a) : a \in I\}} = \overline{\bigcup \{\varphi(a) : \varphi(a) \subseteq U\}} = \overline{U} = \mathrm{Spec}(L).$$

Thus, $\bigvee I = 1$. Consequently, $I = L$, yielding that $U = \mathrm{Spec}(L)$. The obtained contradiction proves that each $\mathfrak{p} \in \mathrm{Min}(L)$ is an isolated point.

(3)⇒(4): Suppose U is a dense open upset. Since U is dense and each $\mathfrak{p} \in \mathrm{Min}(L)$ is isolated, $\mathrm{Min}(L) \subseteq U$. Therefore, as U is an upset, $\mathrm{Spec}(L) = {\uparrow}\mathrm{Min}(L) = U$.

(4)⇒(1): Suppose $\bigvee S = 1$. By [BB08, Lem. 2.3], $\overline{\bigcup \{\varphi(a) : a \in S\}} = \mathrm{Spec}(L)$. Let $U = \bigcup \{\varphi(a) : a \in S\}$. Then U is a dense open upset. Therefore, $U = \mathrm{Spec}(L)$, so $\bigcup \{\varphi(a) : a \in S\} = \mathrm{Spec}(L)$. Since $\mathrm{Spec}(L)$ is compact, there is a finite $T \subseteq S$ such that $\bigcup \{\varphi(a) : a \in T\} = \mathrm{Spec}(L)$. Thus, $\bigvee T = 1$, and hence L is compact. ∎

3.2. Remark. The equivalence of (1) and (4) of Lemma 3.1 was first established in [PS88, Thm. 3.5].

3.3. Lemma. *Let L be a frame and let $a, b \in L$. Then $b \prec a$ iff ${\downarrow}\varphi(b) \subseteq \varphi(a)$.*

Proof. We have:

$$\begin{array}{rll} b \prec a & \text{iff} & \neg b \vee a = 1 \\ & \text{iff} & \varphi(\neg b) \cup \varphi(a) = \mathrm{Spec}(L) \\ & \text{iff} & (\mathrm{Spec}(L) \smallsetminus {\downarrow}\varphi(b)) \cup \varphi(a) = \mathrm{Spec}(L) \\ & \text{iff} & {\downarrow}\varphi(b) \subseteq \varphi(a). \end{array}$$

∎

3.4. Definition. *For a frame L and $a \in L$, let*

$$R_a := \bigcup \{\varphi(b) : {\downarrow}\varphi(b) \subseteq \varphi(a)\}.$$

Clearly R_a is an open upset of $\mathrm{Spec}(L)$ *contained in $\varphi(a)$. We call R_a the* regular part *of $\varphi(a)$.*

3.5. Lemma. *Let L be a frame and $X = \mathrm{Spec}(L)$ be the spectrum of L. For $a \in L$, we have $R_a = X \smallsetminus {\downarrow}{\uparrow}(X \smallsetminus \varphi(a))$.*

PROOF. Since $R_a = \bigcup\{\varphi(b) : {\downarrow}\varphi(b) \subseteq \varphi(a)\}$, we have

$$X \smallsetminus R_a = \bigcap\{X \smallsetminus \varphi(b) : {\downarrow}\varphi(b) \subseteq \varphi(a)\}\,.$$

As $X \smallsetminus \varphi(b)$ is a clopen downset and each clopen downset is of this form, $X \smallsetminus R_a = \bigcap\{D : {\downarrow}(X \smallsetminus D) \subseteq \varphi(a)\}$, where D ranges over clopen downsets. But ${\downarrow}(X \smallsetminus D) \subseteq \varphi(a)$ is equivalent to $X \smallsetminus \varphi(a) \subseteq X \smallsetminus {\downarrow}(X \smallsetminus D)$, which in turn is equivalent to ${\uparrow}(X \smallsetminus \varphi(a)) \subseteq D$. Therefore, $X \smallsetminus R_a = \bigcap\{D : {\uparrow}(X \smallsetminus \varphi(a)) \subseteq D\}$. Thus, $X \smallsetminus R_a$ is the least closed downset containing ${\uparrow}(X \smallsetminus \varphi(a))$, so $X \smallsetminus R_a = {\downarrow}{\uparrow}(X \smallsetminus \varphi(a))$. Consequently, $R_a = X \smallsetminus {\downarrow}{\uparrow}(X \smallsetminus \varphi(a))$. ■

3.6. LEMMA. *A frame L is regular iff for each $a \in L$, the regular part of $\varphi(a)$ is dense in $\varphi(a)$.*

PROOF. By [BB08, Lem. 2.3] and Lemma 3.3, we have:

$$\begin{aligned} L \text{ is regular} \quad &\text{iff} \quad a = \bigvee\{b : b \prec a\} \text{ for each } a \in L \\ &\text{iff} \quad \varphi(a) = \overline{\bigcup\{\varphi(b) : {\downarrow}\varphi(b) \subseteq \varphi(a)\}} \text{ for each } a \in L \\ &\text{iff} \quad \varphi(a) = \overline{R_a} \text{ for each } a \in L \\ &\text{iff} \quad R_a \text{ is dense in } \varphi(a) \text{ for each } a \in L. \end{aligned}$$

■

3.7. REMARK. Another characterization of regular frames can be obtained by working with clopen downsets instead of clopen upsets. Let L be a frame and $X = \mathrm{Spec}(L)$ be the spectrum of L. Then clopen downsets of X are precisely of the form $X \smallsetminus \varphi(a)$ for some $a \in L$. Let D be a clopen downset of X. Then $D = X \smallsetminus \varphi(a)$ for some $a \in L$. Therefore, applying Lemma 3.5,

$$\varphi(a) = \overline{R_a} \text{ iff } D = X \smallsetminus \overline{R_a} = \mathrm{Int}(X \smallsetminus R_a) = \mathrm{Int}{\downarrow}{\uparrow}D.$$

Thus, by Lemma 3.6, L is regular iff $D = \mathrm{Int}{\downarrow}{\uparrow}D$ for each clopen downset D of X. Since ${\uparrow}D = {\uparrow}(D \cap \mathrm{Min}(X))$, we see that the last condition is equivalent to $D = \mathrm{Int}{\downarrow}{\uparrow}(D \cap \mathrm{Min}(X))$ for each clopen downset D of X. In particular, in the spectrum of a regular frame, clopen downsets are uniquely determined by their "footprints" on the minimum, i. e. $D \cap \mathrm{Min}(\mathrm{Spec}(L)) = D' \cap \mathrm{Min}(\mathrm{Spec}(L))$ implies $D = D'$ for any clopen downsets D, D' of $\mathrm{Spec}(L)$.

3.8. REMARK. For a slightly different characterization of regular frames we refer to [PS88, Thm. 3.4].

Putting Lemmas 3.1 and 3.6 together, we obtain:

3.9. THEOREM. *A frame L is compact regular iff minimal points of $\mathrm{Spec}(L)$ are isolated and the regular part R_U of each clopen upset U in $\mathrm{Spec}(L)$ is dense in U.*

4. Homomorphisms of compact regular frames

By Priestley duality, bounded lattice homomorphisms between bounded distributive lattices correspond to continuous order preserving maps between their Priestley spaces. More specifically, if $h : L \to M$ is a bounded lattice homomorphism, then its Priestley dual $f = h^{-1} : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$ is continuous and order preserving; and if $f : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$ is continuous and order preserving, then the corresponding bounded lattice homomorphism h is uniquely determined by $\varphi(ha) = f^{-1}\varphi(a)$ for each $a \in L$.

By [PS88, Cor. 2.5], the duals of frame homomorphisms $h : L \to M$ are continuous order preserving maps $f : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$ that in addition satisfy $f^{-1}\left(\overline{U}\right) = \overline{f^{-1}(U)}$ for each open upset U of $\mathrm{Spec}(M)$. Indeed, h is a frame homomorphism iff $h(\bigvee S) = \bigvee\{h(s) : s \in S\}$ for each $S \subseteq L$. This is equivalent to $\varphi(h \bigvee S) = \varphi(\bigvee\{h(s) : s \in S\})$ for each $S \subseteq L$. By [BB08, Lem. 2.3],

$$\varphi(h \bigvee S) = f^{-1}\varphi(\bigvee S) = f^{-1}\overline{\bigcup\{\varphi(s) : s \in S\}} = f^{-1}\overline{U},$$

where $U = \bigcup\{\varphi(s) : s \in S\}$. Similarly,

$$\begin{aligned}\varphi(\bigvee\{h(s) : s \in S\}) &= \overline{\bigcup\{\varphi(hs) : s \in S\}} = \overline{\bigcup\{f^{-1}\varphi(s) : s \in S\}} \\ &= \overline{f^{-1}\bigcup\{\varphi(s) : s \in S\}} = \overline{f^{-1}U}.\end{aligned}$$

Since each open upset is of the above form, the result follows.

As we will see, more can be said about frame homomorphisms between compact regular frames. For a frame homomorphism $h : L \to M$, let $r : M \to L$ be the right adjoint of h given by $r(b) = \bigvee\{a \in L : h(a) \leqslant b\}$. We call h *closed* if the following *Frobenius reciprocity* condition $r(h(a) \vee b) \leqslant a \vee r(b)$ holds for all $a \in L$ and $b \in M$. The next lemma (see also [PP12, Cor. VII.2.2.3]) is the point-free version of the well-known fact that a continuous map from a compact space to a Hausdorff space is closed.

4.1. Lemma. *If L is regular and M is compact, then each frame homomorphism $h : L \to M$ is closed.*

Proof. Since L is regular, it suffices to prove that $x \prec r(h(a) \vee b)$ implies $x \prec a \vee r(b)$ for each $x \in L$. Therefore, it is sufficient to prove that $\neg x \vee r(h(a) \vee b) = 1$ implies $\neg x \vee a \vee r(b) = 1$ for each $x \in L$. Suppose $\neg x \vee r(h(a) \vee b) = 1$. Then

$$\begin{aligned} 1 &= \neg x \vee r(h(a) \vee b) \\ &\leqslant rh(\neg x) \vee r(h(a) \vee b) \\ &\leqslant r(h(\neg x) \vee h(a) \vee b) \\ &= r(h(\neg x) \vee h(a) \vee hr(b) \vee b) \\ &= r(h(\neg x \vee a \vee r(b)) \vee b). \end{aligned}$$

This yields

$$1 = h(1) \leqslant h(\neg x \vee a \vee r(b)) \vee b.$$

As L is regular, $\neg x \vee a \vee r(b) = \bigvee\{y \in L : \neg y \vee \neg x \vee a \vee r(b) = 1\}$. But $\neg y = y \to 0 \leqslant y \to r(b)$, so $\neg y \vee \neg x \vee a \vee r(b) = 1$ implies $(y \to r(b)) \vee \neg x \vee a \vee r(b) = 1$, which is equivalent to $(y \to r(b)) \vee \neg x \vee a = 1$. Moreover, $(y \to r(b)) \vee \neg x \vee a = 1$ implies

$$\begin{aligned} y &= y \wedge ((y \to r(b)) \vee \neg x \vee a) = (y \wedge (y \to r(b)) \vee (y \wedge (\neg x \vee a)) \\ &= (y \wedge r(b)) \vee (y \wedge (\neg x \vee a)) \leqslant r(b) \vee \neg x \vee a. \end{aligned}$$

Thus, $\neg x \vee a \vee r(b) = \bigvee I$, where $I = \{y \in L : (y \to r(b)) \vee \neg x \vee a = 1\}$.

Let J be the ideal of M generated by b and $h[I]$. Since h preserves joins, $\bigvee J = b \vee \bigvee h[I] = b \vee h(\bigvee I) = b \vee h(\neg x \vee a \vee r(b)) = 1$. As M is compact and J is an ideal, we see that $1 \in J$. Therefore, there is $y \in I$ with $b \vee h(y) = 1$. But then

$$h(y \to r(b)) \leqslant h(y) \to hr(b) \leqslant h(y) \to b = (b \vee h(y)) \to b = b.$$

Thus, $y \to r(b) = r(b)$, and hence

$$1 = (y \to r(b)) \vee \neg x \vee a = r(b) \vee \neg x \vee a.$$

■

4.2. Theorem. *Let L, M be compact regular, $X = \mathrm{Spec}(L)$, and $Y = \mathrm{Spec}(M)$. Suppose $h : L \to M$ is a frame homomorphism and $f : Y \to X$ is its dual. If D is a clopen downset of Y, then its image $f[D]$ is a clopen downset of X.*

Proof. Since D is a clopen downset, there is $b \in M$ with $D = Y \setminus \varphi(b)$. Since $\varphi(rb)$ is a clopen upset of X, it is sufficient to prove that $f[Y \setminus \varphi(b)] = X \setminus \varphi(rb)$. For the inclusion $f[Y \setminus \varphi(b)] \subseteq X \setminus \varphi(rb)$, let $y \in Y \setminus \varphi(b)$. Then $b \notin y$, so $hr(b) \notin y$, and hence $r(b) \notin h^{-1}(y)$. Therefore, $f(y) \notin \varphi(rb)$, yielding $f(y) \in X \setminus \varphi(rb)$. Thus, $f[Y \setminus \varphi(b)] \subseteq X \setminus \varphi(rb)$.

For the reverse inclusion, let $x \in X \setminus \varphi(rb)$. Then $r(b) \notin x$. Consider the filter $F := r^{-1}[x]$ of M and the ideal I of M generated by $h[L \setminus x] \cup \{b\}$. If $F \cap I \neq \varnothing$, then there are $a \in F$ and $c \notin x$ such that $a \leqslant h(c) \vee b$. But then $r(a) \leqslant r(h(c) \vee b) \leqslant c \vee r(b)$, where the last inequality follows from Lemma 4.1. Therefore, $c \vee r(b) \in x$, a contradiction since $c, r(b) \notin x$ and x is a prime filter of L. Thus, $F \cap I = \varnothing$, and hence there is a prime filter y of M with $F \subseteq y$ and $y \cap I = \varnothing$. This yields that $b \notin y$ and $h^{-1}[y] = x$. Consequently, $y \in Y \setminus \varphi(b)$ and $f(y) = x$, giving $x \in f[Y \setminus \varphi(b)]$. ■

4.3. Corollary. *Let L, M be compact regular, $X = \mathrm{Spec}(L)$, and $Y = \mathrm{Spec}(M)$. If $h : L \to M$ is a frame homomorphism, then its dual $f : Y \to X$ satisfies $f[{\downarrow} y] = {\downarrow} f(y)$ for each $y \in Y$.*

Proof. Since f is order preserving, $f[\downarrow y] \subseteq \downarrow f(y)$ for each $y \in Y$. For the reverse inclusion, let $x \in \downarrow f(y)$. As Y is a Priestley space, each closed downset is the intersection of clopen downsets containing it. Therefore, $\downarrow y$ is the intersection of clopen downsets containing it. By Theorem 4.2, if D is a clopen downset of Y, then $f[D]$ is a clopen downset of X. Thus, if $\downarrow y \subseteq D$, then $f(y) \in f[D]$, so $x \in f[D]$, and hence $f^{-1}(x) \cap D \neq \varnothing$. Since Y is compact and the collection of clopen downsets containing $\downarrow y$ is down-directed, we conclude that $f^{-1}(x) \cap \bigcap\{D : \downarrow y \subseteq D\} \neq \varnothing$. But $\bigcap\{D : \downarrow y \subseteq D\} = \downarrow y$, so there is $z \leqslant y$ with $x = f(z)$, yielding that $x \in f[\downarrow y]$. ∎

4.4. Remark. On the other hand, $f[\uparrow y] = \uparrow f(y)$ does not hold in general. Let L be compact regular. It is easy to see that for each $a \in L$, the frame $M := [a, 1]$ is also compact regular, and $h_a : L \twoheadrightarrow M$ is a frame homomorphism, where $h_a(x) = a \vee x$. Recall that $a \in L$ is *dense* provided $\neg a = 0$ (equivalently, $\neg\neg a = 1$). Suppose there is a dense element $a \neq 1$ in L. Let $X = \mathrm{Spec}(L)$ and $Y = \mathrm{Spec}(M)$. Then we may identify Y with $X \smallsetminus \varphi(a)$, and the dual $f = h^{-1} : Y \to X$ with the inclusion $X \smallsetminus \varphi(a) \subseteq X$. Since $\varphi(\neg a) = X \smallsetminus \downarrow \varphi(a)$, it is easy to see that a is dense iff $\downarrow \varphi(a) = X$, which happens iff $\mathrm{Max}(X) \subseteq \varphi(a)$. As $a \neq 1$, there is $y \in X \smallsetminus \varphi(a)$. Now, $f[\uparrow y] = \uparrow y \cap (X \smallsetminus \varphi(a))$ while $\uparrow f(y) = \uparrow y$. Since there is $x \in \mathrm{Max}(X)$ with $y \leqslant x$, we see that $x \in \uparrow f(y)$ but $x \notin f[\uparrow y]$. Thus, $f[\uparrow y] \neq \uparrow f(y)$.

5. Minimal and maximal spectra

We next show that for a compact regular frame L, the information about the compact Hausdorff space of points $\mathrm{pt}(L)$ and its Gleason cover $\widehat{\mathrm{pt}(L)}$ is encoded in $\mathrm{Min}(L)$ and $\mathrm{Max}(L)$. The reader might find it useful at this point to consult Examples 6.15 and 6.16 given at the end of the paper. Besides illustrating the complexity of $\mathrm{Spec}(L)$, they could provide some background intuition for the technical development in this section.

In [BB08, Thm. 2.7(2)], a dual characterization of completely join-prime elements of a Heyting algebra was given. If a is completely join-prime, then $\uparrow a$ is a completely prime filter, but not every completely prime filter has this form. We start by giving a dual characterization of completely prime filters.

5.1. Lemma. *Let L be a frame. A filter $\mathfrak{p}$ of L is completely prime iff $\downarrow\mathfrak{p}$ is clopen in* $\mathrm{Spec}(L)$.

Proof. First suppose that $\downarrow\mathfrak{p}$ is clopen in $\mathrm{Spec}(L)$. Let $\bigvee S \in \mathfrak{p}$. By [BB08, Lem. 2.3], $\mathfrak{p} \in \varphi(\bigvee S) = \overline{\bigcup\{\varphi(s) : s \in S\}}$. Since $\downarrow\mathfrak{p}$ is clopen, it is an open neighborhood of $\mathfrak{p}$, so $\downarrow\mathfrak{p} \cap (\bigcup\{\varphi(s) : s \in S\}) \neq \varnothing$. Therefore, there is $s \in S$ with $\downarrow\mathfrak{p} \cap \varphi(s) \neq \varnothing$. Thus, there is $\mathfrak{q} \leqslant \mathfrak{p}$ with $\mathfrak{q} \in \varphi(s)$. As $\varphi(s)$ is an upset, this yields $\mathfrak{p} \in \varphi(s)$. Consequently, $s \in \mathfrak{p}$, and hence $\mathfrak{p}$ is a completely prime filter.

Conversely, suppose that $\mathfrak{p}$ is a completely prime filter. Let $U = \mathrm{Spec}(L) \smallsetminus \downarrow\mathfrak{p}$. Since $\downarrow\mathfrak{p}$ is a closed downset, U is an open upset. Therefore, $U = \bigcup\{\varphi(a) : \varphi(a) \subseteq U\}$. As $\mathrm{Spec}(L)$ is an Esakia space and U is an upset, $\overline{U}$ is an upset. Thus, if $\downarrow\mathfrak{p}$ is not clopen, then ${\downarrow}\mathfrak{p} \cap \overline{U} \neq \varnothing$, and so $\mathfrak{p} \in \overline{U} = \overline{\bigcup\{\varphi(a) : \varphi(a) \subseteq U\}} = \varphi\left(\bigvee\{a : \varphi(a) \subseteq U\}\right)$. This yields $\bigvee\{a : \varphi(a) \subseteq U\} \in \mathfrak{p}$. Since $\mathfrak{p}$ is completely prime, there is $a \in \mathfrak{p}$ with $\varphi(a) \subseteq U$. But $\varphi(a) \subseteq U$ implies $\mathfrak{p} \notin \varphi(a)$, so $a \notin \mathfrak{p}$. The obtained contradiction proves that ${\downarrow}\mathfrak{p}$ is clopen in $\mathrm{Spec}(L)$. ∎

5.2. LEMMA. *If L is a compact frame, then each $\mathfrak{p} \in \mathrm{Min}(L)$ is a completely prime filter of L.*

PROOF. Suppose $\mathfrak{p} \in \mathrm{Min}(L)$. Since L is compact, by Lemma 3.1, $\mathfrak{p}$ is an isolated point of $\mathrm{Spec}(L)$. Therefore, ${\downarrow}\mathfrak{p} = \{\mathfrak{p}\}$ is clopen. Thus, by Lemma 5.1, $\mathfrak{p}$ is a completely prime filter of L. ∎

5.3. LEMMA. *If L is a regular frame, then each completely prime filter of L is minimal prime.*

PROOF. Suppose $\mathfrak{p}$ is a completely prime filter of L. Then ${\downarrow}\mathfrak{p}$ is clopen in $\mathrm{Spec}(L)$ by Lemma 5.1. If $\mathfrak{p}$ is not minimal prime, then there is $\mathfrak{q} \in \mathrm{Spec}(L)$ with $\mathfrak{q} < \mathfrak{p}$. Therefore, there is $a \in L$ with $\mathfrak{p} \in \varphi(a)$ and $\mathfrak{q} \notin \varphi(a)$. Let R_a be the regular part of $\varphi(a)$. Then $\mathfrak{p} \notin R_a$. As R_a is an upset, this yields $R_a \cap {\downarrow}\mathfrak{p} = \varnothing$. Since ${\downarrow}\mathfrak{p}$ is clopen, $\overline{R_a} \cap {\downarrow}\mathfrak{p} = \varnothing$, so $\mathfrak{p} \notin \overline{R_a}$. On the other hand, as L is regular, $\varphi(a) = \overline{R_a}$, so $\mathfrak{p} \in \overline{R_a}$. The obtained contradiction proves that $\mathfrak{p}$ is minimal prime. ∎

Lemmas 5.2 and 5.3 put together give that in a compact regular frame L, completely prime filters are exactly minimal primes. Since completely prime filters correspond to points of L, this gives a 1-1 correspondence between points and minimal primes of $L \in$ KRFrm. We next show that this 1-1 correspondence is in fact a homeomorphism of the corresponding spaces.

5.4. THEOREM. *Let L be a compact regular frame. If we view $\mathrm{Spec}(L)$ as a spectral space, then $\mathrm{Min}(L)$ as a subspace of $\mathrm{Spec}(L)$ is homeomorphic to $\mathrm{pt}(L)$.*

PROOF. It is well known (see, e.g., [J82, Ch.II.1.3]) that points of L are in 1-1 correspondence with completely prime filters of L; namely, for $p \in \mathrm{pt}(L)$, we have that $p^{-1}(1)$ is a completely prime filter of L and each completely prime filter arises this way. By Lemmas 5.2 and 5.3, completely prime filters are minimal primes. Thus, if we define $f : \mathrm{pt}(L) \to \mathrm{Min}(L)$ by $f(p) = p^{-1}(1)$, then f is a 1-1 correspondence. Let $a \in L$ and $p \in \mathrm{pt}(L)$. Then

$$p \in f^{-1}(\varphi(a)) \text{ iff } f(p) \in \varphi(a) \text{ iff } a \in f(p) \text{ iff } p(a) = 1 \text{ iff } p \in O_a$$

and

$$f(p) \in \varphi(a) \text{ iff } a \in f(p) \text{ iff } p(a) = 1 \text{ iff } p \in O_a \text{ iff } f(p) \in f(O_a).$$

Therefore, $f^{-1}(\varphi(a)) = O_a$ and $\varphi(a) = f(O_a)$ for each $a \in L$. Thus, f is a homeomorphism. ∎

5.5. Corollary. *If L is a compact regular frame, then* $\mathrm{Min}(L)$ *is a compact Hausdorff space.*

5.6. Remark. Since each minimal prime of L is completely prime, for $S \subseteq L$, we have

$$\varphi(\bigvee S) \cap \mathrm{Min}(L) = \bigcup \{\varphi(s) : s \in S\} \cap \mathrm{Min}(L).$$

Therefore, each open in $\mathrm{Min}(L)$ is of the form $\varphi(a) \cap \mathrm{Min}(L)$ for some $a \in L$.

We next turn to $\mathrm{Max}(L)$. We recall that a subset U of a topological space is *regular open* provided $\mathrm{Int}\left(\overline{U}\right) = U$. Let $\mathcal{RO}(X)$ be the set of regular open subsets of X. It is well known that $\mathcal{RO}(X)$ is a Boolean frame, where $\bigvee U_i = \mathrm{Int}\left(\overline{\bigcup U_i}\right)$, $U \wedge V = U \cap V$, and $\neg U = \mathrm{Int}(X \smallsetminus U)$. The *Gleason cover* of a compact Hausdorff space X is then the pair (Y, γ), where Y is the Stone space of $\mathcal{RO}(X)$ and $\gamma : Y \to X$ is given by $\gamma(\nabla) = \bigcap\{\overline{U} : U \in \nabla\} = \bigcap \nabla$ [G58].

More generally, we recall [BP96] that the *Booleanization* of a frame L is the Boolean frame $\mathfrak{B}(L)$ of regular elements of L, where $a \in L$ is *regular* if $\neg\neg a = a$. It is well known that $\mathfrak{B}(L)$ is a Boolean frame, where $\bigvee_{\mathfrak{B}(L)} S = \neg\neg\left(\bigvee_L S\right)$, $a \wedge_{\mathfrak{B}(L)} b = a \wedge_L b$, and $\neg_{\mathfrak{B}(L)} a = \neg_L a$. Moreover, if $L = \Omega(X)$, then $\mathfrak{B}(L) = \mathcal{RO}(X)$.

In general, $\mathfrak{B}(L)$ is not a subframe of L. However, $\mathfrak{B}(L)$ is always a homomorphic image of L. In fact, $\neg\neg : L \to \mathfrak{B}(L)$ is an onto frame homomorphism. The kernel of this homomorphism is the filter D of dense elements.

5.7. Lemma. *If L is compact regular, then* $\mathrm{Max}(L)$ *is homeomorphic to the Gleason cover Y of* $\mathrm{pt}(L)$.

Proof. Since $a \in L$ is dense iff $\mathrm{Max}(L) \subseteq \varphi(a)$, we see that $\mathrm{Max}(L) = \bigcap\{\varphi(a) : a \in D\}$. Therefore, $\mathrm{Max}(L)$ is the closed upset of $\mathrm{Spec}(L)$ corresponding to the filter D. Thus, $\mathrm{Max}(L)$ is homeomorphic to $\mathrm{Spec}(\mathfrak{B}(L))$. More precisely, the map $(\neg\neg)^{-1} : \mathrm{Spec}(\mathfrak{B}(L)) \to \mathrm{Spec}(L)$ induced by $\neg\neg : L \to \mathfrak{B}(L)$ is a homeomorphism from $\mathrm{Spec}(\mathfrak{B}(L))$ onto the subspace $\mathrm{Max}(L)$ of $\mathrm{Spec}(L)$. ∎

Consequently, for $L \in \mathsf{KRFrm}$, we have that $\mathrm{Min}(L)$ is homeomorphic to $\mathrm{pt}(L)$ and $\mathrm{Max}(L)$ is homeomorphic to the Gleason cover Y of $\mathrm{Min}(L)$. We next describe the map $\pi : \mathrm{Max}(L) \to \mathrm{Min}(L)$ realizing the Gleason cover.

5.8. Lemma. *Suppose L is a compact regular frame. For each $\mathfrak{p} \in \mathrm{Spec}(L)$, there is a unique $\mathfrak{m} \in \mathrm{Min}(L)$ such that $\mathfrak{m} \leqslant \mathfrak{p}$.*

Proof. As we already pointed out in the preliminaries, for each $\mathfrak{p} \in \mathrm{Spec}(L)$ there is $\mathfrak{m} \in \mathrm{Min}(L)$ with $\mathfrak{m} \leqslant \mathfrak{p}$. Suppose there also exists $\mathfrak{n} \in \mathrm{Min}(L)$ with $\mathfrak{m} \neq \mathfrak{n}$ and $\mathfrak{n} \leqslant \mathfrak{p}$. As $\mathfrak{m} \neq \mathfrak{n}$, there is a clopen upset U of $\mathrm{Spec}(L)$ with $\mathfrak{m} \in U$ and $\mathfrak{n} \notin U$. From $\mathfrak{m} \in U$ it follows that $\mathfrak{p} \in U$, and $\mathfrak{n} \notin U$ implies $\mathfrak{p} \notin R_U$. Therefore, $\mathfrak{m} \notin R_U$. Since L is compact regular, by Theorem 3.9, $\mathfrak{m} \notin \overline{R_U} = U$. The obtained contradiction proves that for each $\mathfrak{p} \in \mathrm{Spec}(L)$ there is a unique $\mathfrak{m} \in \mathrm{Min}(L)$ with $\mathfrak{m} \leqslant \mathfrak{p}$. ∎

5.9. Remark. Since L in Lemma 5.8 is compact regular, it is normal ($a \vee b = 1 \Rightarrow \exists c, d : c \wedge d = 0$, $a \vee d = 1$, and $b \vee c = 1$). Therefore, Lemma 5.8 follows from [J82, Ch. II.3.7], but the proof given above is shorter.

5.10. Remark. We recall that a space X is *normal* if disjoint closed sets can be separated by disjoint open sets, and that X is *hereditarily normal* if every subspace of X is normal. By [J82, Ch. II.3.7], X is normal iff for each $\mathfrak{p} \in \mathrm{Spec}(\Omega X)$ there is a unique $\mathfrak{m} \in \mathrm{Min}(\Omega X)$ such that $\mathfrak{m} \leqslant \mathfrak{p}$. Thus, X is hereditarily normal iff $\downarrow \mathfrak{p}$ is a chain for each $\mathfrak{p} \in \mathrm{Spec}(\Omega X)$.

If X is a non-hereditarily normal compact Hausdorff space, then there is a subspace Y of X which is not normal. Therefore, there are $\mathfrak{p} \in \mathrm{Spec}(\Omega Y)$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Min}(\Omega Y)$ with $\mathfrak{m}_1 \neq \mathfrak{m}_2$ and $\mathfrak{m}_i < \mathfrak{p}$ for $i = 1, 2$. By identifying $\mathrm{Spec}(\Omega Y)$ with a subspace of $\mathrm{Spec}(\Omega X)$, we see that there are $\mathfrak{p}, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Spec}(\Omega X)$ with $\mathfrak{m}_1 \neq \mathfrak{m}_2$ and $\mathfrak{m}_i < \mathfrak{p}$ for $i = 1, 2$.

5.11. Remark. If for each $\mathfrak{p} \in \mathrm{Spec}(L)$, there is a unique $\mathfrak{m} \in \mathrm{Min}(L)$ with $\mathfrak{m} \leqslant \mathfrak{p}$, then for every $a \in L$, the regular part R_a of $\varphi(a)$ is not only an upset, but also a downset. To see this, setting $X = \mathrm{Spec}(L)$, by Lemma 3.5, $R_a = X \smallsetminus \downarrow\uparrow(X \smallsetminus \varphi(a))$. Since $X \smallsetminus \varphi(a)$ is a clopen downset, $\uparrow(X \smallsetminus \varphi(a)) = \uparrow \mathrm{Min}(X \smallsetminus \varphi(a))$. Therefore, $R_a = X \smallsetminus \downarrow\uparrow \mathrm{Min}(X \smallsetminus \varphi(a))$. But $\mathfrak{p} \in \downarrow\uparrow \mathrm{Min}(X \smallsetminus \varphi(a))$ implies there are $\mathfrak{q} \in X$ and $\mathfrak{m} \in \mathrm{Min}(X \smallsetminus \varphi(a))$ with $\mathfrak{p} \leqslant \mathfrak{q}$ and $\mathfrak{m} \leqslant \mathfrak{q}$. Since $\mathfrak{p}, \mathfrak{q}$ are above a unique minimal point, we conclude that $\mathfrak{m} \leqslant \mathfrak{p}$, so $\mathfrak{p} \in \downarrow \mathrm{Min}(X \smallsetminus \varphi(a))$. Thus, $R_a = X \smallsetminus \uparrow \mathrm{Min}(X \smallsetminus \varphi(a))$, and hence R_a is a downset.

Define $\pi : \mathrm{Spec}(L) \to \mathrm{Min}(L)$ by assigning to $\mathfrak{p} \in \mathrm{Spec}(L)$ the unique minimal prime $\mathfrak{m} = \pi(\mathfrak{p})$ contained in $\mathfrak{p}$. It is well known that up to homeomorphism the Gleason cover of a compact Hausdorff space X is a pair (Y, γ), where Y is an extremally disconnected compact Hausdorff space and $\gamma : Y \to X$ is an irreducible map, where we recall that γ is *irreducible* if it is a continuous onto map and the image of each proper closed subset of Y is a proper subset of X.

5.12. Lemma. *The map* $\pi : \mathrm{Max}(L) \to \mathrm{Min}(L)$ *is irreducible.*

Proof. Since each proper filter is contained in a maximal filter, it is clear that π is onto. For continuity, let $a \in L$. We show that $\pi^{-1}(\varphi(a) \cap \mathrm{Min}(L)) = R_a \cap \mathrm{Max}(L)$. If $\mathfrak{p} \in R_a \cap \mathrm{Max}(L)$, then there is $b \in L$ with $\downarrow\varphi(b) \subseteq \varphi(a)$ and $\mathfrak{p} \in \varphi(b)$. Therefore, $\pi(\mathfrak{p}) \in \downarrow\varphi(b) \subseteq \varphi(a)$, and so $\pi(\mathfrak{p}) \in \varphi(a) \cap \mathrm{Min}(L)$. Conversely, if $\pi(\mathfrak{p}) \in \varphi(a) \cap \mathrm{Min}(L)$, then by Lemma 3.6, $\pi(\mathfrak{p}) \in \overline{R_a}$. This yields $\pi(p) \in R_a$ since $\pi(\mathfrak{p})$ is an isolated point by Lemma 3.1. Thus, $\pi^{-1}(\varphi(a) \cap \mathrm{Min}(L)) = R_a \cap \mathrm{Max}(L)$, and hence π is continuous.

For irreducibility, we show that $\pi(\varphi(a) \cap \mathrm{Max}(L)) = \mathrm{Min}(L)$ implies $\mathrm{Max}(L) \subseteq \varphi(a)$ for each $a \in L$. From $\pi(\varphi(a) \cap \mathrm{Max}(L)) = \mathrm{Min}(L)$ it follows that for each $\mathfrak{m} \in \mathrm{Min}(L)$ there is a maximal filter containing both a and $\mathfrak{m}$. Therefore, $a \wedge b \neq 0$ for each $b \in \mathfrak{m}$. To see that a is dense in L, by Remark 5.11, R_b is a downset. Since R_b is dense in $\varphi(b)$, each $b \neq 0$ is contained in some $\mathfrak{m} \in \mathrm{Min}(L)$. Thus, $a \wedge b \neq 0$ for each $b \neq 0$, and hence a is dense in L. This yields $\mathrm{Max}(L) \subseteq \varphi(a)$. As each closed subset of $\max(L)$ is the intersection of clopens containing it, we conclude that π is irreducible. ∎

As an immediate consequence, we obtain:

5.13. Theorem. *Let L be a compact regular frame. Then* $(\mathrm{Max}(L), \pi)$ *is up to homeomorphism the Gleason cover of* $\mathrm{Min}(L) \approx \mathrm{pt}(L)$.

5.14. Remark. That $(\mathrm{Max}(L), \pi)$ is up to homeomorphism the Gleason cover of $\mathrm{Min}(L)$ can alternatively be seen by showing that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Spec}(\mathfrak{B}(L)) & \xrightarrow{\approx} & \mathrm{Max}(L) \\ \gamma \downarrow & & \downarrow \pi \\ \mathrm{pt}(L) & \xrightarrow{\approx} & \mathrm{Min}(L) \end{array}$$

The homeomorphism $\mathrm{Spec}(\mathfrak{B}(L)) \approx \mathrm{Max}(L)$ is given by sending an ultrafilter ∇ of $\mathfrak{B}(L)$ to the maximal filter $\mathfrak{p} := \neg\neg^{-1}(\nabla) \in \mathrm{Max}(L)$, and the homeomorphism $\mathrm{pt}(L) \approx \mathrm{Min}(L)$ is given by sending $p \in \mathrm{pt}(L)$ to the minimal prime $\mathfrak{m} := p^{-1}(1) \in \mathrm{Min}(L)$. The commutativity of the diagram means that for each ultrafilter ∇ of $\mathfrak{B}(L)$, the unique $p \in \mathrm{pt}(L)$ determined by $\bigcap \nabla = \{p\}$ and the unique $\mathfrak{m} \in \mathrm{Min}(L)$ determined by $\mathfrak{m} \subseteq \mathfrak{p}$ satisfy $p^{-1}(1) = \mathfrak{m}$. Now, $p \in \bigcap \nabla$ means that $\nabla \subseteq p^{-1}(1)$. Therefore, $a \in p^{-1}(1)$ implies $\neg a \notin p^{-1}(1)$, so $\neg a \notin \nabla$, and hence $\neg\neg a \in \nabla$. Thus, $p^{-1}(1) \subseteq \mathfrak{p}$. Since $p^{-1}(1)$ is a minimal prime and $\mathfrak{m}$ is a unique minimal prime contained in $\mathfrak{p}$, we conclude that $p^{-1}(1) = \mathfrak{m}$.

6. Zero-dimensional, extremally disconnected, and scattered cases

The category KRFrm has several interesting subcategories such as the categories consisting of zero-dimensional, extremally disconnected, and scattered objects of KRFrm. In this section we study the spectra of zero-dimensional, extremally disconnected, and scattered objects of KRFrm.

Let L be a frame. We recall that $a \in L$ is *complemented* if $a \vee \neg a = 1$, and that the *center* $\mathfrak{Z}(L)$ of L is the set of complemented elements of L. It is well known that $\mathfrak{Z}(L)$ is a sublattice of L and that $\mathfrak{Z}(L)$ is a Boolean algebra. In fact, $\mathfrak{Z}(L)$ is a subalgebra of $\mathfrak{B}(L)$. A frame L is *zero-dimensional* if $a = \bigvee \{b \in \mathfrak{Z}(L) : b \leqslant a\}$ and L is *extremally disconnected* if $\mathfrak{Z}(L) = \mathfrak{B}(L)$.

Let zKFrm be the category of zero-dimensional compact frames and frame homomorphisms. Since each zero-dimensional compact frame is regular, zKFrm is a full subcategory of KRFrm. Let eKRFrm be the full subcategory of KRFrm consisting of extremally disconnected compact regular frames. Since each object of eKRFrm is zero-dimensional, we see that eKRFrm is a full subcategory of zKFrm.

It is well known that zero-dimensional compact frames dually correspond to Stone spaces, while extremally disconnected compact regular frames to extremally disconnected compact Hausdorff spaces.

6.1. Lemma. *An element a of a frame L is complemented iff* $\downarrow\varphi(a) = \varphi(a)$.

Proof. We have:

$$a \text{ is complemented iff } \varphi(a) \cup (\mathrm{Spec}(L) \setminus {\downarrow}\varphi(a)) = \mathrm{Spec}(L) \text{ iff } {\downarrow}\varphi(a) = \varphi(a).$$

■

We call $U \subseteq \mathrm{Spec}(L)$ a *biset* if U is both an upset and a downset. As follows from Lemma 6.1, $a \in L$ is complemented iff $\varphi(a)$ is a biset.

6.2. Definition. *For a clopen upset U of $\mathrm{Spec}(L)$, let*

$$Z_U := \bigcup \{V \subseteq U : V \text{ is a clopen biset}\}.$$

Clearly Z_U is the largest open biset contained in U, and we call Z_U the biregular part *of U. If $U = \varphi(a)$, then we denote Z_U by Z_a.*

6.3. Theorem. *Let L be a frame.*

1. *L is zero-dimensional iff for each $a \in L$, the biregular part of $\varphi(a)$ is dense in $\varphi(a)$.*
2. *L is extremally disconnected iff for each $\mathfrak{p} \in \mathrm{Spec}(L)$ there is a unique $\mathfrak{q} \in \mathrm{Max}(L)$ such that $\mathfrak{p} \leqslant \mathfrak{q}$.*

Proof. (1) For $a \in L$, by [BB08, Lem. 2.3], we have:

$$a = \bigvee \{b \in \mathfrak{Z}(L) : b \leqslant a\} \text{ iff } \varphi(a) = \overline{\bigcup \{\varphi(b) : \varphi(b) \subseteq \varphi(a) \text{ is a biset}\}} \text{ iff } \varphi(a) = \overline{Z_a}.$$

Thus, L is zero-dimensional iff Z_a is dense in $\varphi(a)$ for each $a \in L$.

(2) It is well known (see, e.g., [J82, Ch. III.3.5]) that L is extremally disconnected iff $\neg a \vee \neg\neg a = 1$ for each $a \in L$. It is also well known (see, e.g., [DL59]) that a Heyting algebra L satisfies $\neg a \vee \neg\neg a = 1$ for each $a \in L$ iff for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathrm{Spec}(L)$), if $\mathfrak{p} \leqslant \mathfrak{q}, \mathfrak{r}$, then there is $\mathfrak{s} \in \mathrm{Spec}(L)$ with $\mathfrak{q}, \mathfrak{r} \leqslant \mathfrak{s}$. Since for each $\mathfrak{p} \in \mathrm{Spec}(L)$ there is $\mathfrak{q} \in \mathrm{Max}(L)$ with $\mathfrak{p} \leqslant \mathfrak{q}$, the last condition is equivalent to such a $\mathfrak{q}$ being unique. ■

6.4. Remark. Let L be a frame and U be a clopen upset of $\mathrm{Spec}(L)$. It follows from the definition that $Z_U \subseteq R_U$. We show that a compact regular frame L is zero-dimensional iff $Z_U = R_U$ for each clopen upset U of $\mathrm{Spec}(L)$. Indeed, if $Z_U = R_U$, then by Lemma 3.6, $U = \overline{R_U} = \overline{Z_U}$ for each clopen upset U of $\mathrm{Spec}(L)$. Therefore, by Lemma 6.3(1), L is zero-dimensional. Conversely, suppose L is zero-dimensional and $\mathfrak{p} \in R_U$. Then $\mathfrak{p} \in V$ for some clopen upset V satisfying $\overline{{\downarrow}V} \subseteq U$. Let $\mathfrak{m} \in \mathrm{Min}(L)$ be such that $\mathfrak{m} \leqslant \mathfrak{p}$. Clearly $\mathfrak{m} \in U$. Therefore, by Lemma 6.3(1), $\mathfrak{m} \in \overline{Z_U}$. But $\mathfrak{m}$ is an isolated point by Lemma 3.1. Thus, $\mathfrak{m} \in Z_U$, which yields that $\mathfrak{p} \in Z_U$ as Z_U is a biset.

For a frame L and $a \in L$, let D_a be the filter of dense elements of the frame $[a, 1]$. Thus, $b \in D_a$ iff $b \geqslant a$ and $b \to a = a$, which holds iff $b \to a \leqslant b$. In particular, $a \leqslant a'$ implies $D_{a'} \subseteq D_a$.

6.5. Lemma. *Let L be a frame and let $X = \mathrm{Spec}(L)$ be the spectrum of L. Suppose $a, b \in L$ with $a \leqslant b$. Then $b \in D_a$ iff $\mathrm{Max}(X \smallsetminus \varphi(a)) \subseteq \varphi(b)$.*

Proof. By Esakia duality for Heyting algebras [E74], $\varphi(b \to a) = X \smallsetminus {\downarrow}(\varphi(b) \smallsetminus \varphi(a))$. Therefore, $b \in D_a$ iff $X \smallsetminus {\downarrow}(\varphi(b) \smallsetminus \varphi(a)) \subseteq \varphi(a)$, which is equivalent to $X \smallsetminus \varphi(a) \subseteq {\downarrow}(\varphi(b) \smallsetminus \varphi(a))$. Therefore, it is sufficient to show that $X \smallsetminus \varphi(a) \subseteq {\downarrow}(\varphi(b) \smallsetminus \varphi(a))$ iff $\mathrm{Max}(X \smallsetminus \varphi(a)) \subseteq \varphi(b)$.

First suppose that $X \smallsetminus \varphi(a) \subseteq {\downarrow}(\varphi(b) \smallsetminus \varphi(a))$. If $x \in \mathrm{Max}(X \smallsetminus \varphi(a))$, then $x \in X \smallsetminus \varphi(a)$, so $x \in {\downarrow}(\varphi(b) \smallsetminus \varphi(a))$. Therefore, there is $y \in \varphi(b) \smallsetminus \varphi(a)$ with $x \leqslant y$. Since $x \in \mathrm{Max}(X \smallsetminus \varphi(a))$, this yields $x = y$. Thus, $x \in \varphi(b)$, and so $\mathrm{Max}(X \smallsetminus \varphi(a)) \subseteq \varphi(b)$.

Conversely, suppose that $\mathrm{Max}(X \smallsetminus \varphi(a)) \subseteq \varphi(b)$. If $x \in X \smallsetminus \varphi(a)$, then as $X \smallsetminus \varphi(a)$ is closed, there is $y \in \mathrm{Max}(X \smallsetminus \varphi(a))$ with $x \leqslant y$. Therefore, $y \in \varphi(b)$. Since also $y \in X \smallsetminus \varphi(a)$, we see that $y \in \varphi(b) \smallsetminus \varphi(a)$. Thus, $x \in {\downarrow}(\varphi(b) \smallsetminus \varphi(a))$. ∎

Define the *coderivative operator* $\tau : L \to L$ by

$$\tau(a) = \bigwedge D_a.$$

A frame L is *scattered* if D_a is a principal filter for each $a \in L$, in which case D_a is the principal filter generated by τa. By [S82], if L is the frame of opens of a T_0-space, then τ is dual to the Cantor-Bendixson derivative; that is, for any closed set $F \subseteq X$, the set $d(F) := X \smallsetminus \tau(X \smallsetminus F)$ is the set of limit points of F. Consequently, a T_0-space is scattered iff so is its frame of opens.

6.6. Theorem. *For a frame L, the following are equivalent:*

1. *L is scattered.*
2. *The maximum of any clopen downset of* $\mathrm{Spec}(L)$ *is clopen.*
3. *The maximum of any clopen subset of* $\mathrm{Spec}(L)$ *is clopen.*

Proof. (1)⇔(2): First suppose that L is scattered. Let $a \in L$. Since D_a is the principal filter generated by τa, by Lemma 6.5, $\varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a)) \subseteq \varphi(\tau a)$. If $x \notin \varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$, then as $\varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$ is a closed upset of $\mathrm{Spec}(L)$, there is a clopen upset U of $\mathrm{Spec}(L)$ such that $\varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a)) \subseteq U$ and $x \notin U$. But $U = \varphi(b)$ for some $b \in L$. By Lemma 6.5, $b \in D_a$. Therefore, $\tau a \leqslant b$, and so $x \notin \varphi(\tau a)$. This proves that $\varphi(\tau a) = \varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$. Thus, $\varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$ is clopen, and hence so is $\mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$. Conversely, if each $\mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$ is clopen, then so is each $\varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$. Therefore, for each $a \in L$ there is $b \in L$ with $\varphi(b) = \varphi(a) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a))$. By Lemma 6.5, b is the least element of D_a. Thus, L is scattered.

(2)⇔(3): Since L is a Heyting algebra, $\mathrm{Spec}(L)$ is an Esakia space. Therefore, the downset of clopen is clopen, and for U clopen, we have $\mathrm{Max}(U) = \mathrm{Max}({\downarrow}U)$. The result follows. ∎

6.7. Definition. *For a frame L, we define its* height *(or* depth *or* Krull dimension*)* $\mathrm{ht}(L)$ *as follows. If there is a natural number $n \geqslant 0$ such that there is a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in* $\mathrm{Spec}(L)$ *and $k \leqslant n$ for all other chains $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k$ in* $\mathrm{Spec}(L)$*, then* $\mathrm{ht}(L) = n$*. Otherwise* $\mathrm{ht}(L) = \infty$.

6.8. Remark. If $h : L \to M$ is an onto frame homomorphism, then its dual $f : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$ is an embedding. Therefore, $\mathrm{ht}(M) \leqslant \mathrm{ht}(L)$.

The next theorem follows from the main result of [CLR05], but we give an alternative proof based on Esakia duality.

6.9. Theorem. *For a frame L, the following are equivalent for any $n \geqslant 0$:*

- $\mathrm{ht}(L) \geqslant n$.
- *There is a chain $1 > a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n = 0$ in L satisfying $a_{i-1} \in D_{a_i}$ for all $1 \leqslant i \leqslant n$.*

Proof. First suppose that $\mathrm{ht}(L) \geqslant n$. Then there is a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in $\mathrm{Spec}(L)$. Set $0 = a_n \notin \mathfrak{p}_n$, and for $i \in [1, n]$, if $a_i \notin \mathfrak{p}_i$, then find $a_{i-1} \notin \mathfrak{p}_{i-1}$ with $a_{i-1} \in D_{a_i}$ inductively as follows. Since $\mathfrak{p}_i \in \mathrm{Spec}(L) \smallsetminus \varphi(a_i)$ and $\mathrm{Spec}(L) \smallsetminus \varphi(a_i)$ is a downset, $\mathfrak{p}_{i-1} \in \mathrm{Spec}(L) \smallsetminus \varphi(a_i)$. Therefore, $\mathfrak{p}_{i-1} \notin \varphi(a_i) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a_i))$. Since $\varphi(a_i) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a_i))$ is a closed upset, there is $a_{i-1} \in L$ with $\mathfrak{p}_{i-1} \notin \varphi(a_{i-1})$ and $\varphi(a_i) \cup \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a_i)) \subseteq \varphi(a_{i-1})$. Thus, $a_{i-1} \notin \mathfrak{p}_{i-1}$, and by Lemma 6.5, $a_{i-1} \in D_{a_i}$. This yields the desired chain $1 > a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n = 0$ in L.

Conversely, if there is a chain $1 > a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n = 0$ in L satisfying $a_{i-1} \in D_{a_i}$ for all $i \in [1, n]$, then we have to prove that $\mathrm{ht}(L) \geqslant n$. Let $\mathrm{Spec}(L) \supsetneq \varphi(a_0) \supseteq \varphi(a_1) \supseteq \cdots \supseteq \varphi(a_n) = \varnothing$ be the corresponding chain of clopen upsets in $\mathrm{Spec}(L)$. Since $\varphi(a_0) \neq \mathrm{Spec}(L)$, there is $\mathfrak{p}_0 \in \mathrm{Spec}(L)$ with $\mathfrak{p}_0 \in \mathrm{Spec}(L) \smallsetminus \varphi(a_0)$. For $i \in [1, n]$, if $\mathfrak{p}_{i-1} \in \mathrm{Spec}(L) \smallsetminus \varphi(a_{i-1})$ is already found, then find $\mathfrak{p}_i \supsetneq \mathfrak{p}_{i-1}$ inductively as follows. As $\varphi(a_{i-1}) \supseteq \varphi(a_i)$, we see that $\mathfrak{p}_{i-1} \in \mathrm{Spec}(L) \smallsetminus \varphi(a_i)$. Because $\mathrm{Spec}(L) \smallsetminus \varphi(a_i)$ is clopen, there is $\mathfrak{p}_i \in \mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a_i))$ with $\mathfrak{p}_{i-1} \subseteq \mathfrak{p}_i$. Since $a_{i-1} \in D_{a_i}$, by Lemma 6.5, $\mathrm{Max}(\mathrm{Spec}(L) \smallsetminus \varphi(a_i)) \subseteq \varphi(a_{i-1})$. Therefore, $\mathfrak{p}_i \in \varphi(a_{i-1})$. Thus, $\mathfrak{p}_i \neq \mathfrak{p}_{i-1}$ as $\mathfrak{p}_{i-1} \notin \varphi(a_{i-1})$. This yields a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in $\mathrm{Spec}(L)$, so $\mathrm{ht}(L) \geqslant n$. ∎

6.10. Definition. *We say that a frame L is of* rank n *if $\tau^{n+1}(0) = 1$ but $\tau^n(0) \neq 1$.*

6.11. Theorem. *A scattered frame L is of height n iff it is of rank n.*

Proof. First suppose that L is of height n. By Theorem 6.9, there is a chain $1 > a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n = 0$ in L with $a_{i-1} \in D_{a_i}$ for each $i \in [1, n]$. Since $a_{i-1} \in D_{a_i}$ implies $\tau(a_i) \leqslant a_{i-1}$, we see that

$$\begin{array}{rcccl} \tau(0) & \leqslant & \tau(a_n) & \leqslant & a_{n-1} \\ \tau^2(0) & \leqslant & \tau(a_{n-1}) & \leqslant & a_{n-2} \\ & & \vdots & & \\ \tau^n(0) & \leqslant & \tau(a_1) & \leqslant & a_0 < 1. \end{array}$$

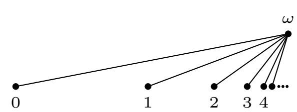
Therefore, $\tau^n(0) \neq 1$. If $\tau^{n+1}(0) \neq 1$, then consider the chain $1 > \tau^{n+1}(0) \geqslant \tau^n(0) \geqslant \cdots \geqslant \tau(0) \geqslant 0$. Since L is scattered, each D_a is the principal filter generated by τa. Thus, $\tau^{i+1}(0) \in D_{\tau^i(0)}$ for each i. Applying Theorem 6.9 then yields a chain in $\mathrm{Spec}(L)$ of height $n+1$, a contradiction. Consequently, $\tau^{n+1}(0) = 1$, and hence L is of rank n.

Conversely, suppose that L is of rank n. Consider the chain $0 < \tau(0) < \tau^2(0) < \cdots < \tau^n(0) < 1$ in L. Since L is scattered, $\tau^{i+1}(0) \in D_{\tau^i(0)}$ for each i. Therefore, by Theorem 6.9, there is a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in $\mathrm{Spec}(L)$. Moreover, if there is a chain in $\mathrm{Spec}(L)$ of length $k > n$, then Theorem 6.9 yields a chain $1 > a_0 \geqslant a_1 \geqslant \cdots \geqslant a_k = 0$ in L with $a_{i-1} \in D_{a_i}$ for each $i \in [1,k]$. Thus, the same argument as in the displayed inequalities above gives $\tau^{n+1}(0) \neq 1$, a contradiction. Consequently, L is of height n. ∎

6.12. Remark. For compact regular frames, the assumption in Theorem 6.11 that L is scattered becomes redundant. To see this, by Isbell duality, a compact regular frame is the frame of open sets of a compact Hausdorff space. By [S71, Thm. 8.5.4], a compact Hausdorff space X is not scattered iff there is a continuous map f from X onto the closed unit interval $[0,1]$. Now, the frame $\Omega[0,1]$ is of infinite height. This follows, for example, from the fact that for each natural number n, the space $[0,1]$ has a (closed) subspace homeomorphic to the ordinal $\omega^n + 1$. Therefore, there is an onto frame homomorphism $h : \Omega[0,1] \to \Omega(\omega^n + 1)$. Thus, by Remark 6.8, $\mathrm{ht}\,\Omega(\omega^n+1) \leqslant \mathrm{ht}\,\Omega[0,1]$. But $\Omega(\omega^n+1)$ is a scattered frame of rank n, so $\mathrm{ht}\,\Omega(\omega^n+1) = n$ by Theorem 6.11. Therefore, by Theorem 6.9, for each $n \geqslant 0$, there is a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in $\mathrm{Spec}(\Omega[0,1])$. But since f is onto, f^{-1} is an embedding of $\Omega[0,1]$ into $\Omega(X)$, and so $(f^{-1})^{-1} : \mathrm{Spec}(\Omega X) \to \mathrm{Spec}(\Omega[0,1])$ is onto. Thus, by Corollary 4.3, for each $n \geqslant 0$, there is a chain $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ in $\mathrm{Spec}(\Omega X)$. This yields that ΩX also has infinite height. Consequently, a compact regular frame of finite height is necessarily scattered.

As the following example shows, regularity is essential in Remark 6.12.

6.13. Example. Let X be the ordinal $\omega + 1$ with its usual interval topology, but ordered as shown below.



It is well known (see, e.g., [E85, Thm. III.2.4]) that X is an Esakia space. In fact, the clopen upsets of X are isomorphic to the frame L of cofinite subsets of ω together with the empty set. Consequently, L is a coherent frame. Clearly $\mathrm{ht}(L) = 1$. But L is not scattered since every nonzero element of L is dense, so the filter of dense elements of L is not principal. This can also be seen by observing that $\mathrm{Max}(X) = \{\omega\}$ is not clopen, so L is not scattered by Theorem 6.6.

Summing up, we have:

6.14. Corollary. *Let L be compact regular. Then:*

1. *L is zero-dimensional iff the biregular part of each clopen upset U of* $\mathrm{Spec}(L)$ *is dense in U.*

2. *L is extremally disconnected iff for each* $\mathfrak{p} \in \mathrm{Spec}(L)$ *there is a unique* $\mathfrak{q} \in \mathrm{Max}(L)$ *with* $\mathfrak{p} \leqslant \mathfrak{q}$.

3. *L is scattered iff* $\mathrm{Max}(U)$ *is clopen for each clopen U of* $\mathrm{Spec}(L)$.

4. *L is of finite height n iff L is of finite rank n.*

We conclude the paper with some examples of spectra of compact regular frames.

6.15. Example. Let L be the frame of opens of $\omega + 1$. Then L is a compact regular scattered frame. The rank of L is 1, so by Theorem 6.11, $\mathrm{ht}(L) = 1$. The minimum of $\mathrm{Spec}(L)$ is homeomorphic to $\omega + 1$, and the maximum to the Gleason cover of $\omega + 1$. But $\omega + 1$ is homeomorphic to the one-point compactification $\alpha\omega$ of ω, while the Gleason cover of $\omega + 1$ is homeomorphic to the Stone-Čech compactification $\beta\omega$ of ω.

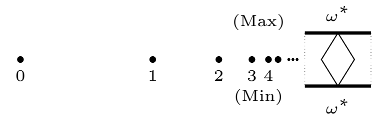
The isolated points of $\omega + 1$, by Lemma 6.1, give rise to clopen bisets in $\mathrm{Spec}(L)$, which appear as simultaneously minimal and maximal points of $\mathrm{Spec}(L)$. The single non-isolated point ω of $\omega + 1$ is the only minimal point of $\mathrm{Spec}(L)$ that is not a maximal point. Since a minimal point $\mathfrak{p}$ is below a maximal point $\mathfrak{q}$ iff $\pi(\mathfrak{q}) = \mathfrak{p}$, we see that the point ω is underneath the entire remainder $\omega^* := \beta\omega \smallsetminus \omega$. Thus, we obtain the following picture:

$\omega + 1$ $\qquad$ $\mathrm{Spec}(L)$

Similar but a more complicated picture arises from the frame L_n of opens of $\omega^n + 1$, $n > 1$. Since L_n is scattered and the rank of L_n is n, by Theorem 6.11, $\mathrm{ht}(L_n) = n$. Thus, increasing n, we get a fractal-like structure: By Theorem 6.6, $\mathrm{Max}(L_n)$ is clopen, and is homeomorphic to the Stone-Čech compactification of the discrete space of isolated points of $\omega^n + 1$. The complement of $\mathrm{Max}(L_n)$ is a clopen downset, which up to isomorphism, is the spectrum of the frame of opens of the space of limit points of $\omega^n + 1$. This subspace is homeomorphic to $\omega^{n-1} + 1$. Thus, $\mathrm{Spec}(L_n)$ has clopen maximum homeomorphic to the Stone-Čech compactification of a countable discrete space, and its complementary clopen downset is up to isomorphism $\mathrm{Spec}(L_{n-1})$.

6.16. Example. Let M be the frame of opens of the Stone-Čech compactification $\beta\omega$ of ω. The spectrum of M is much more complicated than those in the previous example. Since $\beta\omega$ is extremally disconnected, by Lemma 6.3(2), the minimum and maximum of $\mathrm{Spec}(M)$ are homeomorphic. However, the "middle part" of $\mathrm{Spec}(M)$ is rather complicated. For example, since $\beta\omega$ is not hereditarily normal (see, e. g., [E89, Example 3.6.19]), by Remark 5.10, there are some downward branchings in the middle of $\mathrm{Spec}(M)$. In addition, $\mathrm{Spec}(M)$ has infinite height. A rough sketch of $\mathrm{Spec}(M)$ looks as follows:

$\beta\omega$



$\mathrm{Spec}(M)$

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