Theory and Applications of Categories, Vol. 32, No. 43, 2017, pp. 1501–1521.

ACTION REPRESENTABILITY VIA SPANS AND PREFIBRATIONS

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Abstract. We give several reformulations of action representability of a category as well as action representability of its category of morphisms. In particular we show that for a semi-abelian category $\mathbb{C}$, its category of morphisms is action representable if and only if the functor from the category of split extensions in $\mathbb{C}$ to $\mathbb{C}$, sending a split extension to its kernel, is a prefibration. To obtain these reformulations we show that certain conditions are equivalent for right regular spans of categories.

1. Introduction

The aim of this paper is to study action representability, in the sense of F. Borceux, G. Janelidze and G. M. Kelly [2], and to explain that many of its aspects can be understood as special cases of general facts about spans of categories. Let us recall the necessary background to explain this. For a pointed category $\mathbb{C}$, a split extension (of B with kernel X) is a diagram in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd}$$

where κ is the kernel of α, and $\alpha\beta = 1_B$. A morphism of split extensions is a diagram in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, "u"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & B \arrow[l, shift left, "\beta"] \arrow[d, "w"] \\ X' \arrow[r, "\kappa'"] & A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"] \end{tikzcd} \tag{1}$$

where the top and bottom rows are split extensions (the domain and codomain respectively), and $v\kappa = \kappa' u$, $v\beta = \beta' w$ and $w\alpha = \alpha' v$. Let us denote by $\mathbf{SplExt}(\mathbb{C})$ the category of split extensions in $\mathbb{C}$, and by K and P the functors sending a split extension to its kernel and codomain, respectively. These data together form a span

$$\mathbb{C} \xleftarrow{P} \mathbf{SplExt}(\mathbb{C}) \xrightarrow{K} \mathbb{C}. \tag{2}$$

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number 109203)

Received by the editors 2016-11-26 and, in final form, 2017-11-16.

Transmitted by Walter Tholen. Published on 2017-12-11.

2010 Mathematics Subject Classification: 18A05, 18A22, 18A25, 18A40, 18A99, 18B99, 18D99.

Key words and phrases: action representable, semi-abelian, split extension, normalizer, prefibration, regular span.

Note that when the category $\mathbb{C}$ has finite limits the category $\mathbf{SplExt}(\mathbb{C})$ is equivalent to $\mathbf{Pt}(\mathbb{C})$, the category of split epimorphism in $\mathbb{C}$, that is, the category defined in the same way as $\mathbf{SplExt}(\mathbb{C})$ except omitting all kernels from the diagrams involved. The functor $\pi : \mathbf{Pt}(\mathbb{C}) \to \mathbb{C}$ sending a split epimorphism to its codomain was called the fibration of points by D. Bourn in [5], who first noticed its importance in categorical algebra. The composite of the forgetful functor $\mathbf{SplExt}(\mathbb{C}) \to \mathbf{Pt}(\mathbb{C})$ (which is part of the equivalence of categories mentioned above) and π gives the functor P. Recall that a finitely complete category is protomodular [5], when the change of base functors of the fibration of points reflect isomorphisms. Recall also that a pointed finitely complete category is protomodular if and only if the split short five lemma holds. That is, when for each morphism of split extensions of the form (1), if u and w are isomorphisms, then v is as well. Furthermore, a pointed protomodular category is semi-abelian, in the sense of G. Janelidze, L. Marki and W. Tholen [12], when it is (Barr-)exact [1], and has finite colimits.

For a pointed protomodular category $\mathbb{C}$ and for objects X and B in $\mathbb{C}$, let $\mathrm{SplExt}(B, X)$ be the set of split extensions of B with kernel X quotiented by the equivalence relation, under which, two split extensions are equivalent when there is a morphism between them which is identity on both X and B. For a fixed object X, these sets determine the object map of a functor $\mathrm{SplExt}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, defined on morphisms by *pulling back*. It turns out that for certain categories each of these functors is representable. In particular this is the case for the category of groups and for the category of Lie algebras over an arbitrary commutative ring (see [2, 3]). The representability of these functors was shown in [2] to be equivalent to the existence of terminal objects in the fibres of K, which are called *generic split extensions*. In [8] D. Bourn and G. Janelidze showed, under conditions which hold in any semi-abelian category, that the change of base functors between the fibres of the fibration of points are monadic. This leads to an alternative description of split extensions as *object actions*, and hence to a functor $\mathrm{Act}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ which is isomorphic to $\mathrm{SplExt}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ (see [2, 3]). Accordingly, a semi-abelian category $\mathbb{C}$ is called action representable [2, 3], when for each object X in $\mathbb{C}$ the functor $\mathrm{Act}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable.

We can think of (2) as special case of a span of categories $\underline{\mathbf{S}} =$

$$\mathbb{A} \xleftarrow{P} \mathbb{S} \xrightarrow{Q} \mathbb{B} \tag{3}$$

satisfying the condition:

(i) For each object S in $\mathbb{S}$ and each morphism $f : A' \to A$ in $\mathbb{A}$ such that $P(S) = A$, there exists a P-cartesian morphism $s : S' \to S$ in $\mathbb{S}$ such that $P(s) = f$ and $Q(s) = 1_{Q(S)}$.

The construction of SplExt is then a special case of the following construction. For objects A in $\mathbb{A}$ and B in $\mathbb{B}$, let

$$\overline{\underline{\mathbf{S}}}(A, B) = \{S \in \mathbb{S} \mid P(S) = A \text{ and } Q(S) = B\}/\sim$$

where $\sim$ is the smallest equivalence relation such that $S \sim S'$ if there exists a morphism $s : S \to S'$ with $P(s) = 1_A$ and $Q(s) = 1_B$. For each morphism $f : A' \to A$ in $\mathbb{A}$, it can be seen that, the condition (i) gives rise to a map $\overline{\underline{\mathbf{S}}}(A, B) \to \overline{\underline{\mathbf{S}}}(A', B)$, which sends the equivalence class $[S]$ in $\overline{\underline{\mathbf{S}}}(A, B)$ to the equivalence class $[S']$ in $\overline{\underline{\mathbf{S}}}(A', B)$, where $s : S' \to S$ is a P-cartesian morphism such that $P(s) = f$ and $Q(s) = 1_B$, obtained from the condition (i). It is easy to check that these maps make $\overline{\underline{\mathbf{S}}}(-, B)$ into a functor from $\mathbb{A}^{\mathrm{op}}$ to $\mathbf{Set}$. N. Yoneda [14] was the first to consider the condition (i) and called the P-cartesian morphisms which it requires to exist translations. In fact he studied those spans, which he called regular, satisfying the condition (i) together with a condition (i)* which is equivalent, for a span $\underline{\mathbf{S}}$, to requiring that the span

$$\mathbb{B}^{\mathrm{op}} \xleftarrow{Q^{\mathrm{op}}} \mathbb{S}^{\mathrm{op}} \xrightarrow{P^{\mathrm{op}}} \mathbb{A}^{\mathrm{op}} \tag{4}$$

also satisfies the condition (i). For a regular span $\underline{\mathbf{S}}$, for each object A in $\mathbb{A}$, in an essential dual way, one can construct a functor $\overline{\underline{\mathbf{S}}}(A, -) : \mathbb{B} \to \mathbf{Set}$. The families of functors $\overline{\underline{\mathbf{S}}}(A, -)$ and $\overline{\underline{\mathbf{S}}}(-, B)$ are compatible (in the sense of Proposition 1 Chapter II Section 3 of [13]) and hence determine a bifunctor $\overline{\underline{\mathbf{S}}}(-, *) : \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to \mathbf{Set}$. The construction of a bifunctor $\overline{\underline{\mathbf{S}}}$ from a regular span $\underline{\mathbf{S}}$ is also due to N. Yoneda and was used to study the functors $\mathrm{Ext}(-, *)$, which can be obtained in this way from suitable spans of categories. Note that G. Janelidze was the first to study spans satisfying the conditions (i) or (i)* alone, and called them right and left regular, respectively (see [10]). In Section 3 we will see that for a right regular span satisfying two additional conditions, the functor $\overline{\underline{\mathbf{S}}}(-, B)$ is representable if and only if the fibre $Q^{-1}(B)$ has a terminal object. This essentially recovers as a special case, the above mentioned result, that action representability is equivalent to the existence of generic split extensions, and allows us to study action representability by considering the functor K alone. In order to do so we will compare various conditions on an abstract functor related to the existence of terminal objects in its fibres (see Section 2).

In [9] the normalizer of a monomorphism $f : A \to B$ in a pointed category $\mathbb{C}$ was introduced as a triple (N, n, m), where N is an object, $n : A \to N$ is a normal monomorphism (i.e. the kernel of some morphism) and $m : N \to B$ is a monomorphism such that $mn = f$, which is universal amongst such factorizations. The main result of [9] was that for a semi-abelian category $\mathbb{C}$ the following conditions are equivalent:

(a) $\mathbb{C}$ is action representable and $\mathbb{C}$ has normalizers;

(b) The category of monomorphisms of $\mathbb{C}$ is action representable;

(c) The category of morphisms of $\mathbb{C}$ is action representable;

(d) For each finite category $\mathbb{I}$ the functor category $\mathbb{C}^{\mathbb{I}}$ is action representable.

In [7] a different definition of normalizer of a monomorphism, in an arbitrary category $\mathbb{C}$, was given, which agrees with the one from [9] as soon as $\mathbb{C}$ is a pointed exact protomodular category. For a pointed category with finite limits the existence of normalizers, in the sense of [7], was shown to be equivalent to the functor K being a prefibration on monomorphisms.

The main purpose of this paper is to show that the conditions: (e) K is a prefibration, and (f) every morphism admits a construction generalizing that of normalizer, are equivalent to the conditions (a) - (d) above, and to explain that the equivalence of (c), (d) and (e) is a special case of a general fact about spans of categories (Theorem 3.5), which we will obtain as a corollary of a fact about functors (Theorem 2.24).

2. Prefibrations and terminal objects in fibres

Throughout the paper we will denote by $\mathbf{2}$ the category with two objects 0 and 1 and one non-identity morphism $0 \to 1$. As usual each functor $F : \mathbb{C} \to \mathbb{X}$ induces a functor $F^{\mathbf{2}} : \mathbb{C}^{\mathbf{2}} \to \mathbb{X}^{\mathbf{2}}$ where $\mathbb{C}^{\mathbf{2}}$ and $\mathbb{X}^{\mathbf{2}}$ are the functor categories from $\mathbf{2}$ to $\mathbb{C}$ and $\mathbb{X}$, respectively. The category $\mathbb{C}^{\mathbf{2}}$ can be identified with the category of morphisms in $\mathbb{C}$ and its objects will be written as triples (A, B, f), where A and B are objects and $f : A \to B$ is a morphism in $\mathbb{C}$. A morphism $(A, B, f) \to (A', B', f')$ in $\mathbb{C}^{\mathbf{2}}$ will be written as a pair (u, v), where $u : A \to A'$ and $v : B \to B'$ are morphisms in $\mathbb{C}$ with $f'u = vf$. By the codomain functor from $\mathbb{C}^{\mathbf{2}}$ to $\mathbb{C}$ we will mean the functor sending a morphism to its codomain, that is, the functor sending (A, B, f) to B.

Throughout this section $\mathbb{C}$ and $\mathbb{X}$ denote categories and $F : \mathbb{C} \to \mathbb{X}$ denotes a functor. We compare the conditions:

(i) F is a prefibration;

(ii) The fibres of F have terminal objects;

(iii) The fibres of $F^{\mathbf{2}}$ have terminal objects;

(iv) The fibres of $F^{\mathbb{I}}$ for some finite category $\mathbb{I}$ have terminal objects.

Recall that a morphism $f : A \to B$ in $\mathbb{C}$ is F-precartesian if for each morphism $f' : A' \to B$ such that $F(f') = F(f)$ there exists a unique morphism $u : A' \to A$ such that $fu = f'$ and $F(u) = 1_{F(A)}$. Recall also that F is a prefibration when for each object B in $\mathbb{C}$ and each morphism $\theta : X \to F(B)$ there exists an *F-precartesian lifting* of θ to B, that is, there is an F-precartesian morphism $f : A \to B$ such that $F(f) = \theta$. Note that we will write $F^{-1}(X)$ for the fibre of F above X, that is, the subcategory of $\mathbb{C}$ consisting of those objects and morphisms which get mapped by F to X and 1_X, respectively.

Some of the results below are folklore, however not having an appropriate reference we will give full proofs.

The first part of this section is devoted to comparing (i), (ii) and (iii) above. In particular a condition, weaker than (i), is given whose conjunction with (iii) is equivalent to the conjunction of (i) and (ii). We begin by examining under what conditions a prefibration has terminal objects in its fibres.

Essentially by definition we have:

2.1. Lemma. *For an object B in $\mathbb{C}$, an object X in $\mathbb{X}$, and a morphism $\theta : X \to F(B)$ in $\mathbb{X}$, the functor F admits an F-precartesian lifting of $\theta : X \to F(B)$ to B if and only if the functor $F^B : (\mathbb{C} \downarrow B) \to (\mathbb{X} \downarrow F(B))$, which sends (A, f) to $(F(A), F(f))$, has a terminal object in its fibre ${F^B}^{-1}(X, \theta)$.*

On the other hand we have:

2.2. Lemma. *Suppose F preserves terminal objects and T is a terminal object in $\mathbb{C}$. For an object X in $\mathbb{X}$ there is an F-precartesian lifting of the morphism $X \to F(T)$ to T if and only if the fibre $F^{-1}(X)$ has a terminal object.*

Proof. Since T is a terminal object in $\mathbb{C}$ and F preserves terminal objects it follows that for each object X in $\mathbb{X}$ the first projection ${F^T}^{-1}(X, X \to F(T)) \to F^{-1}(X)$ is an isomorphism. The claim now follows by Lemma 2.1. ∎

As an immediate corollary we obtain:

2.3. Proposition. *Suppose $\mathbb{C}$ has a terminal object. If F preserves terminal objects and is a prefibration, then the fibres of F have terminal objects.*

2.4. Remark. *Since F is a prefibration if and only if for each X in $\mathbb{C}$ the inclusion $F^{-1}(X) \to (X \downarrow F)$ has a right adjoint, one obtains another simple proof of the previous proposition by observing that under the above conditions $(X \downarrow F)$ has a terminal object and hence, since right adjoints preserve limits, so does $F^{-1}(X)$.*

Next we show that if F is a prefibration and has terminal objects in its fibres, then $F^{\mathbf{2}}$ has terminal objects in its fibres. In addition we will also see that the existence of terminal objects in the fibres of $F^{\mathbf{2}}$ implies the existence of terminal objects in the fibres of F.

Since for any category $\mathbb{C}$ the diagonal functor $\Delta : \mathbb{C} \to \mathbb{C}^{\mathbf{2}}$ has both a left and a right adjoint, it follows that if $\mathbb{C}^{\mathbf{2}}$ has a terminal object, then so does $\mathbb{C}$, and the image of a terminal object under Δ is terminal in $\mathbb{C}^{\mathbf{2}}$. As an immediate consequence we obtain:

2.5. Lemma. *For each object X in $\mathbb{X}$, the categories ${F^{\mathbf{2}}}^{-1}(X, X, 1_X)$ and $(F^{-1}(X))^{\mathbf{2}}$ are the same and hence if (A, B, f) is a terminal object in ${F^{\mathbf{2}}}^{-1}(X, X, 1_X)$, then f is an isomorphism and A and B are terminal objects in the fibre $F^{-1}(X)$.*

On the other hand we have:

2.6. Lemma. *Let $\theta : X \to Y$ be a morphism in $\mathbb{X}$. For a morphism $f : A \to B$ in $\mathbb{C}$ such that $F(f) = \theta$ the following conditions are equivalent:*

(a) $F^{-1}(Y)$ has a terminal object and the triple (A, B, f) is a terminal object in ${F^{\mathbf{2}}}^{-1}(X, Y, \theta)$;

(b) B and (A, B, f) are terminal objects in $F^{-1}(Y)$ and ${F^{\mathbf{2}}}^{-1}(X, Y, \theta)$, respectively;

(c) B is a terminal object in $F^{-1}(Y)$ and f is an F-precartesian morphism.

Proof. Suppose that (A, B, f) and $\tilde{B}$ are terminal objects in $F^{-1}(X, Y, \theta)$ and $F^{-1}(Y)$, respectively. It follows that there is a unique morphism $u : B \to \tilde{B}$ in $F^{-1}(Y)$ and hence a unique morphism $(s, t) : (A, \tilde{B}, uf) \to (A, B, f)$ in $F^{-1}(X, Y, \theta)$. However, this means that $ut : \tilde{B} \to \tilde{B}$ and $(s, tu) : (A, B, f) \to (A, B, f)$ are morphisms in $F^{-1}(Y)$ and $F^{-1}(X, Y, \theta)$, respectively, and hence are both identity morphisms. This means that B is isomorphic to $\tilde{B}$ in $F^{-1}(Y)$ proving (a) implies (b).

Now, suppose that (A, B, f) and B are terminal objects in $F^{\mathbf{2}^{-1}}(X, Y, \theta)$ and $F^{-1}(Y)$, respectively. If $f' : A' \to B$ is a morphism in $\mathbb{C}$ such that $F(f') = \theta$, then there exists a unique morphism $(p, q) : (A', B, f') \to (A, B, f)$ in $F^{\mathbf{2}^{-1}}(X, Y, \theta)$. However q being a morphism in $F^{-1}(Y)$ is necessarily 1_B and so f is F-precartesian, proving that (b)$\Rightarrow$(c).

To prove that (c)$\Rightarrow$(a), suppose $f : A \to B$ is F-precartesian morphism and B is a terminal object in $F^{-1}(Y)$. If (A', B', f') is an object in $F^{\mathbf{2}^{-1}}(X, Y, \theta)$, then there exists a unique morphism $t : B' \to B$ in $F^{-1}(Y)$. However this means that $tf' : A' \to B$ is mapped by F to θ and so there exists a unique morphism $s : A' \to A$ such that $F(s) = 1_X$ and $fs = tf'$. It is easy to check that (s, t) is the unique morphism from (A', B', f') to (A, B, f) in $F^{\mathbf{2}^{-1}}(X, Y, \theta)$. ∎

As easy consequences of Lemmas 2.5 and 2.6 we have:

2.7. Proposition. *If F has terminal objects in its fibres and is a prefibration, then $F^{\mathbf{2}}$ has terminal objects in its fibres.*

2.8. Proposition. *If $F^{\mathbf{2}}$ has terminal objects in its fibres, then F has terminal objects in its fibres and the codomain functor sends terminal objects in the fibres of $F^{\mathbf{2}}$ to terminal objects in the fibres of F.*

Combining Proposition 2.3 and 2.7 we obtain:

2.9. Proposition. *Suppose that F preserves terminal objects and $\mathbb{C}$ has a terminal object. If F is a prefibration, then $F^{\mathbf{2}}$ has terminal objects in its fibres.*

It follows from Lemma 2.6 that when $F^{\mathbf{2}}$ has terminal objects in its fibres, certain F-precartesian liftings exist. However, in general one cannot obtain all liftings from this condition alone. For instance, if $\mathbb{C}$ is the (partially) ordered set with elements a, a_1, a_2, b and b_1 and order generated by $a_i \leq a \leq b$ and $a_i \leq b_1 \leq b$, considered as a category, $\mathbb{X} = \mathbf{2}$, and F is the functor with $F(a_1) = F(a_2) = F(a) = 0$ and $F(b) = F(b_1) = 1$, then $F^{\mathbf{2}}$ has terminal objects in its fibres, but F is not a prefibration.

Next we introduce a condition on F, which is weaker than being a prefibration, under which we prove that if $F^{\mathbf{2}}$ has terminal objects in its fibres, then F is a prefibration.

Let $D : \mathbb{G} \to \mathbb{C}$ be a diagram (i.e. a morphism of graphs from a graph $\mathbb{G}$ to the underlying graph of $\mathbb{C}$) and let (X, ϕ) be a cone over FD. We call a cone (A, γ) over D an *F-precartesian lifting* of (X, ϕ) to D if $F(A, \gamma) = (X, \phi)$ (i.e. $F(A) = X$ and $F(\gamma_x) = \phi_x$ for each x in $\mathbb{G}_0$), and for each cone (A', γ') over D such that $F(A', \gamma') = (X, \phi)$, there exists a unique cone morphism $u : (A', \gamma') \to (A, \gamma)$ such that $F(u) = 1_X$. If (X, ϕ) is a limit of FD, then we say that F *weakly creates* the limit (X, ϕ) if there is cone (A, γ)

which is the limit of D and $F(A,\gamma) = (X,\phi)$. Note that F weakly creates the limit (X,ϕ) as soon as D has a limit in $\mathbb{C}$, F preserves limits, and F is an isofibration.

2.10. Remark. *Let $D : \mathbb{G} \to \mathbb{C}$ be a diagram, and let (X,ϕ) be a cone over FD.*

(a) If (X,ϕ) is a limit of FD, then the existence of an F-precartesian lifting of (X,ϕ) to D follows from F weakly creating the limit of (X,ϕ). In fact F weakly creates the limit of (X,ϕ) if and only if (X,ϕ) admits an F-cartesian lifting to D i.e. there exist a cone (A,γ) over D such that $F(A,\gamma) = (X,\phi)$ and for each cone (A',γ') over D together with a cone morphism $v : F(A',\gamma') \to (X,\phi)$ there exist a unique cone morphism $u : (A',\gamma') \to (A,\gamma)$ such that $F(u) = v$;

(b) If $\mathbb{G}$ is a graph with one object x and no morphisms, then the existence of an F-precartesian lifting of (X,ϕ) to D is essentially the same as an F-precartesian lifting of $\phi_x : X \to F(D(x))$ to $D(x)$;

(c) If $\mathbb{G}$ is the empty graph, then the existence of an F-precartesian lifting of (X,ϕ) to D is essentially the same as the existence of a terminal object in the fibre of $F^{-1}(X)$;

(d) If F is a topological functor, then (X,ϕ) admits an F-cartesian lifting to D. For discrete graphs this holds by definition, while for an arbitrary graph, since every topological functor is faithful, such a lifting can be constructed as the lifting of a cone over a discrete diagram.

Essentially by definition we have:

2.11. Lemma. *Let $D : \mathbb{G} \to \mathbb{C}$ be a diagram, $\Delta_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^{\mathbb{G}}$ and $\Delta_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^{\mathbb{G}}$ be the diagonal functors, and (X,ϕ) be a cone over FD. There is an F-precartesian lifting of (X,ϕ) to D if and only if the fibre $F^{D^{-1}}(X,\phi)$ of the functor $F^D : (\Delta_{\mathbb{C}} \downarrow D) \to (\Delta_{\mathbb{X}} \downarrow FD)$ which takes a cone (A,γ) to the cone $(F(A), F\gamma)$ has a terminal object.*

Let us denote by $\mathbb{G}_{\vee}$ the graph with three objects x, y and z and two morphisms $i : x \to z$ and $j : y \to z$. We will be interested in the situation where $D : \mathbb{G}_{\vee} \to \mathbb{C}$ is a diagram and (X,ϕ) is a cone over FD such that $D(i)$ is F-precartesian and $\phi_x = 1_X$. In such a situation if $g : A \to D(y)$ is a morphism in $\mathbb{C}$ such that $F(g) = \phi_y$, then since $F(D(j)g) = F(D(i))$ there is a unique morphism $\gamma_x : A \to D(x)$, shown in the display

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, dashed, "\gamma_x"'] \arrow[dd, mapsto, "F"'] & & D(y) \arrow[dr, "D(j)"] \arrow[dd, mapsto, "F"] & \\
& D(x) \arrow[rr, "D(i)"'] \arrow[dd, mapsto, "F"] & & D(z) \arrow[dd, mapsto, "F"] \\
X \arrow[rr, "\phi_y"] \arrow[dr, equal, "\phi_x"'] & & FD(y) \arrow[dr, "FD(j)"'] & \\
& FD(x) \arrow[rr, "FD(i)"'] & & FD(z),
\end{tikzcd}$$

such that $F(\gamma_x) = \phi_x = 1_X$. One easily checks that the pair (A, γ) with $\gamma_y = g$ and $\gamma_z = D(j)g = D(i)\gamma_x$ is a cone over D such that $F(A, \gamma) = (X, \phi)$. This means that the functor sending (A', γ') in $F^{D^{-1}}(X, \phi)$ (where F^D is defined as in the previous lemma) to (A', γ'_y) in $F^{D(y)^{-1}}(X, \phi_y)$ is an isomorphism.

As an easy corollary of this observation via Lemmas 2.1 and 2.11 we obtain:

2.12. Proposition. *For each diagram $D : \mathbb{G}_\vee \to \mathbb{C}$ and each cone (X, ϕ) over FD such that $D(i)$ is F-precartesian and $\phi_x = 1_X$:*

(a) If $g : A \to D(y)$ is an F-precartesian lifting of ϕ_y to $D(y)$, then there is a unique cone (A, γ) with $\gamma_y = g$ which is an F-precartesian lifting of (X, ϕ) to D.

(b) If (A, γ) is an F-precartesian lifting of (X, ϕ) to D, then γ_y is F-precartesian lifting of ϕ_y to $D(y)$.

2.13. Remark. *Lemma 2.7 of [7] can be recovered from (b) of the previous proposition (see Remark 2.10(a) above).*

2.14. Remark. *According to Lemma 2.6 the previous proposition would remain true if we replaced the assumption "$D(i)$ is F-precartesian" by "$(D(x), D(z), D(i))$ and $D(z)$ are terminal objects in $F^{\mathbf{2}^{-1}}(FD(x), FD(z), FD(i))$ and $F^{-1}(FD(z))$, respectively". In particular, this means that an F-precartesian lifting of $FD(i)$ to an object B in $F^{-1}(FD(z))$ can be obtained as a lifting of a limiting cone to such a diagram D where $D(j)$ is the unique morphism from B to $D(z)$ in $F^{-1}(FD(z))$.*

Consider the successively weaker conditions:

2.15. Condition. *For each diagram $D : \mathbb{G}_\vee \to \mathbb{C}$, and each cone (X, ϕ) over FD*

(a) If $D(i)$ is F-precartesian and $\phi_x = 1_X$, then there exists an F-precartesian lifting of (X, ϕ) to D;

(b) If D and (X, ϕ) are as in (a) and $FD(j)$ is a monomorphism, then there exists an F-precartesian lifting of (X, ϕ) to D.

(c) If D and (X, ϕ) are as in (a) and $FD(j) = 1_{FD(y)}$, then there exists an F-precartesian lifting of (X, ϕ) to D.

2.16. Remark. *Note that Conditions 2.15 (b) and (c) hold when F weakly creates pullbacks. In particular they hold when $\mathbb{C}$ has pullbacks, F preserves pullbacks, and F is an isofibration.*

As an immediate corollary of Proposition 2.12 we obtain:

2.17. Proposition. *If F is prefibration, then F satisfies Conditions 2.15 (a), (b) and (c).*

As a corollary of Proposition 2.12 and Lemma 2.6 we obtain:

2.18. Proposition. *The functor F is a prefibration with terminal objects in its fibres if and only if*

(i) The fibres of $F^{\mathbf{2}}$ have terminal objects, and

(ii) F satisfies Condition 2.15 (a), (b) or (c).

Proof. The "only if" part follows from Propositions 2.7 and 2.17, while the "if" part follows from Proposition 2.8 and Proposition 2.12 via Remark 2.14. ∎

2.19. Corollary. *Suppose that F weakly creates pullbacks. The functor F is a prefibration with terminal objects in its fibres if and only if $F^{\mathbf{2}}$ has terminal objects in its fibres.*

2.20. Proposition. *Let $E : \mathbb{C} \to \mathbb{C}$ be a functor, and let $\kappa : 1_{\mathbb{C}} \to E$ be a natural transformation. The functor F is a prefibration if and only if*

(i) for each object B in $\mathbb{C}$ and for each morphism $\theta : X \to FE(B)$ in $\mathbb{X}$ there exists an F-precartesian lifting of θ to $E(B)$, and

(ii) for each diagram $D : \mathbb{G}_{\vee} \to \mathbb{C}$ and each cone (X, ϕ) over FD such that $D(j) = \kappa_B$ for some B in $\mathbb{C}$, $\phi_x = 1_X$ and $D(i)$ is F-precartesian, there is an F-precartesian lifting of (X, ϕ) to D.

Proof. The "only if" part follows from Proposition 2.17, while the "if" part follows from Proposition 2.12 ∎

2.21. Remark. *Condition 2.20 (ii) follows from F weakly creating pullbacks as soon as the components of κ are monomorphisms (see Remark 2.10(a)).*

Next we show, under certain conditions, that if $F^{\mathbf{2}}$ has terminal objects in its fibres, then for each finite category $\mathbb{I}$ the functor $F^{\mathbb{I}}$ also has terminal objects in its fibres.

2.22. Proposition. *Suppose F weakly creates finite connected limits. If the fibres of the functor $F^{\mathbf{2}}$ have terminal objects, then for each finite category $\mathbb{I}$ the fibres of the functor $F^{\mathbb{I}}$ have terminal objects.*

Proof. Let $\mathbb{I}$ be a finite category. We may assume $\mathbb{I}$ is non-empty since the otherwise the claim is trivially true. We will show that the fibres of the functor $F^{\mathbb{I}}$ have terminal objects. To simplify notation let us write T_X for the terminal object in $F^{-1}(X)$, $T_{(X,Y,\theta)} = (T_\theta, T_Y, p_2)$ for the terminal object in ${F^{\mathbf{2}}}^{-1}(X, Y, \theta)$ and $p_1 : T_\theta \to T_X$ for the unique morphism in $F^{-1}(X)$. Note that this notation is consistent since the codomain functor takes terminal objects in the fibres of $F^{\mathbf{2}}$ to terminal objects in the fibres of F (see Proposition 2.8). For each object x in $\mathbb{I}$, let $\mathbb{G}_x$ be the graph with objects

$$\mathbb{G}_{x_0} = (x \downarrow \mathbb{I})_0 \sqcup (x \downarrow \mathbb{I})_1,$$

with morphisms

$$\mathbb{G}_{x_1} = (x \downarrow \mathbb{I})_1 \times \{1, 2\} \cong (x \downarrow \mathbb{I})_1 \sqcup (x \downarrow \mathbb{I})_1$$

and with domain d and codomain c the unique maps making the diagrams

$$\begin{tikzcd} (x\downarrow\mathbb{I})_1 \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & \mathbb{G}_{x_1} \arrow[d, "d"] & (x\downarrow\mathbb{I})_1 \arrow[l, "\iota_2"'] \arrow[dl, "\iota_2"] \\ & \mathbb{G}_{x_0} & \end{tikzcd} \qquad \begin{tikzcd} (x\downarrow\mathbb{I})_1 \arrow[r, "\iota_1"] \arrow[dr, "\iota_1 d"'] & \mathbb{G}_{x_1} \arrow[d, "c"] & (x\downarrow\mathbb{I})_1 \arrow[l, "\iota_2"'] \arrow[dl, "\iota_1 c"] \\ & \mathbb{G}_{x_0} & \end{tikzcd}$$

(in which the ι_i's are the various coproduct inclusions) commute. Each morphism $j : x' \to x$ induces a functor $j_* : (x \downarrow \mathbb{I}) \to (x' \downarrow \mathbb{I})$ which takes an object (y, f) to (y, fj) and a morphism $h : (y, f) \to (z, g)$ to $h : (y, fj) \to (z, gj)$. This functor in turn induces a graph morphism $G_j : \mathbb{G}_x \to \mathbb{G}_{x'}$ with object map G_{j_0} and morphism map G_{j_1} the unique maps making the diagrams

$$\begin{tikzcd} (x\downarrow\mathbb{I})_0 \arrow[r, "\iota_1"] \arrow[d, "j_{*0}"'] & \mathbb{G}_{x_0} \arrow[d, "G_{j_0}"] & (x\downarrow\mathbb{I})_1 \arrow[l, "\iota_2"'] \arrow[d, "j_{*1}"] \\ (x'\downarrow\mathbb{I})_0 \arrow[r, "\iota_1"] & \mathbb{G}_{x'_0} & (x'\downarrow\mathbb{I})_1 \arrow[l, "\iota_2"'] \end{tikzcd} \qquad \begin{tikzcd} (x\downarrow\mathbb{I})_1 \arrow[r, "\iota_1"] \arrow[d, "j_{*1}"'] & \mathbb{G}_{x_1} \arrow[d, "G_{j_1}"] & (x\downarrow\mathbb{I})_1 \arrow[l, "\iota_2"'] \arrow[d, "j_{*1}"] \\ (x'\downarrow\mathbb{I})_1 \arrow[r, "\iota_1"] & \mathbb{G}_{x'_1} & (x'\downarrow\mathbb{I})_1 \arrow[l, "\iota_2"'] \end{tikzcd}$$

commute. For an object X in $\mathbb{X}^{\mathbb{I}}$ we will construct the terminal object in the fibre $F^{\mathbb{I}^{-1}}(X)$. Let $D_x : \mathbb{G}_x \to \mathbb{C}$ be the diagram defined on objects by

$$D_{x_0}(y, f) = T_{X(y)} \text{ and } D_{x_0}(h : (y, f) \to (z, g)) = T_{X(h)},$$

and on morphisms by

$$D_{x_1}(h : (y, f) \to (z, g), 1) = p_1 \text{ and } D_{x_1}(h : (y, f) \to (z, g), 2) = p_2.$$

Since the images of $p_1 : T_{X(h)} \to T_{X(y)}$ and $p_2 : T_{X(h)} \to T_{X(z)}$ under F are $F(p_1) = 1_{X(y)}$ and $F(p_2) = X(h)$, respectively, it follows that the cone $(X(x), \alpha)$ over FD_x defined for each (y, f) in $\mathbb{G}_{x_0}$ by $\alpha_{(y,f)} = X(f)$ and for each $h : (y, f) \to (z, g)$ in $\mathbb{G}_{x_0}$ by $\alpha_h = X(f)$ is the limit of FD_x. Therefore by assumption there is a cone $(T_X(x), \lambda_x)$ over D_x such that $F\lambda_x = \alpha$ which is the limit of D_x. An easy calculation shows that for each morphism $j : x' \to x$ in $\mathbb{I}$ the diagram

$$\begin{tikzcd} \mathbb{G}_x \arrow[rr, "G_j"] \arrow[dr, "D_x"'] & & \mathbb{G}_{x'} \arrow[dl, "D_{x'}"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes. This commutative diagram induces a canonical morphism

$$T_X(x') \to T_X(x)$$

such that for each (y,f) is $\mathbb{G}_{x_0}$ and each $h:(y,f)\to(z,g)$ in $\mathbb{G}_{x_0}$ the diagram

$$\begin{tikzcd}
 & T_{X(y)} & \\
T_X(x') \arrow[ur, "\lambda_{x'_{G_j(y,f)}}"] \arrow[rr] \arrow[dr, "\lambda_{x'_{G_j(h)}}"'] & & T_X(x) \arrow[ul, "\lambda_{x_{(y,f)}}"'] \arrow[dl, "\lambda_{x_h}"] \\
 & T_{X(h)} &
\end{tikzcd}$$

commutes. It can easily be checked that assigning $T_X(j)$ to be this canonical morphism determines a functor $T_X:\mathbb{I}\to\mathbb{C}$. We will show that this functor, considered as an object in $\mathbb{C}^{\mathbb{I}}$, is a terminal object in $F^{\mathbb{I}^{-1}}(X)$. Let H be an object in $F^{\mathbb{I}^{-1}}(X)$ and let x be an object in $\mathbb{I}$. For each morphism $h:(y,f)\to(z,g)$ in $(x\downarrow\mathbb{I})$ the diagram

$$\begin{tikzcd}
 & H(x) \arrow[d, "H(f)"'] \arrow[dr, "H(g)"] & \\
 & H(y) \arrow[r, "H(h)"'] \arrow[dl, "u_y"'] \arrow[d, "v_h"] & H(z) \arrow[d, "u_z"] \\
T_{X(y)} & T_{X(h)} \arrow[l, "p_1"] \arrow[r, "p_2"'] & T_{X(z)},
\end{tikzcd}$$

in which u_y and u_z are the unique morphisms into the terminal objects in the fibres $F^{-1}(X(y))$ and $F^{-1}(X(z))$, respectively, and (v_h,u_z) is the unique morphism into the terminal object in $F^{\mathbf{2}^{-1}}(X(y),X(z),X(h))$, commutes. It follows that β_x with components $\beta_{x_{(y,f)}}=u_yH(f)$ for each (y,f) in $\mathbb{G}_{x_0}$ and $\beta_{x_h}=v_hH(f)$ for each $h:(y,f)\to(z,g)$ in $\mathbb{G}_{x_0}$ determines a cone $(H(x),\beta_x)$ over D_x and hence a unique morphism $\omega_x:(H(x),\beta_x)\to(T_X(x),\lambda_x)$. Let $j:x'\to x$ be a morphism in $\mathbb{I}$. Since for each (y,f) in $\mathbb{G}_{x_0}$ and for each $h:(y,f)\to(z,g)$ in $\mathbb{G}_{0_x}$ each of the triangles in each of the diagrams

$$\begin{tikzcd}
H(x') \arrow[rr, "\omega_{x'}"] \arrow[dd, "H(j)"'] \arrow[dr, "\beta_{x'_{G_j(y,f)}}"] & & T_X(x') \arrow[dd, "T_X(j)"] \arrow[dl, "\lambda_{x'_{G_j(y,f)}}"'] \\
 & T_{X(y)} & \\
H(x) \arrow[rr, "\omega_x"'] \arrow[ur, "\beta_{x_{(y,f)}}"] & & T_X(x) \arrow[ul, "\lambda_{x_{(y,f)}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H(x') \arrow[rr, "\omega_{x'}"] \arrow[dd, "H(j)"'] \arrow[dr, "\beta_{x'_{G_j(h)}}"] & & T_X(x') \arrow[dd, "T_X(j)"] \arrow[dl, "\lambda_{x'_{G_j(h)}}"'] \\
 & T_{X(h)} & \\
H(x) \arrow[rr, "\omega_x"'] \arrow[ur, "\beta_{x_h}"] & & T_X(x) \arrow[ul, "\lambda_{x_h}"']
\end{tikzcd}$$

commutes, it follows that $\omega=(\omega_x)_{x\in\mathbb{I}}$ is a natural transformation H to T_X. To see that ω lies in the fibre $F^{\mathbb{I}^{-1}}(X)$ note that for each x in $\mathbb{I}$, $\lambda_{x_{(x,1_x)}}\omega_x=u_x$ and hence $F(\omega_x)=F(\lambda_{x_{(x,1_x)}})F(\omega_x)=F(u_x)=1_{X(x)}$. Let ω' be a morphism from H to T_X in $F^{\mathbb{I}^{-1}}(X)$. Since for each (y,f) in $\mathbb{G}_{x_0}$

$$\lambda_{x_{(y,f)}}\omega'_x=\lambda_{y_{(y,1_y)}}T_X(f)\omega'_x=\lambda_{y_{(y,1_y)}}\omega'_yH(f)$$

and $\lambda_{y_{(y,1_y)}}\omega'_y$ is a morphism from $H(y)$ to $T_{X(y)}$ in $F^{-1}(X(y))$ it must be u_y it follows that $\lambda_{x_{(y,f)}}\omega'_x = \beta_{x_{(y,f)}}$. A similar calculation shows for each $h : (y, f) \to (z, g)$ in $\mathbb{G}_{x_0}$ that $\lambda_{x_h}\omega_x = \beta_{x_h}$ proving that $\omega' = \omega$. ∎

2.23. Remark. *Note that in the above proof one could replace weak creation of limits by the existence of F-precartesian liftings of suitable cones.*

Combining Propositions 2.9, 2.18 and 2.22, and Corollary 2.19 we obtain:

2.24. Theorem. *Suppose $\mathbb{X}$ has a terminal object and F weakly creates finite limits. The following conditions are equivalent:*

(a) F is a prefibration;

(b) For each finite category $\mathbb{I}$ the functor $F^{\mathbb{I}}$ is a prefibration;

(c) The fibres of the functor $F^{\mathbf{2}}$ have terminal objects;

(d) For each finite category $\mathbb{I}$ the fibres of $F^{\mathbb{I}}$ have terminal objects.

Proof. The implications (a) ⇒ (c), (c) ⇒ (a), and (c) ⇒ (d) follow from Proposition 2.9, Corollary 2.19 and Proposition 2.22, respectively. Since trivially (b) implies (a), and (d) implies (c), it follows that to complete the proof it is sufficient to prove that (d) implies (b). However since for any finite category $\mathbb{I}$ it follows from (d) that the functor $F^{\mathbb{I}\times\mathbf{2}}$, and hence $(F^{\mathbb{I}})^{\mathbf{2}}$, has terminal objects in its fibres, Proposition 2.18 (applied to $F^{\mathbb{I}}$) implies that $F^{\mathbb{I}}$ is a prefibration since Condition 2.15 (c) holds for $F^{\mathbb{I}}$ because it weakly creates those finite limits which are componentwise. ∎

2.25. Proposition. *Let $\langle I, H, \eta, \epsilon\rangle : \mathbb{C} \to \mathbb{D}$ be an adjunction such that $FHI = F$ and $1_F \circ \eta = 1_F$. If $g : D \to D'$ is an FH-precartesian morphism, then $H(g)$ is an F-precartesian morphism.*

Proof. Suppose that $f : A \to H(D')$ is a morphism in $\mathbb{C}$ such that $F(f) = FH(g)$. By adjunction there exists a morphism $\overline{f} : I(A) \to D'$ such that $H(\overline{f})\eta_A = f$ and hence $FH(\overline{f}) = FH(g)$. It follows that there exists a unique morphism $u : I(A) \to D$ such that $FH(u) = 1_{FH(D)}$ and which makes the diagram

$$\begin{tikzcd} & I(A) \arrow[dl, "u"'] \arrow[dr, "\overline{f}"] & \\ D \arrow[rr, "g"'] & & D' \end{tikzcd} \tag{5}$$

commute. This determines by adjunction a morphism $\underline{u} = H(u)\eta_A$ such that $F(\underline{u}) = 1_{FH(D)}$ making the diagram

$$\begin{tikzcd} & A \arrow[dl, "\underline{u}"'] \arrow[dr, "f"] & \\ H(D) \arrow[rr, "H(g)"'] & & H(D') \end{tikzcd} \tag{6}$$

commute. The uniqueness of $\underline{u}$ follows from the fact that the adjunction $\langle I, H, \eta, \epsilon\rangle$ determines a bijection between diagrams of the form (5) and (6) (with $FH(u) = 1_{FH(D)}$ and $F(\underline{u}) = 1_{FH(D)}$, respectively). ∎

2.26. Proposition. *Let $\langle T, \eta, \mu\rangle$ be a monad on $\mathbb{C}$ such that $FT = F$ and $1_F \circ \eta = 1_F$, let $U^T : \mathbb{C}^T \to \mathbb{C}$ be the forgetful functor from the category of algebras over the monad T to $\mathbb{C}$, let (B, β) be an object in $\mathbb{C}^T$, and let $\theta : X \to F(B)$ be a morphism in $\mathbb{X}$.*

(a) If $f : A \to B$ is an F-precartesian lifting of θ to B, then there exists a unique morphism $\alpha : T(A) \to A$ such that (A, α) is a T-algebra and f is a morphism from (A, α) to (B, β) in $\mathbb{C}^T$ which is an FU^T-precartesian lifting of θ to (B, β);

(b) If $f : (A, \alpha) \to (B, \beta)$ is an FU^T-precartesian lifting of θ to (B, β), then f considered as a morphism from A to B is an F-precartesian lifting of θ to B.

Proof. Note that for each object (C, γ) in $\mathbb{C}^T$, since $F(\eta_C) = 1_{F(C)}$ it follows that $F(\gamma) = F(\gamma)F(\eta_C) = F(\gamma\eta_C) = 1_{F(C)}$.
(a) Let $f : A \to B$ be an F-precartesian morphism such that $F(f) = \theta$. Since as noted above $F(\beta) = 1_{F(B)}$ it follows that $F(\beta T(f)) = F(f) = \theta$. Therefore there exists a unique morphism $\alpha : T(A) \to A$ such that $f\alpha = \beta T(f)$ and $F(\alpha) = 1_X$. The fact that (A, α) is a T-algebra follows from the fact (B, β) is a T-algebra, and from the fact that for each pair of parallel morphisms $u, v : W \to A$ if $F(u) = F(v) = 1_{F(A)}$ and $fu = fv$, then $u = v$. The same fact together with the assumption that f is F-precartesian implies that $f : (A, \alpha) \to (B, \beta)$ is an FU^T-precartesian morphism.
(b) The claim follows directly from Proposition 2.25. ∎

As a corollary we obtain:

2.27. Theorem. *Let $\langle T, \eta, \mu, \rangle$ be a monad on $\mathbb{C}$ such that $FT = F$ and $1_F \circ \eta = 1_F$, and let $U^T : \mathbb{C}^T \to \mathbb{C}$ be as in previous proposition. If F is a prefibration, then FU^T is a prefibration. Furthermore if F weakly creates limits or more generally if F satisfies Condition 2.15 (c), then F is prefibration if and only if FU^T is a prefibration.*

Proof. The first claim is a direct corollary of the previous proposition. The second claim now follows from Proposition 2.20 where $E = T$ and $\kappa = \eta$, since by assumption Condition 2.20 (ii) is satisfied and Condition 2.20 (i) follows from Proposition 2.26 (b). ∎

3. Spans

In this section we study right regular spans of categories. As explained in the introduction, these consist of those spans $\underline{\mathbf{S}} =$

$$\mathbb{A} \xleftarrow{P} \mathbb{S} \xrightarrow{Q} \mathbb{B} \tag{7}$$

satisfying the first of the conditions:

3.1. Condition.

(i) For each object S in $\mathbb{S}$ and each morphism $f : A' \to A$ in $\mathbb{A}$ such that $P(S) = A$ there exists a P-cartesian morphism $s : S' \to S$ in $\mathbb{S}$ such that $P(s) = f$ and $Q(s) = 1_{Q(S)}$;

(ii) The induced functor $\langle P, Q\rangle : \mathbb{S} \to \mathbb{A} \times \mathbb{B}$ reflects isomorphisms;

(iii) The induced functor $\langle P, Q\rangle : \mathbb{S} \to \mathbb{A} \times \mathbb{B}$ is faithful.

Note that Condition 3.1 (iii) follows from Condition 3.1 (ii) when $\mathbb{S}$ has equalizers and both P and Q preserve equalizers.

3.2. Proposition. *Let $\underline{\mathbf{S}}$ be a span satisfying Conditions 3.1 (i) and (ii). Let $s : S' \to S$ be a morphism in $\mathbb{S}$. If $Q(s) = 1_{Q(S)}$, then s is P-cartesian.*

Proof. According to Condition 3.1 (i) there exists a P-cartesian morphism $\overline{s} : \overline{S} \to S$ such that $P(\overline{s}) = P(s)$ and $Q(\overline{s}) = 1_{Q(S)}$. Therefore by the universal property of $\overline{s}$ there exists a unique morphism $u : S' \to \overline{S}$ such that $\overline{s}u = s$ and $P(u) = 1_{P(S)}$. Since $1_{Q(S)} = Q(s) = Q(\overline{s}u) = Q(\overline{s})Q(u) = 1_{Q(S)}Q(u) = Q(u)$ it follows from Condition 3.1 (ii) that u is an isomorphism and hence s is P-cartesian. ∎

3.3. Theorem. *Let $\underline{\mathbf{S}}$ be a span satisfying Conditions 3.1 (i) - (iii). For each object B in $\mathbb{B}$ the category $Q^{-1}(B)$ has a terminal object if and only if the functor $\overline{\underline{\mathbf{S}}}(-, B)$ is representable.*

Proof. Since the representability of $\overline{\underline{\mathbf{S}}}(-, B)$ is equivalent to the existence of a terminal in its category of elements, it follows that the proof will be completed if we show $Q^{-1}(B)$ is equivalent to $\mathrm{Ele}(\overline{\underline{\mathbf{S}}}(-, B))$, the category of elements of $\overline{\underline{\mathbf{S}}}(-, B)$. It follows from Proposition 3.2 that for each morphism $s : S' \to S$ in $Q^{-1}(B)$

$$\overline{\underline{\mathbf{S}}}(P(s), B)([S]) = [S']$$

and hence assigning to each S' in $Q^{-1}(B)$, the pair $(P(S'), [S'])$ in $\mathrm{Ele}(\overline{\underline{\mathbf{S}}}(-, B))$ and assigning to each morphism $s : S' \to S$ in $Q^{-1}(B)$ the morphism $P(s) : (P(S'), [S']) \to (P(S), [S])$ in $\mathrm{Ele}(\overline{\underline{\mathbf{S}}}(-, B))$ determines a functor $H : Q^{-1}(B) \to \mathrm{Ele}(\overline{\underline{\mathbf{S}}}(-, B))$ which is surjective on objects. To see that H is full note that if $f : (P(S'), [S']) \to (P(S), [S])$ is a morphism in $\mathrm{Ele}(\overline{\underline{\mathbf{S}}}(-, B))$, then there exists a P-cartesian lifting $\bar{s} : \bar{S} \to S$ of f to S such that $Q(\bar{s}) = 1_B$ and $[\bar{S}] = [S']$. But, as follows from Condition 3.1 (ii), the equality $[\bar{S}] = [S']$ implies the existence of an isomorphism $u : S' \to \bar{S}$ such that both $P(u)$ and $Q(u)$ are identity morphisms, and hence the image under H of $\bar{s}u : S' \to S$ is f as desired. On the other hand, Condition 3.1 (iii) implies that H is faithful, and hence is an equivalence of categories. ∎

3.4. Theorem. *Let $\underline{\mathbf{S}}$ be a span such that $\mathbb{B}$ has a terminal object and Q weakly creates finite limits. For each object B in $\mathbb{B}$ the following conditions are equivalent:*

(a) The category $Q^{-1}(B)$ has a terminal object;

(b) There is a Q-precartesian lifting of the morphism $B \to Q(1)$ to 1.

(c) There is a Q-precartesian lifting of the morphism $B \to Q(1)$ to S in $Q^{-1}(1)$;

If, in addition, $\underline{\mathbf{S}}$ satisfies Conditions 3.1 (i) - (iii), then these conditions are further equivalent to:

(d) The functor $\overline{\underline{\mathbf{S}}}(-, B)$ is representable.

Proof. The equivalence of (a), (b) and (c) follows from Lemma 2.2 and Lemma 2.7 of [7] (or from Proposition 2.12 via Remark 2.13). The claim now follows from the previous theorem. ∎

It is easy to check that if $\underline{\mathbf{S}}$ is a span satisfying Conditions 3.1 (i) - (iii), then for each category $\mathbb{I}$ the span $\underline{\mathbf{S}^{\mathbb{I}}} =$

$$\mathbb{A}^{\mathbb{I}} \xleftarrow{P^{\mathbb{I}}} \mathbb{S}^{\mathbb{I}} \xrightarrow{Q^{\mathbb{I}}} \mathbb{B}^{\mathbb{I}} \tag{8}$$

also satisfies Conditions 3.1 (i) - (iii). Therefore, combining the previous theorem and Theorem 2.24 we obtain:

3.5. Theorem. *Let $\underline{\mathbf{S}}$ be a span such that $\mathbb{B}$ has a terminal object and Q weakly creates finite limits. The following conditions are equivalent:*

(a) The functor Q is a prefibration;

(b) For each finite category $\mathbb{I}$ the functor $Q^{\mathbb{I}}$ is a prefibration;

(c) The fibres of the functor $Q^{\mathbf{2}}$ have terminal objects;

(d) For each finite category $\mathbb{I}$ the fibres of the functor $Q^{\mathbb{I}}$ have terminal objects.

If, in addition, $\underline{\mathbf{S}}$ satisfies Conditions 3.1 (i) - (iii), then these conditions are further equivalent to:

(a) For each object (B, B', β) in $\mathbb{B}^{\mathbf{2}}$ the functor $\overline{\underline{\mathbf{S}^{\mathbf{2}}}}(-, (B, B', \beta))$ is representable;

(b) For each finite category $\mathbb{I}$ and for each object B in $\mathbb{B}^{\mathbb{I}}$ the functor $\overline{\underline{\mathbf{S}^{\mathbb{I}}}}(-, B)$ is representable.

4. Action representability

In this section, for a semi-abelian category $\mathbb{C}$, we add to the conditions equivalent to $\mathbb{C}^{\mathbf{2}}$ being action representable of Theorem 4.8 of [9], which include the condition that $\mathbb{C}$ is action representable and has normalizers. In addition when $\mathbb{C}$ is a pointed category with finite limits we study conditions equivalent to the existence of *generic split extensions* for $\mathbb{C}^{\mathbf{2}}$. In particular we show that this condition is equivalent to each morphism admitting a universal construction generalizing that of normalizer from [7] and also to the functor K being a prefibration. This last part is related to Theorem 1.8 of [7] where, as explained above, it was already shown that $\mathbb{C}$ has normalizers in the sense described there if and only if K is a prefibration on monomorphisms. Some of the results of this section will be obtained by applying Theorem 3.5 to the span $\underline{\mathbf{SE}(\mathbb{C})}$ =

$$\mathbb{C} \xleftarrow{P} \mathbf{SplExt}(\mathbb{C}) \xrightarrow{K} \mathbb{C} \tag{9}$$

where, as explained in the introduction, $\mathbf{SplExt}(\mathbb{C})$ is the category of split extensions, and K and P are the functors which send a split extension to its kernel and codomain, respectively. As mentioned above, when $\mathbb{C}$ is semi-abelian, the existence of generic split extensions is equivalent to action representability [2]. This fact, up to the equivalence of split extensions and actions, can also be seen as a consequence of Theorem 3.3. Indeed, when $\mathbb{C}$ is pointed protomodular with finite limits, Conditions 3.1 (i) - (iii) hold for the span $\underline{\mathbf{SE}(\mathbb{C})}$, and, as explained in the introduction, the functors

$$\mathrm{SplExt}(-,X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set} \text{ and } \overline{\underline{\mathbf{SE}(\mathbb{C})}}(-,X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$$

are the same. Applying Theorem 3.4 to $\underline{\mathbf{SE}}(\mathbb{C})$ we obtain:

4.1. Theorem. *Let $\mathbb{C}$ be a pointed category with finite limits. The following conditions are equivalent:*

(a) The category $\mathbb{C}$ has generic split extensions;

(b) For each object X in $\mathbb{C}$ there is a K-precartesian lifting of $X \to 0$ to the split extension $0 \longrightarrow 0 \rightleftarrows 0$;

(c) For each object X in $\mathbb{C}$ there is a K-precartesian lifting of $X \to 0$ to each split extension with kernel 0.

When $\mathbb{C}$ is semi-abelian these conditions are further equivalent to:

(d) The category $\mathbb{C}$ is action representable.

To explain more clearly part of the connection between action representability and the existence normalizers, discovered in [9], for a pointed category $\mathbb{C}$ with finite limits we will consider a certain monad on $\mathbf{SplExt}(\mathbb{C})$ and apply Theorem 2.27. For arbitrary morphisms $f : X \to Z$ and $g : Y \to Z$ in a category $\mathbb{C}$ let us denote a chosen pullback as follows

$$\begin{tikzcd} \underset{\langle f,g\rangle}{X \times Y} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd}$$

Let $\mathbf{KGpd}(\mathbb{C})$ be the category with objects 8-tuples $(X, G_0, G_1, d, c, e, m, k)$ consisting of objects and morphisms such that the diagram on the left

$$\begin{tikzcd} \underset{\langle d,c\rangle}{G_1 \times G_1} \arrow[r, "m"] & G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "k"] & G_1 \arrow[r, shift left, "d"] & G_0 \arrow[l, shift left, "e"] \end{tikzcd} \tag{10}$$

is a groupoid and the diagram on the right is a split extension. A morphism with domain $(X, G_0, G_1, d, c, e, m, k)$ and codomain $(X', G'_0, G'_1, d', c', e', m', k')$ is a triple (u, v, w) where $u : X \to X'$, $v : G_1 \to G'_1$ and $w : G_0 \to G'_0$ are morphisms in $\mathbb{C}$ such that the diagram on the left is a functor

$$\begin{tikzcd} \underset{\langle d,c\rangle}{G_1 \times G_1} \arrow[r, "m"] \arrow[d, "v\times v"] & G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "v"] & G_0 \arrow[l, "e" description] \arrow[d, "w"] \\ \underset{\langle d',c'\rangle}{G'_1 \times G'_1} \arrow[r, "m'"] & G'_1 \arrow[r, shift left=2, "d'"] \arrow[r, shift right=2, "c'"'] & G'_0 \arrow[l, "e'" description] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "k"] \arrow[d, "u"] & G_1 \arrow[r, shift left, "d"] \arrow[d, "v"] & G_0 \arrow[l, shift left, "e"] \arrow[d, "w"] \\ X \arrow[r, "k"] & G_1 \arrow[r, shift left, "d"] & G_0 \arrow[l, shift left, "e"] \end{tikzcd}$$

and the diagram on the right is a morphism of split extensions. In [4] (see also [6]) D. Bourn showed that the functor $U : \mathbf{KGpd}(\mathbb{C}) \to \mathbf{SplExt}(\mathbb{C})$ which assigns to each object $(X, G_0, G_1, d, c, e, m, k)$ in $\mathbf{KGpd}(\mathbb{C})$ the split extension on the right in (10) is monadic. In fact he showed more generally that for a category $\mathbb{C}$ with finite limits the forgetful functor from the category of groupoids in $\mathbb{C}$ to the category of split epimorphisms in $\mathbb{C}$, defined in essentially the same way (omitting all kernels), is monadic. Let us write $\langle T, \eta, \mu\rangle$ for this monad and recall how it is defined. The functor T assigns to a split extension

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} \tag{11}$$

the split extension

$$\begin{tikzcd} X \arrow[r, "\langle 0,\kappa\rangle"] & \underset{\langle\alpha,\alpha\rangle}{A \times A} \arrow[r, shift left, "\pi_1"] & A. \arrow[l, shift left, "\langle 1,1\rangle"] \end{tikzcd}$$

The natural transformations η and μ have component at (11) the left and right hand diagrams

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, shift left, "\alpha"] \arrow[d, "\langle\beta\alpha,1\rangle"'] & B \arrow[l, shift left, "\beta"] \arrow[d, "\beta"'] \\ X \arrow[r, "\langle 0,\kappa\rangle"] & \underset{\langle\alpha,\alpha\rangle}{A \times A} \arrow[r, shift left, "\pi_1"] & A \arrow[l, shift left, "\langle 1,1\rangle"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\langle 0,\langle 0,\kappa\rangle\rangle"] \arrow[d, equal] & \underset{\langle\pi_1,\pi_1\rangle}{(\underset{\langle\alpha,\alpha\rangle}{A \times A}) \times (\underset{\langle\alpha,\alpha\rangle}{A \times A})} \arrow[r, shift left, "\pi_1"] \arrow[d, "\pi_2\times\pi_2"] & \underset{\langle\alpha,\alpha\rangle}{A \times A} \arrow[l, shift left, "\langle 1,1\rangle"] \arrow[d, "\pi_2"] \\ X \arrow[r, "\langle 0,\kappa\rangle"] & \underset{\langle\alpha,\alpha\rangle}{A \times A} \arrow[r, shift left, "\pi_1"] & A, \arrow[l, shift left, "\langle 1,1\rangle"] \end{tikzcd}$$

respectively. This means that for the monad $\langle T, \eta, \mu\rangle$ we have $KT = K$ and $1_K \circ \eta = 1_K \circ \mu = 1_K$. Furthermore, it is easy to check that the comparison functor from the category $\mathbf{KGpd}(\mathbb{C})$ to the category of algebras $\mathbf{SplExt}(\mathbb{C})^T$ is an isomorphism. This means, writing U^T for the forgetful functor from the category $\mathbf{SplExt}(\mathbb{C})^T$ to the category $\mathbf{SplExt}(\mathbb{C})$, that the functor KU is prefibration whenever the functor KU^T is.

4.2. Theorem. *Let $\mathbb{C}$ be a pointed category with finite limits. The following conditions are equivalent:*

(a) The category $\mathbb{C}^{\mathbf{2}}$ has generic split extensions;

(b) For each finite category $\mathbb{I}$ the category $\mathbb{C}^{\mathbb{I}}$ has generic split extensions;

(c) The functor K is a prefibration;

(d) For each finite category $\mathbb{I}$ the functor $K^{\mathbb{I}}$ is a prefibration;

(e) The functor KU is a prefibration;

(f) For each finite category $\mathbb{I}$ the functor $K^{\mathbb{I}}U^{\mathbb{I}}$ is a prefibration;

(g) For each morphism $f : X' \to X$ there is an K-precartesian lifting of f to the split extension

$$X \xrightarrow{\langle 0,1\rangle} X \times X \underset{\langle 1,1\rangle}{\overset{\pi_1}{\rightleftarrows}} X; \tag{12}$$

(h) For each morphism $f : X' \to X$ there exists a groupoid

$$\begin{tikzcd} G_1 \underset{\langle d,c\rangle}{\times} G_1 \arrow[r, "m"] & G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description] \end{tikzcd} \tag{13}$$

together with morphisms $k : X' \to G_1$ and $h : G_0 \to X$ such that k is the kernel of d and $hck = f$, which is universal amongst such triples.

When in addition $\mathbb{C}$ is semi-abelian, these conditions are further equivalent to:

(i) For each morphism $f : X' \to X$ there exists a pair $((N, X', \zeta, g), h)$ such that (N, X', ζ, g) is an internal crossed module (in the sense of G. Janelidze in [11]) and $h : N \to X$ is a morphism such that $hg = f$, which is universal amongst such;

(j) The category $\mathbb{C}$ is action representable and $\mathbb{C}$ has normalizers;

(k) The category of monomorphisms in $\mathbb{C}$ is action representable;

(l) The category $\mathbb{C}^{\mathbf{2}}$ is action representable;

(m) For each finite category $\mathbb{I}$ the category $\mathbb{C}^{\mathbb{I}}$ is action representable.

Proof. Applying Theorem 3.5 to the span $\mathbf{SE}(\mathbb{C})$ we find that (a), (b), (c) and (d) are equivalent. Next we will prove that (c) is equivalent to (g). To do so let $E : \mathbf{SplExt}(\mathbb{C}) \to \mathbf{SplExt}(\mathbb{C})$ be the functor which sends a split extension (11) to the split extension at the bottom of the diagram

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, "\kappa"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "\langle \beta\alpha, 1\rangle"] & B \arrow[l, shift left, "\beta"] \arrow[d, "\beta"] \\ A \arrow[r, "\langle 0,1\rangle"] & A \times A \arrow[r, shift left, "\pi_1"] & A, \arrow[l, shift left, "\langle 1,1\rangle"] \end{tikzcd} \tag{14}$$

and let $\kappa : 1_{\mathbf{SplExt}(\mathbb{C})} \to E$ be the natural transformation with component at (11) the morphism of split extensions (14). According to Remarks 2.10 (a) and 2.21 and Proposition 2.20, where $F = K$ and E and κ are as above, it follows that (c) and (g) are equivalent. Since when $\mathbb{C}$ is semi-abelian so is $\mathbb{C}^{\mathbf{2}}$, it follows that (l) and (a) are equivalent. The equivalence of (c) and (e) follows from Theorem 2.27 via the remarks proceeding the theorem.

Therefore since (e) and (f) are equivalent by Theorem 2.24, (j), (k), (l) and (m) by Theorem 4.8 of [9], and (h) and (i) are reformulations of each other when $\mathbb{C}$ is semi-abelian (as follows from [11]), to complete the proof we will show that (g) and (h) are equivalent. Note that each split extension of the form (12) is part of the data describing the indiscrete groupoid on X (together with the kernel of its domain) and that the data given in (h) is essentially the same as a KU-precartesian lifting of f to the indiscrete groupoid on X (with chosen kernel of its domain). The equivalence of (g) and (h) follows from these observations and Proposition 2.26. ∎

Let us specifically highlight the fact that Theorem 4.2 characterizes action representability in 'good' categories in standard Grothendieck terms, as follows:

4.3. Corollary. *A semi-abelian category with normalizers is action representable if, and only if, the functor assigning to a split extension its kernel is a prefibration.*

4.4. Remark. *As mentioned above, in [7] a definition of the normalizer of a monomorphism in an arbitrary category $\mathbb{C}$ was given, which is different to the definition in [9] even when $\mathbb{C}$ is pointed. Indeed, when $\mathbb{C}$ is pointed and has finite limits the normalizer of a monomorphism $f : X' \to X$ in the sense of [7] can equivalently be defined as a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\kappa"] \arrow[d, "n"] \arrow[dd, bend right, "f"'] & R \arrow[d, "\langle r_1, r_2\rangle"] \\ N \arrow[r, "\langle 0,1\rangle"] \arrow[d, "m"] & N \times N \\ X & \end{tikzcd}$$

where the upper square is a pullback and the morphisms $r_1, r_2 : R \to N$ are the projections of an equivalence relation, which is universal amongst such commutative diagrams. It can *be seen that the morphisms m and n are necessarily a monomorphism (see Proposition 2.3 [7]) and a Bourn-normal monomorphism, respectively, and furthermore that n is a normal monomorphism (i.e. the kernel of some morphism) when $\mathbb{C}$ is exact.*

Since by Proposition 2.26, the data in (h) determines a K-precartesian morphism

$$\begin{tikzcd} X' \arrow[r, "k"] \arrow[d, "f"'] & G_1 \arrow[r, shift left, "d"] \arrow[d, "\langle hd{,}hc\rangle"] & G_0 \arrow[l, shift left, "d"] \arrow[d, "h"] \\ X \arrow[r, "\langle 0{,}1\rangle"] & X \times X \arrow[r, shift left, "\pi_1"] & X; \arrow[l, shift left, "\langle 1{,}1\rangle"] \end{tikzcd}$$

it can be checked that when f is a monomorphism so are both h and $\langle hd, hc\rangle$, and so the groupoid (13) is an equivalence relation corresponding to the normal subobject ck. From this it follows that the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "k"] \arrow[d, "ck"] \arrow[dd, bend right, "f"'] & G_1 \arrow[d, "\langle d{,}c\rangle"] \\ G_0 \arrow[r, "\langle 0{,}1\rangle"] \arrow[d, "h"] & G_0 \times G_0 \\ X & \end{tikzcd}$$

is the normalizer of f in sense of [7]. This means that when f is an arbitrary morphism the data in (h) or equivalently (i), when $\mathbb{C}$ is semi-abelian, can be thought of as a generalization of the normalizer to an arbitrary morphism. It is easy to check that for a semi-abelian category $\mathbb{C}$ the existence of these "generalized normalizers" is equivalent to requiring that for each object X in $\mathbb{C}$ the forgetful functor from the category of crossed modules with objects of the form (B, X, ζ, f) to the category $(X \downarrow \mathbb{C})$, sending (B, X, ζ, f) to (B, f) has a right adjoint.

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