Fixed point theorems for Ćirić type generalized contractions defined on cyclic representations

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1 Introduction

One of the consistent generalization of the Contraction Principle was given in 2003 by Kirk, Srinivasan and Veeramani, using the concept of cyclic operator. More precisely, they proven in [2] the following result.

Theorem 1.1 ([2, Theorem 2.4]) Let \( \{A_i\}_{i=1}^{m} \) be nonempty subsets of a complete metric space and suppose \( f : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i \) satisfies the following conditions (where \( A_{m+1} = A_1 \)):

1. \( f(A_i) \subseteq A_{i+1} \) for \( 1 \leq i \leq m \);
2. \( d(f(x),f(y)) \leq \psi(d(x,y)), \forall x \in A_i, \forall y \in A_{i+1}, \) for \( 1 \leq i \leq m \),

where \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is upper semi-continuous from the right and satisfies \( 0 \leq \varphi(t) < t \) for \( t > 0 \).

Then \( f \) has a unique fixed point.

The purpose of this paper is to investigate the properties of some Ćirić type generalized contractions defined on cyclic representations in a metric space. Our main results generalize some similar theorems given for Banach, Kannan, Bianchini, Reich, Sehgal, Chatterjea and Zamfirescu type operators (see [4], [10]), in the case of a cyclic condition (see [5]). Also, the main result is a generalization of Theorem 2.1 given by Păcurar and Rus in [3].
2 Preliminaries

We present here some notions and results which will be used in our main section.

There are several conditions upon the comparison function that have been considered in literature. In order to study the convergence of the Picard iteration \( \{x_n\}_{n \geq 0} \) defined by

\[
x_n = f(x_{n-1}), \quad n \geq 1
\]

in this paper we shall refer only to:

**Definition 2.1** ([9]) A function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is called a comparison function if it satisfies:

(i) \( \varphi \) is increasing;

(ii) \( \varphi \) (\( \varphi^n(t) \)\( n \in \mathbb{N} \) converges to 0 as \( n \to \infty \), for all \( t \in \mathbb{R}^+ \).

If the condition (ii) is replaced by the condition:

(iii) \( \sum_{k=0}^{\infty} \varphi^k(t) < \infty \), for any \( t > 0 \),

then \( \varphi \) is called a strong comparison function.

**Lemma 2.1** ([6]) If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a comparison function, then the following hold:

(i) \( \varphi(t) < t \), for any \( t > 0 \);

(ii) \( \varphi(0) = 0 \);

(iii) \( \varphi \) is continuous at 0.

**Lemma 2.2** ([3], [9]) If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a strong comparison function, then the following hold:

(i) \( \varphi \) is a comparison function;

(ii) the function \( s : \mathbb{R}^+ \to \mathbb{R}^+ \), defined by

\[
s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \quad t \in \mathbb{R}^+,
\]

is increasing and continuous at 0.
(iii) there exist \( k_0 \in \mathbb{N}, a \in (0,1) \) and a convergent series of nonnegative terms \( \sum_{k=1}^{\infty} v_k \) such that

\[
\varphi^{k+1}(t) \leq a \varphi^k(t) + v_k, \text{ for } k \geq k_0 \text{ and any } t \in \mathbb{R}_+.
\]

**Remark 2.1** Some authors use the notion of \((c)\)-comparison function defined by the statements (i) and (iii) from Lemma 2.2. Actually, the concept of \((c)\)-comparison function coincides with that of strong comparison function.

**Example 2.1**

(1) \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+, \varphi(t) = at, \) where \( a \in [0,1[ \), is a strong comparison function.

(2) \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+, \varphi(t) = \frac{t}{1+t} \) is a comparison function, but is not a strong comparison function.

(3) \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+, \varphi(t) = \frac{1}{2} t, \) for \( t \in [0,1] \) and \( \varphi(t) = t - \frac{1}{2}, \) for \( t > 1, \) is a strong comparison function.

(4) \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+, \varphi(t) = at + \frac{1}{2} \lfloor t \rfloor, \) where \( a \in \mathbb{R}, \frac{1}{2} \) is a strong comparison function.

For more considerations on comparison functions see [6], [8] and the references therein.

Let \((X,d)\) be a metric space. \( P(X) \) denotes the collection of nonempty subsets of \( X, \) and \( P_{cl}(X) \) denotes the collection of nonempty and closed subsets of \( X. \)

We recall the following notion, introduced in [7], suggested by the considerations in [2].

**Definition 2.2** Let \( X \) be a nonempty set, \( m \) a positive integer and \( f : X \to X \) an operator. By definition, \( \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( f \) if

(i) \( X = \bigcup_{i=1}^{m} A_i, \) with \( A_i \in P(X), \) for \( 1 \leq i \leq m; \)

(ii) \( f(A_i) \subseteq A_{i+1}, \) for \( 1 \leq i \leq m, \) where \( A_{m+1} = A_1.\)
3 Main results

We start this section by presenting the notion of cyclic $\varphi$-contraction of Ćirić type.

**Definition 3.1** Let $(X,d)$ be a metric space, $m$ a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y \in P(X)$ and $f : Y \to Y$ an operator. If

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of $Y$ with respect to $f$;

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

\[
d(f(x), f(y)) \leq \varphi \left( \max\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\} \right),
\]

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$,

then, by definition, we say that $f$ is a cyclic $\varphi$-contraction of Ćirić type.

The main result of this paper is the following.

**Theorem 3.1** Let $(X,d)$ be a complete metric space, $m$ a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y \in P(X)$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function, and $f : Y \to Y$ be an operator. Assume that $f$ is a cyclic $\varphi$-contraction of Ćirić type.

Then:

(1) $f$ has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $\{x_n\}_{n \geq 0}$ given by (2.1) converges to $x^*$ for any starting point $x_0 \in Y$;

(2) the following estimates hold:

\[
d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \ n \geq 1;
\]

\[
d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \ n \geq 1;
\]

(3) for any $x \in Y$:

\[
d(x, x^*) \leq s(d(x, f(x))),
\]

where $s$ is given by (2.2) in Lemma 2.2;
\[ \sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty, \text{ i.e., } f \text{ is a good Picard operator}; \]
\[ \sum_{n=0}^{\infty} d(x_n, x^*) < \infty, \text{ i.e., } f \text{ is a special Picard operator}. \]

Proof. (1) Let \( x_0 \in Y, x_n = f(x_{n-1}), \) for \( n \geq 1. \) Then we have:
\[ d(f(x_{n-1}), f(x_n)) \leq \varphi \left( \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1}) \right\} \right). \]

(3.1)

Using the triangle inequality,
\[ \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \]
\[ \leq \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \]
the inequality becomes:
\[ d(x_n, x_{n+1}) \leq \varphi(\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \]

Supposing that there exists \( p \in \mathbb{N}, p \geq 1 \) such that
\[ d(x_{p-1}, x_p) \leq d(x_p, x_{p+1}), \]
and taking in consideration that \( \varphi \) is a comparison function, from (3.1) we have:
\[ d(x_p, x_{p+1}) \leq \varphi(d(x_p, x_{p+1})) < d(x_p, x_{p+1}), \]
which is a contradiction.

So, \( d(x_{n-1}, x_n) > d(x_n, x_{n+1}), \) for any \( n \geq 1 \) and (3.1) becomes
\[ d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)). \]

(3.2)

Using the monotonicity of \( \varphi, \) we get
\[ d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)). \]

(3.3)

For \( p \geq 1, \) we have
\[ d(x_n, x_{n+p}) \leq \varphi^n(d(x_0, x_1)) + \varphi^{n+1}(d(x_0, x_1)) + \ldots + \varphi^{n+p-1}(d(x_0, x_1)), \]

(3.4)
and denoting $S_n := \sum_{k=0}^{n} \varphi^k(d(x_0, x_1))$, 
\[ d(x_n, x_{n+p}) \leq S_{n+p-1} - S_{n-1}. \]  
(3.5)

As $\varphi$ is a strong comparison function,
\[ \sum_{k=0}^{\infty} \varphi^k(d(x_0, x_1)) < \infty, \]
so there is $S \in \mathbb{R}_+$ such that $\lim_{n \to \infty} S_n = S$.

Using (3.5), $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$, which means that $(x_n)_{n \geq 0}$ is a Cauchy sequence in the complete subspace $Y$. So the sequence $(x_n)_{n \geq 0}$ is convergent to a $p \in Y$.

The sequence $\{x_n\}_{n \geq 0}$ has an infinite number of terms in each $A_i$, $i = 1, \ldots, m$, so from each $A_i$ one, we can extract a subsequence of $\{x_n\}_{n \geq 0}$ which converges to $p = \lim_{n \to \infty} x_n$.

Because $A_i$ are closed, it follows $p \in \bigcap_{i=1}^{m} A_i$. So $\bigcap_{i=1}^{m} A_i \neq \emptyset$. We consider the restriction
\[ f\big|_{\bigcap_{i=1}^{m} A_i} : \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i. \]
$\bigcap_{i=1}^{m} A_i$ is also complete. Using Theorem 1.5.1 from [1], $f\big|_{\bigcap_{i=1}^{m} A_i}$ has a unique fixed point $x^*$, which can be obtained by means of the Picard iteration starting from any initial point.

We still have to prove that the Picard iteration converges to $x^*$ for any initial guess $x \in Y$.
\[ d(x_{n+1}, x^*) = d(f(x_n), f(x^*)) \leq \varphi \left( \max \left\{ d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)), \frac{1}{2} \left[ d(x_n, f(x^*)) + d(x^*, x_{n+1}) \right] \right\} \right). \]

If we denote $a_n = d(x_n, x^*)$, $n \in \mathbb{N}$, then we have
\[ a_{n+1} \leq \varphi \left( \max \left\{ a_n, d(x_n, x_{n+1}), 0, \frac{1}{2} (a_n + a_{n+1}) \right\} \right). \]
Using $\frac{1}{2}(a_n + a_{n+1}) \leq \max\{a_n, a_{n+1}\}$, we have
\[
a_{n+1} \leq \varphi(\max\{a_n, a_{n+1}, d(x_n, x_{n+1})\}) .
\]
But $\max\{a_n, a_{n+1}, d(x_n, x_{n+1})\} \neq a_{n+1}$, otherwise $a_{n+1} \leq \varphi(a_{n+1})$, which is a contradiction with $\varphi(t) < t$, for any $t > 0$. We get
\[
a_{n+1} \leq \varphi(\max\{a_n, d(x_n, x_{n+1})\}), \text{ for any } n \in \mathbb{N} . \tag{3.6}
\]

The following cases need to be analysed:

• There exists a positive integer $k$ such that $a_k < d(x_k, x_{k+1})$.
  
  For $n = k$, the inequality (3.6) becomes
  \[
a_{k+1} \leq \varphi(d(x_k, x_{k+1})).
\]
  For $n = k + 1$, using (3.2), the inequality (3.6) becomes
  \[
  a_{k+2} \leq \varphi(\max\{a_{k+1}, d(x_{k+1}, x_{k+1})\})
  \leq \varphi(\max\{a_{k+1}, \varphi(d(x_k, x_{k+1}))\})
  \leq \varphi^2(d(x_k, x_{k+1})).
  \]
  By induction, we obtain
  \[
a_{k+p} \leq \varphi^p(d(x_k, x_{k+1})) \tag{3.7}
\]
  and by letting $p \to \infty$, the sequence $\{a_n\}_{n \geq 0}$ converges to 0.

• For any $n \in \mathbb{N}^*$, $a_n \geq d(x_n, x_{n+1})$.
  
  The inequality (3.6) becomes
  \[
a_{n+1} \leq \varphi(a_n), \text{ for any } n \in \mathbb{N}^*,
\]
  so $a_n \leq \varphi^n(a_0)$, which implies again $a_n \to 0$, as $n \to \infty$.

(2) By letting $p \to \infty$ in (3.3), we obtain the a priori estimate
\[
d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \text{ for any } n \geq 1.
\]

Using (3.2) and the monotonicity of $\varphi$,
\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p})
\leq \sum_{k=0}^{p-1} \varphi^k(d(x_n, x_{n+1})),
\]
and letting \( p \to \infty \),

\[
d(x_n, x^*) \leq \sum_{k=0}^{\infty} \varphi^k(d(x_n, x_{n+1})), \quad n \geq 0,
\]

(3.8)

which considering the definition of \( s \) yields the a posteriori estimate

\[
d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \quad \text{for any } n \geq 1.
\]

(3) Let \( x \in Y \). From (3.6), for \( x_0 := x \) we have:

\[
d(x, x^*) \leq \sum_{k=0}^{\infty} \varphi^k(d(x, f(x))).
\]

(4) Using the inequality (3.3),

\[
\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \varphi^n(d(x_0, x_1)) = s(d(x_0, x_1)) < \infty.
\]

(5) We use the inequality (3.6), i.e., \( a_{n+1} \leq \varphi(\max\{a_n, d(x_n, x_{n+1})\}) \), for any \( n \in \mathbb{N} \), where \( a_n := d(x_n, x^*) \). It needs to be discussed two cases:

- If there exists \( k \in \mathbb{N} \) such that \( a_k < d(x_k, x_{k+1}) \), then the inequality (3.7), i.e., \( a_k + p \leq \varphi^p(d(x_k, x_{k+1})) \) holds for any \( p \in \mathbb{N} \). Then

\[
\sum_{n=k+1}^{\infty} a_k \leq \sum_{n=k+1}^{\infty} \varphi^n(d(x_k, x_{k+1})) < \infty,
\]

so

\[
\sum_{n=0}^{\infty} d(x_n, x^*) < \infty.
\]

- If \( a_n \geq d(x_n, x_{n+1}) \), for any \( n \in \mathbb{N} \), then (3.6) becomes

\[
a_{n+1} \leq \varphi(a_n), \quad \text{for any } n \in \mathbb{N}
\]

which implies \( a_n \leq \varphi^n(a_0) \).

Then

\[
\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} \varphi^n(a_0) < \infty,
\]

so again

\[
\sum_{n=0}^{\infty} d(x_n, x^*) < \infty.
\]
Theorem 3.2 Let \( f : Y \to Y \) be as in Theorem 3.1. Then the fixed point problem for \( f \) is well posed, that is, assuming there exist \( z_n \in Y, n \in \mathbb{N} \) such that
\[
d(z_n, f(z_n)) \to 0 \text{ as } n \to \infty,
\]
this implies that
\[
z_n \to x^* \text{ as } n \to \infty,
\]
where \( F_f = \{ x^* \} \).

Proof. Using the inequality \( d(x, x^*) \leq s(d(x, f(x))) \) from Theorem 3.1, for \( x := z_n \), we have:
\[
d(z_n, x^*) \leq s(d(z_n, f(z_n))), n \in \mathbb{N},
\]
and letting \( n \to \infty \) we obtain
\[
d(z_n, x^*) \to 0, n \to \infty.
\]

Theorem 3.3 Let \( f : Y \to Y \) be as in Theorem 3.1, and \( g : Y \to Y \) such that:
1. \( g \) has at least one fixed point \( x_g^* \in F_g \);
2. there exists \( \eta > 0 \) such that
\[
d(f(x), g(x)) \leq \eta, \text{ for any } x \in X.
\]
Then \( d(x_f^*, x_g^*) \leq s(\eta) \), where \( F_f = \{ x_f^* \} \) and \( s \) is defined in Lemma 2.2.

Proof. By letting \( x := x_g^* \) in the inequality \( d(x, x^*) \leq s(d(x, f(x))) \), we have
\[
d(x_f^*, x_g^*) \leq s(d(x_g^*, f(x_g^*))) = s(d(g(x_g^*), f(x_g^*)�).
\]
Using the monotonicity of \( s \) we obtain \( d(x_f^*, x_g^*) \leq s(\eta) \).

Theorem 3.4 Let \( f : Y \to Y \) be as in Theorem 3.1 and \( f_n : Y \to Y, n \in \mathbb{N} \) such that:
1. for each \( n \in \mathbb{N} \) there exists \( x_n^* \in F_{f_n} \);
2. \( \{ f_n \}_{n \in \mathbb{N}} \) converges uniformly to \( f \).
Then \( x_n^* \to x^* \) as \( n \to \infty \), where \( F_f = \{ x^* \} \).
Proof. As \( \{ f_n \}_{n \geq 0} \) converges uniformly to \( f \), there exists \( \eta_n \in \mathbb{R}_+ \), \( n \in \mathbb{N} \) such that \( \eta_n \to 0 \), \( n \to \infty \) and \( d(f_n(x), f(x)) \leq \eta_n \), for any \( x \in Y \).

Using Theorem 3.3 for \( g := f_n, n \in \mathbb{N} \), we have

\[
d(x^*_n, x^*) \leq s(\eta_n), \quad n \in \mathbb{N}.
\]

By letting \( \eta_n \to 0 \) as \( n \to \infty \), we get \( d(x_n, x^*) \to 0 \).

The following theorem is a Maia type result regarding Ćirić type generalized contractions defined on cyclic representations.

**Theorem 3.5** Let \( X \) be a nonempty set, \( d \) and \( \rho \) two metric on \( X \), \( m \) a positive integer, \( A_1, \ldots, A_m \in P_c(X), Y \in P(X) \) and \( f : Y \to Y \) be an operator. Assume that:

(i) there exists \( c > 0 \) such that \( d(x, y) \leq c \cdot \rho(x, y) \), for any \( x, y \in Y \);

(ii) \( (Y, d) \) is a complete metric space;

(iii) \( f : (Y, d) \to (Y, d) \) is continuous;

(iv) \( f : (Y, \rho) \to (Y, \rho) \) is a cyclic \( \varphi \)-contraction of Ćirić type.

Then \( f \) has a unique fixed point \( x^* \in \bigcap_{i=1}^m A_i \) and the Picard iteration \( \{ x_n \}_{n \geq 0} \) given by (2.1) converges to \( x^* \) for any starting point \( x_0 \in Y \).

**Proof.** By the same reasoning as in Theorem 3.1, using condition (iv), we obtain that \( \{ x_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( (X, \rho) \).

Using condition (i) it follows that it is Cauchy in \( (X, d) \) as well.

By (ii) and (iii) it is easy to prove that \( \{ x_n \}_{n \in \mathbb{N}} \) converges to \( (X, d) \) to the unique fixed point of \( f \).

**Remark 3.1** It is an open problem in which conditions the operator \( f : Y \to Y \) as in Theorem 3.1 has the limit shadowing property, that is, assuming that there exist \( z_n \in Y, n \in \mathbb{N} \) such that \( d(z_{n+1}, f(z_n)) \to 0 \) as \( n \to \infty \), then there exists \( x \in Y \) such that

\[
d(z_n, f^n(x)) \to 0 \text{ as } n \to \infty.
\]
References


