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A STRUCTURE THEOREM FOR QUASI-HOPF BIMODULE COALGEBRAS

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ABSTRACT. Let H be a quasi-Hopf algebra. We show that any H-bimodule coalgebra C for which there exists an H-bimodule coalgebra morphism $\nu : C \to H$ is isomorphic to what we will call a smash product coalgebra. To this end, we use an explicit monoidal equivalence between the category of two-sided two-cosided Hopf modules over H and the category of left Yetter-Drinfeld modules over H. This categorical method allows also to reobtain the structure theorem for a quasi-Hopf (bi)comodule algebra given in [Panaite, Van Oystaeyen, 2007] and [Dello, Panaite, Van Oystaeyen, Zhang, 2016].

1. Introduction

Two-sided two-cosided Hopf modules were introduced by Woronowicz [Woronowicz, 1989] under the name of bicovariant bimodules, as a tool in the study of non-commutative differential calculus on quantum groups. He also extended the structure theorem of Hopf modules to the category of Hopf bimodules ${}_H\mathcal{M}_H^H$ and two-sided two-cosided Hopf modules ${}_H^H\mathcal{M}_H^H$ over a Hopf algebra H. Later on, Schauenburg proved in [Schauenburg, 1994] that the structure theorems provide the classification of Hopf bimodules and two-sided two-cosided Hopf modules in the form of category equivalences ${}_H\mathcal{M}_H^H \cong {}_H\mathcal{M}$ and ${}_H^H\mathcal{M}_H^H \cong {}_H^H\mathcal{Y}D$, where ${}_H\mathcal{M}$ is the category of left H-representations and ${}_H^H\mathcal{Y}D$ is the category of left Yetter-Drinfeld modules over H. These equivalences are even monoidal and they can be regarded as a coordinate free versions of the classification in [Woronowicz, 1989]. Using categorical techniques, Schauenburg [Schauenburg, 2012] also proved that all the results mentioned above remain valid in the setting provided by quasi-Hopf algebras. Despite the fact that a quasi-Hopf algebra H is not a coassociative coalgebra, and thus we cannot define H-comodules, one can still consider the categories ${}_H\mathcal{M}_H^H$ and ${}_H^H\mathcal{M}_H^H$, by using the framework of monoidal categories. Namely, H is a coalgebra within the monoidal category of H-bimodules ${}_H\mathcal{M}_H$, and therefore we can define ${}_H\mathcal{M}_H^H := ({}_H\mathcal{M}_H)^H$ and ${}_H^H\mathcal{M}_H^H := {}^H({}_H\mathcal{M}_H)^H$, the category of right H-corepresentations and of H-bicomodules, respectively, within ${}_H\mathcal{M}_H$.

Our main goal is to give structure theorems for algebras and coalgebras in ${}_H\mathcal{M}_H^H$, and respectively in ${}_H^H\mathcal{M}_H^H$. On one hand, due to the monoidal category equivalences men-

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tioned above, these are classified by algebras and coalgebras in ${}_H\mathcal{M}$, and respectively in ${}_H^H\mathcal{Y}D$. Since our equivalences are monoidal, and monoidal functors carry (co)algebras to (co)algebras we get that, up to an isomorphism, the algebras are smash product algebras in the sense of [Bulacu, Panaite, Van Oystaeyen, 2000], while coalgebras are H-corings whose structure is completely determined by a cowreath defined by an H-module coalgebra as in [Bulacu, Caenepeel, 2014]. Of course, in the two-sided two-cosided case some additional structures on these objects are required, mainly because of the extra left H-colinear condition. On the other hand, it turns out that algebras in ${}_H\mathcal{M}_H^H$ (resp. ${}_H^H\mathcal{M}_H^H$) are H-(bi)comodule algebras $\mathbf{A}$ in the sense of Hausser and Nill [Hausser, Nill, 1999], equipped with an H-(bi)comodule algebra map $i : H \to \mathbf{A}$. This is nothing but the context considered in [Panaite, Van Oystaeyen, 2007] and [Dello, Panaite, Van Oystaeyen, Zhang, 2016], so the main results in loc. cit. actually classify the algebras in ${}_H\mathcal{M}_H^H$ and ${}_H^H\mathcal{M}_H^H$, respectively. As we have explained, one gets these for free from the above monoidal category equivalences. Furthermore, adapting categorical techniques from [Bespalov, Drabant, 1998], we show that coalgebras in ${}_H^H\mathcal{M}_H^H$ can be also characterized as pairs (C, H) consisting of an H-bimodule coalgebra C and an H-bimodule coalgebra morphism $\pi : C \to H$, but also as smash product coalgebras of a coalgebra in ${}_H^H\mathcal{Y}D$ and H. In particular this leads to a structure theorem for quasi-Hopf bimodule coalgebras.

The paper is organized as follows. In Section 2 we briefly recall the axioms of a quasi-Hopf algebra H and its basic properties, the formalism of monoidal categories, functors and monoidal equivalences, and review the Hausser and Nill monoidal equivalence between ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$, respectively. In Section 3 we give the structure of an algebra in ${}_H\mathcal{M}_H^H$. The main result here is Theorem 3.7, an equivalent version of [Panaite, Van Oystaeyen, 2007, Theorem 2.5]. For its proof we use an alternative monoidal equivalence between ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$. The latter is based on a concrete definition of the space of coinvariants of a quasi-Hopf bimodule and has the advantage that provides an explicit monoidal equivalence between ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{Y}D$, too. In Section 4, we use this alternative monoidal equivalence between ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{Y}D$ to give in Theorem 4.11 the structure of an algebra in ${}_H^H\mathcal{M}_H^H$. We also show that Theorem 4.11 and [Dello, Panaite, Van Oystaeyen, Zhang, 2016, Theorem 1.7] are equivalent. The coalgebra case is treated in Section 5. The main results of this last section are Theorem 5.3 and Theorem 5.6 which in particular give a structure theorem for a quasi-Hopf bimodule coalgebra.

We end our introduction with a philosophical note. Although the definition of quasi-Hopf algebras is - essentially - very natural, the explicit formulas and computations are often quite technical. This time, due to the categorical results we proved, we got some complicated formulas for free and were able to avoid most of the computations involving them.

2. Preliminaries

2.1. Quasi-bialgebras and quasi-Hopf algebras. We work over a field k. All algebras, linear spaces, etc. will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld

[Drinfeld, 1990], a quasi-bialgebra is a quadruple $(H, \Delta, \varepsilon, \Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

$$(\mathrm{Id}_H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \mathrm{Id}_H)(\Delta(h))\Phi^{-1}, \tag{2.1}$$

$$(\mathrm{Id}_H \otimes \varepsilon)(\Delta(h)) = h \quad , \quad (\varepsilon \otimes \mathrm{Id}_H)(\Delta(h)) = h, \tag{2.2}$$

for all $h \in H$, where Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\mathrm{Id}_H \otimes \Delta \otimes \mathrm{Id}_H)(\Phi)(\Phi \otimes 1)$$
$$= (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H)(\Phi), \tag{2.3}$$

$$(\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id}_H)(\Phi) = 1 \otimes 1. \tag{2.4}$$

The map Δ is called the coproduct or the comultiplication, ε is the counit, and Φ is the reassociator. As for Hopf algebras we denote $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes \mathrm{Id}_H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2 \quad , \quad (\mathrm{Id}_H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \cdots$$
$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \cdots$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{2.5}$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2 \beta S(x^3) = 1. \tag{2.6}$$

Our definition of a quasi-Hopf algebra is different from the one given by Drinfeld [Drinfeld, 1990] in the sense that we do not require the antipode to be bijective. In the case where H is finite dimensional or quasi-triangular, bijectivity of the antipode follows from the other axioms, see [Bulacu, Caenepeel, 2003] and [Bulacu, Nauwelaerts, 2003], so the two definitions are equivalent. Anyway, the bijectivity of the antipode S will be implicitly understood in the case when S^{-1}, the inverse of S, appears is formulas or computations.

It is well known that the antipode of a Hopf algebra is an anti-morphism of coalgebras. For a quasi-Hopf algebra, we have the following statement: there exists an invertible element $f = f^1 \otimes f^2 \in H \otimes H$, called the Drinfeld twist or the gauge transformation, such that $\varepsilon(f^1)f^2 = \varepsilon(f^2)f^1 = 1$ and

$$f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\mathrm{cop}}(h)), \tag{2.7}$$

for all $h \in H$. f can be described explicitly: first we define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(x^1X^2)\alpha x^2X_1^3 \otimes S(X^1)\alpha x^3X_2^3 \overset{(2.3,2.5)}{=} S(X^2x_2^1)\alpha X^3x^2 \otimes S(X^1x_1^1)\alpha x^3, \quad (2.8)$$

$$\delta = X_1^1x^1\beta S(X^3) \otimes X_2^1x^2\beta S(X^2x^3) \overset{(2.3,2.5)}{=} x^1\beta S(x_2^3X^3) \otimes x^2X^1\beta S(x_1^3X^2). \quad (2.9)$$

With this notation f and f^{-1} are given by the formulas

$$f = (S \otimes S)(\Delta^{\rm op}(x^1))\gamma\Delta(x^2\beta S(x^3)), \quad (2.10)$$

$$f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\rm cop}(x^3)). \quad (2.11)$$

Moreover, f satisfies the following relations:

$$f\Delta(\alpha) = \gamma \ \ , \ \ \Delta(\beta)f^{-1} = \delta. \quad (2.12)$$

We will need the appropriate generalization of the formula $h_1 \otimes h_2S(h_3) = h \otimes 1$ in classical Hopf algebra theory. Following [Hausser, Nill, 1999] and [Hausser, Nill, unpublished], we define

$$p_R = p^1 \otimes p^2 = x^1 \otimes x^2\beta S(x^3), \quad (2.13)$$

$$q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2, \quad (2.14)$$

$$p_L = \tilde{p}^1 \otimes \tilde{p}^2 = X^2S^{-1}(X^1\beta) \otimes X^3, \quad (2.15)$$

$$q_L = \tilde{q}^1 \otimes \tilde{q}^2 = S(x^1)\alpha x^2 \otimes x^3. \quad (2.16)$$

For all $h \in H$, we then have

$$(1 \otimes S^{-1}(h_2))q_R\Delta(h_1) = (h \otimes 1)q_R, \quad (2.17)$$

$$\Delta(h_2)p_L(S^{-1}(h_1) \otimes 1) = p_L(1 \otimes h). \quad (2.18)$$

We also have that

$$q^1Q_1^1x^1 \otimes q^2Q_2^1x^2 \otimes Q^2x^3 = q^1X_1^1 \otimes S^{-1}(f^2X^3)q_1^2X_{(2,1)}^1 \otimes S^{-1}(f^1X^2)q_2^2X_{(2,2)}^1, \quad (2.19)$$

$$X^1 \otimes S(X^2)\tilde{q}^1X_1^3 \otimes \tilde{q}^2X_2^3 = q^1x_1^1 \otimes S(q^2x_2^1)x^2 \otimes x^3, \quad (2.20)$$

$$q_1^1x^1 \otimes q_2^1x^2 \otimes q^2x^3 = X^1 \otimes q^1X_1^2 \otimes S^{-1}(X^3)q^2X_2^2, \quad (2.21)$$

where $Q^1 \otimes Q^2$ is a second copy for q_R in $H \otimes H$.

2.2. Monoidal categories. For the definition of a (co)algebra in a monoidal category $\mathcal{C}$ and related topics we refer to [Kassel, 1995] and [Majid, 1995]. Usually, for a monoidal category $\mathcal{C}$, we denote by $\otimes$ the tensor product, by $\underline{1}$ the unit object, and by a, l, r the associativity constraint and the left and right unit constraints, respectively.

If H is a quasi-bialgebra, then the category ${}_H\mathcal{M}$ of left H-representations is monoidal. If U, V are left H-modules then the tensor product between U and V is the tensor product over k equipped with the left H-module structure given by Δ, i.e. $h\cdot(u\otimes v) = h_1\cdot u\otimes h_2\cdot v$, for all $h \in H$, $u \in U$ and $v \in V$. The associativity constraint on ${}_H\mathcal{M}$ is the following: for $U, V, W \in {}_H\mathcal{M}$, $a_{U,V,W} : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$ is given by

$$a_{U,V,W}((u \otimes v) \otimes w) = X^1 \cdot u \otimes (X^2 \cdot v \otimes X^3 \cdot w).$$

The unit object is k considered as a left H-module via ε, the counit of H. The left and right unit constraints are the same as for the category ${}_k\mathcal{M}$ of k-vector spaces.

A (co)algebra in ${}_H\mathcal{M}$ is called a left H-module (co)algebra.

${}_H\mathcal{M}_H$, the category of H-bimodules is monoidal as well, since it can be identified with the category of left modules over the quasi-Hopf algebra $H^{\rm op} \otimes H$, where $H^{\rm op}$ is the opposite quasi-bialgebra associated to H. We provide the explicit construction of the monoidal structure on ${}_H\mathcal{M}_H$.

- The associativity constraints $a'_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ are given by
$$a'_{M,N,P}((m \otimes n) \otimes p) = X^1 \cdot m \cdot x^1 \otimes (X^2 \cdot n \cdot x^2 \otimes X^3 \cdot p \cdot x^3); \tag{2.22}$$
- the unit object is k viewed as an H-bimodule via the counit ε of H;
- the left and right unit constraints are given by the natural isomorphisms $k \otimes M \cong M \cong M \otimes k$.

We call a (co)algebra in ${}_H\mathcal{M}_H$ an H-bimodule (co)algebra.

A (op)monoidal functor between two monoidal categories is a functor that respects the two monoidal structures. More precisely:

2.3. Definition. *Let $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$ and $(\mathcal{C}', \otimes', \underline{1}', a', l', r')$ be monoidal categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor.*

i) F is called monoidal if there exists a family of morphisms

$$\varphi_2 = (\varphi_{2,X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y))_{X,Y \in \mathcal{C}},$$

natural in X and Y, and $\varphi_0 : \underline{1}' \to F(\underline{1})$ a morphism in $\mathcal{C}'$ such that, for all $X, Y, Z \in \mathcal{C}$,

$$\begin{aligned}
&\varphi_{2,X,Y\otimes Z}(Id_{F(X)} \otimes' \varphi_{2,Y,Z})a_{F(X),F(Y),F(Z)} = F(a_{X,Y,Z})\varphi_{2,X\otimes Y,Z}(\varphi_{2,X,Y} \otimes' \mathrm{Id}_{F(Z)}),\\
&F(l_X)\varphi_{2,\underline{1},X}(\varphi_0 \otimes' \mathrm{Id}_{F(X)}) = l'_{F(X)},\\
&F(r_X)\varphi_{2,X,\underline{1}}(\mathrm{Id}_{F(X)} \otimes' \varphi_0) = r'_{F(X)}.
\end{aligned}$$

ii) F is called opmonoidal if there exists a family of morphisms

$$\psi_2 = (\psi_{2,X,Y} : F(X \otimes Y) \to F(X) \otimes' F(Y))_{X,Y \in \mathcal{C}},$$

natural in X and Y, and $\psi_0 : F(\underline{1}) \to \underline{1}'$ a morphism in $\mathcal{C}'$ such that, for all $X, Y, Z \in \mathcal{C}$,

$$(Id_{F(X)} \otimes' \psi_{2,Y,Z})\psi_{2,X,Y\otimes Z}F(a_{X,Y,Z}) = a_{F(X),F(Y),F(Z)}(\psi_{2,X,Y} \otimes' \mathrm{Id}_{F(Z)})\psi_{2,X\otimes Y,Z}, \tag{2.23}$$
$$l'_{F(X)}(\psi_0 \otimes' \mathrm{Id}_{F(X)})\psi_{2,\underline{1},X} = F(l_X), \tag{2.24}$$
$$r'_{F(X)}(\mathrm{Id}_{F(X)} \otimes' \psi_0)\psi_{2,X,\underline{1}} = F(r_X). \tag{2.25}$$

iii) F is called a strong monoidal functor if it is monoidal and, moreover, φ_0 and φ_2 are defined by isomorphisms in $\mathcal{C}'$. Equivalently, F is strong monoidal if it is opmonoidal and, moreover, ψ_0 and ψ_2 are defined by isomorphisms in $\mathcal{C}'$.

It is well known that a (op)monoidal functor carries (co)algebras to (co)algebras, see [Majid, 1995].

The notion of natural transformation extends to the monoidal setting as follows.

2.4. Definition. *Let $\mathcal{C}$, $\mathcal{C}'$ be monoidal categories and $(F, \varphi_2^F, \varphi_0^F), (G, \varphi_2^G, \varphi_0^G) : \mathcal{C} \to \mathcal{C}'$ monoidal functors. A natural monoidal transformation ω from $(F, \varphi_2^F, \varphi_0^F)$ to $(G, \varphi_2^G, \varphi_0^G)$ is a natural transformation $\omega : F \to G$ such that, for any objects X, Y of $\mathcal{C}$, the following equalities hold:*

$$\omega_{X\otimes Y}\varphi^F_{2,X,Y} = \varphi^G_{2,X,Y}(\omega_X \otimes' \omega_Y),$$
$$\omega_{\underline{1}}\varphi^F_0 = \varphi^G_0.$$

The transformation ω is called a natural monoidal isomorphism if ω is both a natural monoidal transformation and a natural isomorphism.

Reversing the arrows in the above diagrams we get the definition of a natural opmonoidal transformation between two opmonoidal functors.

We are now able to define the concept of monoidal equivalence, a concept intensively used throughout this paper.

2.5. Definition. *Let $\mathcal{C}$, $\mathcal{C}'$ be monoidal categories and $F : \mathcal{C} \to \mathcal{C}'$ a monoidal (opmonoidal, resp. strong monoidal) functor. We say that F is a monoidal (opmonoidal, resp. strong monoidal) equivalence if there exists a monoidal (opmonoidal, resp. strong monoidal) functor $G : \mathcal{C}' \to \mathcal{C}$ such that FG is naturally monoidally (opmonoidally, resp. strong monoidally) isomorphic to $\mathrm{Id}_{\mathcal{C}'}$ and GF is naturally monoidally (opmonoidally, resp. strongly monoidally) isomorphic to $\mathrm{Id}_{\mathcal{C}}$.*

If a functor $F : \mathcal{C} \to \mathcal{C}'$ defines a monoidal (opmonoidal, resp. strong monoidal) equivalence between $\mathcal{C}$ and $\mathcal{C}'$ we say that the categories $\mathcal{C}$ and $\mathcal{C}'$ are monoidally (opmonoidally, resp. strongly monoidally) equivalent.

2.6. The Hausser and Nill structure theorem for quasi-Hopf bimodules. Throughout this subsection, H is a quasi-bialgebra or a quasi-Hopf algebra with antipode S and distinguished elements α and β.

Although H is not necessarily a coassociative coalgebra, its comultiplication Δ and its counit ε endow H with a coalgebra structure in the monoidal category of H-bimodules, ${}_H\mathcal{M}_H$. Otherwise stated, H is an H-bimodule coalgebra. This allows us to define ${}_H\mathcal{M}_H^H := ({}_H\mathcal{M}_H)^H$ and ${}_H^H\mathcal{M}_H := {}^H({}_H\mathcal{M}_H)$ as the categories of right, respectively left, H-corepresentations within ${}_H\mathcal{M}_H$. These categories were introduced by Hausser and Nill in [Hausser, Nill, unpublished] under the name of the categories of quasi-Hopf H-bimodules.

Explicitly, a right quasi-Hopf H-bimodule M is an H-bimodule together with an H-bimodule map $\rho : M \ni m \mapsto m_{(0)} \otimes m_{(1)} \in M \otimes H$ such that the following relations hold:

$$(\mathrm{Id}_M \otimes \varepsilon) \circ \rho = \mathrm{Id}_M, \tag{2.26}$$
$$\Phi \cdot (\rho \otimes \mathrm{Id}_M)(\rho(m)) = (\mathrm{Id}_M \otimes \Delta)(\rho(m)) \cdot \Phi, \quad \forall \;\; m \in M. \tag{2.27}$$

Up to an isomorphism, a quasi-Hopf H-bimodule is of the form $N \otimes H$, for a certain left H-module N; see the details below.

We have a well defined functor $F : {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$. If $M \in {}_H\mathcal{M}$ then $F(M) = M \otimes H$ regarded as an object in ${}_H\mathcal{M}_H^H$ via

$$h(m \otimes h)h' = h_1 \cdot m \otimes h_2 h h',$$
$$\rho(m \otimes h) = x^1 \cdot m \otimes x^2 h_1 \otimes x^3 h_2,$$

for all $m \in M$ and $h, h' \in H$. F sends a morphism f in ${}_H\mathcal{M}$ to $f \otimes \mathrm{Id}_H$.

The functor F provides a (monoidal) equivalence. To see this we have to recall first the structure theorem for quasi-Hopf bimodules proved by Hausser and Nill [Hausser, Nill, unpublished].

For $M \in {}_H\mathcal{M}_H^H$, define $E : M \to M$ given by

$$E(m) = X^1 \cdot m_{(0)} \cdot \beta S(X^2 m_{(1)}) \alpha X^3, \tag{2.28}$$

for all $m \in M$, where $M \ni m \mapsto \rho_M(m) := m_{(0)} \otimes m_{(1)} \in M \otimes H$ denotes the right coaction of H on M. The space $M^{\mathrm{co}H} = \{n \in M \mid E(n) = n\}$ is called the space of coinvariants of M. By [Hausser, Nill, unpublished, Corollary 3.9] it can be also described as

$$M^{\mathrm{co}H} = \{n \in M \mid \rho(n) = E(x^1 \cdot n) \cdot x^2 \otimes x^3\}. \tag{2.29}$$

$M^{\mathrm{co}H}$ becomes a left H-module under the action given by $h \neg n = E(h \cdot n)$, for all $h \in H$ and $n \in M^{\mathrm{co}H}$. Also, for further use, record that the following relations hold:

$$h \cdot E(m) = [h_1 \neg E(m)] \cdot h_2 \;, \tag{2.30}$$
$$E(m \cdot h) = \varepsilon(h) E(m) \;, \;\; E(h \cdot E(m)) = E(h \cdot m) \;, \tag{2.31}$$
$$E^2 = E, E(m_{(0)}) \cdot m_{(1)} = m \text{ and } E(E(m)_{(0)}) \otimes E(m)_{(1)} = E(m) \otimes 1, \tag{2.32}$$

for all $m \in M$ and $h \in H$.

The following structure theorem for quasi-Hopf bimodules is [Hausser, Nill, unpublished, Theorem 3.8].

2.7. Theorem. *If M is a right quasi-Hopf H-bimodule then the map*

$$\nu_M : \;\; M^{\mathrm{co}H} \otimes H \to M, \quad \nu_M(n \otimes h) = n \cdot h, \quad \forall \;\; n \in M^{\mathrm{co}H}, \; h \in H$$

is an isomorphism of right quasi-Hopf H-bimodules, where $M^{\mathrm{co}H} \otimes H$ is a right quasi-Hopf H-bimodule via the structure defined by

$$a \cdot (n \otimes h) \cdot b = E(a_1 \cdot n) \otimes a_2 h b \;\; \text{and} \;\; \rho(n \otimes h) = E(x^1 \cdot n) \otimes x^2 h_1 \otimes x^3 h_2,$$

for all $n \in N$*,* $a, h, b \in H$*. The inverse of* ν *is given by*

$$\nu_M^{-1}(m) = E(m_{(0)}) \otimes m_{(1)}, \quad \forall \quad m \in M. \tag{2.33}$$

As we already mentioned, the structure theorem presented above was used by Hausser and Nill in [Hausser, Nill, unpublished] in order to extend to the quasi-Hopf algebra setting a result of Schauenburg [Schauenburg, 1994, Theorem 5.7], which says that ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$ are monoidally equivalent categories. Here ${}_H\mathcal{M}_H^H$ is viewed as a (strict) monoidal category with tensor product $\otimes_H$, the usual tensor product over H, unit object H considered as a quasi-Hopf H-bimodule under its regular multiplication and comultiplication, and canonical associativity constraint and left and right unit constraints, respectively.

Actually, for M, N two right quasi-Hopf H-bimodules, $M \otimes_H N$ is a right quasi-Hopf H-bimodule with the structure given by

$$h \cdot (m \otimes_H n) \cdot h' = h \cdot m \otimes_H n \cdot h', \tag{2.34}$$

$$\rho_{M\otimes_H N} : M \otimes_H N \ni m \otimes_H n \mapsto m_{(0)} \otimes_H n_{(0)} \otimes m_{(1)}n_{(1)} \in M \otimes_H N \otimes H, \tag{2.35}$$

for all $m \in M$, $n \in N$ and $h, h' \in H$.

The equivalence functor between the categories ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$ is $G : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}$, defined as follows. If $M \in {}_H\mathcal{M}_H^H$ then $G(M) = M^{\mathrm{co}H}$, regarded as a left H-module via the action $\neg$. G sends a morphism $f : M \to N$ in ${}_H\mathcal{M}_H^H$ to its restriction at $M^{\mathrm{co}H}$ and corestriction at $N^{\mathrm{co}H}$, a well defined left H-linear morphism.

If we consider the maps $i_{M,N} : M^{\mathrm{co}H} \otimes N^{\mathrm{co}H} \to M \otimes_H N$ and $j_{M,N} : M \otimes_H N \to M^{\mathrm{co}H} \otimes N^{\mathrm{co}H}$, determined by $i_{M,N}(m \otimes n) = E_M(X^1 \cdot m) \otimes_H E_N(X^2 \cdot n) \cdot X^3$ and $j_{M,N}(m \otimes_H n) = E_M(m_{(0)}) \otimes E_N(m_{(1)} \cdot n)$, for all $m \in M$ and $n \in N$, we then have $j_{M,N} i_{M,N} = \mathrm{Id}_{M^{\mathrm{co}H} \otimes N^{\mathrm{co}H}}$ and $i_{M,N} j_{M,N} = E_{M\otimes_H N}$. Consequently, the image of $i_{M,N}$ is $(M \otimes_H N)^{\mathrm{co}H}$, and so $i_{M,N}$ induces a left H-module isomorphism between $M^{\mathrm{co}H} \otimes N^{\mathrm{co}H}$ and $(M \otimes_H N)^{\mathrm{co}H}$. We will denote it by

$$\phi_{2,M,N} : M^{\mathrm{co}H} \otimes N^{\mathrm{co}H} \to (M \otimes_H N)^{\mathrm{co}H}.$$

If $\iota : (M \otimes_H N)^{\mathrm{co}H} \hookrightarrow M \otimes_H N$ is the inclusion map, then the inverse of $\phi_{2,M,N}$ is $\phi_{2,M,N}^{-1} := j_{M,N}\iota : (M \otimes_H N)^{\mathrm{co}H} \to M^{\mathrm{co}H} \otimes N^{\mathrm{co}H}$, which is well defined. Finally, if $\phi_0 : k \to G(H) = k1_H$ is the canonical isomorphism, then the functor $G : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}$ is strong monoidal with the structure provided by (ϕ_2, ϕ_0).

By [Hausser, Nill, unpublished, Proposition 3.11] we then have the following.

2.8. Theorem. *If* H *is a quasi-Hopf algebra then the functor* $G : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}$ *is an equivalence of monoidal categories. The quasi-inverse functor of* G *is* $F = \bullet \otimes H : {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$*;* F *is a strong monoidal functor via the structure given by* φ_2 *defined by the following composition of isomorphisms:*

$$F(M) \otimes_H F(N) = (M \otimes H) \otimes_H (N \otimes H) \cong M \otimes (N \otimes H) \cong (M \otimes N) \otimes H = F(M \otimes N),$$

for all $M, N \in {}_H\mathcal{M}$*. Explicitly, we have that*

$$\varphi_{2,M,N}((m \otimes h) \otimes_H (n \otimes h')) = (x^1 \cdot m \otimes x^2 h_1 \cdot n) \otimes x^3 h_2 h', \tag{2.36}$$

for all $m \in M$, $h, h' \in H$ *and* $n \in N$. *The second morphism is* $\varphi_0 = \mathrm{Id}_H : H \to F(k) = k \otimes H \cong H$.

Note that a different approach for the monoidal equivalence in Theorem 2.8 was proposed by Schauenburg in [Schauenburg, 2012, Theorem 3.10]. In the sequel, a third approach will be derived from the structure theorem for quasi-Hopf bimodules proved in [Bulacu, Torrecillas, 2006].

3. A structure theorem for quasi-Hopf comodule algebras

In this section we will see that the structure theorem for quasi-Hopf comodule algebras given in [Panaite, Van Oystaeyen, 2007] can be easily obtained from the monoidal equivalence between ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$. Towards this end, we prefer to make use of a second pair of functors that define a monoidal equivalence between ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$. In other words, we prefer to work with the alternative structure theorem for quasi-Hopf bimodules from [Bulacu, Torrecillas, 2006, Remark 2.4].

3.1. An alternative structure theorem for quasi-Hopf bimodules. In the Hopf algebra case, the set of coinvariants of a Hopf module over H is defined as being the set of those elements on which H coacts trivially. When H is a Hopf algebra we can always define a projection onto the set of coinvariants of a Hopf module and, moreover, it covers the natural inclusion and is closed under the adjoint action. In the quasi-Hopf case, Hausser and Nill walked backwards through these facts: they first defined the projection and then, using it, the set of coinvariants of a quasi-Hopf bimodule.

An approach closely related to what we have in the Hopf case was proposed in [Bulacu, Torrecillas, 2006]. We recall it below; for full details we refer to [Bulacu, Torrecillas, 2006, Remark 2.4].

For $M \in {}_H\mathcal{M}_H^H$ we define $M^{\overline{co(H)}}$, the set of alternative coinvariants of M, as being the set

$$M^{\overline{co(H)}} := \{m \in M \mid \rho(m) = x^1 \cdot m \cdot S(x_2^3X^3)f^1 \otimes x^2X^1\beta S(x_1^3X^2)f^2\},$$

where f is the Drinfeld's twist. $M^{\overline{co(H)}}$ is a left H-module under the action given by

$$h \triangleright m := h_1 \cdot m \cdot S(h_2),\ \forall\ h \in H,\quad m \in M.$$

Furthermore, if $\overline{E} :\ M \to M$ is given by

$$\overline{E}(m) := m_{(0)} \cdot \beta S(m_{(1)}),\quad \forall\ \ m \in M,$$

then it can be proved that $M^{\overline{co(H)}} = \{\overline{E}(m) \mid m \in M\}$. In addition, we have that

$$\overline{E} :\ M^{\mathrm{co}H} \longrightarrow M^{\overline{co(H)}} \quad \text{and} \quad E :\ M^{\overline{co(H)}} \longrightarrow M^{\mathrm{co}H}$$

are inverse each other and at the same time left H-linear morphisms. Consequently, for all $M \in {}_H\mathcal{M}_H^H$, the morphism

$$M \ni m \mapsto \overline{E}(m_{(0)}) \otimes m_{(1)} \in M^{\overline{co(H)}} \otimes H$$

is an isomorphism in ${}_H\mathcal{M}_H^H$ with inverse given by

$$M^{\overline{co(H)}} \otimes H \ni m \otimes h \mapsto X^1 \cdot m \cdot S(X^2)\alpha X^3 h \in M,$$

and this provides an alternative structure theorem for quasi-Hopf bimodules.

At this point it is clear that a second category equivalence between ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$ is produced by the functors $\mathcal{F} : {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$ and $\mathcal{G} : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}$ defined as follows. If $M \in {}_H\mathcal{M}_H^H$ then $\mathcal{G}(M) = M^{\overline{co(H)}}$, and if $f : M \to N$ is a morphism in ${}_H\mathcal{M}_H^H$ then $\mathcal{G}(f)$ is the restriction and corestriction of f at $M^{\overline{co(H)}}$ and $N^{\overline{co(H)}}$, respectively, a well defined morphism in ${}_H\mathcal{M}$. The functor $\mathcal{F}$ equals to F, the functor defined in Subsection 2.6.

3.2. Corollary. *Let H be a quasi-Hopf algebra and $\mathcal{F} : {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$ and $\mathcal{G} : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}$ the functors defined above. Then $\mathcal{F}$, $\mathcal{G}$ are strong monoidal functors and they induce a monoidal equivalence between ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$.*

Proof. The functors $G, \mathcal{G}$ are naturally isomorphic. The natural isomorphism between them is given by the natural transformation

$$\overline{E} = \left(\overline{E}_M : G(M) = M^{\mathrm{co}(H)} \to \mathcal{G}(M) = M^{\overline{\mathrm{co}}(H)}\right)_{M \in {}_H\mathcal{M}_H^H}.$$

Note that the inverse natural transformation of $\overline{E}$ is E, and that $F = \mathcal{F}$. Thus, by Theorem 2.8 it follows that $\mathcal{F}$, $\mathcal{G}$ are strong monoidal functors, and that they provide a monoidal equivalence of categories.

We end by pointing out that the strong monoidal structure of $\mathcal{G}$ is given by $\overline{\phi}_{2,M,N} : M^{\overline{\mathrm{co}(H)}} \otimes N^{\overline{\mathrm{co}(H)}} \to (M \otimes_H N)^{\overline{\mathrm{co}(H)}}$ determined by the composition

$$M^{\overline{\mathrm{co}(H)}} \otimes N^{\overline{\mathrm{co}(H)}} \xrightarrow{E_M \otimes E_N} M^{\mathrm{co}(H)} \otimes N^{\mathrm{co}(H)} \xrightarrow{\phi_{2,M,N}} (M \otimes_H N)^{\mathrm{co}(H)} \xrightarrow{\overline{E}_{M \otimes_H N}} (M \otimes_H N)^{\overline{\mathrm{co}(H)}},$$

and $\overline{\phi}_0 : k \to \mathcal{G}(H) = k\beta$ defined by $\overline{\phi}_0(\kappa) = \kappa\beta$, for all $\kappa \in k$. Explicitly, we have

$$\overline{\phi}_{2,M,N}(m \otimes n) = q^1 x^1_1 \cdot m \cdot S(q^2 x^1_2) x^2 \otimes_H n \cdot S(x^3), \tag{3.1}$$

for all $M, N \in {}_H\mathcal{M}_H^H$ and $m \in M^{\overline{\mathrm{co}(H)}}$, $n \in N^{\overline{\mathrm{co}(H)}}$. Indeed, by taking into account the above definitions and structures we compute that

$$\begin{aligned}
&\overline{\phi}_{2,M,N}(m \otimes n)\\
&\quad = \overline{E}_{M \otimes_H N}\phi_{2,M,N}(E_M(m) \otimes E_N(n))\\
&\quad = \overline{E}_{M \otimes_H N}(E_M(X^1 \cdot E_M(m)) \otimes_H E_N(X^2 \cdot E_N(n)) \cdot X^3)\\
&\quad \overset{(2.31)}{=} \overline{E}_{M \otimes_H N}(E_M(X^1 \cdot m) \otimes_H E_N(X^2 \cdot n) \cdot X^3)
\end{aligned}$$

$$\begin{aligned}
&= E_M(X^1\cdot m)_{(0)}\otimes_H E_N(X^2\cdot n)_{(0)}\cdot X^3_1\beta S(E_M(X^1\cdot m)_{(1)}E_N(X^2\cdot n)_{(1)}X^3_2)\\
&\overset{(2.5)}{=} E_M(m)_{(0)}\otimes_H \overline{E}E(n)\cdot S(E_M(m)_{(1)})\\
&\overset{(2.29)}{=} E_M(x^1\cdot E_M(m))\cdot x^2\otimes_H n\cdot S(x^3)\\
&\overset{(2.31)}{=} E_M(x^1\cdot m)\cdot x^2\otimes_H n\cdot S(x^3)\\
&= q^1x^1_1\cdot m_{(0)}\cdot \beta S(q^2x^1_2m_{(1)})x^2\otimes_H n\cdot S(x^3)\\
&= q^1x^1_1\cdot \overline{E}_M(m)\cdot S(q^2x^1_2)x^2\otimes_H n\cdot S(x^3)\\
&= q^1x^1_1\cdot m\cdot S(q^2x^1_2)x^2\otimes_H n\cdot S(x^3),
\end{aligned}$$

for all $m\in M^{\overline{\mathrm{co}(H)}}$ and $n\in N^{\overline{\mathrm{co}(H)}}$, as stated. ∎

3.3. A STRUCTURE THEOREM FOR COMODULE ALGEBRAS. We will see that the alternative structure theorem for quasi-Hopf bimodules provides a structure theorem for algebras within categories of quasi-Hopf bimodules.

The category of H-modules is monoidal, and an H-module (co)algebra is a (co)algebra in this category. This categorical definition cannot be used to introduce H-comodule (co)algebras, since we do not have H-comodules. However, to introduce the concept of comodule algebra over a quasi-bialgebra Hausser and Nill [Hausser, Nill, 1999] generalize the property of the comultiplication Δ of H to an arbitrary H-coaction $\rho:\mathbf{A}\to\mathbf{A}\otimes H$ on an associative k-algebra $\mathbf{A}$. More precisely, we have the following.

3.4. DEFINITION. *Let H be a quasi-bialgebra. A unital associative algebra $\mathbf{A}$ is called a right H-comodule algebra if there exist an algebra morphism $\rho:\mathbf{A}\to\mathbf{A}\otimes H$ and an invertible element $\Phi_\rho\in\mathbf{A}\otimes H\otimes H$ such that:*

$$\Phi_\rho(\rho\otimes\mathrm{Id}_H)(\rho(\mathbf{a}))=(\mathrm{Id}_{\mathbf{A}}\otimes\Delta)(\rho(\mathbf{a}))\Phi_\rho,\quad\forall\ \ \mathbf{a}\in\mathbf{A},\tag{3.2}$$

$$(1_{\mathbf{A}}\otimes\Phi)(\mathrm{Id}_{\mathbf{A}}\otimes\Delta\otimes\mathrm{Id}_H)(\Phi_\rho)(\Phi_\rho\otimes 1_H)=(\mathrm{Id}_{\mathbf{A}}\otimes\mathrm{Id}_H\otimes\Delta)(\Phi_\rho)(\rho\otimes\mathrm{Id}_H\otimes\mathrm{Id}_H)(\Phi_\rho),\tag{3.3}$$

$$(\mathrm{Id}_{\mathbf{A}}\otimes\varepsilon)\circ\rho=\mathrm{Id}_{\mathbf{A}},\tag{3.4}$$

$$(\mathrm{Id}_{\mathbf{A}}\otimes\varepsilon\otimes\mathrm{Id}_H)(\Phi_\rho)=(\mathrm{Id}_{\mathbf{A}}\otimes\mathrm{Id}_H\otimes\varepsilon)(\Phi_\rho)=1_{\mathbf{A}}\otimes 1_H.\tag{3.5}$$

In a similar manner we can introduce the notion of left comodule algebra over a quasi-bialgebra.

We begin with a lemma of independent interest. As before, ${}_H\mathcal{M}^H_H$ is the category of right quasi-Hopf bimodules over H equipped with the monoidal structure presented in Subsection 2.6.

3.5. LEMMA. *Let H be a quasi-bialgebra. Then giving an algebra $\mathbf{A}$ in ${}_H\mathcal{M}^H_H$ is equivalent to giving a triple $(\mathbf{A},\rho,i)$ consisting of an associative k-algebra $\mathbf{A}$, a k-linear map $\rho:\mathbf{A}\to\mathbf{A}\otimes H$ and a k-algebra morphism $i:H\to\mathbf{A}$ such that $(\mathbf{A},\rho,\Phi_\rho:=i(X^1)\otimes X^2\otimes X^3)$ is a right H-comodule algebra and i is a right H-comodule morphism, i.e., in addition,*

$$\rho(i(h))=i(h_1)\otimes h_2,\ \forall\ h\in H.$$

PROOF. Assume that $(\mathbf{A}, \underline{m} : \mathbf{A} \otimes_H \mathbf{A} \to \mathbf{A}, i : H \to \mathbf{A})$ is an algebra in ${}_H\mathcal{M}_H^H$. Since the forgetful functor from $({}_H\mathcal{M}_H^H, \otimes_H, H)$ to $({}_H\mathcal{M}_H, \otimes_H, H)$ is strong monoidal, we get that $(\mathbf{A}, \underline{m}, i)$ is an algebra in $({}_H\mathcal{M}_H, \otimes_H, H)$, too. Otherwise stated, $(\mathbf{A}, \underline{m}, i)$ is an H-ring. Thus, we obtain that $\mathbf{A}$ is a k-algebra with multiplication $m = q^H_{\mathbf{A},\mathbf{A}}\underline{m}$ and unit $1_{\mathbf{A}} = i(1_H)$. Furthermore, the input H-bimodule structure of $\mathbf{A}$ is completely determined by $h \cdot \mathbf{a} \cdot h' = i(h)\mathbf{a}i(h')$, for all $h, h' \in H$ and $\mathbf{a} \in \mathbf{A}$, and $i : H \to \mathbf{A}$ becomes a k-algebra morphism.

Now, since $\mathbf{A}$ is an object in ${}_H\mathcal{M}_H^H$ we have a k-linear map $\rho : \mathbf{A} \ni \mathbf{a} \mapsto \mathbf{a}_{\langle 0\rangle} \otimes \mathbf{a}_{\langle 1\rangle} \in \mathbf{A} \otimes H$ such that $\varepsilon(\mathbf{a}_{\langle 1\rangle})\mathbf{a}_{\langle 0\rangle} = \mathbf{a}$ and

$$i(X^1)\mathbf{a}_{\langle 0,0\rangle} \otimes X^2\mathbf{a}_{\langle 0,1\rangle} \otimes X^3\mathbf{a}_{\langle 1\rangle} = \mathbf{a}_{\langle 0\rangle}i(X^1) \otimes \mathbf{a}_{\langle 1\rangle_1}X^2 \otimes \mathbf{a}_{\langle 1\rangle_2}X^3,\ \forall\ \mathbf{a} \in \mathbf{a},$$

that is (3.4) and (3.2) hold. Furthermore, ρ is an H-bimodule morphism, and so

$$\rho(i(h)\mathbf{a}i(h')) = i(h_1)\mathbf{a}_{\langle 0\rangle}i(h'_1) \otimes h_2\mathbf{a}_{\langle 1\rangle}h'_2,\ \forall\ \mathbf{a} \in \mathbf{A} \text{ and } h,\ h' \in H.$$

Clearly, this implies $\rho(i(h)) = i(h_1) \otimes h_2$, for all $h \in H$. The latter equality allows to show that $\Phi_\rho := i(X^1) \otimes X^2 \otimes X^3$ satisfies (3.3). (3.5) is automatic.

It remains to prove that ρ is a k-algebra morphism. This follows easily from the fact that $\underline{m}$ and i are right H-colinear morphisms, we leave the verification of this detail to the reader. So we have shown that $(\mathbf{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra and $i : H \to \mathbf{A}$ is a right H-comodule algebra morphism.

For the converse, assume that we have a datum $(\mathbf{A}, \rho, i)$ as in the statement. First, $\mathbf{A}$ becomes an H-bimodule via i, i.e. $h \cdot \mathbf{a} \cdot h' = i(h)\mathbf{a}i(h')$, for all $h, h' \in H$ and $\mathbf{a} \in \mathbf{A}$. Together with ρ this turns $\mathbf{A}$ into an object in ${}_H\mathcal{M}_H^H$, see [Panaite, Van Oystaeyen, 2007, Lemma 2.3]. Since $\mathbf{A}$ is an associative k-algebra and $i : H \to \mathbf{A}$ is a k-algebra morphism, it follows that $(\mathbf{A}, \underline{m}, i)$ with $\underline{m} : \mathbf{A} \otimes_H \mathbf{A} \to \mathbf{A}$ given by $\underline{m}(\mathbf{a} \otimes_H \mathbf{a}') = \mathbf{a}\mathbf{a}'$, for all $\mathbf{a}, \mathbf{a}' \in \mathbf{A}$, is an algebra in $({}_H\mathcal{M}_H, \otimes_H, H)$. A simple inspection shows that $(\mathbf{A}, \underline{m}, i)$ is, moreover, an algebra in ${}_H\mathcal{M}_H^H$, where ${}_H\mathcal{M}_H^H$ has the monoidal structure mentioned above. ■

It was proved in [Panaite, Van Oystaeyen, 2007] that particular examples of algebras within ${}_H\mathcal{M}_H^H$ are given by the smash product construction in [Bulacu, Panaite, Van Oystaeyen, 2000]. It associates to an algebra A in ${}_H\mathcal{M}$ (i.e. to a left H-module algebra A) an associative unital algebra $A\#H$ built on the k-vector space $A \otimes H$ as follows. The multiplication is given by

$$(a\#h)(b\#g) = (x^1 \cdot a)(x^2h_1 \cdot b)\#x^3h_2h',$$

for all $a, b \in A$ and $h, h' \in H$, where we write $a\#h$ instead of $a \otimes h$ in order to distinguish this new multiplication on $A \otimes H$. The unit is $1_A\#1_H$.

The fact that $A\#H$ is an algebra in ${}_H\mathcal{M}_H^H$ can be also obtained from the following monoidal categorical arguments.

3.6. COROLLARY. *Let H be a quasi-bialgebra and A a left H-module algebra. Then $A\#H$, the smash product between A and H, is an algebra in ${}_H\mathcal{M}_H^H$.*

PROOF. Since $\mathcal{F} : {}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$ is a strong monoidal functor, $\mathcal{F}$ maps an algebra A in ${}_H\mathcal{M}_H^H$ to an algebra in ${}_H\mathcal{M}_H^H$. The multiplication on $\mathcal{F}(A) = A \otimes H$ is determined by

$$m_{\mathcal{F}(A)} : \mathcal{F}(A) \otimes_H \mathcal{F}(A) \stackrel{\varphi_{2,A,A}}{\longrightarrow} \mathcal{F}(A \otimes A) \stackrel{\mathcal{F}(m_A)}{\longrightarrow} \mathcal{F}(A),$$

where φ_2 is from (2.36). The unit of $\mathcal{F}(A)$ is $i : H \stackrel{\varphi_0}{\longrightarrow} F(k) \stackrel{\eta_A}{\longrightarrow} \mathcal{F}(A)$. Thus

$$(a \otimes_H h)(b \otimes_H h') = (x^1 \cdot a)(x^2 h_1 \cdot b) \otimes_H x^3 h_2 h'$$

defines a unital algebra structure on $A \otimes H$ within ${}_H\mathcal{M}_H^H$, with unit $i : H \ni h \mapsto 1_A \otimes h \in A \otimes H$. According to Lemma 3.5 this is equivalent to the fact that $A \otimes H$ is a unital associative k-algebra under the multiplication

$$(a \otimes h)(b \otimes h') = (x^1 \cdot a)(x^2 h_1 \cdot b) \otimes x^3 h_2 h'$$

and unit $1_A \otimes 1_H$, and that $A \otimes H = F(A)$ is a right H-comodule algebra via $\rho : a \otimes h \mapsto x^1 \cdot a \otimes x^2 h_1 \otimes x^3 h_2 \in A \otimes H \otimes H$, an algebra morphism, and $\Phi_\rho = i(X^1) \otimes X^2 \otimes X^3 = (1_A \otimes X^1) \otimes X^2 \otimes X^3$. Furthermore,

$$\rho(i(h)) = \rho(1_A \otimes h) = x^1 \cdot 1_A \otimes x^2 h_1 \otimes x^3 h_2 = 1_A \otimes h_1 \otimes h_2 = i(h_1) \otimes h_2,$$

since $h \cdot 1_A = \varepsilon(h) 1_A$, for all $h \in H$, by the definition of a left H-module algebra. Otherwise stated, $\mathcal{F}(A) = A\#H$ as a k-algebra is, moreover, an algebra within ${}_H\mathcal{M}_H^H$ modulo the structure we just described. This finishes the proof. ∎

We next show that in the case when H is a quasi-Hopf algebra, any algebra $\mathbf{A}$ in ${}_H\mathcal{M}_H^H$ is of the form presented in Corollary 3.6, for a certain algebra A in ${}_H\mathcal{M}$.

The next result is an equivalent version of [Panaite, Van Oystaeyen, 2007, Theorem 2.5].

3.7. THEOREM. *Let H be a quasi-Hopf algebra and $\mathbf{A}$ an algebra in ${}_H\mathcal{M}_H^H$. Then there exists a left H-module algebra A such that $\mathbf{A} \simeq A\#H$, as algebras in ${}_H\mathcal{M}_H^H$.*

PROOF. We know from Corollary 3.2 that we have a strong monoidal functor $\mathcal{G} : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}$, so an algebra $\mathcal{G}(\mathbf{A}) = \mathbf{A}^{\overline{\mathrm{co}(H)}}$ in ${}_H\mathcal{M}$ corresponds to the algebra $\mathbf{A}$ in ${}_H\mathcal{M}_H^H$. Denote $\mathcal{G}(\mathbf{A}) = A$. By the definition of $\mathcal{G}$ we deduce that A is a left H-module via the action given by

$$h \to a = h_1 \cdot a \cdot S(h_2) = i(h_1) a i(S(h_2)) := h \triangleright_i a,$$

for all $h \in H$ and $a \in A \subseteq \mathbf{A}$. Keeping in mind the strong monoidal structure of $\mathcal{G}$, we deduce that the multiplication of A in ${}_H\mathcal{M}$ is

$$\begin{aligned} a * a' &= \mathcal{G}(\underline{m})\overline{\phi}_{2,\mathbf{A},\mathbf{A}}(a \otimes a') \\ &= (X^1 x_1^1 \cdot a \cdot S(X^2 x_2^1) \alpha X^3 x^2)(a' \cdot S(x^3)) \\ &= i(X^1 x_1^1) a i(S(X^2 x_2^1) \alpha X^3 x^2) a' i(S(x^3)), \end{aligned}$$

for all $a, a' \in A$, while its unit is given by $\mathcal{G}(i)\overline{\phi}_0(1_k) = \mathcal{G}(i)(\beta) = i(\beta)$. But, using (2.3) and (2.5) we deduce easily that

$$a * a' = i(X^1)ai(S(x^1X^2)\alpha x^2X^3_1)a'i(S(x^3X^3_2) := a \circ a', \tag{3.6}$$

for all $a, a' \in H$. The notations $\triangleright_i$ and $\circ$ are imposed by the analogy with the structure in [Bulacu, Panaite, Van Oystaeyen, 2000]. Hence, summing up, the multiplication in $A\#H$ is given by

$$(a\#h)(a'\#h') = i(X^1x^1_1)bi(S(y^1X^2x^1_2)\alpha y^2X^3_1x^2_1h_{(1,1)})b'i(S(y^3X^3_2x^2_2h_{(1,2)}))\#x^3h_2h', \tag{3.7}$$

for all $a, a' \in A$ and $h, h' \in H$.

On the other hand, by the alternative structure theorem for quasi-Hopf bimodules we get that $\chi : A \otimes H \to \mathbf{A}$ given by $\chi(a \otimes h) = X^1 \cdot a \cdot S(X^2)\alpha X^3h = i(X^1)ai(S(X^2)\alpha X^3h)$, for all $a \in A$ and $h \in H$, is an isomorphism in ${}_H\mathcal{M}_H^H$ with inverse $\chi^{-1} : \mathbf{A} \to A \otimes H$ defined by

$$\begin{aligned}\chi^{-1}(\mathbf{a}) &= \overline{E}(\mathbf{a}_{\langle 0\rangle}) \otimes \mathbf{a}_{\langle 1\rangle}\\ &= \mathbf{a}_{\langle 0,0\rangle} \cdot \beta S(\mathbf{a}_{\langle 0,1\rangle}) \otimes \mathbf{a}_{\langle 1\rangle}\\ &= \mathbf{a}_{\langle 0,0\rangle} i(\beta S(\mathbf{a}_{\langle 0,1\rangle})) \otimes \mathbf{a}_{\langle 1\rangle},\end{aligned}$$

for all $\mathbf{a} \in \mathbf{A}$. Thus to end the proof it suffices to show that χ is an algebra morphism in ${}_H\mathcal{M}_H^H$, provided that it is considered as a morphism between $A\#H$ and $\mathbf{A}$. This follows easily from the following general result: if the functors $\mathcal{S} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{R} : \mathcal{D} \to \mathcal{C}$ define a monoidal category equivalence, then $\mathcal{RS}(\mathbf{A}) \cong \mathbf{A}$ is an algebra isomorphism in $\mathcal{C}$, for any algebra $\mathbf{A}$ within $\mathcal{C}$, where $\mathcal{RS}(\mathbf{A})$ has the algebra structure provided by the monoidal structure of $\mathcal{RS}$ and the algebra structure of $\mathbf{A}$, respectively. ∎

4. A structure theorem for quasi-Hopf bicomodule algebras

The goal of this section is to prove that the functors defined in Corollary 3.2 restrict to a category equivalence between two-sided two-cosided Hopf modules and Yetter-Drinfeld modules. This fact will allow us to give a categorical proof for the structure theorem for a quasi-Hopf bicomodule algebra obtained in [Dello, Panaite, Van Oystaeyen, Zhang, 2016].

4.1. Two-sided two-cosided Hopf modules versus Yetter-Drinfeld modules. Recall that ${}_H\mathcal{M}_H$ is a monoidal category and that the underlying quasi-coalgebra structure of H provides a monoidal coalgebra structure for H in ${}_H\mathcal{M}_H$. Thus we can define ${}_H^H\mathcal{M}_H^H$, the category of two-sided two-cosided Hopf modules, as being the category of H-bicomodules within ${}_H\mathcal{M}_H$. Explicitly, we have the following.

4.2. Definition. *Let H be a quasi-bialgebra. A two-sided two-cosided Hopf module over H is a k-vector space M with the following additional structure.*

i) M is a right quasi-Hopf H-bimodule; as before, we write $\cdot$ both for the left and right H-actions, and $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ for the right H-coaction on $m \in M$.

ii) M is a left quasi-Hopf H-bimodule under the same H-bimodule structure as in i) and $\lambda_M : M \to H \otimes M$, $\lambda_M(m) = m_{\{-1\}} \otimes m_{\{0\}}$, called the left H-coaction on M; that is $\varepsilon(m_{\{-1\}})m_{\{0\}} = m$ and

$$\Phi \cdot (\underline{\Delta} \otimes \mathrm{Id}_M)(\lambda_M(m)) = (\mathrm{Id}_H \otimes \lambda_M)(\lambda_M(m)) \cdot \Phi, \tag{4.1}$$

for all $m \in M$.

iii) M is an H-"bicomodule", in the sense that, for all $m \in M$,

$$\Phi \cdot (\lambda_M \otimes \mathrm{Id}_H)(\rho_M(m)) = (\mathrm{Id}_H \otimes \rho_M)(\lambda_M(m)) \cdot \Phi. \tag{4.2}$$

iv) The following compatibility relations hold:

$$\lambda_M(h \cdot m) = h_1 \cdot m_{\{-1\}} \otimes h_2 \cdot m_{\{0\}} \tag{4.3}$$
$$\lambda_M(m \cdot h) = m_{\{-1\}} \cdot h_1 \otimes m_{\{0\}} \cdot h_2 \tag{4.4}$$

for all $h \in H$ and $m \in M$.

${}^H_H\mathcal{M}^H_H$ will be then the category of two-sided two-cosided Hopf modules over H and maps preserving the actions by H and the coactions by H.

It was proved by Schauenburg in [Schauenburg, 2012, Thorem 5.3] that ${}^H_H\mathcal{M}^H_H$ is equivalent to ${}^H_H\mathcal{Y}_2\mathcal{D}$, the so-called category of left Yetter-Drinfeld modules of the second kind, and at the same time with ${}^H_H\mathcal{Y}D$, the category of left Yetter-Drinfeld modules over H defined by Majid in [Majid, 1998]. A generalization of the equivalence between ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{Y}D$, based on the alternative structure theorem for quasi-Hopf H-bimodules, was given in [Bulacu, Torrecillas, 2006]. We will recall it in what follows.

4.3. Definition. *Let H be a quasi-bialgebra, with reassociator Φ. A left Yetter-Drinfeld module over H is a left H-module M together with a k-linear map (called the left H-coaction)*

$$\lambda_M : M \longrightarrow H \otimes M, \quad \lambda_M(m) = m_{[-1]} \otimes m_{[0]}$$

such that the following conditions hold, for all $h \in H$ and $m \in M$:

$$\begin{aligned} &X^1 m_{[-1]} \otimes (X^2 \cdot m_{[0]})_{[-1]} X^3 \otimes (X^2 \cdot m_{[0]})_{[0]} \\ &\qquad = X^1 (Y^1 \cdot m)_{[-1]_1} Y^2 \otimes X^2 (Y^1 \cdot m)_{[-1]_2} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{[0]}, \end{aligned} \tag{4.5}$$

$$\varepsilon(m_{[-1]}) m_{[0]} = m, \tag{4.6}$$

$$h_1 m_{[-1]} \otimes h_2 \cdot m_{[0]} = (h_1 \cdot m)_{[-1]} h_2 \otimes (h_1 \cdot m)_{[0]}. \tag{4.7}$$

The category of left Yetter-Drinfeld modules and k-linear maps that preserve the H-action and H-coaction is denoted by ${}^H_H\mathcal{Y}D$.

By the general result presented at the beginning of the Section 4 in [Bulacu, Torrecillas, 2006] we get the following.

4.4. Proposition. *Consider the functors* $\mathcal{F} : {}_H^H\mathcal{Y}D \to {}_H^H\mathcal{M}_H^H$ *and* $\mathcal{G} : {}_H^H\mathcal{M}_H^H \to {}_H^H\mathcal{Y}D$ *defined as follows:*

– For $M \in {}_H^H\mathcal{Y}D$ *we have* $\mathcal{F}(M) = M \otimes H \in {}_H^H\mathcal{M}_H^H$ *with the structure given by*

$$h \cdot (m \otimes h') \cdot h'' = h_1 \cdot m \otimes h_2 h' h'', \tag{4.8}$$
$$\lambda_{M\otimes H}(m \otimes h) = X^1 \cdot (x^1 \cdot m)_{[-1]} \cdot x^2 h_1 \otimes \left(X^2 \cdot (x^1 \cdot m)_{[0]} \otimes X^3 x^3 h_2\right), \tag{4.9}$$
$$\rho_{M\otimes H}(m \otimes h) = (x^1 \cdot m \otimes x^2 h_1) \otimes x^3 h_2, \tag{4.10}$$

for all $h, h', h'' \in H$ *and* $m \in M$*. If* $f : M \to N$ *is a morphisms in* ${}_H^H\mathcal{Y}D$ *then* $\mathcal{F}(f) = f \otimes \mathrm{Id}_H$*.*

– If $M \in {}_H^H\mathcal{M}_H^H$ *then* $\mathcal{G}(M) = M^{\overline{co(H)}}$*, the set of alternative coinvariants of* M*, which belongs to* ${}_H^H\mathcal{Y}D$ *via the structure defined by*

$$h \to m = h_1 \cdot m \cdot S(h_2), \tag{4.11}$$
$$\lambda_{M^{\overline{co(H)}}}(m) = X^1 Y_1^1 m_{\{-1\}} g^1 S(Z^2 Y_2^2) \alpha Z^3 Y^3 \otimes X^2 Y_2^1 \cdot m_{\{0\}} \cdot g^2 S(X^3 Z^1 Y_1^2), \tag{4.12}$$

for all $h \in H$ *and* $m \in M^{\overline{co(H)}}$*, where* $f^{-1} = g^1 \otimes g^2$ *is the inverse of the Drinfeld's twist* f*. On morphisms we have that* $\mathcal{G}(f) = f|_{M^{\overline{co(H)}}}$*, for any morphism* $f : M \to N$ *in* ${}_H^H\mathcal{M}_H^H$*.*

Then $\mathcal{F}$ *and* $\mathcal{G}$ *are inverse equivalence functors.*

Remark that $\mathcal{F}$ and $\mathcal{G}$ are the inverse equivalence functors defined in Corollary 3.2, restricted and corestricted to ${}_H^H\mathcal{Y}D$ and ${}_H^H\mathcal{M}_H^H$, respectively. This is why we decided to keep for them the same notations as in the previous section.

In the Hopf case, Schauenburg proved in [Schauenburg, 1994, Theorem 5.7] that $\mathcal{F}$ and $\mathcal{G}$ provide a monoidal category equivalence between ${}_H^H\mathcal{Y}D$ and ${}_H^H\mathcal{M}_H^H$. Afterwards, he generalized this result to quasi-Hopf algebras in [Schauenburg, 2012, Corollary 8.3], but without giving the explicit strong monoidal structure of the functors that provide the category equivalence. For the sake of completeness we will do this now. Towards this end, we need first some preliminary results.

The category ${}_H^H\mathcal{M}_H^H$ is monoidal via the monoidal structure on ${}_H\mathcal{M}_H^H$ defined by (2.34), (2.35), and

$$\lambda_{M\otimes_H N} : M \otimes_H N \ni m \otimes_H n \mapsto m_{\{-1\}} n_{\{-1\}} \otimes m_{\{0\}} \otimes_H n_{\{0\}} \in H \otimes M \otimes_H N. \tag{4.13}$$

${}_H^H\mathcal{Y}D$ is identified with the left center of the monoidal category ${}_H\mathcal{M}$, and therefore is a braided monoidal category. The pre-braided monoidal structure on the left weak center of ${}_H\mathcal{M}$ induces a monoidal structure on ${}_H^H\mathcal{Y}D$. This structure is such that the forgetful functor ${}_H^H\mathcal{Y}D \longrightarrow {}_H\mathcal{M}$ is monoidal. According to [Majid, 1998] and [Bulacu, Caenepeel, Panaite, 2006], we find that the H-coaction on the tensor product $\mathbf{M} \otimes \mathbf{N}$ of two left Yetter-Drinfeld modules $\mathbf{M}$ and $\mathbf{N}$ is given, for all $m \in \mathbf{M}$ and $n \in \mathbf{N}$, by

$$\lambda_{\mathbf{M}\otimes\mathbf{N}}(m \otimes n) = X^1 (x^1 Y^1 \cdot m)_{(-1)} x^2 (Y^2 \cdot n)_{(-1)} Y^3 \otimes X^2 \cdot (x^1 Y^1 \cdot m)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot n)_{(0)}. \tag{4.14}$$

4.5. Lemma. *In any quasi-Hopf algebra H the following equality holds:*

$$q^1Q^1_1z^1y^1_1\otimes S(Q^2z^3y^1_{(2,2)})y^2\otimes S(q^2Q^1_2z^2y^1_{(2,1)})y^3 = X^1\otimes S(q^2x^1_2X^2_2)x^2X^3_1\otimes S(q^1x^1_1X^2_1)\alpha x^3X^3_2, \tag{4.15}$$

where $q^1\otimes q^2=Q^1\otimes Q^2$ are two copies of the element $q_R=X^1\otimes S^{-1}(\alpha X^3)X^2\in H\otimes H$.

Proof. The formula (4.15) is a consequence of the following computation:

$$\begin{aligned}
&q^1Q^1_1z^1y^1_1\otimes S(Q^2z^3y^1_{(2,2)})y^2\otimes S(q^2Q^1_2z^2y^1_{(2,1)})y^3\\
\overset{(2.1)}{=}\ &q^1(Q^1y^1_1)_1z^1\otimes S(Q^2y^1_2z^3)y^2\otimes S(q^2(Q^1y^1_1)_2z^2)y^3\\
\overset{(2.20)}{=}\ &q^1X^1_1z^1\otimes S(X^2z^3)\tilde{q}^1X^3_1\otimes S(q^2X^1_2z^2)\tilde{q}^2X^3_2\\
\overset{(2.14),(2.18)}{=}\ &Y^1X^1_1z^1\otimes S(Y^3_1X^2z^3)\tilde{q}^1(Y^3_2X^3)_1\otimes S(Y^2X^1_2z^2)\alpha\tilde{q}^2(Y^3_2X^3)_2\\
\overset{(2.3)}{=}\ &Y^1\otimes S(X^2Y^2_2)\tilde{q}^1X^3_1Y^3_1\otimes S(X^1Y^2_1)\alpha\tilde{q}^2X^3_2Y^3_2\\
\overset{(2.20)}{=}\ &Y^1\otimes S(q^2x^1_2Y^2_2)x^2Y^3_1\otimes S(q^1x^1_1Y^2_1)\alpha x^3Y^3_2,
\end{aligned}$$

as we stated. ■

4.6. Theorem. *If H is a quasi-Hopf algebra then the categories ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{YD}$ are monoidally equivalent.*

Proof. By Corollary 3.2, we have that the functors $\mathcal{F}$ and $\mathcal{G}$ that provide the equivalence between ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{YD}$ in Proposition 4.4, yield a monoidal equivalence between ${}_H\mathcal{M}^H_H$ and ${}_H\mathcal{M}$. Thus it is enough to prove that the strong monoidal structures on $\mathcal{F}$ and $\mathcal{G}$ (as they were considered as functors between ${}_H\mathcal{M}^H_H$ and ${}_H\mathcal{M}$) extend to strong monoidal structures when they are considered as functors between ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{YD}$. Otherwise stated, it suffices to prove that $\overline{\phi}_{2,M,N}$ defined in (3.1) is left H-colinear, for all $M,N\in {}^H_H\mathcal{M}^H_H$, and that $\varphi_{2,\mathbf{M},\mathbf{N}}$ in (2.36) is left H-colinear, for all $\mathbf{M},\mathbf{N}\in {}^H_H\mathcal{YD}$.

As before, for an object M in ${}^H_H\mathcal{M}^H_H$ we denote by $M\ni m\mapsto\lambda_M(m)=m_{\{-1\}}\otimes m_{\{0\}}\in H\otimes M$ its left H-coaction and by $M\ni m\mapsto\rho_M(m)=m_{(0)}\otimes m_{(1)}\in M\otimes H$ its right H-coaction, respectively. If $\mathbf{M}$ is a left Yetter-Drinfeld module over H, then its left H-action will be denoted by $\lambda_{\mathbf{M}}(m)=m_{[-1]}\otimes m_{[0]}\in H\otimes\mathbf{M}$, for all $m\in\mathbf{M}$.

With these notations, we have that, for all $m\in M^{\overline{\mathrm{co}(H)}}$ and $n\in N^{\overline{\mathrm{co}(H)}}$,

$$\begin{aligned}
&\lambda_{(M\otimes_HN)^{\overline{\mathrm{co}(H)}}}\overline{\phi}_{2,M,N}(m\otimes_H n)\\
\overset{(3.1)}{=}\ &\lambda_{(M\otimes_HN)^{\overline{\mathrm{co}(H)}}}(q^1x^1_1\cdot m\cdot S(q^2x^1_2)x^2\otimes_H n\cdot S(x^3))\\
\overset{(4.12)}{=}\ &X^1Y^1_1(q^1x^1_1\cdot m\cdot S(q^2x^1_2)x^2)_{\{-1\}}(n\cdot S(x^3))_{\{-1\}}g^1S(Q^2Y^2_2)Y^3\\
&\quad\otimes X^2Y^1_2\cdot(q^1x^1_1\cdot m\cdot S(q^2x^1_2)x^2)_{\{0\}}\otimes_H(n\cdot S(x^3))_{\{0\}}\cdot g^2S(X^3Q^1Y^2_1)\\
\overset{(2.7),(2.19)}{=}\ &X^1Y^1_1(q^1\mathbf{Q}^1_1z^1y^1_1x^1_1)_1m_{\{-1\}}G^1S(\mathbf{Q}^2z^3y^1_{(2,2)}x^1_{(2,2)})y^2x^2_1n_{\{-1\}}g^1S(Q^2Y^2_2x^3_2)Y^3\\
&\quad\otimes X^2Y^1_2(q^1\mathbf{Q}^1_1z^1y^1_1x^1_1)_2\cdot m_{\{0\}}\cdot G^2S(q^2\mathbf{Q}^1_2z^2y^1_{(2,1)}x^1_{(2,1)}y^3x^2_2\\
&\quad\otimes_H n_{\{0\}}\cdot g^2S(X^3Q^1Y^2_1x^3_1)\\
\overset{(4.15)}{=}\ &X^1Y^1_1Z^1_1x^1_{(1,1)}m_{\{-1\}}G^1S(q^2y^1_2Z^2_2x^1_{(2,2)})y^2Z^3_1x^2_1n_{\{-1\}}g^1S(Q^2Y^2_2x^3_2)Y^3
\end{aligned}$$

$$\otimes X^2Y_2^1Z_2^1x^1_{(1,2)}\cdot m_{\{0\}}\cdot G^2S(q^1y^1_1Z^2_1x^1_{(2,1)})\alpha y^3Z^3_2x^2_2$$
$$\otimes_H n_{\{0\}}\cdot g^2S(X^3Q^1Y^2_1x^3_1).$$

On the other hand, we compute that

$$\begin{aligned}
&(\mathrm{Id}_H\otimes\overline{\phi}_{2,M,N})\lambda_{M^{\overline{\mathrm{co}(H)}}\otimes N^{\overline{\mathrm{co}(H)}}}(m\otimes n)\\
&\overset{(4.14)}{=} X^1(x^1Y^1\to m)_{[-1]}x^2(Y^2\to n)_{[-1]}Y^3\\
&\quad\otimes\overline{\phi}_{2,M,N}(X^2\to(x^1Y^1\to m)_{[0]}\otimes X^3x^3\to(Y^2\to n)_{[0]})\\
&\overset{(4.12)}{=} X^1U^1Z^1_1(x^1Y^1\to m)_{\{-1\}}g^1S(q^2Z^2_2)Z^3x^2T^1V^1_1(Y^2\to n)_{\{-1\}}G^1S(Q^2V^2_2)V^3Y^3\\
&\quad\otimes\overline{\phi}_{2,M,N}\big(X^2\to(U^2Z^1_2\cdot(x^1Y^1\to m)_{\{0\}}\cdot g^2S(U^3q^1Z^2_1))\\
&\quad\otimes X^3x^3\to(T^2V^1_2\cdot(Y^2\to n)_{\{0\}}\cdot G^2S(T^3Q^1V^2_1))\big)\\
&\overset{(4.11),(4.15)}{=} X^1U^1Z^1_1(x^1Y^1)_{(1,1)}m_{\{-1\}}g^1S(q^2Z^2_2(x^1Y^1)_{(2,2)})Z^3x^2T^1V^1_1Y^2_{(1,1)}n_{\{-1\}}G^1\\
&\quad S(Q^2V^2_2Y^2_{(2,2)})V^3Y^3\otimes W^1X^2_1U^2Z^1_2(x^1Y^1)_{(1,2)}\cdot m_{\{0\}}\cdot\\
&\quad g^2S(t^1W^2X^2_2U^3q^1Z^2_1(x^1Y^1)_{(2,1)})\alpha t^2W^3_1X^3_1T^2V^1_2Y^2_{(1,2)}\\
&\quad\otimes_H n_{\{0\}}\cdot G^2S(t^3W^3_2X^3_2x^3_2T^3Q^1V^2_1Y^2_{(2,1)})\\
&\overset{(2.3)}{=} X^1U^1_1Z^1_1(x^1Y^1)_{(1,1)}m_{\{-1\}}g^1S(q^2Z^2_2(x^1Y^1)_{(2,2)})Z^3x^2T^1V^1_1Y^2_{(1,1)}n_{\{-1\}}G^1\\
&\quad S(Q^2V^2_2Y^2_{(2,2)})V^3Y^3\otimes X^2U^1_2Z^1_2(x^1Y^1)_{(1,2)}\cdot m_{\{0\}}\\
&\quad\cdot g^2S(t^1X^3_1U^2q^1Z^2_1(x^1Y^1)_{(2,1)})\alpha t^2(X^3_2U^3)_1x^3_1T^2V^1_2Y^2_{(1,2)}\\
&\quad\otimes_H n_{\{0\}}\cdot G^2S(t^3(X^3_2U^3)_2x^3_2T^3Q^1V^2_1Y^2_{(2,1)})\\
&\overset{(2.17),(2.3)}{=} X^1Z^1_1Y^1_{(1,1)}m_{\{-1\}}g^1S(q^2(x^1Z^2Y^1_2)_2)x^2Z^3_1T^1V^1_1Y^2_{(1,1)}n_{\{-1\}}G^1S(Q^2V^2_2Y^2_{(2,2)})\\
&\quad V^3Y^3\otimes X^2Z^1_2Y^1_{(1,2)}\cdot m_{\{0\}}\cdot g^2S(t^1q^1(x^1Z^2Y^1_2)_1)\alpha t^2(x^3Z^3_2)_1T^2V^1_2Y^2_{(1,2)}\\
&\quad\otimes_H n_{\{0\}}\cdot G^2S(X^3t^3(x^3Z^3_2)_2T^3Q^1V^2_1Y^2_{(2,1)})\\
&\overset{(2.17),(2.1),(2.3)}{=} X^1Z^1_1Y^1_{(1,1)}m_{\{-1\}}g^1S(q^2(t^1x^1Z^2Y^1_2)_2)t^2x^2_1Z^3_{(1,1)}V^1_1Y^2_{(1,1)}n_{\{-1\}}G^1S(Q^2V^2_2Y^2_{(2,2)})\\
&\quad V^3Y^3\otimes X^2Z^1_2Y^1_{(1,2)}\cdot m_{\{0\}}\cdot g^2S(q^1(t^1x^1Z^2Y^1_2)_1)\alpha t^3x^2_2Z^3_{(1,2)}V^1_2Y^2_{(1,2)}\\
&\quad\otimes_H n_{\{0\}}\cdot G^2S(X^3x^3Z^3_2Q^1V^2_1Y^2_{(2,1)})\\
&\overset{(2.3),(2.17),(2.3)}{=} X^1(Z^1V^1(Y^1_1x^1)_1)_1m_{\{-1\}}g^1S(q^2(t^1Z^2_1V^2(Y^1_1x^1)_2)_2)t^2(Z^2_2V^3)_1(Y^1_2x^2)_1n_{\{-1\}}\\
&\quad G^1S(Q^2(Y^2x^3)_2)Y^3\otimes X^2(Z^1V^1(Y^1_1x^1)_1)_2\cdot m_{\{0\}}\\
&\quad\cdot g^2S(q^1(t^1Z^2_1V^2(Y^1_1x^1)_2)_1)\alpha t^3(Z^3_2V^3)_2(Y^1_2x^2)_2\\
&\quad\otimes_H n_{\{0\}}\cdot G^2S(X^3Z^3Q^1(Y^2x^3)_1)\\
&\overset{(2.1),(2.1),(2.17),(2.5)}{=} X^1Y^1_1V^1_1x^1_{(1,1)}m_{\{-1\}}g^1S(q^2(t^1V^2x^1_2)_2)t^2V^3_1x^2_1n_{\{-1\}}G^1S(Q^2Y^2_2x^3_2)Y^3\\
&\quad\otimes X^2Y^1_2V^1_2x^1_{(1,2)}\cdot m_{\{0\}}\cdot g^2S(q^1(t^1V^2x^1_2)_1)\alpha t^3V^3_2x^2_2\\
&\quad\otimes_H n_{\{0\}}\cdot G^2S(X^3Q^1Y^2_1x^3_1).
\end{aligned}$$

Hence, by comparing the two computations performed above we get that $\overline{\phi}_{2,M,N}$ is left H-colinear, as needed.

In a similar manner one can show that $\varphi_{2,\mathbf{M},\mathbf{N}}$ is left H-colinear, for all $\mathbf{M},\mathbf{N}\in {}^H_H\mathcal{Y}D$. Indeed, on one hand we have

$$
\begin{aligned}
&\lambda_{(\mathbf{M}\otimes\mathbf{N})\otimes H}\varphi_{2,\mathbf{M},\mathbf{N}}((m\otimes h)\otimes_H(n\otimes h'))\\
\overset{(2.36)}{=}\ & \lambda_{(\mathbf{M}\otimes\mathbf{N})\otimes H}((x^1\cdot m\otimes x^2h_1\cdot n)\otimes x^3h_2h')\\
\overset{(4.9)}{=}\ & X^1(y^1\cdot(x^1\cdot m\otimes x^2h_1\cdot n))_{[-1]}y^2x^3_1h_{(2,1)}h'_1\\
&\otimes\big(X^2\cdot(y^1\cdot(x^1\cdot m\otimes x^2h_1\cdot n))_{[0]}\otimes X^3y^3x^3_2h_{(2,2)}h'_2\big)\\
\overset{(4.8),(4.14)}{=}\ & X^1Z^1(z^1T^1y^1_1x^1\cdot m)_{[-1]}z^2(T^2y^1_2x^2h_1\cdot n)_{[-1]}T^3y^2x^3_1h_{(2,1)}h'_1\\
&\big((X^2_1Z^2\cdot(z^1T^1y^1_1x^1\cdot m)_{[0]}\otimes X^2_2Z^3z^3\cdot(T^2y^1_2x^2h_1\cdot n)_{[0]})\otimes X^3y^3x^3_2h_{(2,2)}h'_2\big)\\
\overset{(2.3),(2.1)}{=}\ & X^1Z^1(z^1y^1\cdot m)_{[-1]}z^2((y^2h_1)_1x^1\cdot n)_{[-1]}(y^2h_1)_2x^2h'_1\\
&\otimes\big((X^2_1Z^2\cdot(z^1y^1\cdot m)_{[0]}\otimes X^2_2Z^3z^3\cdot((y^2h_1)_1x^1\cdot n)_{[0]})\otimes X^3y^3h_2x^3h'_2\big)\\
\overset{(4.7)}{=}\ & X^1Z^1(z^1y^1\cdot m)_{[-1]}z^2y^2_1h_{(1,1)}(x^1\cdot n)_{[-1]}x^2h'_1\\
&\otimes\big((X^2_1Z^2\cdot(z^1y^1\cdot m)_{[0]}\otimes X^2_2Z^3z^3y^2_2h_{(1,2)}\cdot(x^1\cdot n)_{[0]})\otimes X^3y^3h_2x^3h'_2\big)\\
\overset{(2.1),(4.7)}{=}\ & X^1Z^1y^1_1(z^1\cdot m)_{[-1]}z^2Y^1h_{(1,1)}(x^1\cdot n)_{[-1]}x^2h'_1\\
&\otimes\big((X^2_1Z^2y^1_2\cdot(z^1\cdot m)_{[0]}\otimes X^2_2Z^3y^2z^3_1Y^2h_{(1,2)}\cdot(x^1\cdot n)_{[0]})\otimes X^3y^3z^3_2Y^3h_2x^3h'_2\big)\\
\overset{(2.3)}{=}\ & X^1(z^1\cdot m)_{[-1]}z^2Y^1h_{(1,1)}(x^1\cdot n)_{[-1]}x^2h'_1\\
&\otimes\big((y^1X^2\cdot(z^1\cdot m)_{[0]}\otimes y^2X^3_1z^3_1Y^2h_{(1,2)}\cdot(x^1\cdot n)_{[0]})\otimes y^3X^3_2z^3_2Y^3h_2x^3h'_2\big),
\end{aligned}
$$

for all $m\in\mathbf{M}$, $n\in\mathbf{N}$ and $h,h'\in H$. On the other hand we compute, again for all $m\in\mathbf{M}$, $n\in\mathbf{N}$ and $h,h'\in H$, that

$$
\begin{aligned}
&\varphi_{2,\mathbf{M},\mathbf{N}}\lambda_{(\mathbf{M}\otimes H)\otimes_H(\mathbf{N}\otimes H)}((m\otimes h)\otimes_H(n\otimes h'))\\
=\ & (m\otimes h)_{\{-1\}}(n\otimes h)_{\{-1\}}\otimes\varphi_{2,\mathbf{M},\mathbf{N}}((m\otimes h)_{\{0\}}\otimes_H(n\otimes h)_{\{0\}})\\
\overset{(4.9)}{=}\ & X^1(x^1\cdot m)_{[-1]}x^2h_1Y^1(y^1\cdot n)_{[-1]}y^2h'_1\\
&\otimes\varphi_{2,\mathbf{M},\mathbf{N}}((X^2\cdot(x^1\cdot m)_{[0]}\otimes X^3x^3h_2)\otimes_H(Y^2\cdot(y^1\cdot n)_{[0]}\otimes Y^3y^3h'_2))\\
\overset{(2.36),(2.1)}{=}\ & X^1\cdot(x^1\cdot m)_{[-1]}x^2Y^1h_{(1,1)}(y^1\cdot n)_{[-1]}y^2h'_1\otimes\\
&\otimes\big((z^1X^2\cdot(x^1\cdot m)_{[0]}\otimes z^2X^3_1x^3_1Y^2h_{(1,2)}\cdot(y^1\cdot n)_{[0]})\otimes z^3X^3_2x^3_2Y^3h_2x^3h'_2\big).
\end{aligned}
$$

This shows that $\varphi_{2,\mathbf{M},\mathbf{N}}$ is left H-colinear as well, completing the proof. ∎

4.7. A structure theorem for bicomodule algebras. We will continue the ideas in Subsection 3.3 in order to give a structure theorem for algebras within the strict monoidal category $({}^H_H\mathcal{M}^H_H,\otimes_H,H)$.

4.8. Definition. *Let H be a quasi-bialgebra and $\mathcal{A}$ an associative unital algebra. By an H-bicomodule algebra structure on $\mathcal{A}$ we mean a quintuple $(\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$, where λ and ρ are left and right H-coactions on $\mathcal{A}$, respectively, and where $\Phi_\lambda\in H\otimes H\otimes\mathcal{A}$, $\Phi_\rho\in\mathcal{A}\otimes H\otimes H$ and $\Phi_{\lambda,\rho}\in H\otimes\mathcal{A}\otimes H$ are invertible elements, such that:*

(i) $(\mathcal{A},\lambda,\Phi_\lambda)$ *is a left H-comodule algebra;*

(ii) $(\mathcal{A},\rho,\Phi_\rho)$ *is a right H-comodule algebra;*

(iii) the following compatibility relations hold:

$$\Phi_{\lambda,\rho}(\lambda\otimes \mathrm{Id}_H)(\rho(u)) = (\mathrm{Id}_H\otimes\rho)(\lambda(u))\Phi_{\lambda,\rho},\ \ \forall\ u\in\mathcal{A}, \tag{4.16}$$

$$(1_H\otimes\Phi_{\lambda,\rho})(\mathrm{Id}_H\otimes\lambda\otimes\mathrm{Id}_H)(\Phi_{\lambda,\rho})(\Phi_\lambda\otimes 1_H) = (\mathrm{Id}_H\otimes\mathrm{Id}_H\otimes\rho)(\Phi_\lambda)(\Delta\otimes\mathrm{Id}_{\mathcal{A}}\otimes\mathrm{Id}_H)(\Phi_{\lambda,\rho}), \tag{4.17}$$

$$(1_H\otimes\Phi_\rho)(\mathrm{Id}_H\otimes\rho\otimes\mathrm{Id}_H)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho}\otimes 1_H) = (\mathrm{Id}_H\otimes\mathrm{Id}_{\mathcal{A}}\otimes\Delta)(\Phi_{\lambda,\rho})(\lambda\otimes\mathrm{Id}_H\otimes\mathrm{Id}_H)(\Phi_\rho). \tag{4.18}$$

As we will see, the structure theorem for algebras within ${}^H_H\mathcal{M}^H_H$ is nothing but the structure theorem for quasi-Hopf bicomodule algebras given in [Dello, Panaite, Van Oystaeyen, Zhang, 2016]. To this end, we start by describing an algebra in ${}^H_H\mathcal{M}^H_H$.

4.9. Lemma. *Let H be a quasi-bialgebra. An algebra in ${}^H_H\mathcal{M}^H_H$ is a quadruple $(\mathcal{A},\lambda,\rho,i)$ consisting of a k-algebra $\mathcal{A}$, a k-algebra morphism $i: H\to\mathcal{A}$, and k-linear maps $\lambda:\mathcal{A}\to H\otimes\mathcal{A}$ and $\rho:\mathcal{A}\to\mathcal{A}\otimes H$ such that the following conditions hold:*
- *$\lambda(i(h)) = h_1\otimes i(h_2)$ and $\rho(i(h)) = i(h_1)\otimes h_2$, for all $h\in H$;*
- *$(\mathcal{A},\lambda,\rho,\Phi_\lambda := X^1\otimes X^2\otimes i(X^3), \Phi_\rho := i(X^1)\otimes X^2\otimes X^3, \Phi_{\lambda,\rho} := X^1\otimes i(X^2)\otimes X^3)$ is an H-bicomodule algebra, where $\Phi = X^1\otimes X^2\otimes X^3$ is the reassociator of H.*

Proof. Let $\mathcal{A}$ be an algebra in ${}^H_H\mathcal{M}^H_H$. Since the forgetful functors ${}^H_H\mathcal{M}^H_H\to {}_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{M}^H_H\to {}^H_H\mathcal{M}_H$ are strong monoidals, we get that $\mathcal{A}$, with the same H-bimodule structure, is both an algebra in ${}_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{M}_H$. Hence, by Lemma 3.5 and its left version, we deduce that $\mathcal{A}$ is a k-algebra and there exist $i: H\to\mathcal{A}$ an algebra morphism, and $\lambda:\mathcal{A}\to H\otimes\mathcal{A}$ and $\rho:\mathcal{A}\to\mathcal{A}\otimes H$ algebra maps such that
- $\lambda(i(h)) = h_1\otimes i(h_2)$ and $\rho(i(h)) = i(h_1)\otimes h_2$, for all $h\in H$;
- $(\mathcal{A},\lambda,\Phi_\lambda := X^1\otimes X^2\otimes i(X^3))$ is a left H-comodule algebra and $(\mathcal{A},\rho,\Phi_\rho := i(X^1)\otimes X^2\otimes X^3)$ is a right H-comodule algebra.

There is only one property of $\mathcal{A}$ that we did not explored yet. Namely, the compatibility between the left and right H-coactions on $\mathcal{A}$. More precisely, by (4.2) we have, for all $u\in\mathcal{A}$, that

$$X^1u_{(0)_{\{-1\}}}\otimes i(X^2)u_{(0)_{\{0\}}}\otimes X^3u_{(1)} = u_{\{-1\}}X^1\otimes u_{\{0\}_{(0)}}i(X^2)\otimes u_{\{0\}_{(1)}}X^3,$$

and this means that $(\mathcal{A},\lambda,\rho,\Phi_\lambda := X^1\otimes X^2\otimes i(X^3), \Phi_\rho := i(X^1)\otimes X^2\otimes X^3, \Phi_{\lambda,\rho} := X^1\otimes i(X^2)\otimes X^3)$ is an H-bicomodule algebra.

The converse follows from Lemma 3.5 and its left version. ■

As it was pointed out in [Dello, Panaite, Van Oystaeyen, Zhang, 2016, Proposition 1.3], examples of algebras in ${}^H_H\mathcal{M}^H_H$ are given by particular cases of smash product algebras. At this point we can get a more conceptual proof for [Dello, Panaite, Van Oystaeyen, Zhang, 2016, Proposition 1.3] as follows.

4.10. Proposition. *Let H be a quasi-bialgebra and A an algebra in ${}_H^H\mathcal{Y}D$, with coaction denoted by $A \to H \otimes A$, $a \mapsto a_{[-1]} \otimes a_{[0]}$. Then $(A\#H, \lambda, \rho, i)$ is an algebra in ${}_H^H\mathcal{M}_H^H$, where $i : H \to A\#H$ is the canonical inclusion map and*

$$\lambda : A\#H \to H \otimes (A\#H),\ \lambda(a\#h) = X^1(x^1 \cdot a)_{[-1]}x^2h_1 \otimes (X^2 \cdot (x^1 \cdot a)_{[0]}\#X^3x^3h_2),$$
$$\rho : A\#H \to (A\#H) \otimes H,\ \rho(a\#h) = (x^1 \cdot a\#x^2h_1) \otimes x^3h_2,$$

for all $a \in A$ and $h \in H$.

Proof. By Corollary 3.6 we know that $(\mathcal{A}, \rho, i)$ is a right H-comodule algebra. Also, since A is an algebra in ${}_H^H\mathcal{Y}D$ and the functor $\mathcal{F}$ from the proof of Theorem 4.6 is strong monoidal we obtain that $\mathcal{F}(A) = A \otimes H$ is an algebra in ${}_H^H\mathcal{M}_H^H$. Firstly, by (4.8), (4.9) and (4.10) we deduce that $A \otimes H$ is an object in ${}_H^H\mathcal{M}_H^H$ via the structure given by

$$h' \cdot (a \otimes h) \cdot h'' = h_1 \cdot a \otimes h'_2hh'',$$
$$\lambda(a \otimes h) = X^1(x^1 \cdot a)_{[-1]}x^2h_1 \otimes (X^2 \cdot (x^1 \cdot a)_{[0]} \otimes X^3x^3h_2),$$
$$\rho(a \otimes h) = (x^1 \cdot a \otimes x^2h_1) \otimes x^3h_2,$$

for all $a \in A$ and $h, h', h'' \in H$. Secondly, as we have seen the multiplication on $\mathcal{F}(A)$ is given by

$$(a \otimes h)(a' \otimes h') = \varphi_{2,A,A}((a \otimes h) \otimes_H (a' \otimes h')) \overset{(2.36)}{=} (x^1 \cdot a)(x^2h_1 \cdot a') \otimes x^3h_2h',$$

for all $a, a' \in A$ and $h, h' \in H$, and the unit is $(\mathrm{Id}_H \otimes 1_A)\varphi_0(1_H) = 1_H \otimes 1_A$. In other words, $\mathcal{F}(A) = A\#H$ with the algebra structure in ${}_H^H\mathcal{M}_H^H$ provided by $i : H \ni h \mapsto 1_A\#h \in A\#H$ and λ, ρ as in the statement, and we are done. ∎

The next result can be viewed as a two-sided two-cosided version of Theorem 3.7. It is also an equivalent version of [Dello, Panaite, Van Oystaeyen, Zhang, 2016, Theorem 1.7].

4.11. Theorem. *Let H be a quasi-Hopf algebra and $(\mathcal{A}, \rho, \lambda, i)$ an algebra in ${}_H^H\mathcal{M}_H^H$. Then there exists an algebra A in ${}_H^H\mathcal{Y}D$ such that $\mathcal{A}$ is isomorphic to $A\#H$, as algebras in ${}_H^H\mathcal{M}_H^H$.*

Proof. $(\mathcal{A}, \rho, i)$ is an algebra in ${}_H\mathcal{M}_H^H$. If $A := \mathcal{A}^{\overline{\mathrm{co}(H)}}$, then by Theorem 3.7 we know that A is a left H-module algebra and $\mathcal{A} \cong A\#H$, as algebras in ${}_H\mathcal{M}_H^H$. Furthermore, since $\mathcal{A}$ is actually an algebra in ${}_H^H\mathcal{M}_H^H$ and $\mathcal{G}$ from the proof of Theorem 4.6 is a strong monoidal functor, we get that A is an algebra in ${}_H^H\mathcal{Y}D$. Consequently, $A\#H$ is an algebra in ${}_H^H\mathcal{M}_H^H$ via the structure described in Proposition 4.10.

Thus it only remains to check that the isomorphism χ in the proof of Theorem 3.7 intertwines the left H-coactions of $\mathcal{A}$ and $A\#H$. But this follows from Proposition 4.4, since any $M \in {}_H^H\mathcal{M}_H^H$ decomposes as $M^{\overline{\mathrm{co}H}} \otimes H$ in ${}_H^H\mathcal{M}_M^H$ via an isomorphism, say χ_M, and $\chi = \chi_{\mathcal{A}}$. ∎

5. The structure of a coalgebra in ${}^H_H\mathcal{M}^H_H$

We move now to the coalgebra case. The quasi-Hopf algebra notion is not selfdual. Indeed, the dual space H^* of a finite dimensional quasi-Hopf algebra H is a coassociative coalgebra and a non-associative algebra, and it is an example of what is called a dual quasi-Hopf algebra. Dual quasi-Hopf algebras have their own theory. Thus the results of this section cannot be viewed as the formal dual of the ones proved for quasi-Hopf (bi)comodule algebras. But we should stress the fact that in both situations the key role is played by the monoidal equivalence between ${}_H\mathcal{M}^H_H$ and ${}_H\mathcal{M}$, and ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{Y}D$, respectively.

Recall that, for any k-algebra A, the category of A-bimodules ${}_A\mathcal{M}_A$ is strict monoidal under the structure given by $\otimes_A$; the unit object is A itself. A (co)algebra in ${}_A\mathcal{M}_A$ is called an A-(co)ring. It is well known that giving an A-ring R is equivalent to giving an algebra morphism $i : A \to R$.

The next result describes the structure of a coalgebra within ${}_H\mathcal{M}^H_H$. For the notion of cowreath and its connection to the structure of certain corings we refer to [Bulacu, Caenepeel, 2014].

By an H-coring defined by a left H-module coalgebra we mean a coring of the form $C \otimes H$ with the coalgebra structure in $({}_H\mathcal{M}_H, \otimes_H, H)$ given by the cowreath (H, C) in ${}_k\mathcal{M}$, as it is defined by the "op"-version of [Bulacu, Caenepeel, 2014, Corollary 6.4], specialized for the H-comodule algebra equals to H.

5.1. Proposition. *Let H be a quasi-Hopf algebra. Then there exists a one to one correspondence between*

i) coalgebra structures in ${}_H\mathcal{M}^H_H$;

ii) coalgebra structures in ${}_H\mathcal{M}$;

iii) H-coring structures defined by left H-module coalgebras.

Proof. The one to one correspondence between i) and ii) is established by the monoidal category equivalence between ${}_H\mathcal{M}^H_H$ and ${}_H\mathcal{M}$. Up to an isomorphism, any coalgebra $\mathbf{C}$ in ${}_H\mathcal{M}^H_H$ is of the form $C \otimes H$ for a suitable coalgebra C in ${}_H\mathcal{M}$. Once more, remark that $C \otimes H$ is an object in ${}_H\mathcal{M}^H_H$ via the structure determined by

$$h \cdot (c \otimes h') \cdot h'' = h_1 \cdot c \otimes h_2 h' h'',$$
$$\rho_{C\otimes H}(c \otimes h) = (x^1 \cdot c \otimes x^2 h_1) \otimes x^3 h_2,$$

for all $c \in H$ and $h, h', h'' \in H$. Furthermore, by the strong monoidal structure of the functor $\mathcal{F}$, we deduce that $C \otimes H$ is a coalgebra in ${}_H\mathcal{M}^H_H$ with comultiplication and counit given by

$$\underline{\Delta} : C \otimes H = \mathcal{F}(C) \xrightarrow{\mathcal{F}(\Delta_C)} \mathcal{F}(C \otimes C) \xrightarrow{\varphi^{-1}_{2,C,C}} \mathcal{F}(C) \otimes_H \mathcal{F}(C) = (C \otimes H) \otimes_H (C \otimes H)$$

and $\underline{\varepsilon} : C \otimes H = \mathcal{F}(C) \xrightarrow{\mathcal{F}(\varepsilon_C)} \mathcal{F}(k) \xrightarrow{\varphi_0^{-1}} H$, where Δ_C and ε_C are the comultiplication and the counit of the coalgebra C in ${}_H\mathcal{M}$. According to Theorem 2.8, we have that

$$\underline{\Delta}(c \otimes h) = (X^1 \cdot c_1 \otimes 1_H) \otimes_H (X^2 \cdot c_2 \otimes X^3 h) \text{ and } \underline{\varepsilon}(c \otimes h) = \varepsilon_C(c)h, \tag{5.1}$$

for all $c \in C$ and $h \in H$, where this time $\Delta_C(c) := c_1 \otimes c_2$, for all $c \in C$, and $\cdot$ is the left action of H on C.

Since the forgetful functor from ${}_H\mathcal{M}_H^H$ to ${}_H\mathcal{M}_H$ is strong monoidal, it follows that a coalgebra in ${}_H\mathcal{M}_H^H$ is nothing but an H-coring $(\mathbf{C}, \underline{\Delta}_{\mathbf{C}}, \underline{\varepsilon}_{\mathbf{C}})$ for which the comultiplication $\underline{\Delta}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \otimes_H \mathbf{C}$ and the counit $\underline{\varepsilon}_{\mathbf{C}}$ are right H-colinear maps. Since $\mathbf{C} \equiv C \otimes H$ in ${}_H\mathcal{M}_H^H$, with $C = \mathbf{C}^{\overline{\mathrm{co(H)}}}$ a left H-module coalgebra, we deduce from [Bulacu, Caenepeel, 2014, Proposition 5.1] that $\mathbf{C} \equiv C \otimes H$ is the H-coring in ${}_k\mathcal{M}$ completely determined by the triple (ψ, δ, f) consisting of $(c \in C,\ h \in H)$

- $\psi : H \otimes C \to C \otimes H$, $\psi(h \otimes c) = h_1 \cdot c \otimes h_2$;
- $\delta : C \to C \otimes C \otimes H$, $\delta(c) = X^1 \cdot c \otimes X^2 h_1 \otimes X^3 h_2$;
- $f : C \to H$, $f(c) = \varepsilon_C(c)1_H$.

In other words, (H, C) is the "op"-version of the cowreath considered in [Bulacu, Caenepeel, 2014, Corollary 6.4], specialized for the H-comodule algebra equals to H. As $\mathbf{C} \equiv C \otimes H = \mathcal{F}(C)$ in ${}_H^H\mathcal{M}_H^H$, we deduce that $\underline{\Delta}_{\mathbf{C}}$ and $\underline{\varepsilon}_{\mathbf{C}}$ are as in (5.1), and therefore right H-colinear maps. Thus the one to one correspondence between i) and iii) is established, too. ■

Proposition 5.1 does not say that, up to an isomorphism, a coalgebra in ${}_H\mathcal{M}_H^H$ is a sort of smash product coalgebra, a construction due to Molnar [Molnar, 1977] in the Hopf algebra case, and which does not exist in the quasi-Hopf algebra case. Otherwise stated, even in the Hopf case the smash product construction does not characterize coalgebras in ${}_H\mathcal{M}_H^H$. To have such a dual result, we must work with coalgebras within the category ${}_H^H\mathcal{M}^H$, because ${}_H^H\mathcal{M}^H$ can be regarded as the formal dual version of the category ${}_H\mathcal{M}_H^H$. This has been already done in [Wang, Zhang, Niu, 2013], for the dual context provided by the dual quasi-Hopf algebras. To make a long story short, the result in Proposition 5.1 cannot be seen as the formal dual result in Theorem 3.7.

We pass now to the study of a structure of a coalgebra in ${}_H^H\mathcal{M}_H^H$. Due to the extra corner that we have in this case, this time we can characterize coalgebras in ${}_H^H\mathcal{M}_H^H$ as some sort of smash product coalgebras. To achieve this, we use as a source of inspiration some results obtained in [Bespalov, Drabant, 1998] for the category of Hopf bimodules (two sided two cosided Hopf modules in our terminology) in braided monoidal categories. Note that, our results are not particular cases of some results shown in [Bespalov, Drabant, 1998], since a quasi-Hopf algebra cannot be viewed as a Hopf algebra in a suitable braided monoidal category.

We start by proving the following key result. In order to avoid any confusion, we denote by ${}_H\overline{\mathcal{M}}_H$ the category of bimodules over a quasi-bialgebra H, endowed with the monoidal structure defined by the structure of H as in Subsection 2.2.

5.2. Lemma. *Let H be a quasi-bialgebra. Then the forgetful functor*

$$\mathcal{U}: {}_H^H\mathcal{M}_H^H = ({}_H^H\mathcal{M}_H^H, \otimes_H, H) \to {}_H\overline{\mathcal{M}}_H = ({}_H\mathcal{M}_H, \otimes, k, a', l', r')$$

is opmonoidal under the structure given, for all $M, N \in {}_H^H\mathcal{M}_H^H$, by

$$\psi_{2,M,N}: \mathcal{U}(M \otimes_H N) \ni m \otimes_H n \mapsto m_{(0)} \cdot n_{\{-1\}} \otimes m_{(1)} \cdot n_{(0)} \in \mathcal{U}(M) \otimes \mathcal{U}(N)$$

and $\psi_0 = \varepsilon : \mathcal{U}(H) = H \to k$.

Proof. $\psi_{2,M,N}$ is well defined since, for all $m \in M$, $h \in H$ and $n \in N$ we have that

$$\begin{aligned}
\psi_{2,M,N}(m \cdot h \otimes_H n) &= (m \cdot h)_{(0)} \cdot n_{\{-1\}} \otimes (m \cdot h)_{(1)} \cdot n_{\{0\}} \\
&= m_{(0)} \cdot h_1 n_{\{-1\}} \otimes m_{(1)} h_2 \cdot n_{\{0\}} \\
&= m_{(0)} \cdot (h \cdot n)_{\{-1\}} \otimes m_{(1)} \cdot (h \cdot n)_{\{0\}} \\
&= \psi_{2,M,N}(m \otimes_H h \cdot n).
\end{aligned}$$

Also, it can be easily checked that $\psi_{2,M,N}$ is an H-bilinear map.

We next show that ψ_2 fulfills the relations in (2.23)-(2.25). Indeed, for any $M, N, P \in {}_H^H\mathcal{M}_H^H$ we have

$$\begin{aligned}
& a'_{\mathcal{U}(M),\mathcal{U}(N),\mathcal{U}(P)}(\psi_{2,M,N} \otimes \mathrm{Id}_{\mathcal{U}(P)})\psi_{2,M\otimes_H N,P}(m \otimes_H n \otimes_h p) \\
&= a'_{\mathcal{U}(M),\mathcal{U}(N),\mathcal{U}(P)}(\psi_{2,M,N}((m \otimes_H n)_{(0)} \cdot p_{\{-1\}}) \otimes (m \otimes_H n)_{(1)} \cdot p_{\{0\}}) \\
&= a'_{\mathcal{U}(M),\mathcal{U}(N),\mathcal{U}(P)}(\psi_{2,M,N}(m_{(0)} \otimes_H n_{(0)} \cdot p_{\{-1\}}) \otimes m_{(1)} n_{(1)} \cdot p_{\{0\}}) \\
&= X^1 \cdot m_{(0,0)} \cdot n_{(0)_{\{-1\}}} p_{\{-1\}_1} x^1 \otimes (X^2 m_{(0,1)} \cdot n_{(0)_{\{0\}}} p_{\{-1\}_2} x^2 \\
&\qquad \otimes X^3 m_{(1)} n_{(1)} \cdot p_{\{0\}} \cdot x^3) \\
&\overset{(2.27)}{\underset{(4.2)}{=}} m_{(0)} \cdot n_{\{-1\}} X^1 p_{\{-1\}_1} x^1 \otimes (m_{(1)_1} \cdot n_{\{0\}_{(0)}} \cdot X^2 p_{\{-1\}_2} x^2 \\
&\qquad \otimes m_{(1)_2} n_{\{0\}_{(1)}} X^3 \cdot p_{\{0\}} \cdot x^3) \\
&\overset{(4.1)}{=} m_{(0)} \cdot n_{\{-1\}} p_{\{-1\}} \otimes (m_{(1)_1} \cdot n_{\{0\}_{(0)}} \cdot p_{\{0,-1\}} \otimes m_{(1)_2} n_{\{0\}_{(1)}} \cdot p_{\{0,0\}}) \\
&= m_{(0)} \cdot n_{\{-1\}} p_{\{-1\}} \otimes ((m_{(1)} \cdot n_{\{0\}})_{(0)} \cdot p_{\{0,-1\}} \otimes (m_{(1)} \cdot n_{\{0\}})_{(1)} \cdot p_{\{0,0\}}) \\
&= m_{(0)} \cdot (n \otimes_H p)_{\{-1\}} \otimes \psi_{2,N,P}(m_{(1)} \cdot (n \otimes_H p)_{\{0\}}) \\
&= (\mathrm{Id}_{\mathcal{U}(M)} \otimes \psi_{2,N,P})\psi_{2,M,N\otimes_H P}(m \otimes_H n \otimes_H p),
\end{aligned}$$

for all $m \in M$, $n \in N$ and $p \in P$, as required. We leave it to the reader to check that the relations (2.24)-(2.25) are obeyed by our ψ, too. ■

At this point we can prove one of the main results of this paper.

5.3. Theorem. *Let H be a quasi-bialgebra. Then giving a coalgebra in ${}_H^H\mathcal{M}_H^H$ is equivalent to giving a pair (C, π) consisting of an H-bimodule coalgebra C and an H-bimodule coalgebra morphism $\pi : C \to H$.*

PROOF. Let $(C, \underline{\Delta}, \underline{\varepsilon})$ be a coalgebra in ${}_H^H\mathcal{M}_H^H$. Since the forgetful functor $\mathcal{U}$ in Lemma 5.2 is opmonoidal, it follows that C is an H-bimodule coalgebra via the original H-bimodule structure, but with comultiplication Δ_C and counit ε_C defined by

$$\Delta_C : C = \mathcal{U}(C) \xrightarrow{\mathcal{U}(\underline{\Delta})} \mathcal{U}(C \otimes_H C) \xrightarrow{\psi_{2,C,C}} \mathcal{U}(C) \otimes \mathcal{U}(C) = C \otimes C$$

and $\varepsilon_C : C = \mathcal{U}(C) \xrightarrow{\mathcal{U}(\underline{\varepsilon})} \mathcal{U}(H) \xrightarrow{\psi_0} k$. Explicitly, for all $c \in C$ we have

$$\Delta_C(c) = c_{\underline{1}(0)} \cdot c_{\underline{2}\{-1\}} \otimes c_{\underline{1}(1)} \cdot c_{\underline{2}\{0\}} \ \text{ and } \ \varepsilon_C = \varepsilon\underline{\varepsilon} : C \to k, \tag{5.2}$$

where we denoted $\underline{\Delta}(c) := c_{\underline{1}} \otimes c_{\underline{2}}$. If we take $\pi = \underline{\varepsilon} : C \to H$, then π is a morphism in ${}_H^H\mathcal{M}_H^H$, and so in particular H-bilinear. The left and right H-colinearity of π read as

$$\Delta(\pi(c)) = c_{\{-1\}} \otimes \pi(c_{\{0\}}) = \pi(c_{(0)}) \otimes c_{(1)},$$

for all $c \in C$. These equalities allow us to compute that

$$\begin{aligned} (\pi \otimes \pi)\Delta_C(c) &= \pi(c_{\underline{1}(0)} \cdot c_{\underline{2}\{-1\}}) \otimes \pi(c_{\underline{1}(1)} \cdot c_{\underline{2}\{0\}}) \\ &= \pi(c_{\underline{1}(0)})c_{\underline{2}\{-1\}} \otimes c_{\underline{1}(1)}\pi(c_{\underline{2}\{0\}}) \\ &= \pi(c_{\underline{1}})_1\pi(c_{\underline{2}})_1 \otimes \pi(c_{\underline{1}})_2\pi(c_{\underline{2}})_2 \\ &= \Delta(\pi(c_{\underline{1}})\pi(c_{\underline{2}})) \\ &= \Delta(\pi(\pi(c_{\underline{1}})c_{\underline{2}})) = \Delta(\pi(c)), \end{aligned}$$

for all $c \in C$, where we freely used that π is an H-bimodule morphism and the counit of $\underline{\Delta}$. Hence we have shown that C is a coalgebra in ${}_H\overline{\mathcal{M}}_H$, and that $\pi : C \to H$ is a morphism of coalgebras within ${}_H\overline{\mathcal{M}}_H$.

Conversely, let (C, π) be a pair consisting of an H-bimodule coalgebra C and an H-bimodule coalgebra morphism $\pi : C \to H$. As above, denote by $(\Delta_C, \varepsilon_C)$ the coalgebra structure of C in ${}_H\overline{\mathcal{M}}_H$. We claim that C becomes a coalgebra in ${}_H^H\mathcal{M}_H^H$ via the original H-bimodule structure of C, H-coactions given by

$$\lambda_C(c) = c_{\{-1\}} \otimes c_{\{0\}} := \pi(c_1) \otimes c_2 \in H \otimes C \ \ , \ \ \rho_C(c) = c_{(0)} \otimes c_{(1)} := c_1 \otimes \pi(c_2) \in C \otimes H, \tag{5.3}$$

for all $c \in C$, and coalgebra structure determined by

$$\underline{\Delta}(c) = E(c_1) \otimes_H c_2 \ \text{ and } \ \underline{\varepsilon} = \pi, \tag{5.4}$$

for all $c \in C$, where E is the projection in (2.28) specialized for the object C, considered in ${}_H\mathcal{M}_H^H$ with the structure above.

Indeed, the fact that C is an object in ${}_H^H\mathcal{M}_H^H$ modulo its regular H-actions and (5.3) follows easily from the defining properties of the pair (C, π), as well as the fact that $\pi : C \to H$ becomes a morphism in ${}_H^H\mathcal{M}_H^H$. The comultiplication $\underline{\Delta}$ in (5.4) is an H-bimodule morphisms since

$$\underline{\Delta}(h \cdot c) = E(h_1 \cdot c_1) \otimes_H h_2 \cdot c_2 = E(h_1 \cdot c_1) \cdot h_2 \otimes_H c_2 \overset{(2.30)}{=} h \cdot E(c_1) \otimes_H c_2 = h \cdot \underline{\Delta}(c)$$

and $\underline{\Delta}(c\otimes h) = E(c_1\cdot h_1)\otimes_H c_2\cdot h_2 \overset{(2.31)}{=} E(c_1)\otimes_H c_2\cdot h = \underline{\Delta}(c)\cdot h$, for all $c\in C$ and $h\in H$. The computation

$$\begin{aligned}\rho_{C\otimes_H C}\underline{\Delta}(c) &= E(c_1)_{(0)}\otimes_H c_{(2,1)}\otimes E(c_1)_{(1)}\pi(c_{(2,2)})\\ &\overset{(2.29)}{=} E(x^1\cdot E(c_1))\otimes_H c_{(2,1)}\otimes x^3\pi(c_{(2,2)})\\ &\overset{(2.31)}{=} E(x^1\cdot c_1)\otimes_H x^2\cdot c_{(2,1)}\otimes \pi(x^3\cdot c_{(2,2)})\\ &= E(c_{(1,1)}\cdot x^1)\otimes_H c_{(1,2)}\cdot x^2\otimes \pi(c_2\cdot x^3)\\ &\overset{(2.31)}{=} E(c_{(1,1)})\otimes_H c_{(1,2)}\otimes \pi(c_2)\\ &= (\underline{\Delta}\otimes \mathrm{Id}_H)\rho_C(c),\end{aligned}$$

valid for all $c\in C$, shows that $\underline{\Delta}$ in (5.4) is right H-colinear. It is also left H-colinear. To see this, observe that, for all $c\in C$, we have

$$E(X^1\cdot c_1)_1\cdot X^2\pi(c_2)\otimes E(X^1\cdot c_1)_2\cdot X^3 = q^1_1\cdot c_{(1,1)}\cdot p^1\otimes q^1_2\cdot c_{(1,2)}\cdot p^2S(q^2\pi(c_2)). \tag{5.5}$$

Indeed, since

$$q^1\cdot c_{(1,1)}\otimes S(q^2\pi(c_{(1,2)}))\pi(c_2) = c_1\cdot X^1\otimes S(\pi(c_{(2,1)})X^2)\alpha\pi(c_{(2,2)})X^3 = c\cdot q^1\otimes S(q^2),$$

for all $c\in C$ we compute that

$$\begin{aligned}&E(X^1\cdot c_1)_1\cdot X^2\pi(c_2)\otimes E(X^1\cdot c_1)_2\cdot X^3\\ &= (q^1X^1_1\cdot c_{(1,1)}\cdot \beta S(q^2X^1_2\pi(c_{(1,2)})))_1\cdot X^2\pi(c_2)\\ &\qquad\otimes(q^1X^1_1\cdot c_{(1,1)}\cdot \beta S(q^2X^1_2\pi(c_{(1,2)})))_2\cdot X^3\\ &\overset{(2.7)}{\underset{(2.12)}{=}} (q^1X^1_1\cdot c_{(1,1)})_1\cdot \delta^1S(q^2_2X^1_{(2,2)}\pi(c_{(1,2)_2}))f^1X^2\pi(c_2)\\ &\qquad\otimes(q^1X^1_1\cdot c_{(1,1)})_2\cdot \delta^2S(q^2_1X^1_{(2,1)}\pi(c_{(1,2)_1}))f^2X^3\\ &\overset{(2.19)}{=} (q^1\cdot (Q^1\cdot c_{(1,1)})_1)_1\cdot x^1_1\delta^1S(Q^2\pi(c_{(1,2)})x^3)\pi(c_2)\\ &\qquad\otimes(q^1\cdot (Q^1\cdot c_{(1,1)})_1)_2\cdot x^1_2\delta^2S(q^2\pi((Q^1\cdot c_{(1,1)})_2)x^2)\\ &\overset{(2.9)}{=} (q^1\cdot c_1)_1\cdot Q^1_{(1,1)}p^1\beta S(Q^2)\otimes (q^1\cdot c_1)_2\cdot Q^1_{(1,2)}p^2S(q^2\pi(c_2)Q^1_2)\\ &\overset{(2.17),(2.6)}{=} q^1_1\cdot c_{(1,1)}\cdot p^1\otimes q^1_2\cdot c_{(1,2)}\cdot p^2S(q^2\pi(c_2)),\end{aligned}$$

as desired. With the help of this relation we have that

$$\begin{aligned}\lambda_{C\otimes_H C}\underline{\Delta}(c) &= \lambda_{C\otimes_H C}(E(c_1)\otimes_H c_2)\\ &= \pi(E(c_1)_1)\pi(c_{(2,1)})\otimes E(c_1)_2\otimes_H c_{(2,2)}\\ &= \pi(E(X^1\cdot c_{(1,1)}\cdot x^1)_1\cdot \pi(X^2\cdot c_{(1,2)}\cdot x^2))\\ &\qquad\otimes E(X^1\cdot c_{(1,1)}\cdot x^1)_2\otimes_H X^3\cdot c_2\cdot x^3\\ &\overset{(2.31)}{=} \pi(E(X^1\cdot c_{(1,1)})_1\cdot X^2\pi(c_{(1,2)}))\otimes E(X^1\cdot c_{(1,1)})_2\cdot X^3\otimes_H c_2\\ &= \pi(q^1_1\cdot (c_1)_{(1,1)}\cdot p^1)\otimes q^1_2\cdot (c_1)_{(1,2)}\cdot p^2S(q^2\pi((c_1)_2))\otimes_H c_2\\ &\overset{(2.13)}{=} \pi(q^1_1x^1\cdot (c_1)_1)\otimes q^1_2x^2\cdot (c_1)_{(2,1)}\cdot \beta S(q^2x^3\pi((c_1)_{(2,2)}))\otimes_H c_2\\ &\overset{(2.21)}{=} \pi(X^1\cdot c_{(1,1)})\otimes E(X^2\cdot c_{(1,2)})\otimes_H X^3\cdot c_2\\ &\overset{(2.31)}{=} \pi(c_1)\otimes E(c_{(2,1)})\otimes_H c_{(2,2)}\\ &= (\mathrm{Id}_H\otimes \underline{\Delta})\lambda_C(c),\end{aligned}$$

for all $c \in C$, and therefore $\underline{\Delta}$ in (5.4) is left H-colinear, as stated. So it remains to show that $\underline{\Delta}$ is coassociative in ${}^H_H\mathcal{M}^H_H$, and that $\underline{\varepsilon}$ is a counit for it. To this end, let us note that, for all $c \in C$,

$$\begin{aligned} E(E(c)_1) \otimes E(c)_2 &= E(q^1_1 \cdot c_{(1,1)} \cdot (\beta S(q^2\pi(c_2)))_1) \otimes q^1_2 \cdot c_{(1,2)} \cdot (\beta S(q^2\pi(c_2)))_2 \\ &\overset{(2.31)}{=} E(q^1_1 \cdot c_{(1,1)}) \otimes q^1_2 \cdot c_{(1,2)} \cdot \beta S(q^2\pi(c_2)). \end{aligned}$$

Therefore, we get that, for all $c \in C$,

$$\begin{aligned} (\underline{\Delta} \otimes \mathrm{Id}_C)\underline{\Delta}(c) &= E(E(c_1)_1) \otimes_H E(c_1)_2 \otimes_H c_2 \\ &= E(q^1_1 \cdot (c_1)_{(1,1)}) \otimes_H q^1_2 \cdot (c_1)_{(1,2)} \cdot \beta S(q^2\pi((c_1)_2)) \otimes_H c_2 \\ &\overset{(2.31)}{=} E(q^1_1 x^1 \cdot (c_1)_1) \otimes_H q^1_2 x^2 \cdot (c_1)_{(2,1)} \cdot \beta S(q^2 x^3 \cdot \pi((c_1)_{(2,2)})) \otimes_H c_2 \\ &\overset{(2.21)}{=} E(X^1 \cdot c_{(1,1)}) \otimes_H E(X^2 \cdot c_{(1,2)}) \otimes_H X^3 \cdot c_2 \\ &\overset{(2.31)}{=} E(c_1) \otimes_H E(c_{(2,1)}) \otimes_H c_{(2,2)} \\ &= (\mathrm{Id}_C \otimes \underline{\Delta})\underline{\Delta}(c), \end{aligned}$$

i.e. $\underline{\Delta}$ is coassociative in ${}^H_H\mathcal{M}^H_H$, as desired. Finally, π is a counit for $\underline{\Delta}$ since $E(c_1)\cdot\pi(c_2) = E(c_{(0)}) \cdot c_{(1)} \overset{(2.32)}{=} c$ and

$$\begin{aligned} \pi(E(c_1)) \cdot c_2 &= q^1\pi(c_{(1,1)})\beta S(q^2\pi(c_{(1,2)})) \cdot c_2 \\ &= X^1\beta S(X^2)\alpha X^3 \cdot \varepsilon_C(c_1)c_2 \overset{(2.6)}{=} c, \end{aligned}$$

for all $c \in C$. One can check that the two correspondences defined above are inverses of each other, so we are done. ■

Denote by $H - \mathrm{BimCoalg}(\pi)$ the category whose objects are pairs (C, π) consisting of an H-bimodule coalgebra C and an H-bimodule morphism $\pi : C \to H$. A morphism $\tau : (C, \pi) \to (C', \pi')$ in $H - \mathrm{BimCoalg}(\pi)$ is a morphism of coalgebras $\tau : C \to C'$ within ${}_H\overline{\mathcal{M}}_H$ such that $\pi'\tau = \tau$. Also, by $\mathrm{Coalg}({}^H_H\mathcal{M}^H_H)$ we denote the category of coalgebras and coalgebra morphisms within ${}^H_H\mathcal{M}^H_H$.

5.4. Corollary. *The categories* $H - \mathrm{BimCoalg}(\pi)$ *and* $\mathrm{Coalg}({}^H_H\mathcal{M}^H_H)$ *are isomorphic.*

Proof. By Theorem 5.3, the desired isomorphism is given by the functors $\mathcal{T} : H - \mathrm{BimCoalg}(\pi) \to \mathrm{Coalg}({}_H\mathcal{M}^H_H)$ and $\mathcal{V} : \mathrm{Coalg}({}_H\mathcal{M}^H_H) \to H - \mathrm{BimCoalg}(\pi)$ defined as follows. $\mathcal{T}$ sends (C, π) to C, viewed as coalgebra in ${}^H_H\mathcal{M}^H_H$ under the structure given by (5.3) and (5.4). $\mathcal{T}$ sends a morphism to itself. If $(C, \underline{\Delta}, \underline{\varepsilon})$ is a coalgebra in ${}^H_H\mathcal{M}^H_H$ then $\mathcal{V}(C) = C$, considered as a coalgebra in ${}_H\overline{\mathcal{M}}_H$ with the structure in (5.2). $\mathcal{V}$ acts as identity on morphisms.

We leave the verification of all these details to the reader. ■

5.5. Definition. *For a coalgebra B in ${}_H^H\mathcal{Y}D$ denote by $B\ltimes H$ the k-vector space $B\otimes H$ endowed with the comultiplication*

$$\begin{aligned}\Delta(b\ltimes h)=y^1X^1\cdot b_1\ltimes y^2Y^1(x^1X^2\cdot b_2)_{[-1]}x^2X^3_1h_1\\ \otimes y^3_1Y^2\cdot(x^1X^2\cdot b_2)_{[0]}\ltimes y^3_2Y^3x^3X^3_2h_2,\end{aligned}\tag{5.6}$$

and counit $\varepsilon(b\ltimes h)=\varepsilon_B(b)\varepsilon(h)$, for all $b\in B$ and $h\in H$. As before, $b\mapsto b_{[-1]}\otimes b_{[0]}$ is the left coaction of H on B, $\Delta_B(b)=b_1\otimes b_2$ is the comultiplication of B in ${}_H^H\mathcal{Y}D$ and ε_B is its counit. We call $B\ltimes H$ the smash product coalgebra of B and H.

We have now all the necessary ingredients for the proof of the main result of this paper. In particular, it says that a smash product coalgebra is indeed a coalgebra, but within ${}_H\overline{\mathcal{M}}_H$. In the Hopf case it is just the smash product coalgebra defined by Molnar in [Molnar, 1977]. Note that in this case we don't need the H-module structure on B, and that $B\ltimes H$ is an ordinary k-coalgebra, too.

5.6. Theorem. *Let H be a quasi-Hopf algebra, C an H-bimodule coalgebra and $\pi:C\to H$ an H-bimodule morphism. Then there exists a coalgebra B in ${}_H^H\mathcal{Y}D$ such that C is isomorphic to $B\ltimes H$, as an H-bimodule coalgebra.*

Proof. Consider $C=\mathcal{T}(\underline{(C,\pi)})$ as a coalgebra in ${}_H^H\mathcal{M}_H^H$ with the structure given by (5.3) and (5.4). Then $B=C^{\rm co(H)}$ is a coalgebra in ${}_H^H\mathcal{Y}D$ and C is isomorphic to $B\otimes H$ as coalgebras in ${}_H^H\mathcal{M}_H^H$. The fact that C and $B\otimes H$ are isomorphic objects in ${}_H^H\mathcal{M}_H^H$ follows from the structure theorem for two-sided two-cosided Hopf modules over H. That they are, moreover, isomorphic as coalgebras in ${}_H^H\mathcal{M}_H^H$ is a consequence of a more general result, somehow dual to the one uncovered at the end of the proof of Theorem 3.7. Namely, if the functors $\mathcal{S}:\mathcal{C}\to\mathcal{D}$ and $\mathcal{R}:\mathcal{D}\to\mathcal{C}$ define a monoidal category equivalence then $\mathcal{RS}(\mathbf{C})\cong\mathbf{C}$ is a coalgebra isomorphism in $\mathcal{C}$, for any coalgebra $\mathbf{C}$ within $\mathcal{C}$, where $\mathcal{RS}(\mathbf{C})$ has the coalgebra structure provided by the monoidal structure of $\mathcal{RS}$ and the coalgebra structure of $\mathbf{C}$, respectively.

The structure that makes $B\otimes H$ an object in ${}_H^H\mathcal{M}_H^H$ is the one in (4.8)-(4.10), while the coalgebra structure of $B\otimes H$ in ${}_H^H\mathcal{M}_H^H$ is obtained from (5.1). With these structures, $\mathcal{T}((C,\pi))$ and $B\otimes H$ are isomorphic as coalgebras in ${}_H^H\mathcal{M}_H^H$. By Corollary 5.4 we deduce that $(C,\pi)=\mathcal{VT}(C)$ is isomorphic to $\mathcal{V}(B\otimes H)$ as objects in $H-\mathrm{BimCoalg}(\pi)$, and consequently as H-bimodule coalgebras. To end the proof it suffices to show that $\mathcal{V}(\underline{B\otimes H})=(B\ltimes H,\varepsilon_B\otimes\mathrm{Id}_H)$. As a byproduct, we get that $B\ltimes H$ is indeed a coalgebra in ${}_H\overline{\mathcal{M}}_H$, as claimed.

The latest assertion follows from the following computation:

$$\begin{aligned}\Delta(b\otimes h)&\overset{(5.2)}{=}(b\otimes h)_{\underline{1}_{(0)}}\cdot(b\otimes h)_{\underline{2}_{\{-1\}}}\otimes(b\otimes h)_{\underline{1}_{(1)}}\cdot(b\otimes h)_{\underline{2}_{\{-1\}}}\\ &\overset{(5.1)}{=}(X^1\cdot b_1\otimes 1_H)_{(0)}\cdot(X^2\cdot b_2\otimes X^3h)_{\{-1\}}\\ &\qquad\otimes(X^1b_1\otimes 1_H)_{(1)}\cdot(X^3\cdot b_2\otimes X^3h)_{\{0\}}\\ &\underset{(4.10)}{\overset{(4.9)}{=}}(y^1X^1\cdot b_1\otimes y^2)\cdot Y^1(x^1X^2\cdot b_2)_{[-1]}x^2X^3_1h_1\\ &\qquad\otimes y^3\cdot(Y^2\cdot(x^1X^2\cdot b_2)_{[0]}\otimes Y^3x^3X^3_2h_2)\end{aligned}$$

$$\stackrel{(4.8)}{=} \quad (y^1X^1\cdot b_1\otimes y^2Y^1(x^1X^2\cdot b_2)_{[-1]}x^2X^3_1h_1) \\ \otimes(y^3_1Y^2\cdot(x^1X^2\cdot b_2)_{[0]}\otimes y^3_2Y^3x^3X^3_2h_2),$$

valid for any $b\in B$ and $h\in H$, and the fact that $\varepsilon(b\otimes h)=\varepsilon\underline{\varepsilon}(b\otimes h)=\varepsilon_B(b)\varepsilon(h)$. This finishes the proof of the theorem. ∎

5.7. Remark. The comultiplication on $B\otimes H$ defined in (5.6) and its counit appear for the first time in [Bulacu, Nauwelaerts, 2002] as the coalgebra part of the Radford's biproduct construction for quasi-Hopf algebras. At that time we had no clue how to introduce a smash product coalgebra, and by hard computations we proved that it is coassociative up to conjugation by an invertible element. At this point it is clear that this coassociativity is nothing but a reformulation of the fact that $B\bowtie H$ is a coalgebra in ${}_H\overline{\mathcal{M}}_H$, provided that B is a coalgebra in ${}_H^H\mathcal{Y}D$. So we don't need B to be a bialgebra in ${}_H^H\mathcal{Y}D$, as it was assumed in [Bulacu, Nauwelaerts, 2002].

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